Dual complexes of semistable varieties

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to those who have tried to restart

Contents

Introduction

> In your life there are a few places, or maybe only the one place, where something happened, and then there are all the other places.
>
> —Alice Munro, *Too Much Happiness*

This thesis consists of three chapters, each of which covers a problem related to applications of dual complexes of semistable varieties. Such varieties are those that are, étale locally at singular points, of the form

$$\mathcal{O}_S[x_1, \ldots, x_n]/(x_1 \cdots x_l - b),$$

where S is some base scheme, $l \leq n$, and $b \in \mathcal{O}_S$ is not a unit. Chapter 1 shows how the combinatorial properties of the dual graph of a semistable variety can be used to determine whether or not a Néron model for the associated Picard space exists. In Chapter 2 we investigate how the dual Δ-complex of a semistable morphism behaves in families, and how the dual Δ-complex can act as a higher-dimensional generalisation of a tropical curve associated to a semistable curve. Finally, in Chapter 3 we delve into logarithmic geometry, in which we discuss how to construct important families of logarithmic curves, as well as how the Artin fan of a logarithmic curve relates to the dual graph of the underlying curve when the irreducible components of the curve are smooth.

In Chapter 1, we draw motivation from the case of semistable curves. Given a semistable curve $f : X \to S$, where S is regular, such that f is smooth over a dense open subscheme $U \subset S$, the Jacobian of X_U/U admitting a Néron model over S is equivalent to a combinatorial condition on the dual graphs of the fibres of X/S, as shown in [Hol16]. The motivating problem is then what happens if $f : X \to S$ is no longer a semistable curve, but a (possibly) higher-dimensional semistable variety. Replacing the Jacobian of X_U/U with the fibrewise connected component of the Picard space of X_U/U, we ask if the combinatorics of the fibres of X/S provide a means of determining whether a Néron model exists for $\mathrm{Pic}^0_{X_U/U}$.

Various difficulties are present when working in higher dimensions. The notion of alignment used when looking at dual graphs was originally used by Holmes to determine whether the closure of the unit section of the Picard scheme was flat over S. In the higher-dimensional case, we use a notion of a dual graph of a fibre where edges represent connected components of the singular locus of the fibre. This turns out to be the correct generalisation insofar as one can show that alignment is equivalent to the flatness of the closure of the unit section of $\mathrm{Pic}^0_{X_U/U}$ over S. After this is done, establishing the existence of a Néron model follows by showing the smooth locus of $\mathrm{Pic}_{X/S}$ contains $\mathrm{Pic}^0_{X/S}$ and is both open and closed. This requires additional assumptions on the Picard space. This chapter concludes with Theorem 1.56, which states that if X is regular, S is excellent, and $\mathrm{Pic}_{X/S}$ is smooth over S along the unit section, then a Néron model for $\mathrm{Pic}^0_{X_U/U}$ exists if and only if the closure of the unit section is étale over S, which is true if and only if X/S is aligned.

Chapter 2 is motivated by the following problem: If R is a discrete valuation ring with uniformizer π, suppose $f : X \to \mathrm{Spec}(R)$ is a relative semistable curve with smooth generic fibre. By this we mean that at a non-smooth point of the special fibre, f factors étale locally on X through an étale morphism

$$X \to \mathrm{Spec}(R[x, y]/(xy - \pi^k)). \tag{1}$$

One can then construct a tropical curve associated to this morphism. Namely, the underlying graph is the dual graph of the special fibre of X, and to any edge associated to a non-smooth point, the edge is labelled with the unique integer k such that f factors étale locally at the point through a morphism as in (1). Such a construction is well-known, and it is shown in Section 7 of [CCUW16] that a similar construction can be performed for semistable curves over more general bases equipped with a log structure.

Being a semistable curve has a natural generalisation to higher-dimensions, and one can then ask whether or not an analogue of a tropical curve exists in this case. For example, one might expect that the intersection of the coordinate hyperplanes in $\mathbb{A}^3$ should have a dual complex that is the standard 2-simplex, as this would capture the combinatorics of the underlying topological space. Motivated by the results of [CCUW16], we show that one can use the dual Δ-complex as a reasonable generalisation after adding labels to it analogous to those on tropical curves. As cone complexes lack certain morphisms, we then embed the Δ-complexes into the larger category of generalised cone complexes. We show in Theorem 2.63 that the generalised dual cone complex behaves in families of semistable varieties as does a tropical curve for families of semistable curves.

Chapter 2 can be read independently of Chapter 1, though some of the results expand on those of Chapter 1.

The last chapter, Chapter 3, focuses on logarithmic geometry. Log geometry is concerned with sheaves of monoids on schemes, and provides a good framework to look at dual complexes of semistable varieties. This chapter is independent of the previous two. The goal of this chapter is to show that the dual graph of a semistable curve over a field, all of whose irreducible components are smooth, can be found by looking at the Artin fan of the curve. First, in order to relate a semistable curve to a log curve, we follow [Kat00] and show that to any semistable curve we can find a canonical log structure on the source and target so that the resulting morphism of logarithmic schemes is that of a log curve. To any logarithmic scheme we have an associated object called a Artin fan, and in the case of a log curve arising from a semistable curve we show at the end of the chapter in Theorem 3.55 that the underlying topological space of the closed fibre of the resulting morphism of Artin fans is, in a natural way, the dual graph of the semistable curve.

A large section of this chapter is devoted to building an appropriate family of stable curves that has as a closed fibre the original semistable curve over a field. That is, given $f : X \to S = \mathrm{Spec}(k)$, we construct a family of stable curves $F : \mathcal{X} \to \mathcal{S}$, where $\mathcal{S} \to \mathbb{A}^l$ is étale for some positive integer l, such that the fibre of F over $0 \in \mathbb{A}^l$ is isomorphic to X. This family is special in that the singular points of X/S correspond bijectively to the coordinate hyperplanes of $\mathbb{A}^l$. This allows us to show that the association of an Artin fan to a logarithmic scheme is functorial in the case of our semistable curve with its logarithmic structure, which in general is not true.

Chapter 1

Separated quotients of Picard schemes

> I wanted real adventures to happen to myself. But real adventures, I reflected, do not happen to people who remain at home: they must be sought abroad.
>
> James Joyce, *Dubliners*

1.1 Introduction

Néron models were introduced in 1964 by Néron [Nér64] for abelian varieties over the fraction field of a Dedekind scheme. Their existence has been used to show seminal results in the study of abelian varieties, such as the Serre-Tate theorem on good reduction of abelian varieties [ST68]. Raynaud [Ray70] showed in 1970 that Néron models are well-suited to the study of Jacobians of curves: given a family of curves X over a Dedekind scheme S, smooth over a dense open subscheme $U \subset S$, the Néron model of the relative Jacobian $J(X_U)$ of X_U/U is a quotient of the relative Jacobian $J(X)$ of X/S. Contemporary results have expanded on this theory to the case of an arbitrary regular base scheme [Hol16], nodal curves over traits [Ore16], and constructing universal Néron models for the moduli stack $\mathcal{M}_{g,n}$ of n-marked genus g curves [Mel17], among other examples. Recall the definition of a Néron model:

Definition 1.1. Let S be a scheme, $U \subset S$ a dense open subscheme, and A/U an abelian scheme. A Néron model of A/U over S is a smooth, separated algebraic space N over S together with an isomorphism $N_U \cong A$, that satisfies the Néron mapping property: Given any smooth algebraic space T/S along with a morphism $f_U : T_U \to A$ over U, there exists a unique extension of f_U to a morphism $f : T \to N$ over S

Remark 1.2. The usual definition of Néron models also requires them to be of finite type over S. The above definition is more commonly referred to as a Néron *lft* model, where the *lft* stands for locally of finite type. We shall not use this terminology.

Let $X \to S$ be a semistable curve, where S is a regular base scheme. That is, X/S is proper, flat, and the fibres are semistable curves. Let $U \subset S$ be a dense open subscheme, and suppose furthermore that X is smooth over U. Then Holmes showed in Section 5 of [Hol16] that the Jacobian of X_U/U admits a Néron model over S if and only if the geometric fibres of X/S are aligned, a combinatorial condition on the labelled dual graphs of the geometric fibres.

The aim of this chapter is to generalise this to higher dimensional semistable morphisms. One must replace the Jacobian by $\mathrm{Pic}^0_{X_U/U}$, the connected component of the unit section of the Picard space of X/S. We extend the notion of alignment to higher-dimensions in Section 1.2, and that alignment of a semistable morphism implies the the closure of the unit section of $\mathrm{Pic}^0_{X_U/U}$ in $\mathrm{Pic}_{X/S}$ is étale over S.

Theorem 1.3 (Theorem 1.51 and Theorem 1.56)**.** *Let $X \to S$ be a semistable morphism over a regular base scheme S, and assume X is regular. Then $X \to S$ is aligned if and only if the closure clo(e) of the unit section of $Pic^0_{X_U/U}$ in $Pic_{X/S}$ is étale over S.*

Furthermore, if the Picard scheme of X/S is smooth along the unit section, such as when one restricts to characteristic 0, and if X/S is projective then one has a much stronger result.

Theorem 1.4 (Theorem 1.56)**.** *Let $X \to S$ be a projective semistable morphism with X regular and S an excellent, regular, locally Noetherian scheme. Assume also that $Pic_{X/S}$ is smooth over S along the unit section. Then a Néron model for $Pic^0_{X_U/U}$ exists if and only if X/S is aligned.*

1.1.1 Idea of proof

One shows that the Picard space $\mathrm{Pic}_{X/S}$ satisfies the existence part of the Néron mapping property. Separatedness would imply uniqueness, but this is rarely the case: the failure of $\mathrm{Pic}_{X/S}$ to be separated is equivalent to the failure of the unit section to be a closed immersion. If the closure of the unit section clo(e) is flat over S, one may construct a quotient of $\mathrm{Pic}_{X/S}$ by clo(e) that is separated and satisfies the Néron mapping property.

It is shown in [Hol16] that, for semistable curves, the closure of the unit section is flat over S if and only if X/S is aligned. Namely, alignment is a condition that allows one to construct certain Cartier divisors on X/S that correspond to points in clo(e). We construct a notion of alignment for higher dimensional varieties that generalises the case of curves. This relies on a stratification of the fibres of X/S by certain subschemes. We then construct maps between the dual graphs of the geometric fibres, and finally show that this generalisation is equivalent to the flatness of clo(e).

Finally, we show that the smoothness of the Néron model is ensured if $\mathrm{Pic}_{X/S}$ is smooth along its unit section. The idea is to show that the smooth locus of $\mathrm{Pic}_{X/S}$ is an open and closed subgroup scheme containing $\mathrm{Pic}^0_{X/S}$. In this case one then shows that if a Néron model exists, then the closure of the unit section is flat over S.

1.1.2 Overview of chapter

We introduce the notions of semistable morphisms and the dual graphs of their fibres in Section 1.2, as well as introduce the notion of alignment of a labelled graph. Section 1.3 studies how the fibres of semistable morphisms are related under specialisation and generisation of the points in S. The main result here is the construction of a map of dual graphs $\mathrm{sp} : \Gamma_s \to \Gamma_\eta$ over points s and η in S, where η is a generisation of s and Γ_p is the dual graph of the fibre X_p for a point $p \in S$. This is used in Section 1.4 to show that certain Weil divisors on X/S are also Cartier divisors, and that furthermore these Cartier divisors only exist when X/S is aligned at all geometric points. In Section 1.5 we show our notion of alignment is equivalent to the flatness of the unit section in the Picard scheme of X/S, which in turn allows us to construct a separated quotient of the Picard scheme satisfying the Néron mapping property.

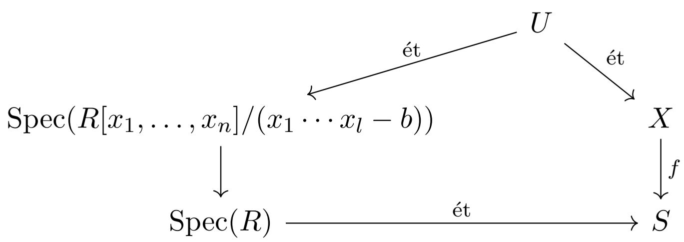
1.2 Semistable morphisms and alignment

1.2.1 Definition of semistable morphism

Definition 1.5. Let S be a locally Noetherian scheme. A morphism $f : X \to S$ is a *semistable morphism* if it is proper with geometrically connected fibres, and such that the following condition holds: For all $x \in X$ with image $s \in S$, there exists

- an étale neighbourhood $(\mathrm{Spec}(R), s')$ of (S, s);
- an étale neighbourhood (U, x') of (X, x) where U is a connected scheme;
- an element $b \in R$;
- an étale morphism $U \to \mathrm{Spec}(R[x_1, \dots, x_n]/(x_1 \cdots x_l - b))$,

making the following diagram commute:



(1.1)

Definition 1.6. Let $f : X \to S$ be a semistable morphism. Let $x \in X$ be a point with image $s \in S$, and suppose we have a diagram as in (1.1), where (U, x') is an étale neighbourhood of (X, x). Let p be the image of x' in $\mathrm{Spec}(R[x_1, \dots, x_n]/(x_1 \cdots x_l - b))$. If $\{x_1, \dots, x_l\}$ are contained in the prime ideal corresponding to p, we say that the data of a diagram as in (1.1) is a *local chart of* x.

We use $U \to \mathrm{Spec}(R[x_1, \dots, x_n]/(x_1 \cdots x_l - b))$ to denote the data of a local chart of x.

Remark 1.7. This definition of local chart is a special case of that given in definition 1.1 of [Li07].

The next lemma is immediately verified by the reader.

Lemma 1.8. *If $f : X \to S$ is a semistable morphism, then every point $x \in X$ admits a local chart.*

If $U \to \mathrm{Spec}(R[x_1, \dots, x_n]/(x_1 \cdots x_l - b))$ is a local chart of a point x, the following lemma of Li shows that the integer l is independent of the local chart.

Lemma 1.9 ([Li07], Lemma 3.2)**.** *Let $f : X \to S$ be a semistable morphism. Let $x \in X$, $s = f(x)$, and let*

$$U \to Spec(R[x_1, \dots x_n]/(x_1 \cdots x_l - b))$$

and

$$U' \to Spec(R'[x_1, \dots x_n]/(x_1 \cdots x_{l'} - b'))$$

be two local charts of f at x. Then

- *$l = l'$,*
- *the images of b and b' in $\mathcal{O}_{S,s}^{sh}$ satisfy $b = ub'$ for some unit $u \in \mathcal{O}_{S,s}^{sh}$.*

Here $\mathcal{O}_{S,s}^{sh}$ denotes the strict henselisation of $\mathcal{O}_{S,s}$ for a fixed separable closure of $k(s)$.

Remark 1.10. When the fibres of X/S are one-dimensional, one can show as in Proposition 2.5 of [Hol16] that if $f : X \to S$ is proper and flat whose fibres are semistable curves, then the completed étale local rings of closed points in the non-smooth locus are of the form $\mathcal{O}_{S,\bar{s}}[[x, y]]/(xy - b)$, where b is in the maximal ideal of $\mathcal{O}_{S,\bar{s}}$. The argument used for showing this does not generalise to higher dimensions, and so we impose the condition of semistability as in Definition 1.5 to ensure the étale local rings have a similar form as in the case of curves.

Definition 1.11. Let $f : X \to \mathrm{Spec}(k)$ be a semistable morphism with k a field, and let $\mathrm{Sing}_f^0(X) = X$, and inductively define $\mathrm{Sing}_f^k(X)$ as the non-smooth locus of $\mathrm{Sing}_f^{k-1}(X)$, viewed as a closed subscheme of X with the reduced induced subscheme structure. The *k-strata of X* are the irreducible components of the smooth locus of $\mathrm{Sing}_f^k(X)$.

Remark 1.12. Let $f : X \to S$ be a semistable morphism, $s \in S$ a point, and X_s the fibre of X over s. The property of being semistable is preserved under base change, and so $f : X_s \to s$ is a semistable morphism. If X_s is locally of the form $k[x_1, \dots, x_n]/(x_1 \cdots x_l)$, then the non-smooth locus of $\mathrm{Sing}_f^k(X)$ is of codimension 1. In the general case it suffices to work étale locally and conclude the same. The closure of a k-stratum is an irreducible component of $\mathrm{Sing}_f^k(X)$, and conversely the smooth locus of an irreducible component of $\mathrm{Sing}_f^k(X)$ is a dense open subscheme. Hence there is a bijection between elements of the set of k-strata and irreducible components of $\mathrm{Sing}_f^k(X)$.

Lemma 1.13. *Let k denote a separably closed field, and let $X \to Spec(k)$ be a semistable morphism. Then the elements of the k-strata of X are geometrically irreducible.*

Proof. This follows immediately by the definition of the k-strata. Namely, each stratum is irreducible over the base field. By Lemma 32.8.8 of [[Sta17], Tag 0364], the property of being geometrically irreducible can be checked over a separably closed field, hence the result.

□

1.2.2 Graphs and alignment

Graphs will play a key role in the notion of alignment of a semistable morphism. We recall some basic definitions.

Definition 1.14. A *graph* is an ordered triple $\Gamma = (V, E, \text{Incidence})$ where V is a finite set (called the *vertices* of the graph Γ), E the (finite) set of *edges*, and Incidence : $E \to (V \times V)/S_2$. Intuitively, the incidence relations tell us which vertices an edge connects. A graph Γ is *connected* if there is a path between any two vertices of Γ. A *cycle* of Γ is a path with at least one edge whose first and last vertex coincide and for which no other vertices or edges are repeated.

Definition 1.15. Let X/S be a semistable morphism, and let X_s be the fibre of X over a point $s \in S$. The *dual graph* Γ_s of X_s is the graph Γ_s defined as follows: For every irreducible component X_i of X_s we have one vertex v_i. For every 1-stratum C_e contained exclusively within components X_i and X_j (where possibly $X_i = X_j$) we have one edge between vertex v_i and v_j.

Remark 1.16. In general the irreducible components and the 1-strata of a fibre are not geometrically irreducible, and so given a geometric point $\overline{s} \to S$ with image s it is not in general true that Γ_s will be isomorphic to $\Gamma_{\overline{s}}$, where $\Gamma_{\overline{s}}$ is the dual graph of the geometric fibre $X_{\overline{s}}$.

However, by Lemma 1.13, the dual graph $\Gamma_{\overline{s}}$ is equivalent to the dual graph defined using elements of the 0- and 1- strata of $X_{k_s^{\text{sep}}}$, where k_s^{sep} is the separable closure of $k(s)$ within $k(\overline{s})$.

Remark 1.17. This generalises the dual graph of a semistable curve as defined in [Hol16]. Namely, in the case of a semistable curve, each irreducible component has an associated vertex, and for each point of intersection between irreducible components C_i and C_j one has an edge between the associated vertices. The set of points of intersection in this case constitute the 1-strata.

1.2.3 Alignment of a semistable morphism

Having defined the strata of any fibre X_s of a semistable morphism $f : X \to S$, we shall attach labels to the strata in such a way that the labels are constant on any connected component of $\mathrm{Sing}_f(X_s)$. These labels are the analogues of those in Definition 2.11 of [Hol16].

Definition 1.18. Let $x \in X_s$ be in the non-smooth locus. Let

$$g : U \to \mathrm{Spec}(R[x_1, \dots x_n]/(x_1 \cdots x_l - b)$$

be a local chart of x. We define the label of x to be $l(x) = (b)$, viewed as a principal ideal of $\mathcal{O}^{\mathrm{sh}}_{S,s}$.

Remark 1.19. By Lemma 1.9 this definition is independent of the local chart we choose.

Lemma 1.20 ([Li07], Lemma 3.2 and Lemma 3.4)**.** *The label $l(x)$ as in the above definition is constant on connected components of the non-smooth locus of X_s.*

Remark 1.21. While the label $l(x)$ is constant on connected components of $\mathrm{Sing}_f(X_s)$, the same is not true of the integer l in Lemma 1.9. For $x \in \mathrm{Sing}^k_f(X_s) \setminus \mathrm{Sing}^{k+1}_f(X_s)$ we will have $l = k + 1$, and hence l will in general vary on a given connected component C of $\mathrm{Sing}_f(X_s)$.

Definition 1.22. Let Y be a k-stratum of a fibre X_s, $k \geq 1$, of a semistable morphism X/S. The *label of Y* is defined to be the label $l(x)$ of any point $x \in Y$.

We now define a labelling of the dual graphs of geometric fibres of X/S. Using these labels we can attach labels to edges of the dual graph $\Gamma_{\overline{s}}$ of $X_{\overline{s}}$, where $\overline{s} \to s$ is a geometric point.

Definition 1.23. Let Γ be a graph with vertex set V and edge set E. Given a monoid M, an *edge-labelling* of Γ is a function $f : E \to M$. We can similarly define a *vertex labelling* as a function $g : V \to M$.

Let C be a cycle of a graph Γ, with an edge-labelling $f : E \to M$. We say that C is *aligned* with respect to the edge-labelling if for every pair of edges e_1 and e_2 in C there exist positive integers n_1 and n_2 such that $f(e_1)^{n_1} = f(e_2)^{n_2}$. We say that an edge-labelled graph (Γ, f) is aligned if every circuit of Γ is aligned with respect to f.

Recall from Lemma 1.13 that the 0- and 1- strata of $X_{\overline{s}}$ are in bijection with the 0- and 1-strata of the special fibre of $X \times_S \mathcal{O}_{S,\overline{s}}$ over $\mathcal{O}_{S,\overline{s}} = \mathcal{O}^{\mathrm{sh}}_{S,s}$ for any geometric point $\overline{s} \to S$, with the closed point of the latter corresponding to the separable closure of $k(s)$ in $k(\overline{s})$.

Definition 1.24. Let X/S be a semistable morphism and let $\overline{s} \to s$ be a geometric point of S. Let $\Gamma_{\overline{s}}$ be the dual graph of $X_{\overline{s}}$, and let M_s be the monoid of principal ideals of the ring $\mathcal{O}_{S,\overline{s}}$. We define a labelling of the edges of $\Gamma_{\overline{s}}$ by sending an edge e to the label of the 1-stratum of the closed fibre of $X \times_S \mathcal{O}_{S,\overline{s}}$ over $\mathcal{O}_{S,\overline{s}}$ that e corresponds to.

We say that X/S *is aligned at* $\overline{s}$ if the dual graph $\Gamma_{\overline{s}}$ is aligned with respect to the above labelling. We say that X/S is *aligned* if it is aligned at all geometric points of S.

1.3 Behaviour of strata under specialisation and generisation

In this section we will study how the dual graph $\Gamma_{\overline{s}}$ of a geometric fibre $X_{\overline{s}}$ of a semistable morphism is related to the dual graph of Γ_η, where η is a generisation of s. The main result is that we will obtain a map $\mathrm{sp} : \Gamma_s \to \Gamma_{\overline{\eta}}$.

1.3.1 Specialisation map on vertices

Lemma 1.25 and Lemma 1.26 can be stated with weaker hypotheses than the morphism f being semistable when the base scheme is integral.

Lemma 1.25. *Let $f : X \to S$ be a proper, flat morphism with reduced fibres to a locally Noetherian integral scheme S with generic point η. Assume that the fibres of f are pure dimensional. Let $Y \subset X_\eta$ be an irreducible component of X_η and let $\overline{Y}$ denote the scheme-theoretic closure of Y in X. Then for every point $s \in S$, $\overline{Y}_s$ is a union of irreducible components of X_s. Moreover, for every irreducible component Z of X_s there exists an irreducible component Y of X_η such that Z is contained in $\overline{Y}_s$.*

Proof. We will show the dimension of each irreducible component of $\overline{Y}_s$ is equal to the dimension of X_s and hence that each irreducible component of $\overline{Y}_s$ is an irreducible component of X_s. To achieve this, we will show

$$\dim(X_s) \geq \dim(\overline{Y}_s) \geq \dim(Y) = \dim(X_\eta) = \dim(X_s).$$

As $\overline{Y}_s \subset X_s$, we have $\dim(X_s) \geq \dim(\overline{Y}_s)$. By 13.1.5 of [Gro66a], fibre dimension is upper semicontinuous for proper morphisms. Hence $\dim(\overline{Y}_s) \geq \dim(Y)$. Furthermore, Y is an irreducible component of X_η and so $\dim(Y) = \dim(X_\eta)$ by the assumption the fibres have pure dimension. By flatness $\dim(X_\eta) = \dim(X_s)$. Thus $\dim(\overline{Y}_s) = \dim(X_s)$.

By Lemma 13.1.1 of [Gro66a], a dominant morphism locally of finite type between irreducible schemes is such that the dimensions of the irreducible components of fibres is greater than or equal to the dimension of the generic fibre. Because $\dim(\overline{Y}_s) = \dim(Y)$, we conclude that $\overline{Y}_s$ is equidimensional. In particular, the underlying topological space of $\overline{Y}_s$ is a union of irreducible components of X_s, and as X_s is reduced we find $\overline{Y}_s$ is a union of irreducible components of X_s.

As f is flat and locally of finite presentation, the image of any non-empty open set in X is open in S, and so contains η. Thus $\overline{X}_\eta = X$, implying the second part of the lemma. □

Lemma 1.26. *Let $f : X \to S$ be as in Lemma 1.25. Suppose in addition that the smooth locus of each fibre is dense in the fibre. If $\eta_1 \neq \eta_2$ are generic points of distinct irreducible components of X_η, then $(\overline{\eta_1})_s$ and $(\overline{\eta_2})_s$ share no irreducible component of X_s in common.*

Proof. Suppose that an irreducible component Z of X_s is contained in $\overline{\eta_1}$ and $\overline{\eta_2}$. Then the local ring $\mathcal{O}_{X,x}$ in X of every smooth point $x \in Z$ over s contains at least two minimal prime ideals, corresponding to the intersection of irreducible components of X. Hence x is not a regular point as $\mathcal{O}_{X,x}$ is not an integral domain. This contradicts the assumption that the smooth locus is dense in the fibres. □

We now focus on the situation where $f : X \to S$ is a semistable morphism, with S the spectrum of a Noetherian strictly henselian local ring. The closed point of S will be denoted by s and we shall use η to denote a generisation of s, which is no longer required to be the generic point of S.

Lemma 1.27. *Let Y be an irreducible component of X_η. Then Y is geometrically irreducible.*

Proof. To show Y/η is geometrically integral it suffices by [[Sta17], Tag 04QM] to show it has a k-rational point with Y/η smooth at that point, where k is the residue field of η. Thus it suffices to show that $\overline{Y}/\overline{\eta}$ has a section through its smooth locus, where $\overline{Y}$ is the closure of Y in X with the reduced induced structure and $\overline{\eta}$ is the closure of η in S.

Define $U = X \setminus \cup_{Y' \neq Y} \overline{Y'}$, where Y' varies over the irreducible components of X_η. Note that U_s is a dense and non-empty open subscheme in each fibre $\overline{Y}_s$ of $\overline{Y}_s$ by Lemma 1.26.

Furthermore, the smooth locus of U/S is open and dense in U by Remark 1.12. Thus $\overline{Y}_s$ admits a non-empty open subscheme of smooth points of X/S that is dense in each irreducible component of $\overline{Y}_s$, and as X_s has sections through the smooth locus of its irreducible components, so too does $\overline{Y}_s$. By 6.2.13 of [Liu02] this section lifts to a section of $\overline{\eta}$ through the smooth locus of $\overline{Y}$. □

Definition 1.28. The *specialisation map on irreducible components*, $\mathrm{sp}_{s\to\eta}$, from the irreducible components of X_s to those of X_η is defined by sending an irreducible component Z of X_s to the unique irreducible component Y of X_η such that $Z \subset \overline{Y}_s$.

Remark 1.29. The specialisation maps are transitive. If η is a generisation of s and ζ is a generisation of η, then $\mathrm{sp}_{s\to\zeta} = \mathrm{sp}_{\eta\to\zeta} \circ \mathrm{sp}_{s\to\eta}$. This follows by construction: if $Z \subset X_s$, $Y \subset X_\eta$, and $W \subset X_\zeta$ are irreducible components of their respective fibres with $Y \subset \overline{W}_\eta$ and $Z \subset \overline{Y}_s$, then $Z \subset \overline{W}_s$.

Remark 1.30. Lemma 1.27 shows that the irreducible components of X_η correspond bijectively to those of $X_{\overline{\eta}}$. This will allow us to define a map on vertices of Γ_s to those of $\Gamma_{\overline{\eta}}$. To define what the map will be on the edges we must establish some additional results.

1.3.2 The specialisation map on edges

Lemma 1.31. *Suppose that $S = Spec(R)$ is the spectrum of a Noetherian strictly henselian local ring with closed point s, and let $f : X \to S$ be a semistable morphism. Let X_1 and X_2 be two irreducible components of X_s. Suppose X_1 and X_2 intersect and contain some element Z of the k-strata of X_s, $k \geq 1$, such that the label (b) of Z is generated by an element that becomes a unit over η. Then $sp_{s\to\eta}(X_1) = sp_{s\to\eta}(X_2)$.*

Proof. Fix a chain of prime ideals $m_s \supset p_1 \supset \ldots \supset p_i \supset p_{i+1} \ldots \supset m_\eta$ of maximum length in R, where m_s and m_η correspond to the points s and η respectively. Let j be such that $b \in p_j$ but $b \notin p_{j+1}$. It suffices to show that $\mathrm{sp}_{s \to p_{j+1}}(X_1) = \mathrm{sp}_{s \to p_{j+1}}(X_2)$, for when this holds Remark 1.29 implies that $\mathrm{sp}_{s \to \eta}(X_1) = \mathrm{sp}_{s \to \eta}(X_2)$ also holds.

