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CHAPTER 2

Skew abelian groups

We classify up to isomorphism triples (A, C, β), where A is a finite abelian group, C is an abelian group, and $\beta : A \times A \to C$ is an antisymmetric perfect pairing. We call these triples *skew abelian groups* and their classification is Theorem 2.28. The main reference, which we summarize and link to our theorems in Section 2.6, is [73] by Wall. Similar results are given in [57] by Poonen and Stoll for the nondivisible part of the Shafarevich–Tate group of an abelian variety over a global field.

2.1 Preliminaries

We state definitions and theorems that are useful for understanding the classification in Section 2.2. Abelian groups are considered with additive notation. As a reference for some definitions and basic results see [32] by Lang.

Definition 2.1 (Pairing or bilinear map)**.** Let A, B, and C be abelian groups. A *pairing* or *bilinear map* from $A \times B$ to C is a map

$$\beta : A \times B \to C,$$

such that for each $a \in A$ the function $B \to C$, $x \mapsto \beta(a, x)$, is a group homomorphism and, similarly, for each $b \in B$ the function $A \to C$, $x \mapsto \beta(x, b)$, is a group homomorphism.

Theorem 2.2. *Let A, B, and C be abelian groups and let $\beta : A \times B \to C$ be a map. Then the following are equivalent.*
 (i) *The map $\beta : A \times B \to C$ is a pairing.*
 (ii) *There is a group homomorphism $A \to \mathrm{Hom}(B, C)$ given by $a \mapsto \beta(a, \cdot)$.*
 (iii) *There is a group homomorphism $B \to \mathrm{Hom}(A, C)$ given by $b \mapsto \beta(\cdot, b)$.*

Proof. See Proposition 5.1 of Chapter XI in [22] by Grillet. □

Definition 2.3 (Perfect pairing)**.** Let A, B, and C be abelian groups. A pairing $\beta : A \times B \to C$ is a *perfect pairing* if the group homomorphisms $A \to \mathrm{Hom}(B, C)$, $a \mapsto \beta(a, \cdot)$, and $B \to \mathrm{Hom}(A, C)$, $b \mapsto \beta(\cdot, b)$, are group isomorphisms.

Definition 2.4 (Alternating pairing, antisymmetric pairing)**.** Let A and C be abelian groups. A pairing $\beta : A \times A \to C$ is
 (a) *alternating* if $\beta(a, a) = 0$ for all $a \in A$,
 (b) *antisymmetric* if $\beta(a, b) = -\beta(b, a)$ for all $a, b \in A$.

Theorem 2.5. *If a pairing is alternating, then it is antisymmetric.*

Proof. Let $\beta : A \times A \to C$ be an alternating pairing. For all $a, b \in A$ we have

$$0 = \beta(a + b, a + b) = \beta(a, a) + \beta(a, b) + \beta(b, a) + \beta(b, b) = \beta(a, b) + \beta(b, a).$$

Hence, the pairing is also antisymmetric. □

Definition 2.6 (Symplectic abelian group)**.** A *symplectic abelian group* is a triple (A, C, β), where A is a finite abelian group, C is an abelian group, and $\beta : A \times A \to C$ is an alternating perfect pairing.

Definition 2.7 (Skew abelian group)**.** A *skew abelian group* is a triple (A, C, β), where A is a finite abelian group, C is an abelian group, and $\beta : A \times A \to C$ is an antisymmetric perfect pairing.

Remark 2.8. By Theorem 2.5 every symplectic abelian group is a skew abelian group.

Definition 2.9 (Similarity of skew abelian groups)**.** Let the triples (A, C, β) and (B, D, γ) be skew abelian groups. Let C' and D' be the groups generated by the images of β in C and of γ in D, respectively. A *similarity of skew abelian groups* is a pair (φ, ψ) of group isomorphisms $\varphi : A \to B$ and $\psi : C' \to D'$ such that for all $x, y \in A$ one has $\psi(\beta(x, y)) = \gamma(\varphi(x), \varphi(y))$.

The diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{\quad \beta \quad} & C' \\ {\scriptstyle \varphi}\downarrow \;\; \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ B \times B & \xrightarrow{\quad \gamma \quad} & D' \end{array}$$

visualizes Definition 2.9.

Definition 2.10 (Isomorphism of skew abelian groups)**.** Let (A, C, β) and (B, C, γ) be skew abelian groups. An *isomorphism of skew abelian groups* is a group isomorphism $\varphi : A \to B$ such that for all $x, y \in A$ one has $\beta(x, y) = \gamma(\varphi(x), \varphi(y))$.

Using Remark 2.8 we give similar definitions for symplectic abelian groups. We say that two symplectic abelian groups are *similar* (*isomorphic*) if they are similar (isomorphic) as skew abelian groups.

Theorem 2.11. *Let (A, C, β) be a skew abelian group. Then there exists a unique $g \in A$ such that $\beta(g, a) = \beta(a, a)$ for all $a \in A$.*

Proof. See Theorem 2.42. □

Definition 2.12 (Skew element)**.** Let (A, C, β) be a skew abelian group. The *skew element* of (A, C, β) is the element $g \in A$ such that $\beta(g, a) = \beta(a, a)$ for all $a \in A$.

Theorem 2.13. *Let (A, C, β) be a skew abelian group. Then the skew element of (A, C, β) has order dividing 2.*

Proof. Let $g \in A$ be the skew element of (A, C, β). For all $a \in A$ we have

$$\beta(2g, a) = \beta(g, a) + \beta(g, a) = \beta(a, a) + \beta(a, a) = 0,$$

because $\beta(g, a) = \beta(a, a)$. Since the group homomorphism $\beta(2g, \cdot) : A \to C$ is trivial and by definition of perfect pairing the map $A \to \mathrm{Hom}(B, C)$, $a \mapsto \beta(a, \cdot)$, is a group isomorphism, we get $2g = 0$. □

Corollary 2.14. *Let $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ be a skew abelian group and let g be its skew element. Then one has either $\beta(g, g) = 1/2 + \mathbb{Z}$ or $\beta(g, g) = 0$.*

Proof. The result follows from Theorem 2.13. □

Theorem 2.15. *Let (A, C, β) be a skew abelian group. Then the following are equivalent.*

(i) *The pairing $\beta : A \times A \to C$ is alternating, that is, the triple (A, C, β) is a symplectic abelian group.*

(ii) *The skew element of (A, C, β) is the zero element of A.*

Proof. Let g be the skew element of (A, C, β). Since $\beta : A \times A \to C$ is a perfect pairing, we have $g = 0$ if and only if the group homomorphism $\beta(g, \cdot) : A \to C$, $a \mapsto \beta(g, a)$, is trivial. This is equivalent to the pairing $\beta : A \times A \to C$ being alternating, because for all $a \in A$ we have $\beta(a, a) = \beta(g, a)$. □

Definition 2.16 (Dual group)**.** Let A be a finite abelian group. The *dual group* $\widehat{A}$ of A is the group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ of homomorphisms from A to $\mathbb{Q}/\mathbb{Z}$.