The properties of being Noetherian and henselian local are preserved under localisation and quotients, and so we may assume that $s = p_j$, $\eta = p_{j+1}$, and that $\dim(R) = 1$. By the theorem of Krull-Akizuki (see [[Sta17], Tag 00P7]), the integral closure of R in its field of fractions is a normal noetherian ring of dimension 1. Localising at a prime ideal lying over the point s, we obtain a discrete valuation ring denoted by F.

Fix a closed point $x \in Z$. Let X_F denote the base change of X with $\mathrm{Spec}(F)$, and let $x' \in X_F$ be any closed point lying over $x \in Z$.

As X_1 and X_2 are geometrically integral and the generic fibre of X_F is isomorphic to that of X, it suffices to show that the $\mathrm{sp}_{s' \to \eta'}(X_{i,F})$ agree, where s' and η' are the closed and generic points of $\mathrm{Spec}(F)$, respectively, and $X_{i,F}$ is the component of $X_{F,s'}$ lying over X_i. Hence we reduce to the case where $S = \mathrm{Spec}(R)$ is the spectrum of a discrete valuation ring.

Let $Y_i = \mathrm{sp}_{s \to \eta}(X_i)$. Let $U \to \mathrm{Spec}(R'[x_1, \ldots, x_n]/(x_1 \cdots x_l - b))$ be a local chart of $x \in Z$, with $g : U \to X$ an étale neighbourhood of x, and R' an étale ring over R. By restricting to an open subscheme of U, we may assume that U is connected. As b is a unit over η we find that the generic fibre of U is smooth.

Because $g(U)$ is an open set containing the generic points of X_1 and X_2, so too must it contain the generic points of Y_1 and Y_2. Furthermore, $g(U)$ is connected, being the continuous image of a connected space, and flat over S as it is open in X. The étale local rings points of X_s are reduced by Lemma 15.42.4 of [[Sta17], Tag 07QL], and the same lemma implies that X_s is reduced.

Hence $g(U)$ is connected, flat over S, and has reduced special fibre, so by Lemma 36.26.7 of [[Sta17], Tag 055C] the generic fibre of $g(U)$ over S is connected, and as it is smooth it is also irreducible. Hence the generic points of Y_1 and Y_2 are equivalent, whence $\mathrm{sp}_{s \to \eta}(X_1) = \mathrm{sp}_{s \to \eta}(X_2)$. □

Definition 1.32. Let $f : X \to S$ be a finite-type morphism. Then $\mathrm{Sing}_f(X)$ is the non-smooth locus of f, viewed as a closed subscheme of X

with the reduced induced structure. As a set, $\mathrm{Sing}_f(X) = \cup_{i\geq 1}\mathrm{Sing}^i_f(X)$.

As $f : X \to S$ is flat and of finite presentation, f is smooth at a point x if and only if x is a smooth point of the fibre $X_{f(x)}$. By Lemma 3.5 of [Li07], the set of connected components of the non-smooth locus $\mathrm{Sing}_f(X)$ of X/S is in bijection with the set of connected components of the singular locus of X_s via taking the fibre over the closed point. Let C be a connected component of $\mathrm{Sing}_f(X)$, and let η be any generisation of s with C_η non-empty. Lemma 3.15a of [Li07] shows that the label of C_η is the image of the label of C_s in $\mathcal{O}_{S,\overline{\eta}}$, where $\overline{\eta}$ corresponds to a choice of an algebraic closure of $k(\eta)$. In particular, if the label of C_s is 0, then the label of any non-empty fibre C_η of C is 0.

Remark 1.33. By Lemma 30.10.3 of [[Sta17], Tag 0C3H], the non-smooth locus of a flat morphism locally of finite presentation with equidimensional fibres of dimension d is cut out by the d-th Fitting ideal of $\Omega_{X/S}$. Furthermore, the Fitting ideal commutes with arbitrary base change by Lemma 30.10.1 of [[Sta17], Tag 0C3H].

In particular, if X/S is a semistable morphism with n-dimensional fibres then $\mathrm{Sing}_f(X)$ is cut out by the n-th Fitting ideal. If $U \to T = \mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b))$ is a local chart of some closed point $x \in X$, then $\mathrm{Sing}(U) \cong \mathrm{Sing}(X) \times_X U$ and also $\mathrm{Sing}(U) \cong \mathrm{Sing}_(T) \times_T U$.

Lemma 1.34. *Let X/S be a semistable morphism with S a Noetherian strictly henselian local scheme with closed point s, and let C a connected component of the singular locus $\mathrm{Sing}_f(X)$ of X whose label is 0. Then C/S is flat and proper over S.*

Proof. X is proper over S and C is a closed subscheme of X, so C is proper over S.

By definition of semistable, any point $x \in C$ admits a local chart $U \to T = \mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l))$, where $g : U \to X$ is an étale neighbourhood of x. As g and h are open and flatness can be checked étale locally, we may assume that g and h are surjective by replacing X and T with their respective images under g and h.

The non-smooth locus $\mathrm{Sing}(T)$ of T is defined by the ideal I generated by polynomials of the form $x_1 \cdots \hat{x}_i \cdots x_l$, $1 \leq i \leq l$, where $\hat{x}_i$ denotes the exclusion of x_i from the term. Viewed as a module over R, $R[x_1, \ldots, x_n]/I$

is free with basis

$$\{x_1^{i_1}\cdots x_n^{i_l} | i_j \in \mathbb{Z} \,\forall\, i,j \text{ s.t. } \exists k_1, k_2 \in [1,l] \text{ with } i_{k_1}, i_{k_2} = 0\}.$$

In particular, Sing(T) is flat over S.

By Remark 1.33, Sing(T) (resp. Sing(U) and $\text{Sing}_f(X)$) is cut out by the d-th Fitting ideal of the module of relative differentials of T (resp. U and X) over S, where $d = n-1$. As formation of the Fitting ideal commutes with arbitrary base change, we have that Sing(U) is isomorphic to $\text{Sing}(T)\times_T U$, which is étale over Sing(T) and hence flat over S. Finally, as $\text{Sing}(U) \cong \text{Sing}_f(X) \times_X U$ is flat over S and flatness can be checked étale locally, we conclude that $\text{Sing}_f(X)$ is flat over S.

□

Corollary 1.35. *Let X/S be a semistable morphism, with (S,s) the spectrum of an integral Noetherian strictly henselian local ring with generic point η. Let $Y \subset X_\eta$ denote an irreducible component of the non-smooth locus of X_η. Then Y is geometrically irreducible.*

Proof. The label of the connected component C of $\text{Sing}_f(X)$ containing Y is necessarily 0 by the integrality of S, and so by Lemma 1.34 we conclude C is flat over S. The remainder of the proof proceeds in a manner completely analogous to that of Lemma 1.27.

□

Let C be a connected component of the non-smooth locus of X_s with label (b), and suppose the generator b of (b) is not a unit over η for some generisation η of s. After taking a suitable quotient in S we may assume that η is the generic point and that $b = 0$. We now describe an analogue for $\text{sp}_{s\to\eta}$ to the 1-strata of X_s contained in C.

Let $Z \subset C$ be a 1-stratum with generic point η_Z, and let $U \to T = \text{Spec}(R[x_1,\ldots,x_n]/(x_1x_2)$ denote a local chart of a closed point $x \in Z$. Fix a lift p of η_Z to U, necessarily mapping to the point defined by (m_s, x_1, x_2) of T. As (m_s, x_1, x_2) generalises in T to the point (m_η, x_1, x_2), we can find a point $q \in U$ that is a generisation of p lying over the point defined by (m_η, x_1, x_2).

Let η_Y be the image of q in X. This is a generisation of η_Z, and as U is étale over T and X with the generic points of the 1-strata of the fibres of

U lying over the generic points of the 1-strata of the fibres of T and X, we see that η_Y is the generic point of an element Y of the 1-strata of X_η.

Lemma 1.36. *There is a unique point $\eta_Y \in X_\eta$ generalising η_Z that is the generic point of an element of the 1-strata of X_η.*

Proof. The above construction gives existence. Uniqueness follows from the fact the connected component C of $\mathrm{Sing}_f(X)$ is flat over S by Lemma 1.34, and then arguing as in the proof of Lemma 1.27.

□

Definition 1.37. Let η be a generisation of the closed point s. Let Z be a 1-stratum of X_s and assume that the label of Z is not a unit over η. We set $\mathrm{sp}'_{s\to\eta}(Z) = Y$, where Y is the unique 1-stratum of X_η as constructed above.

1.3.3 The specialisation map

We now define the specialisation map from Γ_s to $\Gamma_{\overline{\eta}}$. By Lemma 1.27 and Corollary 1.35 we find that $\Gamma_{\overline{\eta}} \cong \Gamma_\eta$, and so it suffices to define the specialisation map from Γ_s to Γ_η.

We define $\mathrm{sp} : \Gamma_s \to \Gamma_\eta$ as follows: A vertex v of Γ_s corresponds to an irreducible component Y of X_s. Let w be the vertex corresponding to $\mathrm{sp}_{s\to\eta}(Y)$ and set $\mathrm{sp}(v) = w$. If e is an edge between vertices v and w of Γ_s corresponding to an element Z of the 1-strata with label b we have two possibilities:

- b is a unit over η. In this case we have v and w map to the same vertex in Γ_η by Lemma 1.31, and we define $\mathrm{sp}(e)$ to be the vertex v and w map to.
- b is not a unit over η. In this case we set $\mathrm{sp}(e) = f$, where f is associated to the element Y of the 1-strata of X_η such that $\mathrm{sp}'_{s\to\eta}(Z) = Y$. As generisations are transitive we have that f is incident with $\mathrm{sp}(v)$ and $\mathrm{sp}(w)$.

The map of the edge-labellings sends a label (b) to the principal ideal (b) of $\mathcal{O}_{X,\overline{\eta}}$ when sp sends an edge to an edge; otherwise no label is required.

We conclude with a lemma that will be used in the proof of Lemma 1.41.

Lemma 1.38. *Suppose that S is an excellent integral strictly Henselian local ring with closed point s and generic point η. Let $f : X \to S$ be a semistable morphism. Suppose we have two irreducible components Z_1 and Z_2 of X_s that share a common 1-stratum whose label is 0. Then $sp_{s\to\eta}(Z_1) \neq sp_{s\to\eta}(Z_2)$.*

Proof. Let us proceed by contradiction and assume $\mathrm{sp}_{s\to\eta}(Z_1) = \mathrm{sp}_{s\to\eta}(Z_2) = Y$, and let $\overline{Y}$ denote its closure in X. We shall show that the normalisation of $\overline{Y}$ is smooth with irreducible special fibre, and hence that $Z_1 = Z_2$, leading to the desired contradiction.

Consider any point x in the special fibre $\overline{Y}_s$ of $\overline{Y}$ lying in a 1-stratum. Fix a local chart $U \to T = \mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1x_2))$ of x.

The normalisation T^ν of T is the disjoint union of its irreducible components, which are themselves defined by ideals of the form (x_i) for $1 \leq i \leq 2$.

As normalisation commutes with étale base change by [[Sta17], Tag 082F], we have

$$U^\nu = U \times_T T^\nu.$$

Hence U^ν is étale over T^ν, and as T^ν is smooth over S, so too is U^ν.

Again using the fact that normalisation commutes with étale base change, we note that U^ν is étale over X^ν. Thus as smoothness is étale local on the source we conclude that $(\overline{Y})^\nu$ is smooth over S.

It remains to show that $(\overline{Y})^\nu$ is proper over S. As $\overline{Y}$ is excellent (being of finite type over S) this follows by Theorem 8.2.39 of [Liu02].

Hence $(\overline{Y})^\nu$ is proper and smooth over S, and so by Corollary 15.5.4 of [Gro66a] its special fibre is connected and smooth, hence irreducible. The morphism $(\overline{Y})^\nu \to Y$ is surjective by Lemma 28.51.5 of [[Sta17], Tag 035E], and so the generic point of the special fibre of $(\overline{Y})^\nu$ must map to a generic point of one of the irreducible components of Y_s, say η_{Z_1}. As η_{Z_2} is also in the image, it is necessarily a specialisation of η_{Z_1}, contradicting the assumption $Z_1 \neq Z_2$. □

1.4 Cartier divisors and alignment

1.4.1 Constructing Cartier divisors on X

Having defined the sp map we are ready to construct Cartier divisors on X in a manner analogous to the construction in section 5.2 of [Hol16]. In this subsection we shall consider semistable morphisms $X \to S$ where $S = \mathrm{Spec}(R)$ is the spectrum of an excellent regular strictly henselian local ring. We shall denote the closed point of S by s and let Γ_s denote the dual graph. The following definition mirrors that of Definition 5.5 of [Hol16].

Definition 1.39. Let $a \in R$ be non-zero and a non-unit. Let $V(a)$ denote the set of edges of Γ_s whose labels b are such that $(b)^m = (a)$ as ideals in R for some $m \geq 1$. Let $\Gamma_s(a)$ denote the graph obtained from Γ_s by removing all edges in set $V(a)$.

Let $\eta_1, \ldots, \eta_r$ denote the generic points of $\mathrm{Spec}(R/a)$, and let m_i denote the order of vanishing of a at η_i. Let H denote the set of vertices of a connected component of $\Gamma_s(a)$. For every $1 \leq i \leq r$, let $Z_i^1, \cdots, Z_i^{s_i}$ denote the vertices of Γ_{η_i} which are images under sp of vertices in H. Note that each Z_i^j can be viewed as a prime Weil divisor on X.

We now construct a Weil divisor on X analogous to that of Definition 5.6 of [Hol16].

Definition 1.40. Define a Weil divisor $\mathrm{div}(a; H)$ by

$$\mathrm{div}(a; H) = \sum_{i=1}^{r} m_i \sum_{j=1}^{s_i} Z_i^j.$$

Lemma 1.41. *The Weil divisor div$(a; H)$ defined above is a Cartier divisor on X.*

Proof. Our proof shall closely follow that of Lemma 5.7 of [Hol16]. It suffices to check D is Cartier at closed points of the closed fibre by Lemma 5.8 of [Hol16], and in particular at points in the non-smooth locus.

Let x be a closed point in the non-smooth locus of X_s, and consider a local chart

$$U \to \mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b)),$$

where U is an étale neighbourhood of x and where necessarily $b \in m_s$ for m_s the maximal ideal of R. From the description of the local chart we see that x belongs to a unique $(l-1)$-stratum C. Hence there are at most l vertices of Γ_s whose corresponding irreducible components of X_s contain C.

Suppose that k of these vertices, $v_1, \ldots, v_k$ are in H, and r are not in H, say $v_{k+1}, \ldots, v_{k+r}$. If $k = 0$ we have that $\operatorname{div}(a; H)$ is locally cut out by a unit. If $r = 0$ one shows as in [Hol16] shows that $\operatorname{div}(a; H)$ is cut out locally by a. So assume now that k and r are greater than 0.

Let $g : (U, x') \to (X, x)$ an étale neighbourhood of x with $h : U \to T = \operatorname{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b)$ a local chart. It suffices to show that $g^*\operatorname{div}(a; H)$ is a Cartier divisor in a neighbourhood of x' by Lemma 2.9 of [Hol16].

Base changing along $V(x_i, m_s) \to s$ for any $1 \leq i \leq l$, we find that $V(x_i, m_s) \times_T U$ is smooth over s. In particular there exists a unique generisation x'_i of x' in U_s lying over each point (x_i, m_s) for all $1 \leq i \leq l$, as otherwise any point lying over x' in $V(x_i, m_s) \times_T U$ would not be smooth over s.

Each irreducible component $V(x_i, m_s)$ of T_s, $1 \leq i \leq l$, corresponds to one of the $k + r$ vertices of Γ_s containing the point x, though in general $l \geq k + r$ and so the correspondence may not be injective. Let $\alpha \geq k$ be such that $V(x_i, m_s) \subset T_s$ corresponds to a vertex in H for $1 \leq i \leq \alpha$, and for $\alpha + 1 \leq j \leq l$ the irreducible components $V(x_j, m_s)$ correspond to vertices not in H, where $l - \alpha \geq r$.

Let m be such that $(b)^m = (a)$ in R. We'll show that $g^*\operatorname{div}(a; H) = h^*\operatorname{div}(x_1 \cdots x_\alpha)^m$ and hence conclude the proof.

Fix an irreducible component of $R/(a)$, say η_1, and let $\operatorname{sp} : \Gamma_s \to \Gamma_{\eta_1}$ be the specialisation map. Note that sp does not map any of the vertices in H to the image of a vertex not in H, as otherwise there exists an edge whose label is a unit between H and the complement of H by Lemma 1.31 and Lemma 1.38, contradicting how H was chosen.

If $V_i = \operatorname{sp}(v_i)$, then $\cup_{1 \leq i \leq k} \overline{V}_i \supset \cup_{1 \leq i \leq k} v_i$, and the same holds along pullbacks via g. Similarly, $\cup_{k+1 \leq j \leq k+r} \overline{V}_j \supset \cup_{k+1 \leq j \leq k+r} v_j$.

In particular $h^*(x_1 \cdots x_\alpha)^m$ does not vanish on $\cup_{k+1 \leq j \leq k+r} g^*\overline{V_j}$, nor does $h^*(x_{\alpha+1} \cdots x_l)$ vanish on $\cup_{1 \leq i \leq k} g^*\overline{V_i}$. Hence, as we have that $\operatorname{div}(a) =$

$\mathrm{div}(b^m) = \mathrm{div}(x_1 \dots x_\alpha)^m + \mathrm{div}(x_{\alpha+1} \dots x_l)^m$, we conclude that $h^*(x_1 \cdots x_\alpha)^m$ vanishes on $g^*\overline{V_i}$, $1 \leq i \leq k+1$, with the same multiplicity as a. This concludes the proof.

□

1.4.2 Cartier labellings on graphs

Suppose now that $X \to S$ is smooth over a dense open set $U \subset S$. In order to introduce the notion of Cartier divisors on the dual graph Γ_s of X_s we first recall the definition and existence of "test curves" in S from section 5.1 of [Hol16].

Definition 1.42 (Definition 5.1 of [Hol16])**.** Let S be a scheme, $s \in S$, and $U \subset S$ an open subscheme. A *non-degenerate trait* in S through s is a morphism $\phi : \mathcal{T} \to S$ where $\mathcal{T}$ is the spectrum of a discrete valuation ring and ϕ maps the closed point of $\mathcal{T}$ to s and the generic point of $\mathcal{T}$ to a point in U.

Lemma 1.43 (Lemma 5.2 of [Hol16])**.** *Let S be a Noetherian scheme, $s \in S$ and $U \subset S$ a dense open subscheme. Then there exists a non-degenerate trait X in S through s.*

With the existence of non-degenerate traits, we now define Cartier functions on Γ_s. Recall that we have an edge-labeling $l : E \to M_s$ on Γ_s, where M_s denotes the monoid of principal ideals of $\mathcal{O}_{S,s}$.

Definition 1.44 (c.f. Definition 5.3 of [Hol16])**.** Let (S, s) be the spectrum of a Noetherian strictly henselian local ring and let $f : X \to S$ be a semistable morphism that is smooth over a dense open set $U \subset S$. Denote by $l : E \to M_s$ the edge-labelling of Γ_s by elements of M_s as in Definition 1.18.

Let $\phi : \mathcal{T} \to S$ be a non-degenerate trait through s, and let ord denote the standard valuation on elements of $\Gamma(\mathcal{T})$. We say that a vertex-labelling $m : V \to \mathbb{Z}$ of Γ_s is $\mathcal{T}$*-Cartier* if for every edge $e \in \Gamma_s$ incident with vertices v_1 and v_2 we have that $m(v_1) - m(v_2)$ is divisible by $\mathrm{ord}(\phi^*(l(e)))$, where $l(e)$ is the label of edge e.

Given a fibral Weil divisor D on $X_\mathcal{T}$, we define a vertex labelling m of Γ_s by attaching to v the multiplicity of D along the irreducible component of X_s corresponding to v.

The significance of this definition is found in the following lemma:

Lemma 1.45. *With the notation as above, a vertex labelling m is $\mathcal{T}$-Cartier if and only if there exists a fibral Cartier divisor D on $X_{\mathcal{T}}$ whose labelling is m.*

Proof. Let $\mathcal{T} = \mathrm{Spec}(A)$ have uniformiser π and standard valuation ord on elements of A.

Assume first that we have a vertical Cartier divisor D on $X_{\mathcal{T}}$ with associated vertex labelling m. Let e be an edge of Γ_s, incident with vertices v_1 and v_2 and with label $(b) \subset \mathcal{O}_{S,s}$, and let $x \in X_s$ be a closed point in the 1-stratum associated to e. By the existence of local charts we see that the $k(s)$-rational points of the 1-stratum are dense in the 1-stratum, and so we may assume that x is $k(s)$-rational.

Let $U \to \mathrm{Spec}(R[x_1, \dots, x_n]/(x_1x_2 - b)$ be a local chart of x. Via base change over S we obtain a local chart $U' \to \mathrm{Spec}(A[x_1, \dots, x_n]/(x_1x_2 - \pi^f)$ of an (A/π)-rational point $x' \in X_{\mathcal{T}}$ lying over $x \in X$. Here $f = \mathrm{ord}\phi^*b$.

In particular, as x' is a rational point in the special fibre, the completion of the étale local ring of x' in $X_{\mathcal{T}}$ is isomorphic to $\hat{A}[[x_1, \dots, x_n]]/(x_1x_2 - \pi^f)$. As $\mathrm{Spec}(\hat{A}[[x_1, \dots, x_n]]/(x_1x_2 - \pi^f))$ is flat over $X_{\mathcal{T}}$, D pulls back to a vertical Cartier divisor on $\mathrm{Spec}(\hat{A}[[x_1, \dots, x_n]]/(x_1x_2 - \pi^f))$. The irreducible components of the special fibre of $\mathrm{Spec}(\hat{A}[[x_1, \dots, x_n]]/(x_1x_2 - \pi^f))$ are defined by ideals of the form (x_i, π) for $1 \le i \le 2$, and the function x_i vanishes on (x_i, π) to order f. After possibly reordering the x_i's, we may assume that (x_1, π) lies over v_1 and (x_2, π) lies over v_2. From this we see that $f | m(v_1) - m(v_2)$.

Conversely, let m be a $\mathcal{T}$-Cartier vertex labelling. Let D be the associated Weil divisor on $X_{\mathcal{T}}$. By adding or subtracting a multiple of $\mathrm{div}\pi$ we may assume that D is effective and that at least one of the v_i's has coefficient 0. By Lemma 5.8 of [Hol16], to show D is Cartier it suffices to show it is Cartier at closed points in the closed fibre, and in particular at points in the non-smooth locus. Let x be such a point, and fix a local chart $h : U \to \mathrm{Spec}(A[x_1, \dots, x_n]/(x_1 \cdots x_l - \pi^f)$, where $g : (U, x') \to (X, x)$ is an étale neighbourhood of x. By Lemma 2.9 of [Hol16], to show D is Cartier at x it suffices to show g^*D is Cartier at x'.

Let $g^*D = \sum_i m_i w_i$, where the w_i's are prime Weil divisors supported in the special fibre of U. We may assume each w_i contains x' by choosing a smaller neighbourhood of x' as needed.

As h is étale, each w_i lies over some prime Weil divisor in the special fibre of $\mathrm{Spec}(A[x_1,\ldots,x_n]/(x_1\cdots x_l-\pi^f)$. The irreducible components of the special fibre of $\mathrm{Spec}(A[x_1,\ldots,x_n]/(x_1\cdots x_l-\pi^f)$ are smooth over $\mathrm{Spec}(A/\pi)$, and so $h(w_i)\neq h(w_j)$ for $i\neq j$, and after rearranging we may write the image of $h(w_i)$ as the point defined by (π,x_i) for $1\leq i\leq l$. Then the rational function $\Pi_{1\leq i\leq l}x_i^{m_i}$ pulls back to the divisor g^*D on U, whence g^*D and hence D are vertical Cartier divisors.

□

1.4.3 Alignment and Cartier divisors

Recall that $f: X\to S$ is said to be aligned at a geometric point $\overline{s}\to S$ if the dual graph $\Gamma_{\overline{s}}$ of $X_{\overline{s}}$ is aligned with respect to the edge-labelling $l: E\to M_{\overline{s}}$, as in Definition 1.24. The following two lemmas will be used in proving Theorem 1.51. They imply the existence and non-existence of certain Cartier divisors on X when X/S is aligned and not aligned, respectively.

Lemma 1.46 (c.f. Lemma 5.12 of [Hol16])**.** *Let (S,s) be the spectrum of an excellent regular strictly henselian local ring, and $f: X\to S$ a semistable morphism that is smooth over a dense open subscheme $U\subset S$. Suppose that X/S is aligned at s, and let $\phi:\mathcal{T}\to S$ denote a non-degenerate trait through s. Let m denote a $\mathcal{T}$-Cartier vertex labelling of Γ_s that takes the value 0 on some fixed vertex v_0. Then there exists a Cartier divisor D on X/S, trivial over the generic point of S, such that m is the vertex labelling corresponding to ϕ^*D.*

Our proof will closely follow that of Lemma 5.10 and Lemma 5.12 of [Hol16].

Proof. We begin by defining three labellings on the edges of Γ_s. The first labelling, denoted l_{orig}, is the usual labelling by the monoid M_s of principal ideals of $\mathcal{O}_{S,s}$.

The second labelling is the quotient of l_{orig} by the equivalence relation that $[a]=[b]$ if and only if $(a)^n=(b)^m$ as ideals of $\mathcal{O}_{S,s}$ for some positive integers n and m. This labelling is denoted by l_Q, where Q denotes the quotient of M_s this equivalence relation.

The final labelling on the edges is determined by sending a label in M_s to $\mathrm{ord}\phi^*(l(e))\in\mathbb{N}\cap\{\infty\}$, where ord denotes the usual valuation on $\Gamma(\mathcal{T})$.

This labelling is denoted by $l_{\mathcal{T}}$. By Lemma 1.45, a vertex labelling is $\mathcal{T}$-Cartier with respect to $l_{\mathcal{T}}$ if and only if it arises from a vertical Cartier divisor on $X_{\mathcal{T}}$.

As l_Q is constant on circuits by the assumption X/S is aligned at s, Lemma 5.11 of [Hol16] implies that we may write the $\mathcal{T}$-Cartier vertex labelling m as a finite sum of functions $m_i : V \to \mathbb{Z}$ such that

- All of the m_i's take the value 0 at v_0, where v_0 is the fixed vertex on which m is 0;
- Each m_i is $\mathcal{T}$-Cartier;
- Each m_i satisfies the following: There exists a subset $B \subset E$ of edges whose labels in l_Q are the same, and such that m_i is constant on connected components of the graph obtained by deleting all edges in set B.

Hence we may reduce to the case where m satisfies the above three properties. In particular there exists an element $h \in Q$ such that m is constant on connected components of Γ_s obtained by deleting every edge with label h. By definition of Q, there exists an element $a \in \mathcal{O}_{S,s}$ such that for each edge e with $l_Q(e) = h$, we have $(a) = (l_{orig}(e))^{n_e}$ for some positive integer n_e depending on e.

As S is factorial, we may decompose a into a product of prime elements to find that there exists some element $\alpha \in \mathcal{O}_{S,s}$ such that $l_{orig}(e)$ is a power of α for each e with label $l_Q(e) = h$. Because m is constant on connected components of Γ_s obtained by deleting all edges with $l_Q(e) = h$, we have that $\mathrm{ord}\phi^*(\alpha)$ divides $l_{\mathcal{T}}(e)$ for every edge e of Γ_s. Moreover, as m takes the value 0 at at least one vertex, we conclude that $\mathrm{ord}\phi^*(\alpha)$ divides $m(v)$ for all v.

Let H be a connected component of Γ_s with edges of label $l_Q(e) = h$ removed on which m is non-zero. Then $m(v) = r\mathrm{ord}\phi^*(\alpha)$ on all vertices of H for some $r > 0$, and we may assume by the decomposition of m into the m_i's as above that m is 0 outside of H.

Let $D = \mathrm{div}(\alpha^r, H)$. This is Cartier by Lemma 1.41, trivial over the generic point by construction. We shall show that the vertex labelling associated to ϕ^*D is m, concluding the proof.

To see this, note that ϕ^*D is 0 outside of H by construction. The proof of Lemma 1.41 showed that ϕ^*D is cut out by α^r near the generic points of vertices in H, and hence that

$$\mathrm{ord}\phi^*\alpha^r = r\mathrm{ord}\phi^*\alpha = m(v)$$

on vertices of H, as required. □

Lemma 1.47 (c.f. Lemma 5.13 of [Hol16]). *Let S be an excellent regular strictly Henselian local scheme with closed point s, $U \subset S$ a dense open subscheme, and $X \to S$ a semistable morphism that is smooth over U. If X/S is not aligned at some $s \in S$, then for every non-degenerate trait $\phi : \mathcal{T} \to S$ through s we can find a Cartier divisor D on $X_{\mathcal{T}}$, trivial over the generic point of $\mathcal{T}$, such that there does not exist a Cartier divisor E on X/S, trivial over U, with ϕ^*E linearly equivalent to D.*

Proof. Let Γ_s be the dual graph of X_s, and let $\phi : \mathcal{T} \to S$ be a non-degenerate trait through s. By assumption there exists a circuit with vertices $v_0, v_1, \ldots, v_N$, in order, with edge e_i incident with v_i and v_{i+1} (taken modulo N) in Γ_s, and where if (b_i) is the label of e_i, then (without loss of generality) $(b_0)^{k_0} \neq (b_1)^{k_1}$ for all $(k_0, k_1) \in \mathbb{N}^2$.