Theorem 2.17. *A finite abelian group is isomorphic to its own dual group.*

Proof. See Theorem 9.1 of Chapter I in [32] by Lang. □

Corollary 2.18. *Let A be a finite abelian group. Then the natural map $A \to \mathrm{Hom}(\widehat{A}, \mathbb{Q}/\mathbb{Z})$, $a \mapsto (f \mapsto f(a))$, is a group isomorphism.*

Proof. See Corollary 3.2 of Chapter 3 in [74]. □

Corollary 2.19. *Let A be a finite abelian group and let $\widehat{A}$ be the dual group of A. Then the pairing $A \times \widehat{A} \to \mathbb{Q}/\mathbb{Z}$, $(a, b) \mapsto b(a)$, is a perfect pairing.*

Proof. The result follows from Corollary 2.18. □

Corollary 2.20. *Let A be an abelian group, let B be a finite abelian group, and let $\beta : A \times B \to \mathbb{Q}/\mathbb{Z}$ be a map. If there is a group isomorphism $A \to \widehat{B}$ given by $a \mapsto \beta(a, \cdot)$, then β is a perfect pairing.*

Proof. Suppose there is a group isomorphism $A \to \widehat{B}$ given by $a \mapsto \beta(a, \cdot)$. By Theorem 2.2 the map β is a pairing. By Corollary 2.18 the map $B \to \mathrm{Hom}(\widehat{B}, \mathbb{Q}/\mathbb{Z})$, $b \mapsto (f \mapsto f(b))$, is a group isomorphism. Using the group isomorphism $A \to \widehat{B}$, $a \mapsto \beta(a, \cdot)$, we get the group isomorphism $B \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, $b \mapsto \beta(\cdot, b)$. Hence β is a perfect pairing. □

Definition 2.21 (Exponent of a group)**.** Let G be a group and let e_G be its identity element. The *exponent* of G is the smallest positive integer m, if there exists one, such that for every g in G one has $g^m = e_G$. Otherwise, the *exponent* of G is zero.

The following theorem was proved by Prüfer [3] for countable groups. The general case is due to Baer [3].

Theorem 2.22 (Prüfer [58], Baer [3])**.** *Let A be an abelian group of positive exponent. Then A is isomorphic to a direct sum of cyclic groups.*

Proof. See Corollary 10.37 of Chapter 10 in [60] by Rotman. □

Given a positive integer n and an abelian group A, the subset $\{a \in A : na = 0\}$ forms a subgroup of A. We denote it by $A[n]$.

Theorem 2.23. *Let A be a finite abelian group, let e be the exponent of A, and let C be an abelian group. Then the following are equivalent.*

(i) *There exist an abelian group B and a perfect pairing $\beta : A \times B \to C$.*

(ii) *The subgroup $C[e]$ of C is a cyclic group of order e.*

Proof. (ii) $\Longrightarrow$ (i) Let B be the dual $\widehat{A}$ of A. By Corollary 2.19 we have the perfect pairing $A \times B \to \mathbb{Q}/\mathbb{Z}$, $(a, b) \mapsto b(a)$. Since A has exponent e, the image of this pairing is contained in $\frac{1}{e}\mathbb{Z}/\mathbb{Z}$. Composing with an isomorphism $\frac{1}{e}\mathbb{Z}/\mathbb{Z} \xrightarrow{\sim} C[e]$ gives a perfect pairing $A \times B \to C$.

(i) $\Longrightarrow$ (ii) By definition of perfect pairing the group homomorphism $B \to \mathrm{Hom}(A, C)$, $b \mapsto \beta(\cdot, b)$ is a group isomorphism. Since for all $b \in B$ we have $e\beta(\cdot, b) = 0$, the exponent of B divides e and the image of β is contained in $C[e]$. A similar argument shows that e divides the exponent of B. Hence, they are equal. The group C contains an element c of order e, because $\mathrm{Hom}(A, C)$ has exponent e. By Theorem 2.22 the group B is isomorphic to a direct sum of cyclic groups. Since $\mathrm{Hom}(B, C)$ is finite, the group B is also finite. We have

$$B \cong \mathrm{Hom}(A, C) \supseteq \mathrm{Hom}(A, \langle c \rangle) \cong \widehat{A}$$

and

$$A \cong \mathrm{Hom}(B, C) \supseteq \mathrm{Hom}(B, \langle c \rangle) \cong \widehat{B}.$$

Theorem 2.17 implies that the inclusions are equalities. Since every element in $C[e]$ is in the image of at least one homomorphism $B \to C$, the equality $\mathrm{Hom}(B, C) = \mathrm{Hom}(B, \langle c \rangle)$ implies that all elements in $C[e]$ are in the group $\langle c \rangle$. ☐

Remark 2.24. Since for every finite cyclic group C' there is an injective group homomorphism $C' \hookrightarrow \mathbb{Q}/\mathbb{Z}$, by Theorem 2.23 every skew (symplectic) abelian group is similar to a skew (symplectic) abelian group of the form $(A, \mathbb{Q}/\mathbb{Z}, \beta)$, where A is a finite abelian group and β is an antisymmetric (alternating) perfect pairing. Hence, we will give results for skew (symplectic) abelian groups of the form $(A, \mathbb{Q}/\mathbb{Z}, \beta)$. By abuse of notation an element in $\mathbb{Q}/\mathbb{Z}$ will be often denoted only by a rational number.

2.2 The classification

We want to classify skew abelian groups up to similarity. By Remark 2.24 we will consider only skew abelian group of the form $(A, \mathbb{Q}/\mathbb{Z}, \beta)$, where A is a finite abelian group and β is an antisymmetric perfect pairing. Moreover, since the results do not change and isomorphism is a stronger notion than similarity, we will only classify skew abelian groups of the form $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ up to isomorphism.

Theorem 2.25 (Classification of symplectic abelian groups)**.** *There is a bijection*

$$\{\text{finite abelian groups}\}/\cong \; \to \{\text{symplectic abelian groups } (A, \mathbb{Q}/\mathbb{Z}, \beta)\}/\cong,$$
$$[B] \mapsto [(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)],$$

where γ is the map

$$\begin{aligned}\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) &\to \mathbb{Q}/\mathbb{Z},\\ ((b_1, f_1), (b_2, f_2)) &\mapsto f_2(b_1) - f_1(b_2).\end{aligned}$$

Proof. Theorem 2.34 implies that the map is well-defined. By Theorem 2.60 it is surjective. The injectivity follows from Theorem 2.17 and the structure theorem for finite abelian groups. □

Definition 2.26 (p-rank)**.** Let p be a prime and let A be an abelian group. The *p-rank* $\mathrm{rk}_p(A)$ of A is the dimension of A/pA as a vector space over $\mathbb{Z}/p\mathbb{Z}$.

Remark 2.27. In order to simplify the exposition, Theorem 2.28 contains the following abuses of notation. The map

$$\begin{aligned}\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} &\to \mathbb{Q}/\mathbb{Z},\\ (g_1, g_2) &\mapsto \begin{cases} 0 & \text{if } g_1 = 0 \text{ or } g_2 = 0,\\ \frac{1}{2} & \text{if } g_1 = g_2 = 1 \bmod 2,\end{cases}\end{aligned}$$

is denoted by $(g_1, g_2) \mapsto g_1 g_2/2$. In a similar way the map

$$\begin{aligned}\frac{1}{2}\mathbb{Z}/\mathbb{Z} \times \frac{1}{2}\mathbb{Z}/\mathbb{Z} &\to \mathbb{Q}/\mathbb{Z},\\ (c_1, c_2) &\mapsto \begin{cases} 0 & \text{if } c_1 = 0 \text{ or } c_2 = 0,\\ \frac{1}{2} & \text{if } c_1 = c_2 = \frac{1}{2},\end{cases}\end{aligned}$$

is denoted by $(c_1, c_2) \mapsto 2c_1c_2$.

Theorem 2.28 (Classification of skew abelian groups)**.**
(a) *Odd 2-rank: there is a bijection*

$$\begin{aligned}\{\text{finite abelian groups}\}/\cong \;&\to \left\{\begin{matrix}\textit{skew abelian groups } (A, \mathbb{Q}/\mathbb{Z}, \beta)\\ \textit{of odd 2-rank}\end{matrix}\right\}/\cong,\\ [B] &\mapsto [(\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)],\end{aligned}$$

where γ is the map

$$\begin{aligned}\gamma : (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) \times (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) &\to \mathbb{Q}/\mathbb{Z},\\ ((g_1, b_1, f_1), (g_2, b_2, f_2)) &\mapsto \frac{g_1 g_2}{2} + f_2(b_1) - f_1(b_2).\end{aligned}$$

(b) *Even 2-rank: there is a bijection*

$$\begin{aligned}\left\{\begin{matrix}(B, g)\textit{: } B \textit{ is a finite abelian}\\ \textit{group and } g \in B[2]\end{matrix}\right\}/\cong \;&\to \left\{\begin{matrix}\textit{skew abelian groups } (A, \mathbb{Q}/\mathbb{Z}, \beta)\\ \textit{of even 2-rank}\end{matrix}\right\}/\cong,\\ [(B, g)] &\mapsto [(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)],\end{aligned}$$

where two pairs (B_1, g_1) and (B_2, g_2) on the left are defined to be isomorphic if there is a group isomorphism $\varphi : B_1 \to B_2$ with $\varphi(g_1) = g_2$ and where γ is the map

$$\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},$$
$$((b_1, f_1), (b_2, f_2)) \mapsto f_2(b_1) - f_1(b_2) + 2f_1(g)f_2(g).$$

Proof. Theorem 2.36 and Theorem 2.38 imply that the maps are well-defined and the skew elements of $(\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ in (a) and of $(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ in (b) are $(1, 0, 0)$ and $(g, 0)$, respectively. By Remark 2.31, Theorem 2.61, and Theorem 2.62 they are surjective. The injectivity in (a) follows from Theorem 2.17 and the structure theorem for finite abelian groups. Since isomorphisms of skew abelian groups map skew elements to skew elements, the injectivity in (b) follows from Lemma 2.30. □

Definition 2.29 (Heights)**.** Let A be an abelian group and let $a \in A$. The *height* of a is the largest positive integer n, if there exists one, such that the equation $nx = a$ has a solution $x \in A$. Otherwise, the *height* of a is infinite. Let p be a prime number. The *p-height* of a is the largest positive integer n, if there exists one, such that the equation $p^n x = a$ has a solution $x \in A$. Otherwise, the *p-height* of a is infinite.

Lemma 2.30. *Let B_1 and B_2 be finite abelian groups and let $g_1 \in B_1[2]$ and $g_2 \in B_2[2]$. Let t_1 and t_2 be the 2-heights of g_1 and g_2, respectively. Then the following are equivalent.*

(i) *There exists a group isomorphism $\varphi : B_1 \to B_2$ with $\varphi(g_1) = g_2$.*
(ii) *There exists a group isomorphism $B_1 \xrightarrow{\sim} B_2$ and $t_1 = t_2$.*

Proof. (i) $\Longrightarrow$ (ii) Obvious.

(ii) $\Longrightarrow$ (i) If g_1 is the zero element, then the implication is obvious. Hence, we may assume that g_1 is not the zero element. Since the order 2 of g_1 is prime, there exists $b_1 \in B_1$ such that $2^{t_1} b_1 = g_1$ and the subgroup $\langle b_1 \rangle$ is a direct summand of B_1. Similarly, there exists $b_2 \in B_2$ such that $2^{t_2} b_2 = g_2$ and the subgroup $\langle b_2 \rangle$ is a direct summand of B_2. From the equality $t_1 = t_2$ we get the group isomorphism $\langle b_1 \rangle \xrightarrow{\sim} \langle b_2 \rangle$, $b_1 \mapsto b_2$. Let H_1 and H_2 be finite abelian groups such that $B_1 \cong \langle b_1 \rangle \oplus H_1$ and $B_2 \cong \langle b_2 \rangle \oplus H_2$. The structure theorem for finite abelian groups implies the existence of a group isomorphism $H_1 \xrightarrow{\sim} H_2$. Now the result follows by extending the group isomorphism $\langle b_1 \rangle \xrightarrow{\sim} \langle b_2 \rangle$, $b_1 \mapsto b_2$, to a group isomorphism $B_1 \xrightarrow{\sim} B_2$. □

Remark 2.31. Theorem 2.61 and Theorem 2.62 imply that the case distinction in Theorem 2.28 is the same as the one in Corollary 2.14. Given a skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ with skew element g, we have the following facts.

(a) The 2-rank $\mathrm{rk}_2(A)$ is odd if and only if one has $\beta(g, g) = 1/2$.

(b) The 2-rank $\mathrm{rk}_2(A)$ is even if and only if one has $\beta(g,g) = 0$.

Remark 2.32. Since every symplectic abelian group is a skew abelian group, one may wonder where symplectic abelian groups occur in Theorem 2.28. They are the skew abelian groups given by the pairs (B, g) in (b) with $g = 0$, because $(g, 0)$ is the skew element of the skew abelian group $(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$.

2.3 Examples

We show how to construct antisymmetric perfect pairings on finite abelian groups.

Example 2.33. Let p be a prime, let $r \in \mathbb{Z}_{\geq 0}$, and let $A = \mathbb{Z}/p^r\mathbb{Z} \oplus \mathbb{Z}/p^r\mathbb{Z}$. We denote by x and y the elements $(1, 0) \in A$ and $(0, 1) \in A$, respectively. The map $\langle x\rangle \times \langle y\rangle \to A$, $(x_1, x_2) \mapsto x_1 + x_2$, is a group isomorphism. We construct a map $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ by setting

$$\beta(x, x) = \beta(y, y) = 0, \quad \beta(x, y) = -\beta(y, x) = \frac{1}{p^r},$$

and extending by bilinearity, that is, for all $i, j, k, l \in \mathbb{Z}$ we set

$$\beta(ix + jy, kx + ly) = ik\beta(x, x) + il\beta(x, y) + jk\beta(y, x) + jl\beta(y, y) = \frac{il - jk}{p^r}.$$

It is immediate to get $\beta(ix + jy, ix + jy) = 0$ for all $i, j \in \mathbb{Z}$. Hence β is an alternating pairing. Moreover, for each $a \in A \setminus \{(0, 0)\}$ we cannot have both $\beta(a, x) = 0$ and $\beta(a, y) = 0$. Hence, the group homomorphism $A \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, $a \mapsto \beta(a, \cdot)$ is injective. Theorem 2.17 implies it is a group isomorphism. By Corollary 2.20 the map β is a perfect pairing. Hence $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ is a symplectic abelian group. The skew element of $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ is the zero element, because β is alternating.

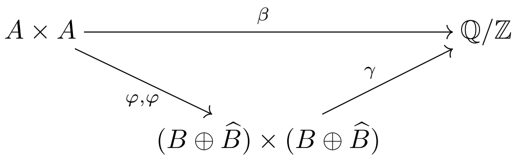
Let $B = \mathbb{Z}/p^r\mathbb{Z}$, let b be a generator of B, and let f be the element in $\widehat{B}$ such that $f(b) = 1/p^r$. Consider the group isomorphism $\varphi : A \to B \oplus \widehat{B}$ defined by setting the images $\varphi(x) = (b, 0)$ and $\varphi(y) = (0, f)$ of the generators x and y of A and extending to the whole group by the homomorphism property. Let γ be the map

$$\begin{aligned} \gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) &\to \mathbb{Q}/\mathbb{Z}, \\ ((b_1, f_1), (b_2, f_2)) &\mapsto f_2(b_1) - f_1(b_2). \end{aligned}$$

By definition it is a pairing and we have

$$\gamma((b, 0), (b, 0)) = \gamma((0, f), (0, f)) = 0,$$
$$\gamma((b, 0), (0, f)) = -\gamma((0, f), (b, 0)) = \frac{1}{p^r}.$$

Hence, we get the following commutative diagram.