Let d be the product of all the values of $\mathrm{ord}_T(\phi^* b_l)$ as b_l runs over the distinct edge labels of Γ_s. Set $D = dv_1$, which is a Cartier divisor on $X_{\mathcal{T}}$ by Lemma 1.45. Suppose there exists a Cartier divisor E on X such that ϕ^*E is linearly equivalent to D. We shall show that this implies the existence of a multiplicative relation between b_0 and b_1, thereby deriving a contradiction.

By 6.2.13 of [Liu02], there exists sections $\sigma_i : S \to X$ through the smooth locus of v_i for all v_i in the circuit. Here we view v_i as both a vertex of Γ_s and the corresponding irreducible component of X_s. Let $(\sigma_0)_{\mathcal{T}}$ denote the section from $\mathcal{T}$ to $X_{\mathcal{T}}$ induced by σ_0. Because D is generically trivial on X_0, $(\sigma_0)^*_{\mathcal{T}} D = 0$. As S is regular, $\sigma_0^* E$ is a Cartier divisor, and the same is true for $\sigma_i^* E$ for all i. We may assume that $\sigma_0^* E = 0$ after multiplying by a suitable Cartier divisor.

As $\pi_* \mathcal{O}_X = \mathcal{O}_S$ (by Exercise 9.3.11 of [FGI+05]) we then have $\phi^* E = D$, as otherwise $\phi^*E - D$ is a multiple of $\mathrm{div}\pi$ for π the uniformizer of $\Gamma(\mathcal{T})$, contradicting the fact that $\sigma_0^* E = 0$ and $(\sigma_0)^*_{\mathcal{T}} D = 0$.

Let $f_i \in \mathcal{O}_{S,s}$ be such that $\mathrm{div}(f_i) = \sigma_i^* E$. Given $a, b \in \mathcal{O}_{S,s}$, we shall write $a \sim b$ if a and b differ by a unit. Thus $f_0 \sim 1$. By Lemma 1.48, we have $f_1 \sim b_0^{d_0} f_0$ for some $d_0 \in \mathbb{Z}$, whence $d_0 = d/\mathrm{ord}_T(b_0)$, and so we can equivalently write $f_1 \sim (b_0)^{\frac{d}{\mathrm{ord}_T(b_0)}} f_0$. Similarly, $f_2 \sim (b_1)^{-\frac{d}{\mathrm{ord}_T(b_1)}} f_1$, and $f_2 \sim \ldots \sim f_N \sim f_0$. We conclude that $(b_1)^{\frac{d}{\mathrm{ord}_T(b_1)}} = (b_0)^{\frac{d}{\mathrm{ord}_T(b_0)}}$, contradicting the non-alignment of Γ_s. □

Lemma 1.48. *Let S be the spectrum of a regular strictly henselian local ring with closed point s, and $X \to S$ a semistable morphism. Let $v_1, \ldots, v_n$ denote the set of irreducible components of X_s, where for each i we have a section $\sigma_i : S \to X$ passing through the smooth locus of v_i. Let E be a Cartier divisor on X, trivial over a dense open subscheme $U \subset S$, and let $f_i \in Frac(\mathcal{O}_{S,s})^*$ be such that $div f_i = \sigma_i^* E$.*

Suppose there is an edge e incident with v_1 and v_2 in the dual graph of X_s. Let $b \in \mathcal{O}_{S,s}$ be the label of the associated 1-stratum. Then, up to multiplication by units in $\mathcal{O}_{S,s}$, we have $f_1 = b^\delta f_2$ for some $\delta \in \mathbb{Z}$.

Proof. Let $p \in X_s$ be a k-rational point on v_1 and v_2 lying in the 1-stratum of X_s associated to e, and fix a local chart $U \to \mathrm{Spec}(\mathcal{O}_{S,s}[x, y, x_1, \ldots, x_n]/(xy-b))$, where $(U, x') \to (X, x)$ is an étale neighbourhood of x. As p is k-rational the image of x' in $\mathrm{Spec}(\mathcal{O}_{S,s}[x, y, x_1, \ldots, x_n]/(xy - b))$ is defined by the ideal $(m_s, x, y, x_1, \ldots, x_n)$ and so the completion of the étale local ring of X at p is of the form $R = \hat{\mathcal{O}}^{\text{ét}}_{S,s}[[x_1, \ldots, x_n]][[x, y]]/(xy - b)$. We may assume E is effective by adding to it the pullback of some effective Cartier divisor on S. Locally near p we have that E is given by an element $r \in \mathrm{Frac}(R)^*$. By Theorem 4.1 of [Hol16] we may write $r = aux^m y^n$, where $a \in \mathcal{O}_{S,s}[[x_1, \ldots, x_n]]$, $u \in R^*$, and $m, n \in \mathbb{Z}_{\geq 0}$.

Suppose without loss of generality that y vanishes on v_2 and x vanishes on v_1. As y is generically invertible on v_1, at the generic point η_1 of v_1 we have that E is defined by ax^m, hence by ab^m. Hence $\sigma_1^* E = \mathrm{div}(ab^m)$. Similarly $\sigma_2^* E = \mathrm{div}(ab^n)$. Thus $\sigma_1^* E = \sigma_2^* E + \mathrm{div} b^{m-n}$. □

1.5 The Picard scheme and alignment

In this section we define the Picard scheme and an object $\mathrm{Pic}^0_{X/S}$ that will serve as a generalisation of the Jacobian of a curve in higher dimensions. We shall then show that a generically smooth semistable morphism $X \to S$ where S is an excellent regular scheme is aligned if and only if the closure of the unit section in $\mathrm{Pic}^{[0]}_{X/S} \supset \mathrm{Pic}^0_{X/S}$ is flat over S. This will allow us to show that $\mathrm{Pic}^0_{X/S}$ permits a Néron model if X/S is aligned.

Definition 1.49. Let S be a scheme, and X be an S-scheme. The functor $P_{X/S} : (\mathrm{Sch/S})^0 \to (\mathrm{Sets})$ is defined as

$$P_{X/S}(T) = \mathrm{Pic}(X \times_S T).$$

The *relative Picard functor of X over S* is the $fppf$-sheaf associated to $P_{X/S}$. It is denoted by $\mathrm{Pic}_{X/S}$.

Remark 1.50. By Proposition 8.1.4 of [BLR90] when $f : X \to S$ is quasi-compact and quasi-separated, satisfies $f_*(\mathcal{O}_X) = \mathcal{O}_S$ universally, and f admits a section, there exists an exact sequence

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X \times_S T) \to \mathrm{Pic}_{X/S}(T) \to 0$$

for any flat S-scheme T. Thus one obtains an easy way to represent elements of $\mathrm{Pic}_{X/S}(T)$. The condition that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally is satisfied when f is proper and flat and has geometrically connected and reduced fibres by Exercise 9.3.11 of [FGI+05]. In particular, it holds in the case of semistable morphisms, and so in particular semistable morphisms are cohomologically flat in codimension 0.

Theorem 8.3.1 of [BLR90] informs us that if $f : X \to S$ is a proper, flat, finitely presented morphism of schemes that is cohomologically flat in dimension 0, then $\mathrm{Pic}_{X/S}$ is representable by a locally separated algebraic space locally of finite presentation over S. Denote by $\mathrm{Pic}^0_{X_U/U}$ the fibre-wise connected component of the identity, and $\mathrm{Pic}^{[0]}_{X/S}$ its closure in $\mathrm{Pic}_{X/S}$.

Theorem 1.51. *Let $X \to S$ be a semistable morphism, where S is an excellent regular scheme. Let $U \subset S$ be a dense open subscheme, and assume X is smooth over U. Then the following are equivalent:*

1. *X/S is aligned.*
2. *The closure of the unit section in $\mathrm{Pic}^{[0]}_{X/S}$ is étale over S.*
3. *The closure of the unit section in $\mathrm{Pic}^{[0]}_{X/S}$ is flat over S.*

First we give a useful lemma, which itself uses the following result of Holmes:

Lemma 1.52 (Lemma 5.17 of [Hol16])**.** *Let S be a Noetherian scheme and $U \subset S$ dense open. Let $f : X \to S$ be a morphism of schemes locally of finite type and which is an isomorphism over U, and such that $f^{-1}U$ is schematically dense in X. Let $x \in X$ be a point. The following are equivalent:*

1. *f is étale at x;*

2. *f is flat at x;*

3. *there exists an open neighbourhood V of $f(x)$ in S and a section $\sigma : V \to X$ through x.*

Lemma 1.53. *Let $S = Spec(R)$ be Noetherian strictly henselian local scheme with closed point s, and let $U \subset S$ a dense open subscheme. Let $f : X \to S$ be a morphism of algebraic spaces that is locally separated and of finite type, and suppose further that X/S is an isomorphism over U. Then the following are equivalent:*

1. *f is étale at points of the fibre X_s.*

2. *X/S is flat at points of the fibre X_s.*

3. *For every point x in the special fibre there exists a section $\sigma : S \to X$ through x.*

Proof. It is immediate that (1) implies (2). We first show (2) implies (3).

Suppose that X/S is flat at points in the special fibre. As the flat locus is open by [[Sta17], Tag 05WU] and contains X_s and X_U, we may replace X by this open subspace and assume X/S is flat.

By Lemma 67.31.3 of [[Sta17], Tag 0D4L], the fibre dimension is lower semicontinuous for flat morphisms of algebraic spaces of finite presentation, and so as X/S is of dimension 0 over the dense open subscheme U of S, the fibre dimension is 0 everywhere. Fixing an étale presentation $Y \to X$ of X by an S-scheme Y, we see that Y/S is of relative dimension 0. By Lemma 28.28.5 of [[Sta17], Tag 0397] we conclude that Y/S is locally quasi-finite, and so in particular X/S is locally quasi-finite.

We may now apply Lemma (C.2) of [Ryd11], which says that a locally separated, locally quasi-finite algebraic space X over a strictly henselian local scheme $S = \operatorname{Spec}(R)$ is such that every geometric point x in the special fibre X_s admits an affine neighbourhood $\operatorname{Spec}(\mathcal{O}_{X,x})$ with $\operatorname{Spec}(\mathcal{O}_{X,x}) \to S$ a finite morphism. In particular, x admits an affine open neighbourhood. Hence we conclude by Lemma 1.52 that there exists a section $\sigma : S \to X$ through x, showing that (2) implies (3)

Now suppose X/S admits a section through $x \in X_s$. Choose an étale presentation $Y \to X$ of X by a scheme Y. Every point in Y_s has finite residue field over s, and hence is closed in Y_s. By Lemma 28.19.6 of

[[Sta17], Tag 01TC] we conclude that Y/S is locally quasi-finite at points of Y_s, and hence so too is X/S locally quasi-finite at points of X_s. Lemma (C.2) of [Ryd11] now implies each point of X_s has an affine neighbourhood. Hence we may apply Lemma 1.52 to find that X/S is étale at points of X_s. $\square$

Proof of Theorem 1.51. Properties (i) to (iii) are étale local on S, so without loss of generality we may assume that S is strictly henselian local with closed point s.

Suppose first that X/S is aligned. We shall show (2). Let $\mathrm{clo}(e)$ denote the closure of the unit section in $\mathrm{Pic}^{[0]}_{X/S}$, and $p \in \mathrm{clo}(e)$ a point. To show $\mathrm{clo}(e)$ is étale over S at p it suffices by Lemma 1.53 construct a section from S through p in $\mathrm{clo}(e)$. By Lemma 1.43 there exists a non-degenerate trait $\phi : T \to \mathrm{clo}(e)$ through p, and by Remark 1.50 this corresponds to a line bundle $\mathcal{L}$ on X_T/T that is trivial over $\phi_S^*(U)$, where $\phi_S : T \to S$ is the composition of ϕ with the structure map to S and X_T/T is a semistable morphism obtained by pull-back. By choosing a rational section of $\mathcal{L}$ that is trivial over $\phi_S^*(U)$, we obtain a vertical Cartier divisor D on X_T whose restriction to the generic fibre is 0.

As D is Cartier we obtain a T-Cartier vertex labelling on Γ_s, and we may assume that it takes the value 0 at some vertex v. By Lemma 1.46 we obtain a Cartier divisor $\overline{D}$ on X/S that pulls back to D over T. This corresponds to a section $\sigma : S \to \mathrm{clo}(e)$ through p, whence we conclude (2) holds.

Clearly (2) $\Rightarrow$ (3), so it remains to show (3) $\Rightarrow$ (1). Equivalently, we'll show that if X/S is not aligned then the closure of the unit section in $\mathrm{Pic}^{[0]}_{X/S}$ is not flat over S. Hence we assume that X/S is not aligned at s. By Lemma 1.47 there exists a non-degenerate trait $f : T \to S$ through s along with a Cartier divisor D, trivial on the generic fibre of X_T, such that there does not exist a Cartier divisor $\overline{D}$ on X/S, trivial over U, that pulls back to D. By the definition of $\mathrm{Pic}^{[0]}_{X/S}$ we have that D gives a morphism $\phi : T \to \mathrm{Pic}^{[0]}_{X/S}$, and in fact the image of T lands in $\mathrm{clo}(e)$. To see this last point we note that the generic point of T lands in the image of the unit section and hence T lands in $\mathrm{clo}(e)$ as $\mathrm{clo}(e)$ is closed in $\mathrm{Pic}^{[0]}_{X/S}$.

If $\mathrm{clo}(e)$ is flat over S then by Lemma 1.53 there exists a section $\sigma : S \to \mathrm{clo}(e)$ of $\mathrm{clo}(e) \to S$ through the point $\phi(t)$, where t is the closed point of T. This morphism corresponds to a Cartier divisor $\overline{D}$ on X/S, trivial over U. As T is reduced we conclude that $\phi : T \to \mathrm{clo}(e)$ factors through $f : T \to S$, and hence that the pullback of $\overline{D}$ to T is linearly equivalent to D, contradicting how D was chosen. □

Under additional hypotheses on X/S, Theorem 1.51 allows us to show that alignment of X/S is equivalent to the existence of a Néron model for $\mathrm{Pic}^0_{X_U/U}$. When $f : X \to S$ is a projective semistable morphism, smooth over a dense open subscheme $U \subset S$, then Theorem 8.2.2 of [BLR90] informs us that $\mathrm{Pic}_{X/S}$ exists as an S-scheme locally of finite presentation over S.

Lemma 1.54. *Let G be a group scheme over a base scheme S such that $G \to S$ is locally of finite presentation and $G \to S$ is smooth along the unit section. Let H denote the smooth locus of $G \to S$. Then H is an open and closed subgroup scheme of G.*

Proof. Clearly H is open in G. We first show it is a subgroup scheme of G. Note that the fibres of G/S are group schemes admitting smooth open sets, and as such are smooth everywhere. As G/S is of finite presentation, it follows that the smooth locus of G/S is exactly the flat locus. By the *critère de platitude par fibres* the image of $H \times H \to G$ in G is flat and hence the image is contained in H. As H is closed under inversion it follows that H is a subgroup scheme.

It remains to show H is closed in G. To accomplish this we will show that the étale locus of G/H over S is closed in G/H, where the quotient G/H exists as an algebraic space over S by Tag 071R of [Sta17].

Let $(G/H)_{fppf}$ denote the quotient $fppf$-sheaf on S of H acting on G. The action of H on G is free, and transitive on the fibres of G over G/H, whence we see that G is an H-torsor over (G/H) in the category of $fppf$-sheaves on S.

Choose an $fppf$-cover $T \to G/H$ of G/H such that there exists a section from T to G. One obtains the following product diagram:

$$\begin{array}{ccc} H \times_S T & \longrightarrow & G \\ \downarrow & & \downarrow \\ T & \longrightarrow & (G/H). \end{array} \tag{1.2}$$

As smoothness is $fppf$-local on the target, we conclude that $G \to G/H$ is smooth. From this we also see that sections of $G \to G/H$ exist étale locally on G/H, and hence that G is an étale H-torsor over G/H.

It remains to show that the étale locus of G/H over S is closed in G/H. First we show G/H is unramified over S.

By Theorem 3.2 in Exposé IVa of [GD62] the quotient G_s/H_s exists as a smooth group scheme over s for any point $s \in S$. Furthermore, if G_s^0 denotes the connected component of the identity in G_s, the quotient G_s/G_s^0 is of dimension 0 over s by Theorem 5.5.1 of [GD62], and we conclude by Remark 5.3.2 of [GD62] that G_s/H_s is étale over s as $G_s^0 \subset H_s$.

As sheaves on the big étale site of S the fibres of the quotient are the quotients of the fibres, and so by Theorem 5.5.1 of Exposé VIa of [GD62] the fibres of G/H are étale schemes over their images in S. In particular they are unramified, and so Lemma 28.33.12 of Tag 02G3 of [Sta17] informs us that G/H is unramified over S.

We conclude by showing the étale locus of G/H over S is closed. Choose an étale presentation $P \to G/H$ of G/H. P is unramified over S, and as being a closed immersion is étale local on the base by Tag 02L6 of [Sta17] it suffices to show that the étale locus of $P \to S$ is closed.

Let $x \in P$ have image $s \in S$. After taking a sufficiently small étale neighbourhood of s, Tag 04GL of [Sta17] tells us that we can find a Zariski open neighbourhood U of x such that $U \to S$ is a closed immersion. Thus $U \to S$ is either étale or the étale locus is empty. Either way, we find that the étale locus is closed, and hence that its pullback H in G is closed. □

Corollary 1.55. *Let S be an excellent regular locally Noetherian scheme, $U \subset S$ a dense open subscheme, and $f : X \to S$ a projective semistable morphism smooth over U. Let e denote the unit section of $Pic_{X/S}$. If $Pic_{X/S}$ is smooth over S along e, then $Pic^0_{X/S}$ is a smooth open subgroup scheme of $Pic_{X/S}$, and the closure $Pic^{[0]}_{X/S}$ of $Pic^0_{X_U/U}$ in $Pic_{X/S}$ is smooth over S.*

Proof. By the assumption that $\mathrm{Pic}_{X/S}$ is smooth along e, 15.6.5 of [Gro66a] implies that $\mathrm{Pic}^0_{X/S}$ is an open subscheme of $\mathrm{Pic}_{X/S}$. By Lemma 1.54 the smooth locus of $\mathrm{Pic}_{X/S}$ is an open and closed subgroup scheme. In particular, as $\mathrm{Pic}^0_{X/S}$ is the fibrewise connected component of e it is contained in the smooth locus, and hence is smooth over S.

The restriction of $\mathrm{Pic}^0_{X/S}$ over U is an open subscheme of the smooth locus, and so its closure in $\mathrm{Pic}_{X/S}$ is a union of connected components of the smooth locus. In particular $\mathrm{Pic}^{[0]}_{X/S}$ is also smooth over S. □

Theorem 1.56 (c.f. Theorem 6.2 of [Hol16])**.** *Let S be an excellent regular locally Noetherian scheme, $U \subset S$ a dense open subscheme, and $f : X \to S$ a projective semistable morphism smooth over U. Let e denote the unit section of $Pic^0_{X_U/U}$ and $clo(e)$ for the closure of e in $Pic^{[0]}_{X/S}$. Assume that $Pic_{X/S}$ is smooth over S along the unit section.*

1. *If X is regular and $clo(e)$ is étale over S, then a Néron model for $Pic^0_{X_U/U}$ exists.*
2. *If a Néron model for $Pic^0_{X_U/U}$ exists, then $clo(e)$ is étale over S.*

Lemma 1.57. *With the assumptions as in Theorem 1.56, one has that $Pic^{[0]}_{X/S} \times_S U \cong Pic^0_{X_U/U}$.*

Proof. We first show that $\mathrm{Pic}^{[0]}_{X/S} \times_S U$ is quasi-compact over U. As quasi-compactness can be checked locally, we may assume for now that U is connected. By Theorem 8.2.5 of [BLR90], we can cover U by open sets V_i such that $\mathrm{Pic}_{X_{V_i}/V_i}$ is a disjoint union of quasi-projective schemes over V_i. In particular, $\mathrm{Pic}^{[0]}_{X/S} \times_S V_i$ is the closure of a connected open subscheme in $\mathrm{Pic}_{X_{V_i}/V_i}$, and so is quasi-projective over V_i, hence quasi-compact over V_i.

Thus $\mathrm{Pic}^{[0]}_{X/S} \times_S U$ is quasi-compact over U. By Theorem 8.4.3 of [BLR90], we conclude that $\mathrm{Pic}^{[0]}_{X/S} \times_S U$ is proper over U, and so by Lemma 15.5.4 of [Gro66a] its fibres are connected over U. Hence we find that the fibres of $\mathrm{Pic}^{[0]}_{X/S} \times_S U$ over U are equal to those of $\mathrm{Pic}^0_{X_U/U}$, and the conclusion follows by the reducedness of both schemes. □

The proof of Theorem 1.56 uses the following result from [Hol16].

Lemma 1.58 (Lemma 6.1 of [Hol16])**.** *Let S be a scheme, $U \subset S$ a dense open subscheme with $U \to S$ quasi-compact, and $f : S' \to S$ a smooth, surjective morphism. Let A/U be an abelian scheme, and suppose f^*A has a Néron model N' over S'. Then A has a Néron model N over S, and $f^*N = N'$.*

Proof of Theorem 1.56. As $\mathrm{clo}(e)$ is flat over S, the quotient N of $\mathrm{Pic}^{[0]}_{X/S}$ by $\mathrm{clo}(e)$ exists as a separated group algebraic space, with separatedness following from the unit section of N being a closed immersion. As N permits an $fppf$-cover by $\mathrm{Pic}^{[0]}_{X/S}$, which itself is smooth over S by Corollary 1.55, Corollary 6.5.2 of [Gro66b] implies that N is smooth. Using the separatedness of N and the reduced-to-separated theorem (10.2.2 of [Vak17]), the uniqueness part of the Néron mapping property is automatic, so we must show existence.

Let $T \to S$ be a smooth morphism of algebraic spaces and $T_U \to N_U$ be an S-morphism. By choosing a presentation of T we can find a surjective étale morphism $T' \to T$ where T' is a scheme; by base change we have a morphism $T'_U \to \mathrm{Pic}^{[0]}_{X/S} \times_S U$.

Because X/S admits a section and is cohomologically flat in dimension 0, by Proposition 8.1.4 of [BLR90] there exists a line bundle $\mathcal{F}$ on $T'_U \times_U X_U$ such that $T'_U \to \mathrm{Pic}^{[0]}_{X/S} \times_S U$ is given by $\mathcal{F}$. Let $D = \sum_i n_i p_i$ be a Cartier divisor on $T'_U \times_U X_U$ with $\mathcal{O}(D) \cong \mathcal{F}$, where the p_i are prime Weil divisors. Define $\overline{D} = \sum_i n_i \overline{p}_i$, where $\overline{p_i}$ denotes the scheme-theoretic closure of p_i in $T' \times_S X$. We claim $T' \times_S X$ is regular, and hence $\overline{D}$ is a Cartier divisor. This follows as $T' \to S$ is smooth and X is regular; by [[Sta17], Tag 036D] regularity is local on the base in the smooth topology. Hence $\overline{D}$ is a Cartier divisor. This defines a morphism $T' \to \mathrm{Pic}^{[0]}_{X/S} \to N$, whose restriction to U coincides with $T'_U \to N_U$. This descends to a morphism $T \to N$ by étale descent, necessarily unique by the separatedness of N, concluding the proof of (1).

We will now prove (2). Let N be the Néron model of $\mathrm{Pic}^0_{X_U/U}$. Theorem 8.2.2 of [BLR90] implies that $\mathrm{Pic}_{X/S}$ is a scheme. Note that $\mathrm{Pic}^{[0]}_{X/S} \times_S U = \mathrm{Pic}^0_{X_U/U}$ by Lemma 1.57, and so by the Néron mapping property we obtain a morphism $\phi : \mathrm{Pic}^{[0]}_{X/S} \to N$. If K denotes the kernel of ϕ, we would like to show that K is flat over S and equal to $\mathrm{clo}(e)$. We shall do this by showing ϕ is flat, for if ϕ were flat then so too would K be flat over S by property of base change. Furthermore the map $K \to \mathrm{Pic}^{[0]}_{X/S}$ would be a closed immersion as N is separated over S, whence its unit section is a closed immersion. Then, as the preimage of U in K is schematically dense by Theorem 11.10.5 of [Gro66a] and $K_U = \mathrm{clo}(e)_U$, we could conclude $K = \mathrm{clo}(e)$.

To show ϕ is flat, it suffices to show that the fibre ϕ_s of ϕ over s is flat for all $s \in S$ by [[Sta17], Tag 05X0]. In fact, we will show that the restriction $\phi_s^0 : \mathrm{Pic}^0_{X_s/s} \to N_s^0$ of ϕ to the connected components of e is flat, as one can then cover $\mathrm{Pic}^{[0]}_{X_s/s}$ by translates of $\mathrm{Pic}^0_{X_s/s}$, the former being a scheme as it is a group algebraic space over a field (see [Art70]). The restriction ϕ^0 of ϕ to the fibrewise-connected components of the identity is an isomorphism over U. Note that $\mathrm{Pic}^0_{X/U}$ is an open subscheme of $\mathrm{Pic}_{X/S}$ by 8.4 of [BLR90], hence it is flat and locally of finite type over S. Hence Corollary 5.5 of [GD62] implies $\mathrm{Pic}^0_{X/S}$ is separated over S, and so we may apply Proposition 3.1(e) of [GRR72] to find that ϕ^0 is an open immersion, and hence flat.

□

Chapter 2

Families of dual complexes of normal crossings varieties

> Go away, the old buildings said. There is no place for you here. You are not wanted.
>
> Harper Lee, *Go Set a Watchman*

2.1 Introduction

Let $S = \mathrm{Spec}(R)$ be the spectrum of a discrete valuation ring with uniformizer π. Let $f : X \to S$ be a relative semistable curve over S with smooth generic fibre. It is well known (see Section 7 of [CCUW16]) that to such a family we may associate a tropical curve whose underlying graph is the dual graph of the special fibre X_s. The labelling of the edges arises from the étale local structure of the morphism: at the non-smooth points of the special fibre, f factors étale locally through an étale S-morphism

$$f : X \to \mathrm{Spec}(R[x, y]/(xy - b)),$$

where $b \in R$ is unique up to a unit. The label of the corresponding edge in the dual graph is then $b \pmod{R^*}$.

This association can be generalised further. Suppose S is an arbitrary logarithmic scheme whose underlying scheme is locally Noetherian with sheaf of monoids M. In Definition 7.4 of [CCUW16], the notion of a *tropical curve over S* is introduced:

Definition 2.1 (Definition 7.4 of [CCUW16])**.** A *tropical curve over S* consists of

- A tropical curve Γ_s for every geometric point $s \in S$ with labels in $M_{S,s}$;
- A weighted edge-contraction $\Gamma_s \to \Gamma_t$ for each geometric specialisation $t \to s$;
- If $E(\Gamma)$ denotes the set of edges of a graph Γ, then the contraction of weighted graphs $\Gamma_s \to \Gamma_t$ where Γ_t is metrised via the composition

$$E(\Gamma_s) \to M_{S,s} \to M_{S,t},$$

where edges of label 0 under specialisation are contracted.

Given a family of semistable curves over S, Cavalieri et al. showed in Section 7 of [CCUW16] that the association of the labelled dual graph Γ_s to each geometric point $s \in S$ yields a tropical curve over S.

It is natural to ask if an analogous process can be done under more general circumstances. If one takes a normal crossings divisor in a smooth scheme over a field k, Chan et al. show in [CGP16] that one can construct a combinatorial object called a generalised Δ-complex associated to divisor. Inspired by this, we will associate such a complex to a semistable morphism, by which we mean a morphism that factors étale locally through $k[x_1, \ldots, x_n]/(x_1 \cdots x_l)$.

In this chapter we will study how the dual Δ-complexes associated to semistable morphisms behave in families of semistable varieties over a general base scheme. These varieties are natural analogues of semistable curves over an arbitrary base.

In Section 2.2 we introduce the notion of a labelled generalised Δ-complex over a base scheme in a way that mirrors the notion of a tropical curve over a logarithmic scheme. The main result is as follows:

Theorem 2.2 (Theorem 2.63). *Let $f\colon X \to S$ be a semistable morphism. For each geometric point $s \in S$, let Σ_s denote the associated generalised Δ-complex with labels given by the principal ideals of $\mathcal{O}^{ét}_{S,s}$. Then the collection $\{\Sigma_s\}_{s\in S}$ is a labelled generalised Δ-complex over S.*

2.1.1 Overview of chapter

In Section 2.2 we cover the definition of semistable morphisms to an arbitrary locally Noetherian base scheme. We recall a stratification of the non-smooth locus of the geometric fibres given in Chapter 1, as well as provide some results on how the strata change under geometric specialisation on the base.

Section 2.3 begins by introducing basic concepts from simplicial geometry. This closely follows section 3 of [CGP16]. In the latter part of this section we introduce the notion of a labelled generalised Δ-complex, and how such combinatorial objects are closely tied to the stratification of the non-smooth locus of the geometric fibres from Section 2.2. We conclude by demonstrating how these labelled complexes behave in families, namely in how they mirror the labelled dual graphs of semistable curves.

2.2 Semistable morphisms

We begin this section by defining semistable morphisms, which are a special case of the definition given by Li in [Li07]. The fibres of these morphisms come with a natural stratification that will later be used to interpret associated combinatorial structures.

2.2.1 Basic definitions

An important concept throughout this chapter is that of *étale neighbourhoods*. By this we mean the following: Given a scheme X and a point $x \in X$, an étale neighbourhood (U, x') of (X, x) is an étale morphism $\phi : U \to X$ from a scheme U to X such that $\phi(x') = x$. We repeat here the definitions of semistable morphism and local chart from Chapter 1.

Definition 2.3. (Definition 1.5 and Definition 1.6) Let S be a locally Noetherian scheme and $f\colon X \to S$ a morphism of finite type. Let $x \in X$ be a point with image $s \in S$. A *local chart of f at x* consists of the following data:

- an étale neighbourhood $(\mathrm{Spec}(R), s')$ of (S, s);
- an étale neighbourhood (U, x') of (X, x) where U is a connected scheme;
- an element $b \in R$;
- an étale morphism $U \to \mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b))$ with $n \geq l \geq 1$;

such that if p is the image of x' in $\operatorname{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b))$ then $\{x_1, \ldots, x_l\}$ are contained in the prime ideal corresponding to p, the point p maps to $s' \in \operatorname{Spec}(R)$, and making the following diagram commute:

$$\begin{array}{ccc} & & U \\ & \swarrow^{\text{ét}} & & \searrow^{\text{ét}} \\ \operatorname{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b)) & & & & X \\ \downarrow & & & & \downarrow f \\ \operatorname{Spec}(R) & & \xrightarrow{\text{ét}} & & S. \end{array} \tag{2.1}$$

Let S be a locally Noetherian scheme. A morphism $f \colon X \to S$ is said to be a *semistable* morphism if it is proper with geometrically connected fibres, and such that any point $x \in X$ admits a local chart of f at x.

We use $U \to \operatorname{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l - b))$ to denote the data of a local chart of x.

Example 2.4. Any semistable curve over a field gives rise to a semistable morphism by taking the structure morphism. For example, taking two lines in the projective plane that intersect transversally yields such an example.

Remark 2.5. As shown in Definition 1.9, for a fixed x the integer l is independent of the local chart.

2.2.2 Strata of fibres

After defining the stratification of the fibres of a semistable morphism we shall explore how the strata behave under specialisation and generisation on the base.

Definition 2.6. [Definition 1.11] Let $f \colon X \to \operatorname{Spec}(k)$ be a semistable morphism with k a field. Set $\operatorname{Sing}^0_f(X) = X$ and inductively define $\operatorname{Sing}^r_f(X)$ as the non-smooth locus of $\operatorname{Sing}^{r-1}_f(X)$, viewed as a closed subscheme of X with the reduced induced subscheme structure. The *r-strata of X* are the irreducible components of the smooth locus of $\operatorname{Sing}^r_f(X)$.

Remark 2.7. If $f : X \to \operatorname{Spec}(k)$ is a geometrically connected scheme that admits local charts over k, one may still define $\operatorname{Sing}^r_f(X)$ as above. Given a local chart $U \to \operatorname{Spec}(k[x_1, \ldots, x_n]/(x_1 \cdots x_l)$, the generic points of the irreducible components of $\mathrm{Sing}^r(\mathrm{Spec}(k[x_1,\dots,x_n]/(x_1\cdots x_l))$ of the morphism $\mathrm{Spec}(k[x_1,\dots,x_n]/(x_1\cdots x_l) \to \mathrm{Spec}(k)$ are cut out by ideals of the form $(x_{i_1},\dots,x_{i_{r+1}})$ for $\{i_1,\dots,i_{r+1}\} \subset \{1,\dots,l\}$.

Lemma 2.8. *[Lemma 1.13] Let k denote a separably closed field, and let $X \to Spec(k)$ be a semistable morphism. Then the r-strata of X are geometrically irreducible.*

We return to the case where S is an arbitrary locally Noetherian base scheme. Let $s \in S$ be a point and X_s the fibre of $f\colon X \to S$ over s.

Definition 2.9. [Definition 1.18] Let $x \in X_s$ be in the non-smooth locus. Let

$$g : U \to \mathrm{Spec}(R[x_1,\dots x_n]/(x_1\cdots x_l - b))$$

be a local chart of x. We define the *label* of x to be $l(x) = (b)$, viewed as a principal ideal of $\mathcal{O}^{\mathrm{sh}}_{S,s}$.

Remark 2.10. The ideal generated by the image of b in $\mathcal{O}^{sh}_{S,s}$ is independent of the local chart by Lemma 3.1 of [Li07]. As x varies over the singular locus of $f\colon X_s \to s$, the ideal (b) in $\mathcal{O}^{sh}_{S,s}$ is constant on connected components of the singular locus, and in particular on the r-strata for $r > 0$.

Remark 2.11. By Lemma 2.8 it will often be convenient to work with geometric points of S. By this we mean points of the small étale site $\text{ét}(S)$ of S, or a functor $\overline{s}^* : \text{ét}(S) \to \mathbf{Sets}$ induced from a morphism $\overline{s} : \mathrm{Spec}(k) \to S$ for k a separably closed field. Given two geometric points $\overline{s}$ and $\overline{t}$ of S, a(n étale) *specialisation* from $\overline{t}$ to $\overline{s}$ is a natural transformation $\overline{s}^* \to \overline{t}^*$.

As discussed in Appendix A of [CCUW16], an étale specialisation from $\overline{t}$ to $\overline{s}$ gives a factorisation of the morphism $\overline{t}$ through $\mathcal{O}^{sh}_{S,\overline{s}}$. Conversely, every such factorisation yields an étale specialisation. Étale specialisations induce Zariski specialisations.

We now provide some results on how the r-strata of geometric fibres change under étale specialisation. These will prove important later on. Most of these results are from Chapter 1. As a consequence of Lemma 2.8 and Remark 2.11 we may reduce to the case where S is the spectrum of a strictly henselian local ring with s its closed point.

Lemma 2.12. *[Lemmas 1.26, 1.27, 1.31, and 1.35 of 1] Let $f\colon X \to S$ be a semistable morphism to the spectrum of a strictly Henselian local ring* *with closed point s, and let η be a generisation of s with $\overline{\eta}$ a fixed geometric point of S with image η.*

1. *The r-strata of X_η are geometrically irreducible.*
2. *Given an r-stratum Z of X_η, let $\overline{Z}$ denote its closure in X and let $\overline{Z}_s$ denote the fibre of $\overline{Z}_s$ over s. Then $\overline{Z}_s$ is a union the closures of r-strata of X_s.*
3. *If the label b of a r-stratum Y of X_s is not a unit over η for $r \geq 1$, there exists a unique r-stratum Z of X_η such that $\overline{Y} \subset \overline{Z}_s$.*
4. *If X_1 and X_2 are two irreducible components of X_s that both contain a 1-stratum with label b such that b is a unit over η, then there exists a unique irreducible component Y of X_η with $X_1 \cup X_2 \subset \overline{Y}$.*

Remark 2.13. The first part of Lemma 2.12 allows us to go back and forth between the r-strata of X_η and $X_{\overline{\eta}}$ when the base is the spectrum of a strictly Henselian local ring and η is a generisation of the closed point s.

We reintroduce the specialisation map from Chapter 1, albeit in a more general context of also acting on higher strata.

Definition 2.14. Let $f\colon X \to S$ be a semistable morphism, $\overline{s} \in S$ a geometric point and $\overline{\eta}$ an étale generalisation of $\overline{s}$ with image $\eta \in S$. If $r = 0$, the *specialisation map* from the r-strata of $X_{\overline{s}}$ whose labels are non-units over η to the r-strata of $X_{\overline{\eta}}$ sends a r-stratum Y_s of $X_{\overline{s}}$ to the unique r-stratum Z of $X_{\overline{\eta}}$ such that $Y \subset \overline{Z}_s$.

When $r > 0$ we extend this to a subset of the higher strata as follows: The *specialisation map* sp from the set of r-strata, $r > 0$, of $X_{\overline{s}}$ whose labels are non-units over η to the r-strata of $X_{\overline{\eta}}$ sends a r-stratum Y_s of $X_{\overline{s}}$ to the unique r-stratum Z of $X_{\overline{\eta}}$ such that $Y \subset \overline{Z}_s$.

Lemma 2.15. *Retain the assumptions as in Lemma 2.12. In particular, S is assumed to be strictly Henselian with closed point s.*

If Y_1 and Y_2 are two distinct r-strata of X_s, $r \geq 1$, both of whose labels are non-units over η, then $sp(Y_1) \neq sp(Y_2)$.

Lemma 2.16. *[c.f. Lemma 1.38] Retain the assumptions as in Lemma 2.12.*

Let Y_1 and Y_2 be two irreducible components of X_s. Then $sp(Y_1) = sp(Y_2)$

if and only if there exists a chain of irreducible components $Y_1 = Z_1 = \ldots = Z_i = \ldots Z_q = Y_2$ *of* X_s *along with a point* $x_i \in Z_i \cap Z_{i+1}$ *whose label is a unit over* η.

The proofs of the above two lemmas are broken into a number of steps. As the r-strata of X_η are geometrically irreducible, we may base change to $\overline{\{\eta\}}$ and assume that S is integral and strictly henselian with generic point η and closed point s.

Definition 2.17. Let $f\colon X \to S$ be a semistable morphism with S integral. Denote by $B \subset X$ the largest open subset of the singular locus of f that is flat over S. Denote by $B^r \subset \mathrm{Sing}_f^{r+1}$ the subset of the singular locus of Sing_f^r that is flat over S.

We will need Lemma 3.5 of [Li07], which states that connected components of the special fibre of a proper morphism over a strictly Henselian local ring are in one-to-one correspondance with connected components of the source scheme. Let $x \in X$ be a singular point of f with image $p \in S$. Let C be the connected component of the singular locus of f containing x. Then C_s is a connected component of the singular locus of X_s, and the label of C_s, viewed as an element of $\mathcal{O}_{S,s}$ and defined up to a unit, is the label of x via the composition $\mathcal{O}_{S,s} \to \mathcal{O}_{S,p} \to \mathcal{O}_{S,p}^{\text{ét}}$.

Lemma 2.18. *B is a closed subscheme in X.*

Proof. It suffices to show that B is a union of connected components of the singular locus of f. We'll show that B is the set of points whose label is 0. If this holds then, by the above discussion, it will necessarily be equal to a union of connected components of the singular locus of $X \to S$.

If the label of a connected component of the singular locus is 0, the connected component is flat over S by Lemma 1.34.

Conversely, if the label of a connected component C of the singular locus is non-zero then $C \to S$ is not dominant and so cannot be flat (recall that S is assumed to be integral). □

Remark 2.19. Let C be a connected component of the singular locus with label 0. Then the restriction of Sing_f^r to C is flat over S for all $r > 1$. This can be verified étale locally: any local chart of a point with label 0 is of the form $U \to \mathrm{Spec}(R[x_1, \ldots x_n]/(x_1 \cdots x_l))$, with Sing_f^r defined by the ideal generated by all elements of the form $x_{i_1} \cdots x_{i_{l-r}}$,

$\{i_1, \ldots, i_{l-r}\} \subset \{1, \ldots, l\}$. This is flat over $\operatorname{Spec}(R)$ as it is a free R-module with basis elements $x_{i_1} \cdots x_{i_{l-r}}$.

Let $\beta : \tilde{X} \to X$ be the blow-up of X along B. Then β is a proper morphism by Lemma 63.17.11 of [[Sta17], Tag 085P]. Via composition with f we may view $\tilde{X}$ as an S-scheme.

Lemma 2.20. *Let $W_1, \ldots, W_m$ be the connected components of $\tilde{X}_\eta$. Then*

1. *$\overline{W_1}, \ldots, \overline{W_m}$ are the connected components of $\tilde{X}$.*
2. *Each $\overline{W}_i$ is flat over S.*
3. *If $\mathring{\overline{W}}_i := \overline{W}_s \setminus \beta^{-1}(B)$ and $\mathring{W}_i := W_i \setminus \beta^{-1}(B)$, then $(\mathring{\overline{W}}_i)_\eta = \mathring{W}_i$, and these are dense open sets in $\overline{W_i}$ and W_i, respectively.*

Proof. Note that $\tilde{X} \to S$ is flat: this can be verified étale locally by blowing-up $\operatorname{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l))$ along $I = (x_1 \cdots \hat{x_i} \cdots x_l)_{i=1,\ldots,l}$ and noting this is the disjoint union of the irreducible components defined by the ideals (x_i). Note that this relies on S being integral and hence reduced: blowing-up $\operatorname{Spec}(k[x_1, \ldots, x_n]/(x_1 \cdots x_l))$ along B is equivalent to blowing-up along the ideal I.

As the composition of proper morphisms is proper, $\tilde{X} \to S$ is proper. If some point x were not in the closure of $\tilde{X}_\eta$, it would lie in an open set $U = \tilde{X} \setminus \overline{\tilde{X}_\eta}$ whose image in S is both open (by flatness) and non-empty, but such that it does not contain η. This is impossible, and hence $\overline{\tilde{X}_\eta} = \tilde{X}$.

Suppose that $\overline{W_1} \cap \overline{W_2} \neq \emptyset$ with $W_1 \neq W_2$. If there exists some $x \in \overline{W_1} \cap \overline{W_2}$ with $\beta(x) \notin B$ then there exists an étale neighbourhood of x isomorphic to an étale neighbourhood of $\beta(x)$ which we may choose to be a local chart, necessarily with a non-zero label. As the generic fibre of such a neighourhood is smooth and irreducible, this implies $W_1 = W_2$, a contradiction. Hence $\beta(x) \in B$, and x itself lies in the exceptional divisor.

Choose a local chart $U \to \operatorname{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_l))$ of $\beta(x)$. As blowing-up commutes with étale base change, $\tilde{X}$ is necessarily smooth in a neighbourhood of x, contradicting the fact x is contained in the intersection of two distinct irreducible components of $\tilde{X}$.

Hence $\overline{W_i}$ is disjoint from $\overline{W_j}$ for $j \neq i$, and so $\tilde{X} = \coprod \overline{W_i}$, which combined with the fact $\overline{\tilde{X}_\eta} \cong \tilde{X}$ proves (i). Part (ii) of the lemma follows immediately: $\tilde{X} \to S$ is flat, and hence its connected components are also flat over S.

As blow-ups commute with flat base change, $(\overline{\mathring{W}}_i)_\eta = \mathring{W}_i$. That $(\overline{\mathring{W}}_i)_\eta$ and $\mathring{W}_i$ are open in $\overline{W_i}$ and W_i, respectively, follows by definition of the exceptional divisor being closed, and density is verified étale locally, proving (iii). □

Corollary 2.21. *Let $C \subset X$ denote a connected component of Sing_f^r, $r > 0$, with label* 0. *Let $\beta_r : \tilde{C} \to C$ denote the blow up of C along $B^r \cap C$. Then $\tilde{C}$ is smooth and proper over S. Furthermore, if $W_1, \dots, W_m$ denote the irreducible components of $\tilde{C}_\eta$, then the $\overline{W_i}$ are the connected components of $\tilde{C}$.*

Proof. That $\tilde{C} \to S$ is proper follows from $\tilde{C} \to C$ being proper and compositions of proper morphisms are proper.

We verify smoothness étale locally, using the fact that blowing-up commutes with étale base change. If $x \in C$ is any point, then viewed as a point in X we can find a local chart $U \to \mathrm{Spec}(R[x_1, \dots, x_n]/(x_1 \cdots x_l)$. After base changing to C we find that x admits a local chart $U_C \to \mathrm{Spec}(R[x_1, \dots, x_n]/J)$ where J is the ideal defined by all elements of the form $x_{i_1} \cdots x_{i_{l-r}}$, $\{i_1, \dots, i_{l-r}\} \subset \{1, \dots, l\}$. One verifies that the blow up of this along the ideal I defined by all elements of the form $x_{i_1} \cdots x_{i_{l-r-1}}$, $\{i_1, \dots, i_{l-r-1}\} \subset \{1, \dots, l\}$ is the disjoint union of the components defined by the ideals $(x_{i_1}, \dots, x_{i_{l-r}})$, and hence is smooth over the base.

Hence $\tilde{C}$ is proper and smooth over S, and the connected components of the generic fibre are the irreducible components of the generic fibre. If two such components W_1 and W_2 satisfy $\overline{W_1} \cap \overline{W_2} \neq \emptyset$, then any point x in the intersection satisfies $\beta_r(x) \in B^r$, a contradiction to how the blow-up was defined. Hence the $\overline{W_i}$ are the connected components of $\tilde{C}$. □

Proof of Lemma 2.15. Suppose there exists a r-stratum Z of X_η with $Y_1 \cup Y_2 \subset \overline{Z} \cap X_s$. Note that $\overline{Z}$ is proper over $\overline{\{\eta\}}$, as $f \colon X \to S$ is proper. After base change to $\overline{\{\eta\}}$ we may assume that the label of any point in $\overline{Z}$, viewed as a point in X, is 0. Hence each point $x \in \overline{Z}$ admits a local chart $U \to \mathrm{Spec}(R[x_1, \dots, x_n]/(x_1 \cdots x_l))$, where $S = \mathrm{Spec}(\mathrm{R})$.

Let C denote the connected component of Sing_f^r containing Z and hence the Y_i. Let $\beta_r : \tilde{C} \to C$ denote the blow-up as in Corollary 2.21, and $\tilde{Z}$

be the irreducible component of $\tilde{C}_\eta$ lying over Z. Then $\overline{\tilde{Z}} \to S$ is proper, smooth, and irreducible, and so it has irreducible special fibre. But then $Y_1 = Y_2$, contradicting our initial assumption. Hence $\mathrm{sp}(Y_1) \neq \mathrm{sp}(Y_2)$. □

Remark 2.22. As $\cup_i \overset{\circ}{\overline{W_i}} \cong X \setminus B$, we have $\beta : \cup_i \overset{\circ}{\overline{W_i}} \to X$ is an open immersion.

Proof of Lemma 2.16. Suppose there exists a chain of irreducible components $Y_1 = Z_1 = \ldots = Z_q = Y_2$ of X_s, along with a point $x_i \in Z_i \cap Z_{i+1}$ whose label is a unit over η for all i. Part (4) of Lemma 2.12 informs us then that $\mathrm{sp}(Z_i)$ agree for all i, and in particular $\mathrm{sp}(Y_1) = \mathrm{sp}(Y_2)$.

Conversely, suppose $\mathrm{sp}(Y_1) = \mathrm{sp}(Y_2)$ and consider $\beta : \tilde{X} \to X$ as in Lemma 2.20, and let $Y = \mathrm{sp}(Y_1) = \mathrm{sp}(Y_2)$. Note that $\overline{Y} \setminus B$ contains the generic points of both Y_1 and Y_2 as the Y_i's are generically smooth.

Let W be an irreducible component of $\tilde{X}_\eta$ lying over Y. By Lemma 2.20 $\overline{W}$ is a connected component of $\tilde{X}$, $\overline{W}_s$ is connected as $\overline{W}$ is proper and flat over S, and $\overline{Y} \setminus B \cong \overline{W} \setminus \beta^{-1}(B)$.

Let Y_1' and Y_2' be irreducible components of $\tilde{X}_s$ whose generic points lies over the generic points of Y_1 and Y_2, respectively. As $\overline{W}_s$ is connected, there exists a chain $Y_1' = Z_1' = \ldots = Z_q' = Y_2'$ of irreducible components of $\overline{W}_s$ such that $Z_i' \cap Z_{i+1}' \neq \emptyset$. Any point $x_i \in Z_i' \cap Z_{i+1}'$ necessarily has a non-zero label and lies in $\overset{\circ}{\overline{W}} \subset X$ by Remark 2.22.

Letting Z_i be the irreducible component of X_s each Z_i' lies over, we have found a chain as required. □

2.3 Dual complexes of semistable varieties

In this section we shall associate generalised cone complexes to the geometric fibres of a semistable morphism. These will serve as a generalisation of the dual graph of a semistable curve. We will conclude by constructing labelled dual cone complexes that behave well in families over a base.

Such generalised cone complexes arise as direct limits of diagrams of a special class of cone morphisms. The small categories defining such diagrams arise from an object called a generalised Δ-complex, studied in Section 3 of [CGP16]. The non-generalised versions of Δ-complexes appeared much earlier, for example in [Hat05]. We begin this section by defining generalised Δ-complexes, then showing how they are associated to the fibres of a semistable morphism. We shall then show how generalised Δ-complexes yield generalised cone complexes, and conclude by assigning labels to the generalised cone complex in a way that mirrors tropical curves.

2.3.1 Generalised Δ-complexes

This subsection closely follows Section 3 of [CGP16].

Definition 2.23. Let S be any finite set. Define

$$\Delta^S = \{f : S \to [0,1] | \sum_{s\in S} f(s) = 1\}.$$

In the special case where $S = [p] = \{0, \dots, p\}$, we say that Δ^p is the *standard p-simplex.*

Let S and T be finite ordered sets. Given any injective, order-preserving map $\theta : S \to T$, we have an associated map $\theta_* : \Delta^S \to \Delta^T$ sending $f \in \Delta^S$ to $\theta_* f$ defined by $\theta_* f(t) = \sum_{\theta(s)=t} f(s)$.

Definition 2.24. Let Δ_{inj} be the category whose objects are the sets $[n] = \{0, \dots, n\}$ for each $n \in \mathbb{N}$, and whose arrows are all injective, order-preserving maps. A Δ*-complex* is a functor from Δ_{inj}^{op} to $Sets$.

If X is a Δ-complex, $\sigma \in X([p])$ and $\theta : [p] \to [q]$ an arrow in Δ_{inj}, the image $\theta^*\sigma$ of σ under $\theta^* = X(\theta)$ is called a *face* of σ.

Example 2.25. Let Δ^1 and Δ^2 be the standard 1- and 2-simplices. Δ^1 may be viewed as the subset of $\mathbb{R}^2$ given by $\{(x, y) : x + y = 1, x, y \geq 0\}$. Namely, a map $f : [1] \to \mathbb{R}$ in Δ^1 determines the x-coordinate as the image of $0 \in [1]$, and the y-coordinate as the image of $1 \in [1]$.

Similarly, Δ^2 may be viewed as the subset of $\mathbb{R}^3$ given by $\{(x, y, z) : x + y + z = 1, x, y, z \geq 0\}$ as follows: to a map $f : [2] \to \mathbb{R}$ we take the x-, y-, and z-coordinates as the images under f of 0, 1, and 2, respectively, i.e. the point $(f(0), f(1), f(2))$.

Let $\theta : [1] \to [2]$ be given by $\theta(0) = 0$ and $\theta(1) = 2$. Given any map $f \in \Delta^1$, $\theta_*(f) : [2] \to \mathbb{R}$ is defined as $\theta_*(f)(0) = f(0)$, $\theta_*(f)(1) = 0$, and $\theta_*(f)(2) = f(1)$. In terms of coordinates, given any point $(x, y) \in \Delta^1$ we have $\theta_*(x, y) = (x, 0, y)$.

Hence θ_* maps Δ^1 to the face of Δ^2 lying in the xz-plane. The other two arrows from $[1] \to [2]$ in Δ_{inj} determine the other two faces of Δ^2.

More generally in the case of standard simplices, a map θ_* arising from a map $\theta : [p] \to [q]$ for $p < q$ may be seen as uniquely identifying Δ^p as a p-face of Δ^q.

Example 2.26. The standard p-simplices may be viewed as Δ-complexes. We demonstrate this for the case Δ^2; the general case follows similarly.

Set

$$\Delta^2([0]) = \{v_0, v_1, v_2\},$$
$$\Delta^2([1]) = \{e_{01}, e_{02}, e_{12}\},$$
$$\Delta^2([2]) = \{t\},$$

and $\Delta^2([p]) = \emptyset$ for all $p \geq 3$. Intuitively, $\Delta^2([0])$ is the set of vertices, $\Delta^2([1])$ is the set of edges, and $\Delta^2([2])$ is the 2-simplex.

The maps $\theta_i : [0] \to [1]$ for $i = 0, 1$ sending 0 to $i \in [1]$ give rise to the maps $\theta_i^* : \Delta^2([1]) \to \Delta^2([0])$ defined by

$$\theta_i^*(e_{pq}) = \begin{cases} v_p, & \text{if } i = 0 \\ v_q, & \text{if } i = 1. \end{cases}$$

Similarly, the maps $\theta_{ij} : [1] \to [2]$, $0 \leq i < j \leq 2$ defined by sending 0 to i and 1 to j give rise to the maps θ_{ij}^* defined by $\theta_{ij}^*(t) = e_{ij}$.

Remark 2.27. The standard p-simplex Δ^p is the Δ-complex mapping $[q]$ to the set of q-faces of Δ^p if $q \leq p$, or to the empty set if $q > p$.

Remark 2.28. There is a natural geometric realisation of a Δ-complex X. Namely, $|X|$ is the topological space

$$|X| = \bigsqcup_{[p] \in \Delta_{inj}} X([p]) \times \Delta^p / \sim,$$

where $\sim$ is the equivalence relation $(x, \theta_* f) \sim (\theta^* x, f)$ for $x \in X([q])$, $\theta : [p] \to [q]$ an arrow in Δ_{inj}, and $f \in \Delta^p$.

We shall associate to a semistable variety over a field a Δ-complex when all irreducible components are smooth. Unfortunately, this process is more complicated when we have monodromy, as in general coequalisers of diagrams do not exist in the category of Δ-complexes. To account for this, Chan et al. introduced *generalised* Δ-complexes in [CGP16], which allow for coequalisers of diagrams of generalised Δ-complexes.

Definition 2.29. Let I be the category whose objects are the same as those of Δ_{inj}, but whose arrows are *all* injective maps $[p] \to [q]$. A *generalised Δ-complex* is a presheaf of sets on I. A *morphism* between two generalised Δ-complexes is a natural transformation between the two functors.

If $X : I \to Sets$ is a generalised Δ-complex such that the action of the symmetric $(p+1)$-group $\mathrm{Sym}(p+1)$ on $X([p])$ is free, then we say furthermore that X is an *unordered generalised Δ-complex*, or just an *unordered Δ-complex.*

Remark 2.30. That the category of generalised Δ-complexes has small colimits is discussed after Example 3.5 of [CGP16], and so in particular one may consider coequalisers of diagrams of the form $X \rightrightarrows Y$.

Example 2.31. Let $X : \Delta_{inj} \to Sets$ be any Δ-complex. To such an X we obtain an unordered Δ-complex X' by setting $X'([p]) = \mathrm{Sym}(p+1) \times X([p])$. Here $\mathrm{Sym}(p+1)$ acts only on the left side of the product.

The geometric realisation of a generalised Δ-complex is defined in the same manner as in Remark 2.28, but the equivalence relation $\sim$ now covers all morphisms in I.

Example 2.32. Let us consider generalised Δ-complexes that do not arise from Δ-complexes. Consider first the functor $X : I \to Sets$ defined by $X([0]) = \{v\}$ and $X([1]) = \{e_1, e_2\}$, where if $\sigma \in S_2$ is the nontrivial automorphism then $X(\sigma)(e_1) = e_2$. The geometric realisation of such a generalised Δ-complex is a loop edge on one vertex.

Suppose instead that $X([1]) = \{e\}$. Then, by Example 3.5 of [CGP16], X is uniquely determined and $|X|$ is identified with the half interval $[0, 0.5] \subset \mathbb{R}$ with a single vertex at 0. Moreover, the action of $\mathrm{Sym}(2)$ on $X([1])$ is not free, so X is not an unordered generalised Δ-complex.

The following lemma will be important in the next subsection:

Lemma 2.33 (Lemma 3.6 of [CGP16])**.** *Let $X \rightrightarrows Y$ be a diagram of generalised Δ-complexes, and let Z be the coequaliser. Then the natural map from the coequaliser of the diagram $|X| \rightrightarrows |Y|$ in topological spaces to $|Z|$ is a homeomorphism.*

2.3.2 Dual delta complexes associated to a semistable morphism

Let k be a separably closed field, and $f : X \to \mathrm{Spec}(k)$ be a morphism such that the points in X admit local charts as in Section 2.2. Assume furthermore that the irreducible components of X have a fixed global ordering $X_1, \ldots, X_l$ and that the X_i are smooth. Note that the constructions here do not require properness as in the definition of semistable morphisms. We shall associate to X a Δ-complex as follows:

As a functor from Δ_{inj} to $Sets$, $\Delta_X([p])$ consists of all tuples $(X_{Y_1}, \ldots, X_{Y_{p+1}}, Y)$ where Y is a p-stratum and X_{Y_i} are the irreducible components of X containing Y, where the Y_i are integers such that $Y_1 \leq \ldots \leq Y_{p+1}$. The morphisms from $\Delta_X([q]) \to \Delta_X([p])$ are induced by inclusions of (closures of) strata: if $\theta : [p] \to [q]$ is an arrow in Δ_{inj}, then $\theta^*((X_{Y_1}, \ldots, X_{Y_{q+1}}, Y)) = (X_{Y_{\theta(0)}}, \ldots, X_{Y_{\theta(p)}}, Z)$, where Z is the unique p-stratum containing Y that is contained in the intersection of $X_{Y_{\theta(0)}}, \ldots, X_{Y_{\theta(p)}}$.