The situation described in Example 2.33 is very general, as we can see in Theorem 2.34 and in Theorem 2.60.

Theorem 2.34. *Let B be a finite abelian group and γ be the map*

$$\begin{aligned}\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) &\to \mathbb{Q}/\mathbb{Z},\\ ((b_1, f_1), (b_2, f_2)) &\mapsto f_2(b_1) - f_1(b_2).\end{aligned}$$

Then the triple $(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ is a symplectic abelian group.

Proof. Corollary 2.19 implies that there are group homomorphisms

$$\begin{aligned}B \oplus \widehat{B} &\to \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}), & B \oplus \widehat{B} &\to \mathrm{Hom}(\widehat{B}, \mathbb{Q}/\mathbb{Z})\\ (b_1, f_1) &\mapsto (b_2 \mapsto -f_1(b_2)), & (b_1, f_1) &\mapsto (f_2 \mapsto f_2(b_1)),\end{aligned}$$

and their kernels are the groups $B \oplus \{0\}$ and $\{0\} \oplus \widehat{B}$, respectively. Using the natural group isomorphism

$$\mathrm{Hom}(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) \times \mathrm{Hom}(\widehat{B}, \mathbb{Q}/\mathbb{Z}),$$

we get an injective group homomorphism $B \oplus \widehat{B} \to \mathrm{Hom}(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z})$, $(b_1, f_1) \mapsto ((b_2, f_2) \mapsto f_2(b_1) - f_1(b_2))$. Since by Theorem 2.17 the group $B \oplus \widehat{B}$ and its dual group are finite groups of the same cardinality, it is a group isomorphism. Corollary 2.20 implies that the map γ is a perfect pairing. A direct computation shows that it is also alternating. □

Example 2.35. Let $A = \mathbb{Z}/2\mathbb{Z}$ and let $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ be the map defined by setting $\beta(1,1) = 1/2$ and $\beta(0,1) = \beta(1,0) = \beta(0,0) = 0$. We see that β is an antisymmetric perfect pairing and the skew element of the skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ is 1. The perfect pairing β is not alternating, because we have $\beta(1,1) = 1/2$.

Example 2.35 describes the case of a skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ with skew element g such that $\beta(g, g) = 1/2$. More generally, we have Theorem 2.36 and Theorem 2.61.

Theorem 2.36. *Let B be a finite abelian group and γ be the map*

$$\gamma : (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) \times (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},$$
$$((g_1, b_1, f_1), (g_2, b_2, f_2)) \mapsto \frac{g_1 g_2}{2} + f_2(b_1) - f_1(b_2).$$

Then the triple $(\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ is a skew abelian group of odd 2-rank and its skew element is $(1, 0, 0)$.

Proof. Since the pairing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, $(g_1, g_2) \mapsto 2g_1g_2$, is perfect and by Corollary 2.19 the map $B \times \widehat{B} \to \mathbb{Q}/\mathbb{Z}$, $(b, f) \mapsto f(b)$, is also a perfect pairing, as in the proof of Theorem 2.34 we get an injective group homomorphism

$$\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B} \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}),$$
$$(g_1, b_1, f_1) \mapsto ((g_2, b_2, f_2) \mapsto 2g_1g_2 + f_2(b_1) - f_1(b_2)).$$

It is also surjective, because by Theorem 2.17 it is an injective group homomorphism between two finite groups of the same cardinality. Corollary 2.20 implies that the map γ is a perfect pairing. A direct computation shows that γ is also antisymmetric and the skew element of $(\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ is $(1, 0, 0)$. Hence $(\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ is a skew abelian group and by Theorem 2.17 its 2-rank is odd. □

Example 2.37. Let $s \in \mathbb{Z}_{>0}$, and let $A = \mathbb{Z}/2^s\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z}$. We denote by x and y the elements $(1, 0) \in A$ and $(0, 1) \in A$, respectively. The map $\langle x \rangle \times \langle y \rangle \to A$, $(x_1, x_2) \mapsto x_1 + x_2$, is a group isomorphism. We construct a map $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ by setting

$$\beta(x, x) = 0, \quad \beta(x, y) = -\beta(y, x) = \frac{1}{2^s}, \quad \beta(y, y) = \frac{1}{2},$$

and extending by bilinearity as in Example 2.33. For all $i, j, k, l \in \mathbb{Z}$ we set

$$\beta(ix + jy, kx + ly) = \frac{il - jk}{2^s} + \frac{jl}{2}.$$

For all $i, j, k, l \in \mathbb{Z}$ we get

$$\beta(ix+jy, kx+ly) = \frac{il - jk}{2^s} + \frac{jl}{2} = -\left(\frac{kj - li}{2^s} + \frac{lj}{2}\right) = -\beta(kx+ly, ix+jy),$$

because we have $1/2 \equiv -1/2 \bmod \mathbb{Z}$. Hence β is an antisymmetric pairing. Moreover, for each $a \in A \setminus \{(0, 0)\}$ we cannot have both $\beta(a, x) = 0$ and $\beta(a, y) = 0$. Hence, the group homomorphism $A \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, $a \mapsto \beta(a, \cdot)$, is injective. Theorem 2.17 implies it is a group isomorphism. By Corollary 2.20

the map β is a perfect pairing. Hence $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ is a skew abelian group. The skew element of $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ is the element $2^{s-1}x$, because for all $i, j \in \mathbb{Z}$ we have

$$\beta(2^{s-1}x, ix + jy) = 2^{s-1}j\beta(x, y) = \frac{j}{2} = j\beta(y, y) = \beta(ix + jy, ix + jy).$$

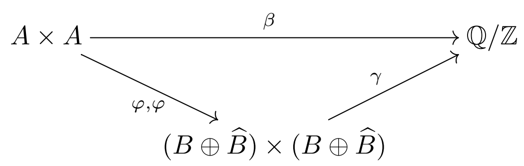
Let $B = \langle x \rangle$ be the subgroup of A generated by x and let f be the element in $\widehat{B}$ such that $f(x) = 1/2^s$. Note that we have $f(2^{s-1}x) = 1/2$. Consider the group isomorphism $\varphi : A \to B \oplus \widehat{B}$ defined by setting the images $\varphi(x) = (x, 0)$ and $\varphi(y) = (0, f)$ of the generators x and y of A and extending to the whole group by the homomorphism property. Let γ be the map

$$\begin{aligned}&\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},\\ &\qquad ((b_1, f_1), (b_2, f_2)) \mapsto f_2(b_1) - f_1(b_2) + \begin{cases} 0 & \text{if } f_1(g) = 0 \text{ or } f_2(g) = 0, \\ \frac{1}{2} & \text{if } f_1(g) = f_2(g) = \frac{1}{2}. \end{cases}\end{aligned}$$

A straightforward computation shows that it is a pairing and we have

$$\begin{aligned}\gamma((x, 0), (x, 0)) = 0, \quad \gamma((0, f), (0, f)) = \frac{1}{2},\\ \gamma((b, 0), (0, f)) = -\gamma((0, f), (b, 0)) = \frac{1}{2^s}.\end{aligned}$$

Hence, we get the following commutative diagram.



Example 2.37 describes the case of a skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ with nontrivial skew element g such that $\beta(g, g) = 0$. More generally, we have Theorem 2.38 and Theorem 2.62.