Remark 2.34. Clearly one could set $\Delta_X([p])$ to be the set of p-strata of X and do away with the tuples. The additional information of which irreducible components contain a given p-stratum (along with their ordering) is there to make the morphisms more intuitive, and will not always be included.

Example 2.35. Let $f : X \to \mathrm{Spec}(k)$ be a semistable curve with smooth components $X_1, \ldots, X_l$. Then $\Delta_X([0]) = \{(X_1, X), \ldots, (X_l, X)\}$ and $\Delta_X([1]) = \{(X_i, X_j, p)\}_{i<j}$ where p is a point of intersection between components X_i and X_j.

The geometric realisation of Δ_X in this case is the dual graph of $f : X \to \mathrm{Spec}(k)$.

Example 2.36. Let $X \subset \mathbb{P}^3 = \mathrm{Proj}(k[x_0, x_1, x_2, x_3])$ by defined by the homogeneous ideal $(x_0 x_1 x_2)$. Then $\Delta_X([0]) = \{X_0, X_1, X_2\}$ is the set of the three irreducible components of X. Each pairwise combination (X_i, X_j) for $i < j$ has a unique line L_{ij} in the intersection of X_i and X_j, whence $\Delta_X([1]) = \{(X_i, X_j, L_{ij})\}$. Finally, there exists a unique 2-stratum lying in the intersection of all three components, so that $\Delta_X([2]) = \{p\}$. The geometric realisation $|\Delta_X|$ is the standard 2-simplex Δ^2.

Consider now the more general case where the irreducible components of X may not be smooth. The following construction is from Section 4 of [Thu07]. Choose a surjective étale morphism $U \to X$ such that

$U \to \operatorname{Spec}(k)$ admits local charts at all the points of U and the irreducible components of U are smooth. For example, we may cover X by local charts of the form $U_i \to \operatorname{Spec}(k[x_1, \ldots, x_n]/(x_1 \cdots x_l)$ and take $U = \sqcup_i U_i$. As $V = U \times_X U$ is étale over X, the points of V also admit local charts via base change and its irreducible components are smooth.

To both U and V we associate Δ-complexes Δ_U and Δ_V, respectively. By the previous subsection these may be viewed as unordered generalised Δ-complexes.

Set

$$\Delta_X = \varinjlim(\Delta_V \rightrightarrows \Delta_U)$$

in the category of generalised Δ-complexes. That this is well-defined is given by the following lemma:

Lemma 2.37 (Proposition 4.6(3) of [Thu07])**.** *In the above construction, Δ_X is independent of the choice of surjective étale morphism $U \to X$, where $U \to S$ admits local charts and has smooth irreducible components.*

Remark 2.38. By Lemma 4.6(4) of [Thu07], the $\operatorname{Sym}(p+1)$ orbits of each element of $\Delta(X)([p])$ correspond bijectively to the p-strata of X. This correspondence can be made very explicit via a geometric approach to the construction of Δ_X given in Section 5 of [CGP16]. In this approach we may take $\Delta_X([p])$ to be the set of equivalence classes $[(x, \sigma)]$, where $x \in X$ is a point in a p-stratum and σ is a total ordering of the local branches of x. Two such pairs $[(x, \sigma)]$ and $[(y, \tau)]$ lie in the same equivalence class if there exists a continuous path $\gamma : [0, 1] \to X$ from x to y within a p-stratum, along with choices of the total-ordering of the local branches at $\gamma(t)$ for all $t \in [0, 1]$, where $\gamma(0) = \sigma$ and $\gamma(1) = \tau$.

With this identification, an orbit under $\operatorname{Sym}(p+1)$ of an equivalence class $[(x, \sigma)]$ is uniquely identified with the p-stratum to which any point x belongs for any point $[(x, \sigma)]$ in the orbit. Under this identification the set of 0-strata $\Delta_X([0])$ may be viewed as the set of irreducible components of X.

This presentation of Δ_X is especially useful for understanding the geometric realisation as a direct limit of topological spaces. In [CGP16], a more direct description of the construction of the dual generalised Δ-complex is given as follows: Let $\beta : \tilde{X} \to X$ be the normalisation of X, and $\tilde{X}^{[p]} = \tilde{X} \times_X \tilde{X} \times_X \ldots \times_X \tilde{X}$ be the $(p+1)$-fold iterated fibre product. Set $\tilde{X}([p]) \subset \tilde{X}^{[p]}$ to be the subset consisting of tuples of pairwise distinct points whose image in X lie in a stratum of codimension p. Then $\Delta_X([p])$ is the set of irreducible components of $\tilde{X}([p])$.

Example 2.39. Let $X \subset \mathbb{P}^2$ be a curve with a single node, where affine locally X is defined by the ideal $(y - x^2(x+1))$. The normalisation $\tilde{X}$ of X is just the projective line, where two distinct points p_1 and p_2 are identified under $\beta : \tilde{X} \to X$.

Using the construction of [CGP16], we have that $\tilde{X}^{[0]} = \tilde{X}$, and $\tilde{X}([0])$ is the set of points of X that lie in the 0-stratum. In particular the unique irreducible component of $\tilde{X}([0])$ is the projective line minus p_1 and p_2, as these are the only points whose images do not lie in the 0-stratum. Thus $\Delta_X([0]) = \{v\}$.

Similarly, $\tilde{X}^{[1]} = \tilde{X} \times_X \tilde{X}$. There are two points in $\tilde{X}([1])$, namely (p_1, p_2) and (p_2, p_1), and these are interchanged under the action of S_2. Hence $\Delta_X([1]) = \{e_1, e_2\}$.

This generalised Δ-complex is isomorphic to the first one discussed in Example 2.32. Its geometric realisation is thus a single vertex with one loop edge.

Remark 2.40. From either description of Δ_X, we see that if $\sigma_Y \in \Delta_X([p])$ and $\sigma_Z \in \Delta_X([q])$, $p < q$, satisfy $\theta^*\sigma_Z = \sigma_Y$ for a map $\theta : [p] \to [q]$ in I, then the associated strata Y and Z satisfy $\overline{Z} \subset \overline{Y}$. That is, inclusions of faces in the geometric realisation of dual Δ-complexes correspond to inclusions of strata in the opposite direction.

2.3.3 Generalised cone complexes

Much of this subsection closely follows Section 2 of [ACP15] and Section 3 of [Uli17].

Definition 2.41. A *polyhedral cone with integral structure* (σ, M) is the data of a topological space σ along with a finitely generated abelian group M of continuous functions $f : \sigma \to \mathbb{R}$ such that the evaluation map $\sigma \to \mathrm{Hom}(M, \mathbb{R})$ is a homeomorphism to a strictly convex polyhedral cone in the $\mathbb{R}$-vector space dual to M. It is *rational* if the image of σ in $\mathrm{Hom}(M, \mathbb{R})$ is rational with respect to the lattice $\mathrm{Hom}(M, \mathbb{Z})$.

A *morphism* of cones $(\sigma, M) \to (\sigma', M')$ is a continuous map of topological spaces $\sigma \to \sigma'$ such that the pullback of any function in M' is in M.

We follow the convention of [ACP15] and use *cone* to refer to rational polyhedral cones with integral structure.

Let (σ, M) be a cone, and $S_\sigma \subset M$ the submonoid of linear functions that are nonnegative on σ. One may then identify σ with the space of monoid homomorphisms $\mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0})$, where $\mathbb{R}_{\geq 0}$ has the structure of a monoid via addition. With this identification, we are able to make the following definition:

Definition 2.42. Let (σ, M) be a monoid. A *face* of σ is a subset $\tau \subset \sigma$ where some function $f \in S_\sigma$ vanishes. A face τ is said to be *proper* if $\tau \neq \sigma$.

Of particular interest to us is the special case of smooth cones.

Example 2.43. Let $C([p])$ be the cone whose topological space is the standard quadrant $\mathbb{R}^{[p]}_{\geq 0} = \Pi^p_0 \mathbb{R}_{\geq 0} \subset \mathbb{R}^p$, along with the data of an abelian group induced from the addition on $\mathbb{R}^p$. A *smooth cone* is a cone isomorphic to $C([p])$ for some p.

For example, $C([1]) = \mathbb{R}^2_{\geq 0} = \{(x, y) | x, y \geq 0\}$. Consider the two projection homomorphisms p_i for $i \in \{0, 1\}$, where $p_0(x, y) = x$ and $p_1(x, y) = y$. The abelian group M generated by these two projections along with the data of the topological space $\mathbb{R}^2_{\geq 0}$ defines $C([1])$.

Similarly, one may consider $C([2]) = \{(x, y, z) | x, y, z \geq 0\}$, and the analogous projections p_i for $i \in \{0, 1, 2\}$.

Let $\theta : \{0, 1\} \to \{0, 1, 2\}$ be any injective map. Any such θ will give us a morphism $C([1]) \to C([2])$ defined by restricting the map sending the i-th basis vector to the $\theta(i)$-th basis vector.

For example, consider θ given by $\theta(0) = 0$ and $\theta(1) = 2$. Then $\theta(x, y) = (x, 0, y)$ as a map of topological spaces. The pullback of p_0 on $C([2])$ is just p_0 on $C([1])$, and the pullback of p_2 on $C([2])$ is p_1 on $C([1])$. The pullback of p_1 on $C([2])$ is 0 on $C([1])$.

Such morphisms fall into an important class of morphisms called *face morphisms* that we shall explore in more depth when discussing generalised cone complexes.

Definition 2.44 (Definition 3.6 of [Uli17])**.** A *cone complex* Σ is the data of a topological space $|\Sigma|$ and a collection (σ_α) of cones with continuous maps $\phi_\alpha : \sigma_\alpha \to |\Sigma|$ such that

- each ϕ_α is injective and induces a bijection $\sqcup_\alpha \mathring{\sigma} \to |\Sigma|$, where $\mathring{\sigma}$ is the relative interior of the cone σ, defined as the complement of all faces of σ of positive codimension;
- given any proper, sharp face $\tau \to \sigma_\beta$ of a cone σ_β, the composition $\tau \to \sigma_\beta \to |\Sigma|$ is also a member of the family (σ_α); and
- a subset $A \subset |\Sigma|$ is closed in $|\Sigma|$ if and only the preimage $\phi_\alpha^{-1}(A) \subset \sigma_\alpha$ is closed in σ_α for all α.

Example 2.45. Any cone may be naturally viewed as a cone complex. We demonstrate this explicitly for the smooth cone $C([2])$.

First, denote by $p = (0, 0, 0)$ the cone point of $C([2])$. It is a proper face of codimension 3.

Denote by L_x the cone $\mathbb{R}_{\geq 0}$, viewed as the line segment $\{(x, 0, 0)|x \geq 0\} \subset C([2])$. One may similarly define the cones L_y and L_z.

Denote by P_{xy} the cone $\mathbb{R}^2_{\geq 0}$, viewed as the face $\{(x, y, 0)|x, y \geq 0\} \subset C([2])$. We may define analogous cones P_{yz} and P_{xz}.

The collection of cones $\{p, L_x, L_y, L_z, P_{xy}, P_{yz}, P_{xz}, C([2])\}$, along with the above identifications of the cones with faces of $C([2])$, constitute a cone complex. Indeed, the first two requirements in the definition of a cone complex are immediate. To see the last point, note that as all the faces of $C([2])$ live in the same ambient space, viz. $\mathbb{R}^3$, they inherit the subspace topology. Each face is closed in $\mathbb{R}^3$, and so the third requirement follows.

We shall associate to a semistable variety over a field a cone complex when all the irreducible components of the variety are smooth. Unfortunately, this process is more complicated when one considers non-smooth irreducible components. To account for this we must allow for spaces arising as direct limits from more general diagrams of cones. We begin by introducing face morphisms, which allow for non-trivial automorphisms of cones.

Definition 2.46. Let σ and τ be cones. A morphism $\phi : \sigma \to \tau$ is said to be a *face morphism* if ϕ induces an isomorphism from σ to any face of τ. Note that this face need not be a proper face of τ.

Example 2.47. Recall that I is the category whose objects are of the form $[p] = \{0, \ldots, p\}$ for each natural number p, and whose arrows are maps of the form $\theta : [q] \to [p]$ for θ an injective map. Consider now any cone of the form $C([p])$, and let $q \leq p$.

Given any arrow $\theta : [q] \to [p]$, one obtains a face morphism $C([q]) \to C([p])$ by restricting the map sending the i-th basis vector to the $\theta(i)$-th basis vector. See Example 2.43 for the specific case of $C([1]) \to C([2])$. All face morphisms between smooth cones are induced from such arrows in I.

Equipped with face morphisms, we can now introduce generalised cone complexes:

Definition 2.48 (Definition 3.6 of [Uli17])**.** A *generalised cone complex* Σ is a topological space $|\Sigma|$ along with a presentation $\Sigma = \varinjlim D$ as a direct limit in the category of topological spaces of a diagram D of face morphisms of cones such that

- Given any proper face τ of a cone σ in the diagram, the proper face morphism $\tau \to \sigma$ is also in the diagram, and
- Any automorphism of a cone $\sigma \in D$ that leaves a proper face τ of σ invariant is such that the restriction of the automorphism to τ is also in D.

A generalised cone complex is said to be *smooth* if each cone in the diagram is a smooth cone.

A *morphism* $\Sigma \to \Sigma'$ of generalised cone complexes is a continuous map of topological spaces $|\Sigma| \to |\Sigma'|$ such that given any cone $\sigma \in \Sigma$, there is a cone $\sigma' \in \Sigma'$ such that the composition $\sigma \to \Sigma \to \Sigma'$ factors through a morphism of cones $\sigma \to \sigma'$.

If the cone morphism $\sigma \to \sigma'$ can be taken to be a face morphism of cones for each $\sigma \in \Sigma$, we say furthermore that the morphism $\Sigma \to \Sigma'$ is a *face morphism.*

Our goal is now to associate a generalised cone complex to a geometric fibre of a semistable morphism. To do so, we follow Section 3.4 of [CGP16] in using the associated generalised Δ-complex to form the needed diagram of cones.

Define a functor $\mu : I \to$ (Smooth cones, face morphisms) by setting $\mu([p]) = C([p])$. As in Section 3.4 of [CGP16], the functor μ is an equivalence of categories between I and the category of smooth cones with face morphisms between them. An arrow $\theta : [q] \to [p]$ in I gives rise to a face morphism $C([q]) \to C([p])$ as described in Example 2.47.

Suppose $\Delta : I^{op} \to Sets$ is a generalised Δ-complex. One can define a smooth generalised cone complex as follows: Let C_Δ be the small category whose objects are $([p], x)$ for $[p] \in I$ and $x \in \Delta([p])$. Furthermore, the Hom-sets are

$$\mathrm{Hom}_{C_\Delta}(([p], x), ([q], y)) = \{\mathrm{Hom}_I([p], [q]) | \theta^*(y) = x\}.$$

If one considers the functor from $C_\Delta \xrightarrow{r} I \xrightarrow{\mu}$ (Cones, face morphisms) that is the composition of the forgetful functor from C_Δ to I with the functor μ, one obtains a diagram of smooth cones with face morphisms, taking the direct limit of which gives a smooth generalised cone complex Σ associated to Δ.

Remark 2.49. As described in Section 3.4 of [CGP16], the association of a smooth generalised cone complex to a generalised Δ-complex admits an inverse. In fact, the category of generalised Δ-complexes is equivalent to the category whose objects are smooth generalised cone complexes and whose morphisms are face morphisms between them.

Let $f : X \to \mathrm{Spec}(k)$ be a semistable morphism, where k is a separably closed field. To such a morphism we have an associated generalised Δ-complex Δ_X, as described in Subsection 2.3.2. From this generalised Δ-complex we obtain a generalised cone complex Σ_X using the above construction.

Remark 2.50.

1. For a smooth generalised cone complex Σ one may form the *link* of its cone point by deleting the cone point and quotienting the remaining space by the natural action of the positive real numbers. As mentioned in [CGP16], if Σ arises from a generalised Δ-complex Δ_X, the link of the cone point of Σ is homeomorphic to the geometric realisation of Δ_X.

2. Note that there are more morphisms in the category of generalised cone complexes than in the category of generalised Δ-complexes. In the next section we shall have to work with morphisms such as the morphism from $C([1])$ to $C([0])$, given in coordinates by sending (x, y) to $x + y$. Such morphisms cannot exist in the category of generalised Δ-complexes, as by Yoneda's lemma they would have to correspond to injective maps $[1] \to [0]$ in I.

3. By Remark 2.38, each element of $\Delta_X([p])$ corresponds to a p-stratum of X along with an ordering of the local branches at some point in

the stratum, modulo an equivalence relation. The symmetric $(p+1)$-group acts by interchanging the ordering of the local branches. Under the above construction, each cone $C([p])$ in the diagram defining Σ_X corresponds to a p-stratum with an ordering of the local branches, with an isomorphism existing between two such cones if and only if the associated elements of $\Delta_X([p])$ are in the same $\mathrm{Sym}(p+1)$-orbit. The stabilizer subgroup of an element of $\Delta_X([p])$ forms the automorphisms of the corresponding cone $C([p])$ in the diagram defining Σ_X.

Example 2.51. This example is a continuation of Example 2.32. Recall the functor $\Delta_X : I \to Sets$ defined by $X([0]) = \{v\}$ and $X([1]) = \{e_1, e_2\}$, where if $\sigma \in S_2$ is the nontrivial automorphism then $X(\sigma)(e_1) = e_2$. The geometric realisation $|\Delta_X|$ is a loop-edge on one vertex.

In the resulting diagram of cones with face morphisms, each e_i gives rise to one copy of $C([1])$, along with an isomorphism between them. We have exactly one copy of $C([0])$, along with face morphisms identifying it with the x- and y-axis of the copies of $C([1])$. The resulting direct limit $|\Sigma_X|$ in topological spaces is $C([1])$ modulo the identification of the x- and y-axis, which, along with the diagram of cones, gives the generalised cone complex Σ_X.

The link of the cone point of Σ_X is clearly homeomorphic to a loop-edge.

Let $f : X \to S$ be the semistable morphism discussed in Example 2.39. As shown in Example 2.39, the dual generalised Δ-complex is the functor Δ_X. We conclude that the dual generalised cone complex associated to X is Σ_X.

We conclude this subsection by discussing reduced diagrams of cones to simplify the work in the next subsection.

Definition 2.52. Let D be a finite diagram of cones with face morphisms. We say that D is *reduced* if every isomorphism is a self-map, and all compositions of arrows in D are arrows in D.

As discussed in Section 2 of [ACP15], given a generalised cone complex $\Sigma = \varinjlim D$, one may construct an isomorphic direct limit in the category of generalised cone complexes that admits a reduced diagram as follows:

Define an equivalence relation on D by setting $\sigma \sim \sigma'$ if there exists an isomorphism in D between σ and σ', and let $\{\sigma_i\}$ be a set of representatives of the equivalence classes. Let D' be the diagram whose objects are the representatives $\{\sigma_i\}$, and whose arrows are all possible maps that are compositions of arrows in D.

Lemma 2.53 (Proposition 2.6.2(1) of [ACP15]). *The diagram D' constructed above is a reduced diagram of cones that satisfies*

$$\varinjlim D' \cong \varinjlim D$$

in the category of generalised cone complexes.

Remark 2.54. If all the cones in D are smooth, then it is clear from the construction of D' that all the cones in D' are also smooth.

In particular, given a semistable morphism $f : X \to \mathrm{Spec}(k)$ we may construct the associated generalised cone complex $\Sigma_X = \varinjlim D$, where D is (not necessarily reduced) diagram of cones constructed earlier in this subsection from Δ_X. By Lemma 2.53, we can form a reduced diagram D' with $\varinjlim D' \cong \Sigma_X$.

Remark 2.55. By Remark 2.50 (3), each cone $C([p])$ in D' corresponds to a unique p-stratum of X. By forming the reduced diagram we lose the information of an ordering of the local branches at some point of the p-stratum, which will greatly simplify the discussion in the next subsection. Furthermore, by Remark 2.38, a proper face morphism in D' corresponds to an inclusion of the associated strata in X.

2.3.4 Labelled cone complexes

We end this section by assigning a natural labelling to generalised cone complexes that arise from the geometric fibres of a semistable morphism in such a way that will generalise the case of dual graphs of semistable curves.

Definition 2.56. Let D be a reduced diagram of smooth cones and $\mathcal{M}$ a monoid. A *labelling* of D by $\mathcal{M}$ is a function

$$l : \bigcup_{p\geq 1, C([p])\in D} C([p]) \to \mathcal{M},$$

such that if $\sigma, \tau \in D$ with σ a face of τ, then $l(\sigma) = l(\tau)$.

We emphasize that the labelling is only defined on cones $C([p])$ of D when $p \geq 1$.

Remark 2.57. A labelling as defined above naturally generalises to a log-semistable morphism $X \to S$ of logarithmic schemes, taking values in the sheaf of monoids $\mathcal{M}_S$ of the log structure on the base.

Remark 2.58. If $f : X \to S$ is a semistable morphism over a field and Σ_X, Δ_X are the associated generalised cone and Δ-complexes, one could have instead labelled the elements of $\Delta_X([p])$ for $p \geq 1$. Viewing a graph as a Δ-complex G whose edges are the set $G([1])$ and whose vertices are $G([0])$, edge-labelled graphs by a monoid are a special case labelling generalised Δ-complexes. The reason we instead work with cones is to allow for more morphisms than Δ-complexes permit.

Definition 2.59. Let $C([p])$ be a smooth cone, $p > 0$. A *contraction* of $C([p])$ is a cone morphism $C([p]) \to C([0])$, given in coordinates by $(x_0, \ldots, x_p) \to (x_0 + \ldots + x_p)$.

Let $\Sigma = \varinjlim D$ and $\Sigma' = \varinjlim D'$ be smooth generalised cone complexes. A *collapse morphism of D to D'* is a morphism of cone complexes such that $|\Sigma| \to |\Sigma'|$ is a surjection and, given $\sigma \in D$, either there exists a cone $\sigma' \in D'$ such that $\sigma \to \sigma'$ is an isomorphism, or else there exists a $\sigma' \cong C([0])$ with a factorisation $\sigma \to \sigma'$ that is a contraction.

Definition 2.60. A *labelled generalised cone complex over a base scheme* S consists of

1. a generalised cone complex $\Sigma_{X_s} = \varinjlim D_s$ for each geometric point $s \in S$ with a labelling of $\varinjlim D$ in $\overline{\mathcal{O}^{\text{ét}}_{S,s}} = \mathcal{O}^{\text{ét}}_{S,s}/\mathcal{O}^*_{S,s}$;
2. a collapse morphism $\Sigma_{X_s} \to \Sigma_{X_\eta}$ for each geometric specialisation $\eta \to s$ in S.

Furthermore, we require that if $\eta \to s$ is a geometric specialisation in S, the collapse morphism $\Sigma_{X_s} \to \Sigma_{X_\eta}$ is such that the labelling of Σ_{X_η} arises from that of Σ_{X_s} via cospecialisation

$$\bigcup_{p \geq 1} \Sigma_{X_s}([p]) \to \mathcal{O}^{\text{ét}}_{S,s} \to \mathcal{O}^{\text{ét}}_{S,\eta},$$

and where cones are contracted if and only if their label is a unit over η.

Remark 2.61. The requirement that a cone is collapsed if and only if it's label is a unit over η ensures that the composition of any two such collapse morphisms is a collapse morphism defining the labelled generalised cone complex.

Proposition 2.62. *Let $f : X \to S$ be a semistable morphism with S a locally Noetherian base scheme. Let $\eta, s \in S$ be geometric points with $\eta \to s$ a geometric specialisation, and let $\Sigma_{X_s} = \varinjlim D_s$ and $\Sigma_{X_\eta} = \varinjlim D_\eta$ be the labelled smooth generalised cone complexes associated to X_s and X_η, respectively. Here we assume D_s and D_η are the reduced diagrams constructed in Subsection 2.3.3.*

There exists a unique collapse morphism $\Sigma_{X_s} \to \Sigma_{X_\eta}$ such that, under the association of p-strata and smooth cones of the form $C([p])$ of Remark 2.55, the morphism agrees with the specialisation map of Definition 2.14.

Proof. Let $Y \subset X_s$ denote a p-stratum of X_s. By Remark 2.55, there exists a cone σ_Y of the form $C([p])$ in D_s corresponding uniquely to Y. We consider two cases:

If $p = 0$, Lemma 2.12 ensures there exists a unique 0-stratum Z of X_η such that $Y \subset \overline{Z}$ in X. Equivalently, using the definition of the specialisation map (Definition 2.14), $\mathrm{sp}(Y) = Z$. Again using Remark 2.55, Z corresponds to a unique cone σ_Z in D_η of the form $C([0])$, and we define $\sigma_Y \to \sigma_Z$ to be the identity.

Suppose now that $p > 0$. There are two subcases to consider:

Suppose first that the label $l(\sigma_Y)$ of σ_Y is a non-unit in $\mathcal{O}_{S,\eta}$. Lemma 2.12 ensures that there exists a unique p-stratum Z of X_η such that $\mathrm{sp}(Y) = Z$. Similarly to the case where $p = 0$, we may take the identity morphism between the cones σ_Y and σ_Z.

Suppose now that the label $l(\sigma_Y)$ is a unit over η. If W is *any* 0-stratum of X_s containing Y in its closure, Lemma 2.16 informs us that $\mathrm{sp}(W) = Z$ for a 0-stratum Z of X_η that does not depend on W. In this case, define $\sigma_Y \to \sigma_Z$ via $(x_0, \ldots, x_p) = (x_0 + \ldots + x_p)$.

This defines the morphism on elements of D_s. It remains to show that it commutes with face morphisms, at which point the universal property of direct limits will ensure we have a collapse morphism $\Sigma_{X_s} \to \Sigma_{X_\eta}$.

Let $\sigma_Y \to \sigma_Z$ be a face morphism in D_s. If this is a proper face morphism, by Remark 2.55 it corresponds to an inclusion of closures of strata, i.e. if $\mathrm{sp}(Y) = Y'$ and $\mathrm{sp}(Z) = Z'$ with $\overline{Y} \subset \overline{Z}$, then necessarily $\overline{Y'} \subset \overline{Z'}$, giving a face morphism $\sigma_{Y'} \to \sigma_{Z'}$.

If this face morphism is an automorphism of σ_Y, i.e. if $\sigma_Y = \sigma_Z$, then such an automorphism corresponds to a permutation of the extremal rays of σ_Y whose corresponding 0-strata are the same by Remark 2.50 (3). In particular, multiple extremal rays corresponding to the same 0-stratum are preserved under the specialisation map, whence the self-maps of σ_Y are also self-maps of the image of σ_Y in D_η.

Hence in both cases the cone morphisms commute with face morphisms, giving a collapse morphism $\Sigma_{X_s} \to \Sigma_{X_\eta}$. Note that surjectivity of the underlying topological spaces follows by surjectivity of the specialisation map of strata. Uniqueness follows by the constraint of being equal to the specialisation map under the strata-cone correspondance. $\square$

Theorem 2.63. *Let $f : X \to S$ be a semistable morphism. The collection $\{\Sigma_{X_s}\}$ of labelled generalised cone complexes for each geometric point $s \in S$, along with the collapse morphisms between them from Proposition 2.62, is the data of a labelled generalised cone complex over S.*

Proof. Points (1) and (2) of the definition of a labelled generalised cone complex over S are immediate by Proposition 2.62. By Proposition 2.10, if Y is a p-stratum of X_s with label (b), $\mathrm{sp}(Y)$ has label $(b) \subset \mathcal{O}^{\text{ét}}_{S,\eta}$ via cospecialisation. The morphisms from Proposition 2.10 contract σ_Y if and only if b is a unit over η, and the result follows. $\square$

Remark 2.64. The idea of a labelled generalised cone complex can be compared with the notion of a *colored polysimplicial complex* introduced by Berkovich in [Ber99], in which polysimplicial sets are labelled (or *colored*) by elements of a monoid. The interested reader is encouraged to consult [Ber99] for more details on the subject.

Chapter 3

Artin fans and dual graphs

Whose life am I living? Whose life am I failing to live?

Margaret Atwood, The Tent

3.1 Introduction

Logarithmic geometry allows one to study certain varieties over a field as though they were smooth, such as those whose singularities étale locally resemble an intersection of hyperplanes in affine space. In the paper [Kat00] by Kato, stable curves are shown to have a unique log structure making them into *basic log curves*, allowing one to construct a moduli space of log curves that recovers the Deligne-Mumford-Knudsen stack of pointed stable curves. More recently, Olsson showed in [Ols03] that any log scheme X comes with a morphism to a logarithmic Artin stack **Log**. In [ACMW14], it is shown that this morphism admits a factorisation through an initial morphism $X \to \mathcal{A}_X$, where $\mathcal{A}_X$ is an *Artin fan*, which will later be defined. The important point of the Artin fan $\mathcal{A}_X$ is that it captures important combinatorial properties of the structure of X.

This chapter is motivated by the following problem: Suppose $f : X \to S = \mathrm{Spec}(k)$ is an n-pointed stable curve of genus g over an algebraically closed field, both equipped with log structures as in [Kat00]. To X and S one may associate Artin fans $\mathcal{A}_X$ and $\mathcal{A}_S$. In what way does the topology of $\mathcal{A}_X$ reflect the combinatorics of X? The final result in this chapter provides answer to this in the following theorem:

Theorem 3.1 (Theorem 3.55)**.** *Let $f : X \to S = Spec(k)$ be a basic stable log curve of type (g, n), where k is an algebraically closed field and X has smooth irreducible components. Let $\mathcal{A}_X$ and $\mathcal{A}_S$ be the Artin fans of X*

and S, respectively, and $\zeta \in \mathcal{A}_S$ the closed point. Then there is a natural morphism $\pi : \mathcal{A}_X \to \mathcal{A}_S$ arising from $f : X \to S$ such that the dual graph Γ_X of $f : X \to S$ can be reconstructed from the underlying topological space of $\pi^{-1}(\zeta)$.

The link between the dual graph of X and the Artin fan of X is not in and of itself a new result - it is shown in Example 4.8 of [Uli19] that when X satisfies satisfies having *no monodromy*, the underlying topological space of $\mathcal{A}_X$ is the well-studied *Kato fan* of X. Rather, we hope to present this fact in a clear manner, and that the intermediate results are of interest.

In order to prove the above theorem, we first need to establish that the association of an Artin fan to both X and S is functorial with respect to the morphism f. In general this fails if the morphism f is not strict, which we do not have in this case. To get around this issue we construct a logarithmically smooth family of stable log curves $F : \mathcal{X} \to \mathcal{S}$ as in the following proposition:

Proposition 3.2 (Proposition 3.40)**.** *Let $f : X \to S = Spec(k)$ be an n-pointed stable curve of genus g with l double points. Then there exists an n-pointed stable curve $\mathcal{X} \to \mathcal{S}$ fitting into a commutative diagram*

$$\begin{array}{ccc} X \cong \mathcal{X} \times_{\mathcal{S}} Spec(k) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & \mathcal{S} \\ & \overset{0}{\searrow} & \downarrow \text{\'et} \\ & & \mathbb{A}^l_k, \end{array}$$

such that

- *$\mathcal{S} \to \mathbb{A}^l_k$ is étale,*
- *the morphism from $Spec(k)$ to $\mathbb{A}^l_k$ maps to $0 \in \mathbb{A}^l_k$, and*
- *if $\mathcal{X}$ and $\mathcal{S}$ are given log structures so that $\mathcal{X} \to \mathcal{S}$ is a basic log curve of type (g, n), the log structure on $\mathcal{S}$ is the pullback of the standard toroidal log structure on $\mathbb{A}^l_k$.*

The proof of this goes through deformation theory. The fact that $\mathcal{X}$ and $\mathcal{S}$ are log smooth over Spec(k) with the trivial log structure is key to establishing a morphism from $\mathcal{A}_{\mathcal{X}}$ to $\mathcal{A}_{\mathcal{S}}$ that is functorial with respect to $F : \mathcal{X} \to \mathcal{S}$. This result seems to be known in some capacity to experts in the field, but a proof of it could not be found and so is produced here. After showing that $\mathcal{A}_X \cong \mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_S \cong \mathcal{A}_{\mathcal{S}}$, we then conclude the chapter by looking at the topology of the Artin fans by studying them étale locally on X.

3.1.1 Overview of chapter

In Section 3.2 we recall the required facts on logarithmic geometry, with a focus on log structures associated to toric varieties. We end the section by defining Artin fans and showing how their topology is related to the topology of Kato cones, which is needed in showing Theorem 3.55.

Section 3.3 provides additional background material more specific to the problems in this chapter. Namely, n-pointed stable curves of genus g and log curves of type (g, n) are defined, and the relation between them as in [Kat00] is laid out in detail. Combined with the relation between Artin fans and Kato cones discussed in 3.2, this will later allow us to completely determine the topology of the Artin fan $\mathcal{A}_X$ of X.

We establish Proposition 3.40 in Section 3.4 through a series of lemmas. This section draws heavily from deformation theory.

The entirety of Section 3.5 is devoted to showing the association of Artin fans to X and S is functorial with respect to the morphism $f : X \to S$. This relies on Proposition 3.40 from the previous section and uses results from [AW18].

Finally, we prove Theorem 3.55 in Section 3.6. This first involves a discussion of the topologies of both $\mathcal{A}_X$ and $\mathcal{A}_S$ which, combined with the morphism from $\mathcal{A}_X$ to $\mathcal{A}_S$ from the previous section, is sufficient for the proof.

3.2 Basics of logarithmic geometry

In this section we recall the basics of logarithmic geometry. This is not intended as a complete introduction to the topic, but rather to set up the notation for further subsections. For a more thorough introduction to the subject, the interested reader is recommended to consult [Ogu18]. Many of the definitions given in this subsection are found there.

3.2.1 Facts on monoids

Monoids are to logarithmic geometry what rings are to algebraic geometry. We start by recalling the definition.

Definition 3.3. A *commutative monoid* $M = (M, \times, e_M)$ is a set M together with a commutative associative binary operation $\times : M \times M \to M$ with an identity e_M. A *morphism* $\phi : M \to N$ of commutative monoids M and N is a map such that $\phi(e_M) = e_N$ and, given two elements m and m' in M, $\phi(m \times m') = \phi(m) \times_N \phi(m')$.

Remark 3.4. Though there are non-commutative monoids, all monoids we consider will be commutative, and so we refer to them simply as monoids. We will almost always refer to a monoid $(M, \times, e_M)$ by M when the identity and operation are understood.

Any abelian group is naturally a monoid, and conversely any monoid whose elements all possess inverses is a group. Furthermore, given a monoid M, there exists a unique group M^{gp} called the *groupification of* M along with a morphism of monoids $M \to M^{gp}$ such that given any morphism of monoids from M to a group G, the morphism factors through $M \to M^{gp}$.

An element $m \in M$ is said to be a *unit* if it has an inverse in M. If M^* denotes the set of units of M, then M^* is a group. A monoid M is *sharp* if $M^* = \{e_M\}$. A morphism $f : M \to N$ of monoids is *sharp* if f induces an isomorphism from M^* to N^*. The quotient M/M^* in the category of monoids is itself a sharp monoid called the *characteristic monoid of* M.

A monoid is *finitely generated* if there exists a finite subset $S \subset M$ such that the smallest submonoid of M containing S is M. We say that a monoid is *integral* if the cancellation law holds in M. That is, given any three elements m, m_1, and m_2 of M such that $m + m_1 = m + m_2$, it holds that $m_1 = m_2$. A monoid M is integral if and only if the map from M to M^{gp} is injective. A finitely generated and integral monoid is said to be *fine*.

Finally, we say that a monoid is *saturated* if it is integral and if, given an element m in M^{gp} and a positive integer n such that $nm \in M$, it holds that $m \in M$.

Example 3.5. One of the simplest examples of a monoid is $\mathbb{N}$, the natural numbers, under addition. In fact, later in this chapter the primary monoids we shall concern ourselves with are those of the form $\mathbb{N}^l$ with $l \geq 1$. Clearly the only unit of such a monoid is the identity, and so $\mathbb{N}^l$ is sharp.

In this case, $(\mathbb{N}^l)^{gp} = \mathbb{Z}^l$, and we see that $\mathbb{N}^l$ is integral as $\mathbb{N}^l$ is included in $\mathbb{Z}^l$. It is immediate that $\mathbb{N}^l$ is finitely generated with the standard generators $e_i = (0, \ldots, 1, \ldots, 0)$ where the only non-zero coordinate the i-th coordinate, and so in particular $\mathbb{N}^l$ is fine. It is trivial to check that $\mathbb{N}^l$ is saturated.

Remark 3.6. Following Ogus, we shall refer to monoids which are fine and saturated as fs-monoids. These play a particularly important role in logarithmic geometry, with many results in the field assuming the monoids in question are fs.

Similar to the case of rings, one may define ideals of a monoid M. Namely, an *ideal* $I \subset M$ is a subset of M such that if $a \in I$ and $m \in M$, then $a + m \in I$. It is a *prime ideal* of M if it is a proper subset of M and if, given m_1 and m_2 in M, then $m_1 + m_2 \in I$ implies that either m_1 or m_2 is in M.

Remark 3.7. The empty set is always the unique minimal prime ideal of a monoid M, contained within all the other prime ideals. The set of non-units is always the unique maximal prime ideal of a monoid M.

Define the *spectrum of a monoid* M, $\mathrm{Spec}(M)$, to be the set of prime ideals of M. If I is an ideal and $Z(I)$ denotes the set of prime ideals of $\mathrm{Spec}(M)$ containing I, one can define the Zariski topology on $\mathrm{Spec}(M)$ where the subsets of the form $Z(I)$ form the closed subsets. Equivalently, one may form the topology by taking as a basis of open sets those of the form $D(f) = \{p \in \mathrm{Spec}(M) | f \notin p\}$, where f is an element in M.

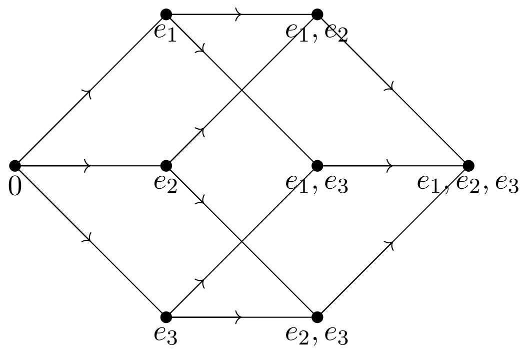
Example 3.8. Let's consider the monoid $M = \mathbb{N}^3$, together with its standard generators $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. A general element of $\mathbb{N}^3$ can be written as (a_1, a_2, a_3), where the a_i are nonnegative integers for all i.

Let $p \in \mathrm{Spec}(M)$ be a prime ideal other than the empty set or the set of non-units, and let $m = (a_1, a_2, a_3) \in p$ be any element where, without loss of generality, $a_1 > 0$. By writing $m = (1, 0, 0) + (a_1 - 1, a_2, a_3)$, we see that either $(1, 0, 0) \in p$ or else $m' = (a_1 - 1, a_2, a_3) \in p$. If $(1, 0, 0)$ is not in p, then by repeating this process we find that $(0, a_2, a_3) \in p$. Repeating this process with $(0, 1, 0)$ if $a_2 > 0$ or $(0, 0, 1)$ if $a_3 > 0$ yields that p must contain at least one of the standard generators of M.

Conversely, if I is any subset of the standard generators, the subset $p_I = \{m+n | m \in M, n \in I\}$ is readily seen to be a prime ideal of M, which we refer to as the prime ideal generated by I, written $p_I = (I)$.

We have shown that the prime ideals of M other than the closed and open point are $\{(e_1), (e_2), (e_3), (e_1, e_2), (e_1, e_3), (e_2, e_3)\}$.

The topology of $\text{Spec}(\mathbb{N}^3)$ is illustrated below. Here the arrows denote specialisation.



Finally, for use in the next subsection we look at localisation of monoids. Given a subset $S \subset M$ of a monoid M, the *localisation of* M *by* S consists of all elements of the form $\{m - s | m \in M, s \in S\}$, wherein $m - s$ is an equivalence class in $M \times S$ such that $m - s$ is equivalent to $m' - s'$ if there is some $t \in S$ such that $m + s' + t = m' + s + t$ holds in M.

To any monoid M, one may define a pre-sheaf of monoids on $\text{Spec}(M)$ via the association $D(f) \to M_f / M_f^*$ with resultant sheaf of monoids $\mathcal{M}_{\text{Spec}(M)}$. When M is fs and sharp, the resulting topological space, together with the sheaf of monoids $\mathcal{M}_{\text{Spec}(M)}$, is referred to as a Kato cone. More formally we have the following definition.

Definition 3.9. A *Kato cone* is topological space S, together with a sheaf of sharp monoids $\mathcal{M}_S$, that is isomorphic to $(\text{Spec}(M), \mathcal{M}_{\text{Spec}(M)})$ for some fs and sharp monoid M. A *morphism of Kato cones* $f : (S, \mathcal{M}_S) \to (T, \mathcal{M}_T)$, is a continuous map of topological spaces $f : S \to T$ together with a sharp morphism of sheaves of monoids $f^b : (\mathcal{M}_T) \to f_* \mathcal{M}_S$.

If N and M are sharp fs-monoids with a sharp morphism $\theta : N \to M$, one obtains a morphism $f : \text{Spec}(M) \to \text{Spec}(N)$. On the level of sets, f sends a prime ideal $p \in \text{Spec}(M)$ to $\theta^{-1}(p)$. If $n \in N$, we have that

$\mathcal{N}_{\mathrm{Spec}(N)}(D(n)) = \overline{N}_n$, and $f_*\mathcal{M}_{\mathrm{Spec}(M)}(D(n)) = \overline{M}_{\theta(n)}$, with the morphism from $\overline{N}_n \to \overline{M}_{\theta(n)}$ induced by $\theta : N \to M$. Hence we have a functor Spec from sharp fs-monoids to Kato cones.

Lemma 3.10. *The functor Spec from sharp fs-monoids to Kato cones is an anti-equivalence of categories.*

Proof. Let M be a sharp fs-monoid. Then $(\mathrm{Spec}(M), \mathcal{M}_{\mathrm{Spec}(M)})$ satisfies $\Gamma(\mathrm{Spec}(M), \mathcal{M}_{\mathrm{Spec}(M)}) = M/M^* = M$. Conversely, if X is a Kato cone, $\mathcal{M}_X(X)$ is a sharp fs-monoid such that $(\mathrm{Spec}(\mathcal{M}_X(X)), \mathcal{M}_{\mathrm{Spec}(\mathcal{M}_X(X))}) = X$.

Let N and M be sharp fs-monoids along with a morphism $\theta : N \to M$. We've seen that the morphism $f : \mathrm{Spec}(M) \to \mathrm{Spec}(N)$ is such that, given $n \in N$, the morphism $f^b : \mathcal{M}_{\mathrm{Spec}(N)}(D(n)) \to f_*\mathcal{M}_{\mathrm{Spec}(M)}(D(\theta(n))$ is the morphism $\overline{N}_n \to \overline{M}_{\theta(n)}$ induced by θ. Taking $n = 0$ recovers θ as both N and M are sharp.

Conversely, let $f : \mathrm{Spec}(M) \to \mathrm{Spec}(N)$ be a morphism of Kato cones. This yields a sharp morphism of sharp fs-monoids $\theta : N \to M$ by considering the morphism f^b on the global sections. To see that f_θ, the morphism from $\mathrm{Spec}(M)$ to $\mathrm{Spec}(N)$ agrees with f, we need only check they agree on the level of sets as the morphism of sheaves of monoids in either case is induced from $\theta : M \to N$, along with the universal property of localisation.

To this end, let $p \in \mathrm{Spec}(N)$. The morphism $f^b_p : \overline{M}_{f(p)} \to \overline{N}_p$ is the morphism arising from localising the morphism $\theta : M \to N$ and quotienting by the units. In particular, $\theta^{-1}(p) = f_\theta(p) = f(p)$. This concludes the proof. □

3.2.2 Logarithmic schemes

Let $\underline{X}$ be a scheme. The notation of underlining a scheme in this chapter will be justified later when $\underline{X}$ will be used to denote the underlying scheme of a logarithmic scheme X. To any such scheme, the structure sheaf under the multiplicative operation may be seen as a sheaf of monoids on $\underline{X}$. Given a scheme $\underline{X}$, let $\mathcal{M}_X$ denote an sheaf of monoids on the small étale site of $\underline{X}$.

Definition 3.11. Let $\mathcal{M}_X$ be an étale sheaf of monoids on a scheme $\underline{X}$.

Given a morphism of sheaves of monoids

$$\sigma : \mathcal{M}_X \to \mathcal{O}_X,$$

the pair $(\mathcal{M}_X, \sigma)$ is a *logarithmic structure on* $\underline{X}$ if $\sigma^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism of sheaves of monoids. We say that $X = (\underline{X}, \mathcal{M}_X, \sigma)$ is a *logarithmic scheme.*

If $\mathcal{M}_X$ is any sheaf of monoids on $\underline{X}$ and we are given a morphism $\sigma : \mathcal{M}_X \to \mathcal{O}_X$, we say that $\mathcal{M}_X$ is a *pre-logarithmic structure on* X. To any such pre-logarithmic structure, we may form a logarithmic structure $\mathcal{M}_X^{log}$ on X, called the *logarithmic structure associated to* $\mathcal{M}_X$, defined as the pushout of the diagram:

$$\begin{array}{ccc} \sigma^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M}_X \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array} \tag{3.1}$$

Lemma 3.12 (Proposition III.1.1.3 of [Ogu18])**.** *Let* $\mathbf{Log}_X$ *and* $\mathbf{pLog}_X$ *denote the category of logarithmic and pre-logarithmic structures on a scheme* X*, respectively. Then the functor from* $\mathbf{pLog}$ *to* $\mathbf{Log}$ *sending* $\mathcal{M}_X$ *to* $\mathcal{M}_X^{log}$ *is left-adjoint to the inclusion functor* $\mathbf{Log}_X \to \mathbf{pLog}_X$.

Remark 3.13. A logarithmic scheme $(X, \mathcal{M}_X, \sigma)$ will often be denoted simply by X when the logarithmic structure is clear.

The *characteristic sheaf* $\overline{\mathcal{M}_X}$ of a logarithmic scheme $(X, \mathcal{M}_X, \sigma)$ is the sheaf of monoids $\mathcal{M}_X/\sigma^{-1}(\mathcal{O}_X^*)$ associated to the pre-sheaf sending $U \subset X$ to $\mathcal{M}_X(U)/\sigma^{-1}(\mathcal{O}_X^*)(U)$, where the quotient is in the category of monoids.

Example 3.14. Let $\underline{X}$ be a smooth scheme over an algebraically closed field and D be an effective divisor on $\underline{X}$. The pre-logarithmic structure on $\underline{X}$ given by $\mathcal{M}_X(U) = \{f \in \mathcal{O}_X(U) | f|_{U \setminus D} \in \mathcal{O}_X(U \setminus D)^*\}$ is readily seen to be a log structure on X, called the *divisorial logarithmic structure associated to* D.

Of particular relevance to us is the case $X = \mathbb{A}^l = \text{Spec}(k[x_1, \ldots, x_l])$ is affine l-space, and $D = (x_1 \cdots x_l)$. This is, equivalently, the logarithmic structure associated to the constant sheaf pre-log structure $\mathbb{N}^l \to \mathcal{O}_{\mathbb{A}^l}$, where, if e_i is the i-th standard generator of $\mathbb{N}^l$, we send e_i to x_i. As this divisorial log structure is concerned with being trivial on the torus $\mathbb{G}_m^l$, it is also called the *toric log structure*.

Closely related to pre-log structures is the notion of a chart. Let $(X, \mathcal{M}_X)$ be a logarithmic scheme. A *chart* of X is a monoid P along with a morphism $P_X \to \mathcal{M}_X$ where P_X is the constant sheaf on X associated to P, such that the natural map from the logarithmic structure associated to the composition $P_X \to \mathcal{M}_X \to \mathcal{O}_X$ to $\mathcal{M}_X$ is an isomorphism. A log scheme X is said to be an *fs-log scheme* if, étale locally on X, one can find a chart where P is an fs-monoid.

One may also consider a morphism between logarithmic structures on a scheme X. Given two logarithmic structures $(\mathcal{M}_X, \sigma)$ and $(\mathcal{N}_X, \rho)$, a morphism from $\mathcal{M}_X$ to $\mathcal{N}_X$ is a morphism of sheaves of monoids on X such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_X & \longrightarrow & \mathcal{N}_X \\ & \searrow^{\sigma} \quad \swarrow_{\rho} & \\ & \mathcal{O}_X & \end{array} \tag{3.2}$$

Finally, we consider morphisms of logarithmic schemes. Suppose first that $\underline{f}\colon \underline{X} \to \underline{S}$ is a morphism of schemes, and $\mathcal{M}_S$ is a logarithmic structure on $\underline{S}$. Then $\underline{f}^*\mathcal{M}_S$ is the logarithmic structure on $\underline{X}$ associated to the pre-log structure $\underline{f}^{-1}\mathcal{M}_S \to \underline{f}^{-1}\mathcal{O}_S \to \mathcal{O}_X$, where $f^{-1}\mathcal{M}_S$ is the inverse image sheaf of $\mathcal{M}_S$ in the category of sheaves of monoids on $\underline{X}$.

Definition 3.15. Let X and S be logarithmic schemes with logarithmic structures $\mathcal{M}_X$ and $\mathcal{M}_S$, respectively. A *morphism f of logarithmic schemes* from X to S is a morphism $\underline{f}\colon \underline{X} \to \underline{S}$ of the underlying schemes along with a morphism of $f^\beta : f^*\mathcal{M}_S \to \mathcal{M}_X$ of logarithmic structures on X. If, furthermore, f^β is an isomorphism, we say the morphism of logarithmic schemes is *strict*.

One may also define a *chart of a morphism* $f : X \to S$ of logarithmic schemes. By this, we mean a triple $(P \to \mathcal{M}_X, Q \to \mathcal{M}_S, Q \to P)$, where P and Q are charts of X and S, respectively, and the morphism of monoids $Q \to P$ makes the following diagram commute:

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ f^*(\mathcal{M}_S) & \longrightarrow & \mathcal{M}_X. \end{array} \tag{3.3}$$

Definition 3.16. Let R be a ring and M be a monoid. The *monoid ring of M over R*, denoted by $R[M]$, is the set of functions $\phi : M \to R$ such that $\phi(m) = 0$ for all but finitely many $m \in M$, with addition defined by addition of functions, and multiplication defined via $(\phi\psi)(m) = \sum_{nt=m} \phi(n)\psi(t)$.

Example 3.17. Let R be a ring and M be a monoid. The monoid ring $R[M]$ admits a natural map from M to $R[M]$ that sends $m \in M$ to the map $\phi_m : M \to R$, where $\phi_m(m) = 1$ and $\phi_m(n) = 0$ for all $n \neq m$. In particular, the scheme $\mathrm{Spec}(R[M])$ admits the pre-log structure associated to this morphism, and hence carries a natural logarithmic structure.

We shall often consider the ring $k[M]$ where k is a field and M is an fs-monoid such that M^{gp} is free. Any fs-monoid M such that M^{gp} is free is called *toric*, but we shall not make use of this terminology. If $T = \mathrm{Spec}(k[M^{gp}])$ and $X = \mathrm{Spec}(k[M])$, then X is a toric variety with torus T. In Example 3.14 we saw the case where $M = \mathbb{N}^l$ and hence $T = \mathbb{G}_m^l$ is the standard torus in affine l-space.

3.2.3 Log smoothness

Morphisms of logarithmic schemes can be logarithmically smooth. The definition as is commonly presented uses an infinitesimal lifting property of log schemes and is analogous to that of the scheme-theoretic definition of smoothness of a morphism. See Chapter 4 of [Ogu18] for a full treatment of the topic. We give an alternative definition that holds in the case of fs-log schemes such that the base has a chart. In fact, an fs-log scheme always admits a chart étale locally, allowing the following to be used as a general definition.

Definition 3.18. Let X and S be fs-log schemes. Let $Q \to \mathcal{M}_S$ be a chart of S with Q an fs-monoid.. Given a morphism $f : X \to S$, we say that f is *logarithmically smooth* (resp. *logarithmically étale*) if étale locally on X there exists a chart $P \to \mathcal{M}_X$ where P is an fs-monoid, such that

- The kernel and the torsion part of the cokernel (resp. the kernel and cokernel) of $Q^{gp} \to P^{gp}$ are finite and of invertible order on X, and
- The morphism $\underline{X} \to \underline{S} \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ is an étale morphism of schemes.

Remark 3.19. That this is equivalent to the normal definition of logarithmic smoothness in the case of fs-log schemes is the content of Theorem 3.5 of [Kat89]. Proposition 3.8 of the same paper shows that if f is a *strict* morphism between fs-log schemes, f is logarithmically smooth (resp. étale) if and only if it is smooth (resp. étale) in the usual sense.

Of importance to us is the following example:

Example 3.20. Let $S = \mathrm{Spec}(k)$ with the trivial log structure, where k is a field. That is, $\mathcal{M}_S(S) = k^*$, with global chart $Q = 0$.

Let $X = \mathbb{A}_k^l = \mathrm{Spec}(k[x_1, \ldots, x_l]) = \mathrm{Spec}(k[\mathbb{N}^l])$, equipped with its toroidal log structure as in Example 3.14. Then $f : X \to S$ is log smooth.

Indeed, both the kernel and torsion part of the cokernel of $Q^{gp} \to \mathbb{Z}^l$ are trivial, and $\underline{X}$ is trivially étale over itself.

Remark 3.21 (Proposition 4.7 of [Thu07])**.** The above example extends to a more general situation. Recall that if k is a field and $\underline{X}$ is a smooth scheme over k, an effective Cartier divisor $D \subset \underline{X}$ is said to be a *strict normal crossings divisor* if at each point $x \in D$, there exists a regular system of parameters $x_1, \ldots, x_n$ in the maximal ideal of $\mathcal{O}_{\underline{X},x}$ and an integer $1 \leq l \leq n$ such that D is defined by $x_1 \cdots x_l$ in $\mathcal{O}_{\underline{X},x}$. We say that D is a *normal crossings divisor* if it is a strict normal crossings divisor étale locally on X at the points of D.

Let $\underline{X}$ be a smooth scheme over k and $D \subset X$ a normal crossings divisor. Then, if $\mathcal{M}_X$ is the divisorial log structure associated to D, $(X, \mathcal{M}_X)$ is log smooth over $\mathrm{Spec}(k)$ equipped with the trivial log structure.

To see this, note that by definition of D being a normal crossings divisor and working étale locally on X, we can find a regular system of parameters $x_1, \ldots, x_n$ in the maximal ideal of $\mathcal{O}_{\underline{X},x}$ and an integer $1 \leq l \leq n$ such that D is defined by $x_1 \cdots x_l$ in $\mathcal{O}_{\underline{X},x}$. Localising if necessary, this yields an étale morphism $X \to \mathrm{Spec}(k[t_1, \ldots, t_n])$ defined by sending $t_i \to x_i$ for all i, whence D is the pullback of the divisor $(t_1 \cdots t_l)$ on $\mathbb{A}_k^n$. In particular, the log structure $\mathcal{M}_X$ admits the chart $\mathbb{N}^n \to \mathcal{O}_{\underline{U}_x}$ that sends $e_i \to x_i$, itself pulled back from the log structure on $\mathbb{A}_k^n$ defined by sending $e_i \to t_i$.

With the trivial log structure on $\underline{S} = \mathrm{Spec}(k)$ admitting the global chart $0 \to k$ sending 0 to 1, the kernel and torsion part of the cokernel from $0 \to \mathbb{N}^n$ are both trivial and hence of invertible order on X. Furthermore, $\underline{U}_x \to \underline{S} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^n]) \cong \mathbb{A}_k^n$ is an étale morphism. We conclude that X is log smooth over S.

3.2.4 Artin fans

The presentation of the definitions and results in this subsection closely follows that of Section 5 of [ACM+16]. Before defining Artin fans, we introduce some terminology.

Let S be an fs-logarithmic scheme. Define $\underline{\mathbf{Log}}_S$ to be the category whose objects are morphisms of log schemes $f : X \to S$, and whose arrows are strict morphisms of log S-schemes. This is a fibred category over $\underline{S}$ schemes via the association $X \to \underline{X}$, and is in fact an algebraic stack by Theorem 1.1 of [Ols03], called the *classifying stack of log structures on S*. Note that id : $S \to S$ induces a morphism $\underline{S} \to \underline{\mathbf{Log}}_S$ that is shown in Proposition 3.19(ii) of [Ols03] to be an open immersion.

Given a morphism $f : X \to S$ of fs-log schemes, there is a tautological morphism of algebraic stacks $\underline{\mathbf{Log}}(f) : \underline{\mathbf{Log}}_X \to \underline{\mathbf{Log}}_S$. This induces a tautological morphism from $\underline{X}$ to $\underline{\mathbf{Log}}_S$ that is the composition of $\underline{X} \to \underline{\mathbf{Log}}_X$ with $\underline{\mathbf{Log}}(f)$, sending $g \in \underline{X}(\underline{T})$ to $(\underline{T}, g^*\mathcal{M}_X)$. As shown in Theorem 4.6(ii) of [Ols03], the morphism $f : X \to S$ is logarithmically smooth (resp. étale) if and only if $\underline{X} \to \underline{\mathbf{Log}}_S$ is smooth (resp. étale). One may impose a universal log structure on $\underline{\mathbf{Log}}_S$ with the property that, given $f : X \to S$, the resulting morphism of log algebraic stacks $X \to \mathbf{Log}_S$ is strict.

Definition 3.22. An *Artin fan* is a logarithmic algebraic stack that has a strict étale cover by a disjoint union of Artin cones.