Theorem 2.38. *Let B be a finite abelian group and let $g \in B[2]$. Then the triple $(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$, where γ is the map*

$$\begin{aligned}&\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},\\ &\qquad ((b_1, f_1), (b_2, f_2)) \mapsto f_2(b_1) - f_1(b_2) + 2f_1(g)f_2(g),\end{aligned}$$

is a skew abelian group of even 2-rank and its skew element is $(g, 0)$.

Proof. We claim that there are group homomorphisms

$$B \oplus \widehat{B} \to \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}), \qquad B \oplus \widehat{B} \to \mathrm{Hom}(\widehat{B}, \mathbb{Q}/\mathbb{Z}),$$
$$(b_1, f_1) \mapsto (b_2 \mapsto -f_1(b_2)), \qquad (b_1, f_1) \mapsto (f_2 \mapsto f_2(b_1) + 2f_1(g)f_2(g)),$$

and the intersection of their kernels is $\{0\}$. A straightforward computation shows that there is a group homomorphism $B \oplus \widehat{B} \to \mathrm{Hom}(\widehat{B}, \mathbb{Q}/\mathbb{Z})$, $(b_1, f_1) \mapsto (f_2 \mapsto 2f_1(g)f_2(g))$. Now the claim follows from Corollary 2.19. As in the proof of Theorem 2.34, from the group homomorphisms of the claim we get an injective group homomorphism

$$B \oplus \widehat{B} \to \mathrm{Hom}(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}),$$
$$((b_1, f_1), (b_2, f_2)) \mapsto ((b_2, f_2) \mapsto f_2(b_1) - f_1(b_2) + 2f_1(g)f_2(g)).$$

Since by Theorem 2.17 the group $B \oplus \widehat{B}$ and its dual group are finite groups of the same cardinality, it is a group isomorphism. Corollary 2.20 implies that the map γ is a perfect pairing. A direct computation shows that γ is also antisymmetric and the skew element of $(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ is $(g, 0)$. Hence $(B \oplus \widehat{B}, \mathbb{Q}/\mathbb{Z}, \gamma)$ is a skew abelian group and by Theorem 2.17 its 2-rank is even. □

2.4 Pairings

We recall common definitions we have not used yet.

Definition 2.39 (Orthogonal or perpendicular)**.** Let A, B, and C be abelian groups and let $\beta : A \times B \to C$ be a pairing. An element $a \in A$ is *orthogonal* or *perpendicular* to a subset B' of B with respect to β if one has $\beta(a, b') = 0$ for all $b' \in B'$.

The set of elements of A orthogonal to a subset B' of B is a subgroup of A and is denoted by ${}^{\perp}B'$. If B' is a set with exactly one element b', then by abuse of notation we write ${}^{\perp}b'$ for ${}^{\perp}B'$. We make similar definitions for elements of B. If A' is a subset of A, then we denote by $A'^{\perp}$ the set of element of B orthogonal to A'. Note that we have ${}^{\perp}B = 0$ if and only if the group homomorphisms $A \to \mathrm{Hom}(B, C)$, $a \mapsto \beta(a, \cdot)$, is injective. Similarly, we have $A^{\perp} = 0$ if and only if the group homomorphism $B \to \mathrm{Hom}(A, C)$, $b \mapsto \beta(\cdot, b)$, is injective.

Remark 2.40. If a pairing $\beta : A \times A \to C$ is antisymmetric, then for each subset A' of A we have ${}^{\perp}A' = A'^{\perp}$. Hence, in this case we will use only the notation $A'^{\perp}$.

Definition 2.41 (Nondegenerate pairing)**.** Let A, B, and C be abelian groups. A pairing $\beta : A \times B \to C$ is a *nondegenerate pairing* if one has both ${}^{\perp}B = 0$ and $A^{\perp} = 0$.

2.5 Antisymmetric pairings

Theorem 2.42. *Let A be a finite abelian group, let C be an abelian group, and let $\beta : A \times A \to C$ be a perfect pairing. Then the following are equivalent.*

(i) *The map $Q : A \to C$, $a \mapsto \beta(a,a)$, is a group homomorphism.*

(ii) *The pairing $\beta : A \times A \to C$ is antisymmetric, that is, the triple (A, C, β) is a skew abelian group.*

(iii) *There exists $g \in A$ such that $\beta(g,a) = \beta(a,a)$ for all $a \in A$.*

(iv) *There exists a unique $g \in A$ such that $\beta(g,a) = \beta(a,a)$ for all $a \in A$.*

Proof. (i) $\implies$ (iv) Since the pairing $\beta : A \times A \to C$ is perfect, the group homomorphism $A \to \mathrm{Hom}(A, C)$, $a \mapsto \beta(a, \cdot)$, is a group isomorphism. Since we have $Q \in \mathrm{Hom}(A, C)$, there exists a unique $g \in A$ such that Q is the map $\beta(g, \cdot) : A \to C$.

(iv) $\implies$ (iii) Obvious.

(iii) $\implies$ (ii) For all $a, b \in A$ we have

$$\begin{aligned}\beta(a,b) + \beta(b,a) &= \beta(a+b, a+b) - \beta(a,a) - \beta(b,b) = \\ &= \beta(g, a+b) - \beta(g,a) - \beta(g,b) = \beta(g,0) = 0.\end{aligned}$$

(ii) $\implies$ (i) For all $a, b \in A$ we see that

$$Q(a+b) = \beta(a,a)+\beta(a,b)+\beta(b,a)+\beta(b,b) = \beta(a,a)+\beta(b,b) = Q(a)+Q(b).$$

Hence, the map $Q : A \to C$, $a \mapsto \beta(a,a)$, is a group homomorphism. □

Definition 2.43 (Orthogonal sum)**.** Let I be an index set. For each $i \in I$ let A_i be a finite abelian group, let C_i be an abelian group, and let $\beta_i : A_i \times A_i \to C_i$ be an antisymmetric pairing. Suppose that for all but finitely many $i \in I$ the group A_i is trivial and that for each $i \in I$ there exists a positive integer e_i such that the group C_i' generated by the image of β_i in C_i is a cyclic group of exponent e_i. Let C be an abelian group such that for each $i \in I$ the subgroup $C[e_i]$ of C is cyclic of order e_i. For each $i \in I$ let $\psi_i : C_i' \to C$ be an injective group homomorphism. The *orthogonal sum*

$$\beta = \perp_{i \in I} \beta_i$$

of the sequence $(\beta_i)_{i \in I}$ of pairings is the pairing

$$\begin{aligned}\beta : \bigoplus_{i \in I} A_i \times \bigoplus_{i \in I} A_i &\to C, \\ ((a_i)_i, (b_i)_i) &\mapsto \sum_{i \in I} \psi_i(\beta_i(a_i, b_i)).\end{aligned}$$

Remark 2.44. Each map $\psi_i : C'_i \to C$ in Definition 2.43 is part of the structure of the orthogonal sum. The orthogonal sum depends on the choice of these maps.

Remark 2.45. In Definition 2.43 the group generated by the image of β in C is a cyclic subgroup of C and can be mapped injectively to $\mathbb{Q}/\mathbb{Z}$. In order to simplify the notation, now we restrict our attention to antisymmetric pairings with image contained in $\mathbb{Q}/\mathbb{Z}$ and to orthogonal sums where the maps $\psi_i : \frac{1}{e_i}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ in Definition 2.43 are the inclusion maps.

Definition 2.46 (Skew pair)**.** A *skew pair* is a pair (A, β), where A is a finite abelian group and $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ is an antisymmetric pairing.

Remark 2.47. Let (A, β) be a skew pair. If β is a perfect pairing, then $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ is a skew abelian group.