Remark 3.23. To simplify notation in the rest of the chapter, we shall write $\mathbf{Log}$ when we mean $\mathbf{Log}_{\mathrm{Spec}(k)}$, where k will denote an algebraically closed field and Spec(k) is endowed with the trivial log structure.

Remark 3.24 (Section 5 of [Ols03])**.** Given $V = \mathrm{Spec}(k[M])$ for a sharp fs-monoid M such that M^{gp} is free with dense torus $T = k[M^{gp}]$, we may form the stack-theoretic quotient $[V/T]$ whose objects over a k-scheme S are T-torsors $P \to S$ over S along with an invariant map $P \to V$, and whose arrows from $P \to S$ to $P' \to S'$ are Cartesian diagrams that are compatible with the maps to V.

The action of T on V extends to an action of T on the log structure associated to $M \to k[M]$, in that given any étale V-scheme Spec(R) and maps $a : M \to R$ and $b : M^{gp} \to R^*$, the log structures associated to both

a and ab are isomorphic, where $(ab)(m) = a(m)b(m) \in R$, under the map

$$M \to R^* \bigoplus M,$$

$$m \mapsto (b(m)^{-1}, m).$$

By Proposition 5.6 of [Ols03], this implies that the toric log structure on V descends to a log structure on $[V/T]$.

More explicitly, by Proposition 5.17 of [Ols03], $[V/T]$ represents the functor on log schemes over k sending S to $\mathrm{Hom}(M, \Gamma(S, \overline{\mathcal{M}}_S))$. In particular, to any toric variety V with torus T over k we obtain an Artin fan of the form $[V/T]$.

Definition 3.25 (Definition 5.3 of [ACM+16])**.** An *Artin cone* is an Artin fan isomorphic to $\mathcal{A}_M = [V/T]$, where $V = \mathrm{Spec}(k[M])$ for a sharp fs-monoid M.

In fact, Artin cones form an étale cover of **Log** by Lemma 5.4 of [ACM+16]. A morphism between Artin cones is a morphism of logarithmic stacks of the Artin fans. The following lemma shows that all such morphisms come from morphisms of the associated monoids, and will be fundamental when proving results later in this chapter. Note that in the lemma we make use of *the underlying topological space of an algebraic stack*, which is explained in depth in [[Sta17], Tag 04XE]. The important point for us is that the points of a quotient stack $[V/T]$ as above are in one-to-one correspondance with the orbits of the points of V under the action of the torus.

Lemma 3.26 (Lemma 5.4 of [ACM+16])**.** *The functor sending a sharp fs-monoid M to $\mathcal{A}_M$ is an equivalence of categories between sharp fs-monoids and Artin cones. In particular, the category of Artin cones is equivalent to that of Kato cones with the topology of a Kato cone Spec(M) being isomorphic to that of $\mathcal{A}_M$.*

Remark 3.27. As in Section 3 of [Uli19], if M is a sharp fs-monoid one may form both the Kato cone $\mathrm{Spec}(M)$ and the Artin cone $\mathcal{A}_M$. The equivalence between the two categories of Artin cones and Kato cone is such that $|\mathcal{A}_M| \cong |\mathrm{Spec}(M)|$.

To associate an Artin fan to a logarithmic scheme, we first consider the case of small log schemes.

Definition 3.28. Let $(X, \mathcal{M}_X)$ be a logarithmic scheme. X is said to be *small* with respect to a point $x \in X$ if the map $\Gamma(X, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$ is an isomorphism, and if the logarithmic stratum $\{y \in X | \overline{\mathcal{M}}_{X,y} \cong \overline{\mathcal{M}}_{X,x}\}$ is connected. X is said to be *small* if it is small with respect to some point.

Such small log schemes are used in the proof of the following result.

Lemma 3.29 (Proposition 4.1 of [Uli19]). *Let X be an fs-logarithmic scheme of finite type over Spec(k), where k is algebraically closed. Then there is a strict morphism from X to an Artin fan $\mathcal{A}_X$ that is initial with respect to morphisms from X to Artin fans for which the tautological strict morphism to **Log** is representable. The association of $\mathcal{A}_X$ to X is functorial with respect to strict morphisms.*

We provide a sketch of the proof to illustrate the construction for later use.

Proof. Let X be a small log scheme with respect to a point $x \in X$. Set $M = \overline{\mathcal{M}}_{X,x}$. By Remark 3.24, the isomorphism $\overline{\mathcal{M}}_{X,x} \cong \Gamma(X, \overline{\mathcal{M}}_X)$ yields a strict morphism from X to $\mathcal{A}_M = [\mathrm{Spec}(k[M])/\mathrm{Spec}(k[M^{gp}])]$ that is shown to be initial in [ACMW14].

Suppose now that X is a general fs-log scheme. Choose an étale presentation $V \rightrightarrows U \to X$ of X, where $V = \bigsqcup V_i$ and $U = \bigsqcup U_i$ are disjoint unions of small log schemes. That this can be done follows from Remark II.2.3.2 of [Ogu18], which states that any point of an fs-log scheme admits a neighbourhood that is small with respect to the point. The Artin fans associated to V and U then exist as $\mathcal{A}_V = \bigsqcup_i \mathcal{A}_{V_i}$ and $\mathcal{A}_U = \bigsqcup_i \mathcal{A}_{U_i}$, respectively. Set $\mathcal{A}_X$ to be the colimit of $\mathcal{A}_V \rightrightarrows \mathcal{A}_U$ in the category of sheaves of **Log** (i.e. the category of representable strict morphisms to **Log**); that there exists a morphism from X to $\mathcal{A}_X$ follows as X is a colimit in the category of log algebraic stacks over k. That this is an initial morphism from $X \to \mathcal{A}_X$ follows by the universal property of colimits. □

Definition 3.30. Let X be a log scheme as in the above lemma, and $\mathcal{A}_X$ the Artin fan such that $X \to \mathcal{A}_X$ is initial. We say that $\mathcal{A}_X$ is the *Artin fan associated to X*.

3.3 Setup

Let g and n be nonnegative integers such that $2g - 2 + n > 0$, and let $\underline{S}$ be an arbitrary locally Noetherian scheme. We are interested in n-pointed stable curves of genus g over S as in [Knu83], by which we mean the following:

Definition 3.31 (Definition 1.1 of [Knu83])**.** An *n-pointed stable curve of genus g* over $\underline{S}$ is a flat and proper morphism $\underline{f} : \underline{X} \to \underline{S}$ along with n sections $s_i : \underline{S} \to \underline{X}$ of f, $i \in \{1, \ldots, n\}$, such that

- the geometric fibres of $\underline{f}$ are reduced, connected curves of arithmetic genus g and with at most double points as singularities;
- the images of the sections s_i all lie in the smooth locus of f; and
- for all $s \in S$, $s_i(s) \neq s_j(s)$ for $i \neq j$; and
- $H^0(\underline{X}, T_{\underline{X}}) = 0$, where $T_{\underline{X}}$ is the tangent sheaf of X.

If k is a separably closed field, let $\underline{f} : \underline{X} \to \underline{S} = \mathrm{Spec}(k)$ be an n-pointed stable curve of genus g with l nodes. We are interested in attaching particular logarithmic structures to both schemes, with the end goal of seeing how the log structures allow us to describe the dual graph of $\underline{X}$. For the next definition we work over a more general log base scheme S.

Definition 3.32 (Definition 1.2 of [Kat00])**.** Let S be an fs-log scheme. A *log curve* over S is a proper, log smooth, integral morphism $f : X \to S$ of fs-log schemes such that each fibre of f is a reduced, connected curve.

Remark 3.33. Being a log curve over S already imposes a great deal of structure on X. Namely, by Theorem 1.3 of [Kat00], if S is the spectrum of a separably closed field then the underlying scheme of X has only double points as singularities. For a more general base scheme S, the étale log structure of X is determined by that of S, as shown by Kato in [Kat00] and summarized succinctly in Theorem 7.1 of [CCUW16].

If $\underline{f} : \underline{X} \to \underline{S}$ is an n-pointed stable curve of genus g over an arbitrary locally Noetherian scheme $\underline{S}$, Kato showed in [Kat00] that one can always attach unique log structures to X and S so that the morphism of log schemes is that of a special type that we will now define.

Definition 3.34 (Definition 1.12 of [Kat00])**.** Let g and n be nonnegative integers, and k be a separably closed field. A log curve $f : X \to S = \mathrm{Spec}(k)$ is *of type* (g, n) if the arithmetic genus of $\underline{X}$ is g, and if there exists n distinct k-points $\{x_1, \ldots, x_n\}$ in the smooth locus of $\underline{f}$ such that

$$\overline{\mathcal{M}}_{X/S} = \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_l} \oplus \mathbb{N}_{x_1} \oplus \ldots \oplus \mathbb{N}_{x_n},$$

where $\{p_1, \ldots, p_l\}$ are the set of double points of $\underline{X}$. If furthermore the underlying curve is an n-marked stable curve of genus g, we say that $f : X \to S$ is a *stable log curve of type* (g, n).

Over a more general base S, a log curve $f : X \to S$ is a *stable log curve of type* (g, n) if all geometric fibres of f are stable log curves of type (g, n).

Remark 3.35. The definition of a log stable curve of type (g, n) as given in Definition 1.12 of [Kat00] is presented differently. Assume S is the spectrum of an algebraically closed field with trivial log structure. Then, rather than requiring the underlying geometric curve is an n-marked stable curve of genus g, it is stated that $H^0(X, \Theta_{X/S}) = 0$ where $\Theta_{X/S}$ is the dual sheaf on X of log differentials.

It is shown in Proposition 3.14(2) of [Kat89] that the condition $H^0(X, \Theta_{X/S}) = 0$ implies the underlying curve is an n-marked stable curve of genus g. Conversely, suppose the underlying curve is an n-marked stable curve of genus g. Let $\{s_1, \ldots, s_n\}$ denote the set of marked points. By Lemma 27.5 in Chapter 4 of [Har09], we have an exact sequence

$$0 \to \varpi_{\underline{X}}(s_1 + \ldots + s_n)^\vee \to T_{\underline{X}} \to \mathcal{O}_Z \to 0$$

of sheaves on $\underline{X}$, where $\varpi_{\underline{X}}$ is the dualizing sheaf, $T_{\underline{X}}$ is the tangent sheaf, and Z is the set of nodes. The assumption that the underlying curve is an n-marked stable curve of genus g implies $H^0(\underline{X}, T_{\underline{X}}) = 0$ and hence that $H^0(X, \varpi_{\underline{X}}(s_1 + \ldots + s_n)^\vee) = 0$. However, by Proposition 1.13 of [Kat00], $\varpi_{\underline{X}}(s_1 + \ldots + s_n) \cong \varpi^\vee_{X/S}$, where by definition $\varpi^\vee_{X/S}$ is the $\mathcal{O}_X$-dual to $\Theta_{X/S}$. Hence the two definitions agree.

Definition 3.36 (Proposition 2.3 of [Kat00])**.** Let $f : X \to S$ be a log curve. We say that $f : X \to S$ is *basic* if it satisfies the following property:

Let $a : \underline{S}' \to \underline{S}$ be a morphism of schemes and $f' : X' \to S'$ a log curve, and suppose $\underline{X}' \cong \underline{S}' \times_{\underline{S}} \underline{X}$ with projection morphism $b : \underline{X}' \to \underline{X}$. Suppose furthermore that the pullback of the divisor on $\underline{X}$ defined by the marked points is the divisor on $\underline{X}'$ defined by its marked points. Then there exist morphisms $\alpha : S' \to S$ and $\beta : X' \to X$ of fs log schemes such that

- $\underline{\alpha} = a$ and $\underline{\beta} = b$ and
- α and β, together with the log curves f and f', form a Cartesian diagram in the category of fs log schemes.

Our interest in basic stable log curves of type (g, n) comes from the following result of Kato:

Theorem 3.37 (Proposition 2.3 of [Kat00]). *Let $\underline{S}$ be a scheme. To any n-pointed stable curve of genus g, $\underline{f} : \underline{X} \to \underline{S}$, the morphism $\underline{f}$ is the underlying morphism of a unique basic stable log curve $f : X \to S$ of type (g, n). Assuming $f : X \to S$ is a basic stable log curve and if $S' \to S$ is a strict morphism of log schemes, then $f' : X \times_S S' \to S'$ is a basic stable log curve of type (g, n), where the fibre product is taken in the category of fs log schemes.*

The form of the logarithmic structure in the above theorem will be of great importance later, so we shall describe it here étale locally. Assume that $\underline{S}$ is the spectrum of a local Noetherian strictly henselian ring A, and $\underline{f} : \underline{X} \to \underline{S}$ a stable curve of type (g, n) with marked points $(s_1, \ldots, s_n)$.

By Lemma 30.10.3 of [[Sta17], Tag 03CH], the non-smooth locus D of the morphism f is cut out by the first Fitting ideal of $\Omega_{X/S}$. Restricted to a geometric fibre, D is the locus of double points of the fibre. Write $D = \bigsqcup_{i=1}^{l} D_i$, where the D_i are the connected components of D, and set $U_i^+ = (\bigsqcup_{j \neq i} D_j)^c$ and $U_i^- = D_i^c$, both of which are open sets in X.

The intersection of U_i^+ with any geometric fibre contains only one double point, and étale locally on U_i^+ around D_i we may find a neighbourhood of the form $\mathrm{Spec}(A[x, y]/(xy - t_i))$, where $t_i \in m_A$, the maximal ideal of A.

On this neighbourhood we may take the log structure associated to the pre-log structure $\mathbb{N}^2 \to A[x, y]/(xy - t_i)$, where $(1, 0) \to x$ and $(0, 1) \to y$. On U_i^- we may instead take the log structure associated to the chart $\mathbb{N} \to \mathcal{O}_{U_i^-}$ sending $q \to t_i^q$. These two log structures glue on the overlap via the diagonal map $\mathbb{N} \to \mathbb{N}^2$. Denote the resulting log structure on $\underline{X}$ by $\mathcal{M}_i$.

Similarly, on S, we have a log structure $\mathcal{L}_i$ associated to the pre-log structure $\mathbb{N} \to A$ sending $q \to t_i^q$. There is a natural map to $\mathcal{M}_i$ from the pullback of this log structure on X, which is the identity on U_i^- and the diagonal morphism on U_i^+.

Finally, let Z be the divisor defined by the sections s_i for $i \in \{1, \ldots, n\}$. This gives a divisorial log structure on D^c that is trivial in an étale neighbourhood of D, and so extends to a log structure $\mathcal{N}$ on X. Set $\mathcal{M}_X = \mathcal{M}_1 \oplus_{\mathcal{O}_X^*} \ldots \oplus_{\mathcal{O}_X^*} \mathcal{M}_l \oplus_{\mathcal{O}_X^*} \mathcal{N}$ on X, and $\mathcal{M}_S = \mathcal{L}_1 \oplus_{\mathcal{O}_S^*} \ldots \oplus_{\mathcal{O}_S^*} \mathcal{L}_l$ on S.

It is shown in [Kat00] that these log structures turn $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ into a basic stable log curve of type (g, n).

The stalks of the charactertistic sheaf of the log structure at geometric points of X are more or less immediate from the construction, and are summarised in the following lemma.

Lemma 3.38 (cf. Theorem 7.1 of [CCUW16]). *Suppose $f : X \to S$ is a basic log curve over $S = Spec(A)$ of type (g, n), where A is a strictly henselian local ring, such that the special fibre of $\underline{f}$ has l double points. The stalk of $\overline{\mathcal{M}_X}$ at a geometric point x of the special fibre is given by*

- $\overline{\mathcal{M}_{X,x}} = \mathbb{N}^l$ *if x is a smooth point of $\underline{f}$;*
- $\overline{\mathcal{M}_{X,x}} = \mathbb{N}^l \bigoplus \mathbb{N}$ *if x is a marked point of the special fibre; and*
- $\overline{\mathcal{M}_{X,x}} = \mathbb{N}^l \bigoplus \mathbb{N}\alpha \bigoplus \mathbb{N}\beta/(\alpha + \beta = e_i) \cong \mathbb{N}^l \bigoplus_{\mathbb{N}} \mathbb{N}^2$ *if x is a double point of the special fibre, where e_i is a standard generator of $\mathbb{N}^l$.*

Proof. We show only the case of x being a double point, with the other cases following analogously. Assume without loss of generality that $x \in D_l$. Then by construction $\mathcal{N}$ is trivial at x, and, for $i = 1, \ldots, l-1$, the log structure $\mathcal{M}_i$ is defined by a chart $\mathbb{N}_i \to \mathcal{O}_X$ sending qe_i to t_i^q for $t_i \in m_A$.

The log structure $\mathcal{M}_l$ is defined at x by the chart $\mathbb{N}^2 = \mathbb{N}\alpha \bigoplus \mathbb{N}\beta \to A[x, y]/(xy - t_l)$, with α sent to x and β to y. Away from x, $\mathcal{M}_l$ is defined by the chart $\mathbb{N} \to \mathcal{O}_X$ sending qe_l to t_l^q, and by construction we glue the two charts on overlaps via the diagonal map from $\mathbb{N} \to \mathbb{N}^2$. Hence $\overline{\mathcal{M}_l} = \mathbb{N} \bigoplus_{\mathbb{N}} \mathbb{N}\alpha \bigoplus \mathbb{N}\beta/(\alpha + \beta = e_l)$.

As $\mathcal{M}_X$ is defined as $\mathcal{M}_1 \oplus_{\mathcal{O}_X^*} \ldots \oplus_{\mathcal{O}_X^*} \mathcal{M}_r \oplus_{\mathcal{O}_X^*} \mathcal{N}$, the result follows. □

Remark 3.39. It is immediate to check that, with the setup of Lemma 3.38, the stalk of $\mathcal{M}_S$ at the closed point s of S is $\mathcal{M}_{S,s} = \mathbb{N}^l$. This follows immediately by the construction of the log structure making $f : X \to S$ into a basic stable log curve of type (g, n), where the log structure $\mathcal{L}_i$ have charts of the form $\mathbb{N} \to A$ sending q to t_i^q for some t_i in the maximal ideal of A.

Moreover, with this log structure, S is a small log scheme. The log scheme X is étale locally small.

3.4 Artin fans and the universal deformation

Consider $\underline{S} = \text{Spec}(k)$, where k is an algebraically closed field, and let $\underline{f} : \underline{X} \to \underline{S}$ be a stable curve of type (g, n). We saw in the previous section how to associate to $\underline{X}$ and $\underline{S}$ log structures so that the resulting morphism $f : X \to S$ of log schemes is a basic stable log curve over S of type (g, n). Certain results we will later use on the Artin fans assume that S is log smooth over k endowed with the trivial log structure, which unfortunately is not the case. To get around this we extend $f : X \to S$ to look at a family of curves.

Constructing an appropriate family of stable curves

We begin by constructing an important family of n-pointed stable curves of genus g such that one of the fibres is our original stable curve. In this subsection we work primarily with schemes rather than logarithmic schemes, and so we temporarily drop the notation of underlining a scheme to signify it is not a log scheme. Note that k is still assumed to be algebraically closed.

Proposition 3.40. *Let $f : X \to S = Spec(k)$ be an n-pointed stable curve of genus g with l double points. Then there exists an n-pointed stable curve $\mathcal{X} \to V$ fitting into a commutative diagram*

$$\begin{array}{ccc}
X \cong \mathcal{X} \times_V Spec(k) & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Spec(k) & \longrightarrow & V \\
& \searrow^{0} & \downarrow {\scriptstyle \acute{e}t} \\
& & \mathbb{A}^l_k,
\end{array}$$

such that

- *$V \to \mathbb{A}^l_k$ is étale,*
- *the morphism from $Spec(k)$ to $\mathbb{A}^l_k$ maps to $0 \in \mathbb{A}^l_k$, and*
- *if $\mathcal{X}$ and V are given log structures so that $\mathcal{X} \to V$ is a basic log curve of type (g, n), the log structure on V is the pullback of the standard toroidal log structure on $\mathbb{A}^l_k$.*

The key point of this is that V is log smooth over $\mathrm{Spec}(k)$ endowed with the trivial log structure, and hence we will later be able to show important results about the Artin fans of $\mathcal{X}$ and V.

We prove Proposition 3.40 in a series of lemmas. We first prove a lemma on the structure of the singular locus of an n-pointed stable curve of genus

g that will be used frequently throughout this subsection. By Lemma 30.10.3 of [[Sta17], Tag 0C3H], the singular locus $Sing(\mathcal{X}/V)$ of an n-pointed stable curve of genus g given by $\mathcal{X} \to V$ is defined by the 1-st Fitting ideal of $\Omega_{\mathcal{X}/V}$.

Lemma 3.41. *Let $\mathcal{X} \to V$ be an n-pointed stable curve of genus g, and $Sing(\mathcal{X}/V)$ the singular locus, viewed as a closed subscheme of $\mathcal{X}$ via the 1-st Fitting ideal of $\Omega_{\mathcal{X}/V}$. If $Sing(\mathcal{X}/V) = \bigsqcup_i D_i$ is a decomposition of $Sing(\mathcal{X}/V)$ into connected components, then at each point $s \in V$ we can find an étale neighbourhood V_s of s such that $D_i \times_V V_s \to V_s$ is a closed immersion.*

Proof. We begin by showing that $Sing(\mathcal{X}/V)$ is finite and unramified over V. To see that it is finite, note that $Sing(\mathcal{X}/V) \to V$ is proper, as it is a composition of proper morphisms. As $Sing(\mathcal{X}/V)$ is, when restricted to any fibre, the double points of an n-pointed stable curve of genus g, we conclude that $Sing(\mathcal{X}/V)$ is proper with finite fibres, and so is finite.

To show $Sing(\mathcal{X}/V)$ is unramified over V, it sufficies to show that, given any $s \in V$, the fibre over s is a finite union of separable field extensions of $k(s)$. Working over $\mathcal{O}^{\text{ét}}_{V,s}$, we can find an étale neighbourhood of any double point of $\mathcal{X}_s$ of the form $\mathcal{O}^{\text{ét}}_{V,s}[x,y]/(xy-a)$, where a is in the maximal ideal of $\mathcal{O}^{\text{ét}}_{V,s}$. In such a neighbourhood we have that $\Omega_{\mathcal{X}/V}$ has generators dx and dy modulo the relation $xdy+ydx$. In particular, the first Fitting ideal is the ideal (x, y), and so the fibre $Sing(\mathcal{X}/V)_s$ consists of finitely many spectra of separable field extensions of $k(s)$. We conclude $Sing(\mathcal{X}/V)$ is finite and unramified over V.

By Lemma 40.17.3 of [[Sta17], Tag 04HJ], as $Sing(\mathcal{X}/V)$ is finite unramified over V, at any point $s \in V$ we can replace V by an étale neighbourhood of s such that $Sing(\mathcal{X}/V)$ is a finite disjoint union $\bigsqcup_i D_i$ with $D_i \to V$ a closed immersion. □

We now reduce the proposition to an easier case.

Lemma 3.42. *Let $f : X \to S = Spec(k)$ be an n-pointed stable curve of genus g with l double points. Then there exists an n-pointed stable curve $\mathcal{X} \to V$ fitting into a commutative diagram*

$$\begin{array}{ccc} X \cong \mathcal{X} \times_V Spec(k) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & V \\ & \overset{0}{\searrow} & \downarrow \text{ét} \\ & & \mathbb{A}^l_k = Spec(k[t_1, \dots, t_l]), \end{array}$$

such that $V \to \mathbb{A}^l_k$ *is étale, and the morphism from* $Spec(k)$ *to* $\mathbb{A}^l_k$ *maps to* $0 \in \mathbb{A}^l_k$. *Furthermore, after possible taking an étale neighbourhood of* $0 \in V$, *given a node* $p_i \in X$, *if* D_i *denotes the connected component of* p_i *in* $Sing(\mathcal{X}/V)$, *then* $Sing(\mathcal{X}/V) = \bigsqcup_i D_i$ *and we have a Cartesian diagram*

$$\begin{array}{ccc} D_i & \longrightarrow & V \\ \downarrow & & \downarrow \\ Z(t_i) & \longrightarrow & \mathbb{A}^l_k, \end{array}$$

for all i.

Proof. (Proof of Proposition 3.40 from Lemma 3.42) By the assumption k is algebraically closed, the fibre $\mathcal{X}_0$ is such that its double points are all k-rational. Replace V by an étale neighbourhood of 0 such that each D_i is closed in V as per Lemma 3.41. It is possible that after base change the components D_i are no longer connected. In this case, one may replace V by a *Zariski* neighbourhood of 0 that is the complement of the image of the connected components of $\mathrm{Sing}(\mathcal{X}/V)$ not passing through one of the points p_i. In particular, we may assume $\mathrm{Sing}(\mathcal{X}/V) = \bigsqcup_i D_i$ and each D_i is closed in V.

Suppose that Lemma 3.42 holds, and let $q \in D_i$ with image $p \in Z(t_i)$. As $\mathcal{X}$ is a stable curve over V, this implies $\mathcal{O}^{\text{ét}}_{\mathcal{X},q} \cong (\mathcal{O}^{\text{ét}}_{\mathbb{A}^l,p}[x,y]_{(x,y)}/(xy-a))^{\text{ét}}$, where necessarily a is in the maximal ideal of $\mathcal{O}^{\text{ét}}_{\mathbb{A}^l,p}$.

From the Cartesian diagram we have that $D_i \cong V \times_{\mathbb{A}^l} Z(t_i)$, and so in particular $(a) = (t_i)$, whence $a = ut_i$ for some unit $u \in \mathcal{O}^{\text{ét}}_{\mathbb{A}^l,p}$. Hence, after a change of coordinates, $\mathcal{O}^{\text{ét}}_{\mathcal{X},q} \cong (\mathcal{O}^{\text{ét}}_{\mathbb{A}^l,p}[x,y]/(xy - t_i))^{\text{ét}}$.

Following the construction of the log structures on $\mathcal{X}$ and V turning $\mathcal{X} \to V$ into a basic log curve, we see that the log structure in an étale neighbourhood of p comes from the pre-log structure on $\mathbb{A}^l_k$ sending $\mathbb{N}$ to $k[t_1, \dots, t_l]$, where $n \to t_i^n$. In particular, each D_i gives a log structure on

V that is the pullback of the divisorial log structure on $\mathbb{A}^l_k$ associated to (t_i). By the assumption that $Sing(\mathcal{X}/V) = \bigsqcup_i D_i$, we conclude that the log structure on V such that $\mathcal{X} \to V$ is a basic stable log curve of type (g, n) is the pullback of the toric log structure on $\mathbb{A}^l_k$. □

We shall prove Lemma 3.42 by proving the following result:

Lemma 3.43. *Let $f : X \to S = Spec(k)$ be an n-pointed stable curve of genus g with l double points. Then there exists an n-pointed stable curve $\mathcal{X} \to V$ fitting into a commutative diagram*

$$\begin{array}{ccc} X \cong \mathcal{X} \times_V Spec(k) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Spec(k) & \longrightarrow & V \\ & \searrow^{0} & \downarrow_{\text{ét}} \\ & & \mathbb{A}^m_k = Spec(k[t_1, \ldots, t_m]), \end{array}$$

where $m = 3g - 3 + n$ and such that $V \to \mathbb{A}^m_k$ is étale, and the morphism from $Spec(k)$ to $\mathbb{A}^m_k$ maps to $0 \in \mathbb{A}^m_k$. Furthermore, after possibly taking an étale neighbourhood of $0 \in V$, given a node $p_i \in X$ for $i = 1, \ldots, l$, if D_i denotes the connected component of p_i in $Sing(\mathcal{X}/V)$, then $Sing(\mathcal{X}/V) = \bigsqcup_{i=1}^l D_i$ and we have a Cartesian diagram

$$\begin{array}{ccc} D_i & \longrightarrow & V \\ \downarrow & & \downarrow \\ Z(t_i) & \longrightarrow & \mathbb{A}^m_k, \end{array}$$

for all i.

Proof. (Proof of Lemma 3.42 from Lemma 3.43) Suppose that Lemma 3.43 holds. Let $Z \subset \mathbb{A}^m$ denote the closed subset defined by the ideal $(t_{l+1}, \ldots, t_m)$. Set $V' = V \times_{\mathbb{A}^m} Z$ and $\mathcal{X}' = \mathcal{X} \times_{\mathbb{A}_m} Z$. It remains to show that $\mathcal{X}' \to V'$ is a stable curve with the desired properties.

That $\mathcal{X}' \to V'$ is an n-marked stable curve of genus g is immediate via pullback, as is that $V' \to \mathbb{A}^l$ is étale. Furthermore, the fibre of $\mathcal{X}'$ over the point Spec(k) mapping to $0 \in \mathbb{A}^l_k$ remains isomorphic to X.

The singular locus of $\mathcal{X}' \to V'$ is cut out by the first Fitting ideal of $\Omega_{\mathcal{X}'/V'}$ by Lemma 30.10.3 of [[Sta17], Tag 0C3H], the formation of which is stable under base change by Lemma 15.8.4 of [[Sta17], Tag 07ZA], whence $Sing(\mathcal{X}'/V') = \coprod_i D_i \times_{\mathbb{A}^m} \mathbb{A}^l$. By base change, this gives a Cartesian diagram

$$\begin{array}{ccc} D_i \times_{\mathbb{A}^m} \mathbb{A}^l & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ Z(t_i) & \longrightarrow & \mathbb{A}^l_k, \end{array}$$

completing the proof. □

In the following proof we will make use of Artin's approximation theorems, for which we refer to Chapter 21 of [Har09].