Definition 2.48 (Isomorphism of skew pairs)**.** Let (A, β) and (B, γ) be skew pairs. An *isomorphism of skew pairs* is a group isomorphism $\varphi : A \to B$ such that for all $x, y \in A$ one has $\beta(x, y) = \gamma(\varphi(x), \varphi(y))$.

Remark 2.49. Let $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ and $(B, \mathbb{Q}/\mathbb{Z}, \gamma)$ be skew abelian groups. They are isomorphic as skew pairs if and only if they are isomorphic as skew abelian groups.

Let A be an abelian group and let H_1 and H_2 be subgroups of A. If the map $H_1 \times H_2 \to A$, $(h_1, h_2) \mapsto h_1 + h_2$, is a group isomorphism, then we write $A = H_1 \times H_2$.

Let (A, β) be a skew pair and let H be a subgroup of A. Then $(H, \beta|_H)$ is also a skew pair, where $\beta|_H : H \times H \to \mathbb{Q}/\mathbb{Z}$ is the restriction of β to $H \times H$. Let H_1 and H_2 be subgroups of A. Then we write $(A, \beta) = (H_1, \beta|_{H_1}) \perp (H_2, \beta|_{H_2})$, if the map $H_1 \times H_2 \to A$, $(h_1, h_2) \mapsto h_1 + h_2$, is an isomorphism of skew pairs. In order to simplify the notation we will often write only $A = H_1 \perp H_2$ if there is no ambiguity on the pairing β.

Lemma 2.50. *Let (A, β) be a skew pair and let H_1 and H_2 be subgroups of A such that $A = H_1 \times H_2$. Then one has $A = H_1 \perp H_2$ if and only if H_1 and H_2 are orthogonal to each other.*

Proof. Let $a \in A$ and let $h_1 \in H_1$ and $h_2 \in H_2$ with $a = h_1 + h_2$. If we have $A = H_1 \perp H_2$, then $\beta(h_1, h_2) = \beta|_{H_1}(h_1, 0) + \beta|_{H_2}(0, h_2) = 0$. Hence H_1 and H_2 are orthogonal to each other.

Now suppose that H_1 and H_2 are orthogonal to each other. For all $g, h \in A$, $g_1, h_1 \in H_1$, $g_2, h_2 \in H_2$, with $g = g_1 + g_2$ and $h = h_1 + h_2$, we have

$$\begin{aligned}\beta(g, h) &= \beta(g_1 + g_2, h_1 + h_2) = \beta(g_1, h_1) + \beta(g_1, h_2) + \beta(g_2, h_1) + \beta(g_2, h_2) \\ &= \beta(g_1, h_1) + \beta(g_2, h_2) = \beta|_{H_1}(g_1, h_1) + \beta|_{H_2}(g_2, h_2).\end{aligned}$$

Hence, we get $A = H_1 \perp H_2$. ☐

Lemma 2.51. *Let (A, β) be a skew pair and let H_1 and H_2 be subgroups of A such that $A = H_1 \perp H_2$. Then β is a perfect pairing if and only if $\beta|_{H_1}$ and $\beta|_{H_2}$ are perfect pairings.*

Proof. The map $\varphi : A \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ may be written as

$$\varphi : H_1 \perp H_2 \to \mathrm{Hom}(H_1, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}(H_2, \mathbb{Q}/\mathbb{Z}).$$

Since H_1 and H_2 are orthogonal to each other, this map preserves the components. The map β is a perfect pairing, which is equivalent to φ being an isomorphism, if and only if φ induces group isomorphisms $H_1 \xrightarrow{\sim} \mathrm{Hom}(H_1, \mathbb{Q}/\mathbb{Z})$ and $H_2 \xrightarrow{\sim} \mathrm{Hom}(H_2, \mathbb{Q}/\mathbb{Z})$, that is, the maps $\beta|_{H_1}$ and $\beta|_{H_2}$ are perfect pairings. $\square$

Lemma 2.52. *Let (A, β) be a skew pair and let H be a subgroup of A such that $(H, \mathbb{Q}/\mathbb{Z}, \beta|_H)$ is a skew abelian group. Then one has $A = H \perp H^\perp$. Moreover β is a perfect pairing if and only if $\beta|_{H^\perp}$ is a perfect pairing.*

Proof. By definition the kernel of the map $\varphi : A \to \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$, $a \mapsto (h \mapsto \beta(a, h))$, is $H^\perp$. Since $(H, \mathbb{Q}/\mathbb{Z}, \beta_H)$ is a skew abelian group, which is equivalent to $\varphi|_H$ being an isomorphism, we have $H \cap H^\perp = \{0\}$ and the natural map $H \to A/H^\perp$ is a group isomorphism. Hence, the subgroups H and $H^\perp$ span A. By Lemma 2.50 we get the decomposition $A = H \perp H^\perp$. The last statement follows from Lemma 2.51. $\square$

Lemma 2.53. *Let $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ and $(B, \mathbb{Q}/\mathbb{Z}, \gamma)$ be skew abelian groups. Then the orthogonal sum $(A, \mathbb{Q}/\mathbb{Z}, \beta) \perp (B, \mathbb{Q}/\mathbb{Z}, \gamma)$ is a skew abelian group.*

Proof. Apply Lemma 2.51 to the orthogonal sum. $\square$

Lemma 2.54. *Let (A, β) be a skew pair and let H_1 and H_2 be subgroups of A such that $A = H_1 \times H_2$. If the exponents of H_1 and H_2 are coprime, then $A = H_1 \perp H_2$.*

Proof. Let e_1 and e_2 be the exponents of H_1 and H_2, respectively. Then for all $h_1 \in H_1$ and $h_2 \in H_2$ we have

$$e_1\beta(h_1, h_2) = e_2\beta(h_1, h_2) = 0.$$

Since e_1 and e_2 are coprime, we get $\beta(h_1, h_2) = 0$ for all $h_1 \in H_1$ and $h_2 \in H_2$. Hence H_1 and H_2 are orthogonal to each other and by Lemma 2.50 we have $A = H_1 \perp H_2$. $\square$

Definition 2.55 (p-primary component of an abelian group)**.** Let p be a prime and let A be an abelian group. The *p-primary component* $A[p^\infty]$ of A is the subgroup

$$A[p^\infty] = \bigcup_{i \in \mathbb{Z}_{\geq 0}} A[p^i]$$

of A.

Theorem 2.56. *Let A be a torsion abelian group and for each prime p let $A[p^\infty]$ be its p-primary component. Then the map*

$$\bigoplus_{p \text{ prime}} A[p^\infty] \to A,$$
$$(a_p)_p \mapsto \sum_{p \text{ prime}} a_p,$$

is an isomorphism of abelian groups.

Proof. See Theorem 4.1.1 in Section 4.1 of Chapter 4 in [59] by Robinson. □

Theorem 2.57. *Let p be a prime, let (A, C, β) be a skew abelian group, and let $A[p^\infty]$ be the p-primary component of A. Then $(A[p^\infty], C, \beta|_{A[p^\infty]})$ is a skew abelian group.*

Proof. By Theorem 2.56 there is a subgroup H of A such that $A = A[p^\infty] \times H$ and the exponents of H and $A[p^\infty]$ are coprime. By Lemma 2.54 we get $A = A[p^\infty] \perp H$. The result follows from Lemma 2.51 . □

Definition 2.58 (p-primary component of a skew abelian group)**.** Let p be a prime, let (A, C, β) be a skew abelian group, and let $A[p^\infty]$ be the p-primary component of A. The *p-primary component* of (A, C, β) is the skew abelian group $(A[p^\infty], C, \beta|_{A[p^\infty]})$.

Theorem 2.59. *Let (A, C, β) be a skew abelian group and for each prime p let $(A[p^\infty], C, \beta|_{A[p^\infty]})$ be its p-primary component. Then the group isomorphism*

$$\varphi : \bigoplus_{p \text{ prime}} A[p^\infty] \to A,$$
$$(a_p)_p \mapsto \sum_{p \text{ prime}} a_p,$$

is an isomorphism

$$\perp_{p \text{ prime}} (A[p^\infty], C, \beta|_{A[p^\infty]}) = (A, C, \beta)$$

of skew abelian groups.

Proof. Since A is a finite abelian group, for all but finitely many primes p the group $A[p^\infty]$ is trivial. Hence, the orthogonal sum in the statement of the theorem is well-defined. The result follows from Theorem 2.56 and Lemma 2.54. □

Theorem 2.60. *Let $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ be a symplectic abelian group. Then there exist a finite abelian group B and a group isomorphism $\varphi : A \xrightarrow{\sim} B \oplus \widehat{B}$ such that the diagram*

$$\begin{array}{ccc} A \times A & \xrightarrow{\quad\beta\quad} & \mathbb{Q}/\mathbb{Z} \\ & {\scriptstyle \varphi,\varphi} \searrow \quad \nearrow {\scriptstyle \gamma} & \\ & (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) & \end{array}$$

commutes, where γ is the map

$$\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},$$
$$((b_1, f_1), (b_2, f_2)) \mapsto f_2(b_1) - f_1(b_2).$$

Proof. We pick an element $a \in A$ of maximal order e. Let $\langle a \rangle$ be the subgroup generated by a. Then the short exact sequence $0 \to \langle a \rangle \to A \to A/\langle a \rangle \to 0$ splits. We choose a splitting of the sequence. Composing the projection of A onto $\langle a \rangle$ with the isomorphism $\langle a \rangle \xrightarrow{\sim} \frac{1}{e}\mathbb{Z}/\mathbb{Z}$ that maps a to $\frac{1}{e} \bmod \mathbb{Z}$ as in the diagram

$$\begin{array}{ccc} A = \langle a \rangle \oplus A/\langle a \rangle & \twoheadrightarrow & \langle a \rangle \\ & {\scriptstyle \chi} \searrow & \sim \big\downarrow {\scriptstyle a \mapsto \frac{1}{e} \bmod \mathbb{Z}} \\ & & \frac{1}{e}\mathbb{Z}/\mathbb{Z} \end{array}$$

gives a homomorphism $\chi : A \to \mathbb{Q}/\mathbb{Z}$. Since the pairing β is perfect, by duality there is an element $b \in A$ such that for all $x \in A$ one has $\chi(x) = \beta(x, b)$. Hence, we have $\beta(a, b) = \frac{1}{e}$ and the order of b is e. Since we have $\langle a \rangle \subset \langle a \rangle^{\perp}$ and $\langle b \rangle \cap \langle a \rangle^{\perp} = \{0\}$, the subgroups $\langle a \rangle$ and $\langle b \rangle$ form a direct sum $H = \langle a \rangle \oplus \langle b \rangle$ in A and the pairing $\beta|_H$ is perfect. By Lemma 2.52 we get the decomposition $A = H \perp H^{\perp}$ and the pairing $\beta|_{H^{\perp}}$ is perfect.

The pairing satisfies $\beta(a, a) = \beta(b, b) = 0$ and $\beta(a, b) = -\beta(b, a) = 1/e$. If we identify the group $\langle b \rangle$ with the dual group of $\langle a \rangle$ as above, the restriction of our pairing to $(\langle a \rangle \oplus \widehat{\langle a \rangle}) \times (\langle a \rangle \oplus \widehat{\langle a \rangle})$ becomes $((a_1, f_1), (a_2, f_2)) \mapsto f_2(a_1) - f_1(a_2)$. By induction on the order of the group A we conclude the proof. □

Theorem 2.61. *Let $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ be a skew abelian group, let g be its skew element, and suppose $\beta(g, g) = 1/2$. Then there exist a finite abelian group B and a group isomorphism $\varphi : A \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}$ such that $\varphi(g) = (1, 0, 0)$*

and the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{\quad\quad\quad\beta\quad\quad\quad} & \mathbb{Q}/\mathbb{Z} \\ & {\searrow^{\varphi,\varphi}} \quad\quad {\nearrow_{\gamma}} & \\ & (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) \times (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) & \end{array}$$

commutes, where γ *is the map*

$$\gamma : (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) \times (\mathbb{Z}/2\mathbb{Z} \oplus B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},$$
$$((g_1, b_1, f_1), (g_2, b_2, f_2)) \mapsto \frac{g_1 g_2}{2} + f_2(b_1) - f_1(b_2).$$

Proof. We have $g \notin \langle g \rangle^\perp$. Hence, the pairing $\beta|_{\langle g \rangle}$ is perfect. By Lemma 2.52 we get the decomposition $A = \langle g \rangle \perp \langle g \rangle^\perp$ and the pairing $\beta|_{\langle g \rangle^\perp}$ is perfect. It is also alternating, because $a \in \langle g \rangle^\perp$ implies $\beta(a, a) = \beta(g, a) = 0$. Hence $(\langle g \rangle^\perp, \mathbb{Q}/\mathbb{Z}, \beta|_{\langle g \rangle^\perp})$ is a symplectic abelian group. We conclude by combining Theorem 2.60 applied to $(\langle g \rangle^\perp, \mathbb{Q}/\mathbb{Z}, \beta|_{\langle g \rangle^\perp})$ and the skew abelian group isomorphism $\langle g \rangle \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$, $g \mapsto 1 \bmod 2$, from $(\langle g \rangle, \mathbb{Q}/\mathbb{Z}, \beta|_{\langle g \rangle})$ to $(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}, (g_1, g_2) \mapsto g_1 g_2/2)$. ☐

Theorem 2.62. *Let* $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ *be a skew abelian group, let* g_A *be its skew element, and suppose* $\beta(g_A, g_A) = 0$. *Then there exist a finite abelian group* B*, an element* $g \in B[2]$*, and a group isomorphism* $\varphi : A \xrightarrow{\sim} B \oplus \widehat{B}$ *such that* $\phi(g_A) = (g, 0)$ *and the diagram*

$$\begin{array}{ccc} A \times A & \xrightarrow{\quad\quad\beta\quad\quad} & \mathbb{Q}/\mathbb{Z} \\ & {\searrow^{\varphi,\varphi}} \quad {\nearrow_{\gamma}} & \\ & (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) & \end{array}$$

commutes, where γ *is the map*

$$\gamma : (B \oplus \widehat{B}) \times (B \oplus \widehat{B}) \to \mathbb{Q}/\mathbb{Z},$$
$$((b_1, f_1), (b_2, f_2)) \mapsto f_2(b_1) - f_1(b_2) + 2f_1(g)f_2(g).$$

Moreover, for $g_A \neq 0$ *there exists a subgroup* H *of* A *such that* $A = H \perp H^\perp$, *the triple* $(H^\perp, \mathbb{Q}/\mathbb{Z}, \beta|_{H^\perp})$ *is a symplectic abelian group, and* $(H, \mathbb{Q}/\mathbb{Z}, \beta|_H)$ *is a skew abelian group isomorphic to* $(\mathbb{Z}/2^s\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z}, \mathbb{Q}/\mathbb{Z}, \delta)$*, where* s *is the largest positive integer such that the equation* $2^{s-1}x = g_A$ *has a solution* $x \in A$ *and* δ *is the map* $\delta : \mathbb{Z}/2^s\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, $\delta((i, j), (k, l)) = (il - jk)/2^s + jl/2$.

Proof. The case $g_A = 0$ is Theorem 2.60. By Theorem 2.13 we are left with the case when g_A has order 2. By Lemma 2.54 we can assume that A is a 2-group. Let s be the largest positive integer such that the equation $2^{s-1}x = g_A$ has a solution $x \in A$. Since the order 2 of g_A is prime, there exists $a \in A$ such that $g_A = 2^{s-1}a$ and the subgroup $\langle a \rangle$ is a direct summand of A. We have $\beta(a,a) = \beta(a, g_A) = 2^{s-1}\beta(a,a) = 0$, because either $s = 1$ and $a = g_A$ or $s > 1$ and an odd multiple of $\beta(a,a)$ equals 0. Now, as in the proof of Theorem 2.60, there is $b \in A$ of order 2^s such that $\beta(a,b) = 1/2^s$, the subgroups $\langle a \rangle$ and $\langle b \rangle$ form a direct sum $H = \langle a \rangle \oplus \langle b \rangle$ in A, and the pairing $\beta|_H$ is perfect. By Lemma 2.52 we get the decomposition $A = H \perp H^\perp$ and the pairing $\beta|_{H^\perp}$ is perfect. It is also alternating, because $c \in H^\perp$ implies $c \in g_A^\perp$ and therefore $\beta(c,c) = \beta(g_A, c) = 0$. Hence, the triple $(H^\perp, \mathbb{Q}/\mathbb{Z}, \beta|_{H^\perp})$ is a symplectic abelian group.

The pairing satisfies $\beta(a,a) = 0$, $\beta(b,b) = 1/2$ and $\beta(a,b) = -\beta(b,a) = 1/2^s$. Hence, the group isomorphism $H \xrightarrow{\sim} \mathbb{Z}/2^s\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z}$ given by $a \mapsto (1,0)$ and $b \mapsto (0,1)$ is an isomorphism between the skew abelian groups $(H, \mathbb{Q}/\mathbb{Z}, \beta|_H)$ and $(\mathbb{Z}/2^s\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z}, \mathbb{Q}/\mathbb{Z}, \delta)$, where δ is the map $\delta : \mathbb{Z}/2^s\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, $\delta((i,j),(k,l)) = (il - jk)/2^s + jl/2$. If we identify the group $\langle b \rangle$ with the dual group $\widehat{\langle a \rangle}$ of $\langle a \rangle$ as in the proof of Theorem 2.60, the restriction of our pairing to $(\langle a \rangle \oplus \widehat{\langle a \rangle}) \times (\langle a \rangle \oplus \widehat{\langle a \rangle})$ becomes

$$((a_1, f_1), (a_2, f_2)) \mapsto f_2(a_1) - f_1(a_2) + 2f_1(g_A)f_2(g_A).$$

If we have $H^\perp = \{0\}$, then the result follows by taking $B = \mathbb{Z}/2^s\mathbb{Z}$, $g = 2^{s-1} \bmod 2^s\mathbb{Z} \in B$, and $\varphi : A \to B \oplus \widehat{B}$ such that $\phi(a) = (1,0)$ and $\varphi(b) = (0,1)$. Otherwise, the result follows by combining this particular case and Theorem 2.60 applied to the symplectic abelian group $(H^\perp, \mathbb{Q}/\mathbb{Z}, \beta|_{H^\perp})$. □

2.6 Wall's results

We summarize the results about isomorphism classes of skew abelian groups of the form $(A, \mathbb{Q}/\mathbb{Z}, \beta)$, where A is a finite abelian group and β is an antisymmetric perfect pairing, in [73] by Wall. We link his notation to our examples and proofs.

Let $\mathfrak{M}$ be the set of isomorphism classes of skew abelian groups of the form $(A, \mathbb{Q}/\mathbb{Z}, \beta)$. By Lemma 2.53 the orthogonal sum induces a commutative and associative operation on $\mathfrak{M}$. The class e of the trivial group is an identity element of this operation. Hence $\mathfrak{M}$ is an abelian monoid. We want to give generators and relations for $\mathfrak{M}$.

For p prime and $r \in \mathbb{Z}_{>0}$ define W_{p^r} as the isomorphism class in $\mathfrak{M}$ of $(A, \mathbb{Q}/\mathbb{Z}, \beta)$, where $A = \langle x \rangle \times \langle y \rangle$, the subgroups $\langle x \rangle$ and $\langle y \rangle$ of A have both order p^r, and $\beta(x,x) = \beta(y,y) = 0$, $\beta(x,y) = -\beta(y,x) = 1/p^r$. This is the isomorphism class given by a cyclic group B of order p^r in Theorem 2.25. It is also the isomorphism class of the skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ in Example 2.33 and of the skew abelian group $(H, \mathbb{Q}/\mathbb{Z}, \beta|_H)$ used in the proof of Theorem 2.60 when H has order p^{2r}.

Define Y_2 as the isomorphism class of the skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$, where $A = \langle g \rangle$ has order 2 and $\beta(g,g) = 1/2$. This is the isomorphism class given by the trivial group B in (a) of Theorem 2.28. It is also the isomorphism class of the skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ in Example 2.35 and of the skew abelian group $(\langle g \rangle, \mathbb{Q}/\mathbb{Z}, \beta|_{\langle g \rangle})$ used in the proof of Theorem 2.61.

For $s \in \mathbb{Z}_{>0}$ define X_{2^s} as the isomorphism class of the skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$, where $A = \langle x \rangle \times \langle y \rangle$, the subgroups $\langle x \rangle$ and $\langle y \rangle$ of A have both order 2^s, and $\beta(x,x) = 0$, $\beta(x,y) = -\beta(y,x) = 1/2^s$, $\beta(y,y) = 1/2$. This is the isomorphism class given by a cyclic group B of order 2^s with $g \neq 0$ in (b) of Theorem 2.28. It is also the isomorphism class of the skew abelian group $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ in Example 2.37 and of the skew abelian group $(H, \mathbb{Q}/\mathbb{Z}, \beta|_H)$ used in the proof of Theorem 2.62 when H has order 2^{2s}.

Theorem 2.63 (Wall [73]). *Let $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ be a skew abelian group, let g be its skew element, and let W be the set $W = \{W_{p^r} : r \in \mathbb{Z}_{>0},\ p \text{ prime}\}$. Then the isomorphism class of $(A, \mathbb{Q}/\mathbb{Z}, \beta)$ in $\mathfrak{M}$ can be written*

(a) *uniquely as a finite sum of elements in W if β is alternating,*

(b) *as a sum of Y_2 and a finite sum of elements in W if $\beta(g,g) = 1/2$,*

(c) *as a sum of X_{2^s} and a finite sum of elements in W if $g \neq 0$, $\beta(g,g) = 0$, and s is the largest positive integer such that the equation $2^{s-1}x = g$ has a solution $x \in A$.*

Proof. Theorem 2.25 and (a) of Theorem 2.28 give (a) and (b), respectively. Combining (b) of Theorem 2.28, Lemma 2.30, and the orthogonal sum decomposition in Theorem 2.62 gives (c). □

Theorem 2.64 (Wall [73]). *The monoid $\mathfrak{M}$ is generated by the elements in the set*

$$M = \{W_{p^r}, Y_2, X_{2^s} : r \in \mathbb{Z}_{>0}, s \in \mathbb{Z}_{>0}, p \text{ prime}\}$$

with relations $Y_2 + Y_2 = X_2$, $Y_2 + X_{2^r} = Y_2 + W_{2^r}$, and $X_{2^r} + X_{2^s} = X_{2^r} + W_{2^s}$, where $r, s \in \mathbb{Z}_{>0}$ and $s \geq r$.

Proof. The result follows from Theorem 2.63. □