Proof. (Proof of Lemma 3.43) By Theorem 21.3 of [Har09], there exists a universal deformation $\overline{X} \to \mathrm{Spec}(k[[t_1, \ldots, t_m]])$ of $X \to \mathrm{Spec}(k)$, along with a scheme S of finite type over k, and a flat and finite type family $\mathcal{X} \to S$ such that there exists $s_0 \in S$ with $\hat{\mathcal{O}}_{S,s_0} \cong k[[t_1, \ldots, t_m]]$ and an isomorphism $\mathcal{X} \times_S \mathcal{O}_{S,s_0} \cong \overline{X}$. At any double point of $\overline{X}_{s_0}$, the completion of the étale local ring in $\overline{X}$ at that point is isomorphic to $k[[t_1, \ldots, t_m]][[x, y]]/(xy - t_i)$, for $1 \leq i \leq l$. In particular, by Lemma 21.4 of [Har09], there is an étale neighbourhood in $\mathcal{X}$ of any such point that itself is étale over $\mathcal{O}_{S,s_0}[x, y]/(xy - t_i)$.

Note that $\hat{\mathcal{O}}_{S,s_0} \cong \hat{\mathcal{O}}_{\mathbb{A}^m,0}$, and so by Lemma 21.4 of [Har09] there exists a scheme S' along with a point $s' \in S'$ such that S' is étale over S and $\mathbb{A}^m$, and s' maps to s_0 and 0, respectively. Replace S by S' and $\mathcal{X}$ by $\mathcal{X} \times_S S'$.

As $X \to \hat{\mathcal{O}}^{\text{ét}}_{S,s_0}$ is a stable curve, there exists a morphism from $\mathrm{Spec}(\hat{\mathcal{O}}^{\text{ét}}_{S,s_0})$ to $\overline{\mathcal{M}_{g,n}}$. Additionally, as $\overline{\mathcal{M}_{g,n}}$ is a Deligne-Mumford stack we can find a surjective étale morphism $U \to \overline{\mathcal{M}_{g,n}}$, where U is a scheme. In particular, if $u \in U$ maps to the same image of $s_0 \in \hat{\mathcal{O}}^{\text{ét}}_{S,s_0}$ in $\overline{\mathcal{M}_{g,n}}$, the completed étale local ring of u must agree with $\hat{\mathcal{O}}^{\text{ét}}_{S,s_0}$, giving a morphism from $\mathrm{Spec}(\hat{\mathcal{O}}^{\text{ét}}_{S,s_0})$ to U.

Note that U is locally of finite type over $\mathrm{Spec}(k)$ as $U \to \overline{\mathcal{M}_{g,n}}$ is surjective with $\overline{\mathcal{M}_{g,n}}$ proper over $\mathrm{Spec}(k)$ as per Lemma 95.16.1 of [[Sta17], Tag 06FM]. By replacing U with a neighbourhood of u, we may assume furthermore that it is of finite type. Applying once again Lemma 21.4 of [Har09] to $s_0 \in S$ and $u \in U$, we can find a scheme V that is étale over both S and U, along with a point $s \in V$ mapping to $s_0 \in S$ and $u \in U$. Note that this implies V is étale over $\mathbb{A}^m$, as S is étale over $\mathbb{A}^m$.

Let $\mathcal{C}_{g,n} \to \overline{\mathcal{M}_{g,n}}$ denote the universal curve over $\overline{\mathcal{M}_{g,n}}$, and consider the n-pointed stable curve $\mathcal{C}_{g,n} \times_{\overline{\mathcal{M}_{g,n}}} V \to V$ of genus g. We also have the scheme $\mathcal{X} \times_S V \to V$, and the base change of both schemes over $\hat{\mathcal{O}}_{V,s}$ agree and are equal to $\overline{X}$.

Replace $\mathcal{X}$ by $\mathcal{C}_{g,n} \times_{\overline{\mathcal{M}_{g,n}}} V$. This is an n-pointed stable curve of genus g, such that V is étale over $\mathbb{A}^m$. Furthermore, the fibre of $\mathcal{X} \to V$ over $s \in V$ is isomorphic to X.

To conclude we still have to show the statements regarding the singular locus. Let $s \in V$ denote the image of Spec(k) whose image in $\mathbb{A}^m$ is the point 0. By Lemma 3.41, $Sing(\mathcal{X}/V) = \bigsqcup_{i \in I} D_i$ is a finite disjoint union of connected subschemes D_i, such that after replacing V by an étale neighbourhood of s, the morphism $D_i \to V$ is a closed immersion. It is possible that after this base change the connected components D_i of $Sing(\mathcal{X}/V)$ passing through the double points p_i of X are no longer connected. In this case, replace V by the Zariski open neighbourhood that is the complement of the images of the connected components of $Sing(\mathcal{X}/V)$ that do not pass through a point p_i, so that $Sing(\mathcal{X}/V) = \bigsqcup_i D_i$.

We first claim that $|I| = l$, the number of double points of $\mathcal{X}_s$. Certainly there is one such D_i for each double point of $\mathcal{X}_s$. Any other D_i necessarily does not contain s in its image in V, and so by base changing to the open set of V not containing these latter closed subschemes we have that $|I|$ is the number of double points of $\mathcal{X}_s$, hence l.

We now establish the existence of the Cartesian diagram as in the statement of the lemma.

Note that each D_i is smooth over Spec(k), which follows from looking at the étale local rings. In particular each D_i is smooth and connected, hence irreducible. Let $\overline{D_i}$ be the scheme-theoretic image of D_i in $\mathbb{A}^m$, where $\overline{D_i} = V(\mathcal{J})$ for some ideal $\mathcal{J} \subset k[t_1, \ldots, t_m]$. If $W = V(\mathcal{J} + (t_i))$, then $W \to \overline{D_i}$ and $W \to Z(t_i)$ are closed immersions. Furthermore, $\overline{D_i} \cong W \cong Z(t_i)$ after base changing to $\hat{\mathcal{O}}^{\text{ét}}_{\mathbb{A}^m,0}$. In particular, by fpqc-descent (Lemma 34.20.17 of [[Sta17], Tag 02YJ]), we conclude that $\overline{D_i} \cong Z(t_i)$ over $\mathcal{O}_{\mathbb{A}^m,0}$, and hence (after possibly base-changing to an open neighbourhood of $0 \in \mathbb{A}^m$) we conclude that $\overline{D_i} = Z(t_i)$. Hence $D_i \to \mathbb{A}^m$ factors through

$Z(t_i)$, and the diagram

$$\begin{array}{ccc} D_i & \longrightarrow & V \\ \downarrow & & \downarrow \\ Z(t_i) & \longrightarrow & \mathbb{A}_k^m, \end{array}$$

is commutative.

Similarly, consider the fibre product $Z(t_i) \times_{\mathbb{A}^m} V$. This is étale over $Z(t_i)$ via base change, and hence smooth over k. Throwing away any connected components that do not contain s in their image in V, we may assume that $Z(t_i) \times_{\mathbb{A}^m} V$ is irreducible. The induced V-morphism from D_i to $Z(t_i) \times_{\mathbb{A}^m} V$ is an isomorphism after base change to $\mathcal{O}_{V,s}^{\text{ét}}$, and hence an isomorphism in an étale neighbourhood of $s \in V$ by Lemma 34.20.17 of [[Sta17], Tag 02YJ]. This concludes the proof. $\square$

We now return to looking at log schemes. Let $\mathcal{X} \to \mathcal{S}$ be the log curve of Proposition 3.40 such that $\mathcal{X}_S \cong X$, where $S = \text{Spec}(k)$ is a point of $\mathcal{S}$. All we have done is replaced V with a different symbol $\mathcal{S}$ to make the next subsection easier to read. That is, we now have a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{S}, \end{array}$$

where the horizontal morphisms are strict. This is clear for $S \to \mathcal{S}$, as the log structure on the former arises from the chart $\mathbb{N}^l \to k$ sending $e_i \to 0$ for all i, and the log structure on $\mathcal{S}$ is pulled back from that the toroidal log structure on $\mathbb{A}^l$, with $S \in \mathcal{S}$ mapping to $0 \in \mathbb{A}^l$. That the morphism from $X \to \mathcal{X}$ is strict now follows from Lemma 3.38.

For use in the next subsection, we need the following easy lemmas:

Lemma 3.44. *Let $F : \mathcal{X} \to \mathcal{S}$ be as above. Then $\mathcal{X}$ and $\mathcal{S}$ are log smooth over Spec(k) with the trivial log structure and, after possibly localising $\mathcal{S}$ around S, $\mathcal{S}$ is a small logarithmic scheme with respect to the point S.*

Proof. As $F : \mathcal{X} \to \mathcal{S}$ is log smooth it suffices to show S is log smooth over the log point. But, by Proposition 3.40, $\mathcal{S}$ has a strict étale morphism to $\mathbb{A}^l$, where $\mathbb{A}^l$ is given the standard toroidal log structure and hence by Remark 3.21 is smooth over Spec(k) with the trivial log structure.

As $(\mathcal{S}, S) \to (\mathbb{A}^l, 0)$ is an étale neighbourhood of 0 endowed with the pullback of the toroidal log structure on $\mathbb{A}^l$, we have that $\overline{\mathcal{M}_{\mathcal{S},S}} \cong \overline{\mathcal{M}_{\mathcal{S}}}$ and may assume after possibly localising that $\overline{\mathcal{M}_{\mathcal{S}}}$ is a global chart. After removing all connected components of the closed logarithmic strata not containing S, we conclude that $\mathcal{S}$ is small with respect to the point S. □

3.5 Artin fans of log schemes

Let $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{S}}$ denote the Artin fans associated to $\mathcal{X}$ and $\mathcal{S}$, respectively. Define in an analogous manner $\mathcal{A}_X$ and $\mathcal{A}_S$. Our goal is to show that there is a morphism from $\mathcal{A}_X$ to $\mathcal{A}_S$ such that the fibre over the closed point of $\mathcal{A}_S$ is, in a natural way, the dual graph of $\underline{X}$.

One of the primary issues in establishing such a map is that, in general, the morphism from X to $\mathcal{A}_X$ is not functorial in X with respect to non-strict morphisms such as $f : X \to S$, as in the example of section 5.4.1 of [ACM+16]. We shall establish that we do have functoriality in the case of $f : X \to S$ by showing that such functoriality exists with respect to the morphism $F : \mathcal{X} \to \mathcal{S}$, and then showing $\mathcal{A}_X \cong \mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_S \cong \mathcal{A}_{\mathcal{S}}$.

Lemma 3.45 (c.f. Corollary 2.5.2 of [AW18]). *Let $F : \mathcal{X} \to \mathcal{S}$ be the morphism of log schemes from the previous subsection, and let $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{S}}$ denote the Artin fans of $\mathcal{X}$ and $\mathcal{S}$, respectively. Then there exists a natural morphism from $\mathcal{A}_{\mathcal{X}} \to \mathcal{A}_{\mathcal{S}}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{A}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathcal{A}_{\mathcal{S}} \end{array}$$

commutes.

Proof. Note that $\mathcal{S}$ is small with respect to the point S by Lemma 3.44, where $\overline{\mathcal{M}_{\mathcal{S},S}} \cong \overline{\mathcal{M}_{\mathcal{S}}} \cong \mathbb{N}^l$, and so its Artin fan is the Artin cone $\mathcal{A}_{\mathbb{N}^l}$. Let $\mathbf{Log}(\mathcal{A}_{\mathcal{S}})$ be the fibred category over schemes whose objects are morphisms $(T, \mathcal{M}_T) \to \mathcal{A}_{\mathcal{S}}$ from a log scheme T to $\mathcal{A}_{\mathcal{S}}$, and whose morphisms are strict morphisms of log schemes over $\mathcal{A}_{\mathcal{S}}$. That this is an algebraic stack is the content of Proposition 5.9 of [Ols03]. By Lemma 2.5.1 of [AW18], the map from $\mathbf{Log}(\mathcal{A}_{\mathcal{S}}) \to \mathbf{Log}$ is strict and étale, where $\mathbf{Log}(\mathcal{A}_{\mathcal{S}}) \to \mathbf{Log}$

sends a log scheme $(X, \mathcal{M}_X)$ and a morphism $(X, \mathcal{M}_X) \to \mathcal{A}_S$ to $(X, \mathcal{M}_X)$, forgetting the morphism to $\mathcal{A}_S$.

Furthermore, as $F : \mathcal{X} \to \mathcal{S}$ is a log smooth morphism of smooth log schemes over the log point endowed with the trivial log structure by Lemma 3.44, the morphism from $\mathcal{X} \to \mathbf{Log}(\mathcal{A}_\mathcal{S})$ is a smooth morphism of stacks.

The construction of the Artin fan of a scheme $\mathcal{X}$ holds in an analogous manner over the log stack $\mathbf{Log}(\mathcal{A}_\mathcal{S})$, where an Artin fan over $\mathbf{Log}(\mathcal{A}_\mathcal{S})$ is a logarithmic algebraic stack that is étale over $\mathbf{Log}(\mathcal{A}_\mathcal{S})$; see Subsection 2.1 of [AW18] for the details. Set $\mathcal{A}'_\mathcal{X}$ to be this relative Artin fan of $\mathcal{X}$ over $\mathcal{S}$. This is a log stack that is étale over $\mathbf{Log}(\mathcal{A}_\mathcal{S})$ with the universal property that the morphism $\mathcal{X} \to \mathcal{A}'_\mathcal{X}$ is initial with respect to morphisms from $\mathcal{X}$ to Artin fans over $\mathbf{Log}(\mathcal{A}_\mathcal{S})$. At this point we have a commutative diagram of algebraic stacks

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{A}'_\mathcal{X} & \longrightarrow & \mathbf{Log}(\mathcal{A}_\mathcal{S}) \\ & \searrow & & & \downarrow \\ & & \mathcal{A}_\mathcal{X} & \longrightarrow & \mathbf{Log}. \end{array}$$

We shall show that $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}'_\mathcal{X}$ satisfy the same universal properties and hence recover the commutative diagram as in the statement of the lemma.

As $\mathbf{Log}(\mathcal{A}_\mathcal{S}) \to \mathbf{Log}$ is étale, the composition of $\mathcal{A}'_\mathcal{X} \to \mathbf{Log}$ is also étale, and so there exists a unique morphism from $\mathcal{A}_\mathcal{X}$ to $\mathcal{A}'_\mathcal{X}$ by the universal property of $\mathcal{A}_\mathcal{X}$.

But now $\mathcal{A}_\mathcal{X} \to \mathcal{A}'_\mathcal{X} \to \mathbf{Log}(\mathcal{A}_\mathcal{S})$ is étale, and so by universal property of $\mathcal{A}'_\mathcal{X}$ there exists a unique étale morphism $\mathcal{A}'_\mathcal{X} \to \mathcal{A}_\mathcal{X}$. As both $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}'_\mathcal{X}$ are étale over $\mathbf{Log}$, and so Lemma 95.34.6 of [[Sta17], Tag 0CIR] ensures the morphism between them is étale.

In particular, we see that both stacks are étale over $\mathbf{Log}$ and $\mathbf{Log}(\mathcal{A}_\mathcal{S})$, and both satisfy the same universal properties with regards to being initial with respect to morphisms from $\mathcal{X}$ to Artin fans over $\mathbf{Log}$ and $\mathbf{Log}(\mathcal{A}_\mathcal{S})$, and so they agree. Hence $\mathcal{A}'_\mathcal{X} \cong \mathcal{A}_\mathcal{X}$.

Consider now the following diagram:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{sm} & \mathbf{Log}(\mathcal{A}_{\mathcal{S}}) \\
\downarrow & & \downarrow \\
\mathcal{S} & \xrightarrow{sm} & \mathcal{A}_{\mathcal{S}}.
\end{array}
\tag{3.4}
$$

That this square is commutative follows as the morphism from $\mathcal{X} \to \mathbf{Log}(\mathcal{A}_{\mathcal{S}})$ is induced by the composition $\mathcal{X} \to \mathcal{S} \to \mathcal{A}_{\mathcal{S}}$. As the morphism $\mathcal{X} \to \mathbf{Log}(\mathcal{A}_{\mathcal{S}})$ factors through $\mathcal{A}'_{\mathcal{X}} \cong \mathcal{A}_{\mathcal{X}}$, we recover the commutative diagram in the statement of the lemma. □

Remark 3.46. The results from [AW18] used in the above proof assume that the log schemes in question are logarithmically smooth. The weaker form of functoriality in associating to a log scheme an Artin fan can be done in a more general context as in Subsection 5.4.2 of [ACM⁺16], though it is unclear if the desired functoriality in the case of a log curve over a field can be recovered without going through the family $\mathcal{X} \to \mathcal{S}$.

Lemma 3.47. *Let $f : X \to S$ and $F : \mathcal{X} \to \mathcal{S}$ be as before. Then $\mathcal{A}_X \cong \mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_S \cong \mathcal{A}_{\mathcal{S}}$.*

Proof. By construction of the logarithmic structures on both $\underline{X}$ and $\underline{S}$, we see that $\overline{\mathcal{M}_S} = \overline{\mathcal{M}_{\mathbb{A}^l}} = \mathbb{N}^l$. As S is trivially small, we see that $\mathcal{A}_S \cong \mathcal{A}_{\mathcal{S}}$.

For showing $\mathcal{A}_X \cong \mathcal{A}_{\mathcal{X}}$ we shall show that $\mathcal{X}$ is étale locally small with respect to the points of its closed fibre X. As $F : \mathcal{X} \to \mathcal{S}$ is proper, an étale cover of X by small log schemes can be assumed to cover $\mathcal{X}$. Indeed, if not, then let U by the union of the open sets of $\mathcal{X}$ covering X. Then U^c is closed in $\mathcal{X}$, and its image $F(U^c)$ is closed in $\mathcal{S}$. As $F(U^c)$ does not contain $S \in \mathcal{S}$, one may base change to $F(U^c)^c$, where U will then cover $\mathcal{X}$.

That $\mathcal{X}$ is étale locally small with respect to points of X follows from the étale local description of the log structure of $\mathcal{X}$ given in Theorem 7.1 of [CCUW16]. For each closed point $x \in \mathcal{X}_S$ we may thus find an étale neighbourhood U_x of x in $\mathcal{X}$ that is small. Let $U_{\mathcal{X}} = \bigsqcup_{x\in X} U_x$. Similarly, let $V_{xy} = U_x \times_X U_y$ for $x, y \in X$ distinct closed points, and let $V_{\mathcal{X}} = \bigsqcup_{x,y\in X} V_{xy}$. By taking the intersection of $U_{\mathcal{X}}$ and $V_{\mathcal{X}}$ with $X = \mathcal{X} \times_{\mathcal{S}} S$, we get a disjoint union of small open sets that cover X, which we denote by U_X and V_X, respectively.

At geometric points x of X, the stalks of $\overline{\mathcal{M}_{\mathcal{X}}}$ and $\overline{\mathcal{M}_X}$ agree, and so we see that the Artin fan of U_X is equal to that of $U_{\mathcal{X}}$ and the Artin fan of V_X is equal to that of $V_{\mathcal{X}}$.

Via the description in Proposition 3.3.1 of [ACMW14] of the Artin fan of X as the colimit of $\mathcal{A}_{V_X} \rightrightarrows \mathcal{A}_{U_X}$ (and similarly for the Artin fan of $\mathcal{X}$) in étale spaces over **Log**, we conclude that $\mathcal{A}_X \cong \mathcal{A}_{\mathcal{X}}$. □

By combining Lemma 3.45 and Lemma 3.47 we immediately get the following corollary.

Corollary 3.48. *Let $f : X \to S$ be a basic log curve of type (g, n), where $S = Spec(k)$ is the spectrum of a separably closed field. Let $\mathcal{A}_X$ and $\mathcal{A}_S$ be the Artin fans of X and S, respectively. Then we have a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{A}_S. \end{array} \tag{3.5}$$

3.6 Dual graph from the Artin fan

From the previous subsection we now have a morphism $\mathcal{A}_X \to \mathcal{A}_S$ compatible with the morphism $f : X \to S$. To see how to find the dual graph of $f : X \to S$ from this morphism of Artin fans, we first offer a more explicit realisation of $\mathcal{A}_X$ and $\mathcal{A}_S$.

As $S = \mathrm{Spec}(k)$ is Zariski small, its Artin fan is simply $\mathcal{A}_{\mathcal{M}_S} = \mathcal{A}_{\mathbb{N}^l} = [\mathbb{A}^l/\mathbb{G}_m^l]$, where $\mathbb{G}_m^l$ acts on $\mathbb{A}^l$ via scaling the coordinates. The k-points of this stack are simply the $\mathbb{G}_m^l$-invariant maps from $\mathbb{G}_m^l$ to $\mathbb{A}^l$, and hence are in bijection with the logarithmic strata of $\mathbb{A}^l$ given its toroidal log structure. Furthermore, the topology of these points is the same as that of the corresponding points in the Kato cone $\mathrm{Spec}(\mathbb{N}^l)$ by Lemma 5.4(iii) of [ACM$^+$16]. As a result, the topology of $\mathcal{A}_S$ when $l = 3$ is illustrated in the figure of Example 3.8.

Remark 3.49. The Artin fan of a general logarithmic scheme is constructed as the colimit of Artin fans arising from Zariski small logarithmic schemes. To this end, let us consider the Artin fans associated to étale neighbourhoods of geometric points of X. Given a geometric point $x \in X$, we know that x is either a smooth point of f, a marked point, or a double point. As in Theorem 7.1 of [CCUW16], these three cases correspond to the existence of an étale neighbourhood U_x of x, strict over X, with $\pi : U_x \to S$ such that

- $U_x \cong \mathrm{Spec}(k)[x]$ if x is a smooth point, with $\mathcal{M}_{U_x} = \pi^*\mathcal{M}_S$;
- $U_x \cong \mathrm{Spec}(k)[x]$ if x is a marked point, with $\mathcal{M}_{U_x} = \pi^*\mathcal{M}_S \bigoplus \mathbb{N}$; or
- $U_x \cong \mathrm{Spec}(k)[x, y]/(xy)$ if x is a double point p_i for $i \in \{1, \ldots, l\}$, with $\mathcal{M}_{U_x} = \pi^*\mathcal{M}_S \bigoplus_{\mathbb{N}} \mathbb{N}\alpha \bigoplus \mathbb{N}\beta/(\alpha + \beta = e_i)$, where e_i is the i-th basis element of $\pi^*\mathcal{M}_S$.

In all cases, U_x is Zariski small, and combined with Lemma 3.38 we conclude that

- $\mathcal{A}_{U_x} = \mathcal{A}_{\mathbb{N}^l}$ if x is smooth;
- $\mathcal{A}_{U_x} = \mathcal{A}_{\mathbb{N}^{l+1}}$ if x is a marked point; or
- $\mathcal{A}_{U_x} = \mathcal{A}_{\mathbb{N}^{l+1}}$ if x is a double point.

By Lemma 5.4(iii) of [ACM+16], the category of Artin cones is equivalent to that of fine, saturated, sharp monoids and hence also to that of Kato cones by Lemma 3.10. In particular, for a given U_x as above, the morphism from $\mathcal{A}_{U_x}$ to $\mathcal{A}_S$ is defined by the morphism from $\pi^*\overline{\mathcal{M}_S}$ to $\overline{\mathcal{M}_{U_x}}$, or equivalently the morphism of Kato cones $\mathrm{Spec}(\overline{\mathcal{M}}_{U_x}) \to \mathrm{Spec}(\overline{\mathcal{M}_S})$. This is crucial in the proof of the following lemma.

Lemma 3.50. *Let $x \in X$ be a geometric point, and let U_x be a small étale neighbourhood of x of the form described in Remark 3.49 with $\pi : U_x \to S$ the morphism to S. Let $\mathcal{A}_{U_x}$ denote the Artin fan of U_x and $\mathcal{A}_S$ that of S, along with the induced morphism $\varpi : \mathcal{A}_{U_x} \to \mathcal{A}_S$ of Artin fans. If ζ denotes the closed point of $\mathcal{A}_S$, then*

- *if x is a smooth point, $\varpi^{-1}(\zeta)$ consists of the closed point ζ_{U_x} of $\mathcal{A}_{U_x}$;*
- *if x is a marked point, $\varpi^{-1}(\zeta)$ consists of the closed point ζ_{U_x} of $\mathcal{A}_{U_x}$ along with a point κ specialising to it; and*
- *if x is a double point, $\varpi^{-1}(\zeta)$ consists of the closed point ζ_{U_x} of $\mathcal{A}_{U_x}$ along with two points κ_x and κ_y that specialise to it.*

Proof. Let $\{e_i\}$ denote the standard generators of $\overline{\mathcal{M}}_S = \mathbb{N}^l$, for $i = 1, \ldots, l$. We consider the cases as in the order of the statement of the lemma:

If x is smooth, $\pi^*\overline{\mathcal{M}}_{U_x} \cong \overline{\mathcal{M}_S}$, and ϖ is the identity, whence $\varpi^{-1}(\zeta) = \zeta_{U_x}$.

If x is a marked point, $\overline{\mathcal{M}}_{U_x} = \overline{\mathcal{M}}_S \bigoplus \mathbb{N}$ with a morphism of monoids $\pi^*\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_{U_x}$ sending e_i to e_i for all i. In particular, the only points of $\mathrm{Spec}(\overline{\mathcal{M}}_{U_x})$ lying over the maximal ideal of $\mathrm{Spec}(\mathbb{N}^l)$ are those defined by $(e_1, \dots, e_l)$ and $(e_1, \dots, e_l, e_{l+1})$, of which the former specialises to the latter. Note that by $(e_1, \dots, e_l)$ we mean the ideal of $\mathbb{N}^l$ generated by the e_i's. Hence $\varpi^{-1}(\zeta) = \{(e_1, \dots, e_l), (e_1, \dots, e_l, e_{l+1})\}$.

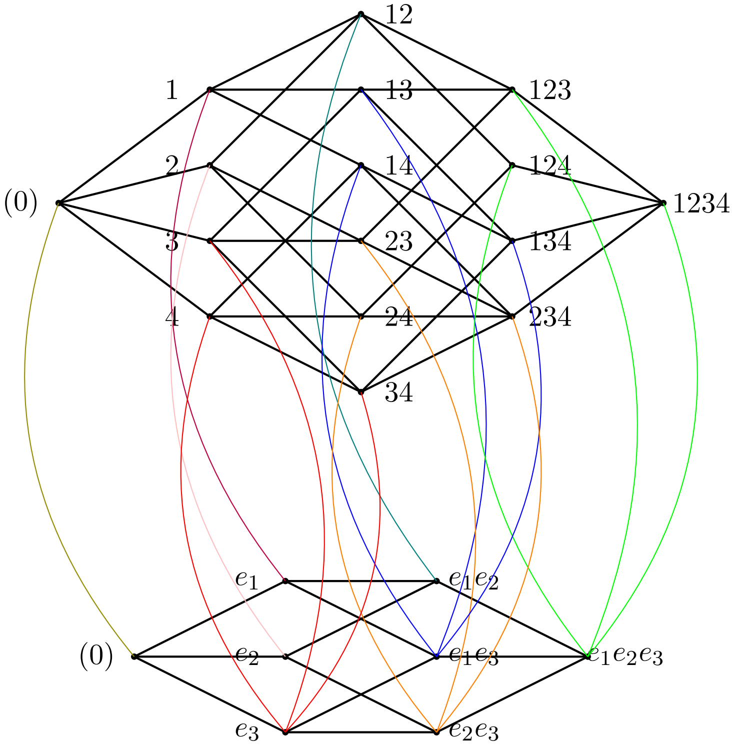
If x is a double point p_j for $j \in \{1, \dots, l\}$, then $\overline{\mathcal{M}}_{U_x} = \overline{\mathcal{M}}_S \bigoplus_{\mathbb{N}} \mathbb{N}^2$. That is, given an element $(n_1, \dots, n_j, \dots n_l) \bigoplus_{\mathbb{N}}(a, b)$, we may rewrite it as the element $(n_1, \dots, 0, \dots n_l) \bigoplus_{\mathbb{N}}(a + n_j, b + n_j)$, and so in particular $\overline{\mathcal{M}}_{U_x} \cong \mathbb{N}^{l+1}$, and the morphism from $\pi^*\overline{\mathcal{M}}_S$ to $\overline{\mathcal{M}}_{U_x}$ maps e_i to e_i when $i \neq j$. When $i = j$, the element e_i is mapped to $(0, \dots, 0) \bigoplus_{\mathbb{N}}(1, 1) = (0, \dots, 1, \dots, 0) \bigoplus_{\mathbb{N}}(0, 0)$.

The points lying over the maximal ideal of $\mathrm{Spec}(\mathbb{N}^l)$ are thus the maximal ideal of $\overline{\mathcal{M}}_{U_x}$ and the points defined by the ideals $(e_1, \dots, 0, \dots, e_l) \bigoplus_{\mathbb{N}}(1, 0)$ and $(e_1, \dots, 0, \dots, e_l) \bigoplus_{\mathbb{N}}(0, 1)$, where the j-th coordinate is 0, both of which specialise to the closed point. □

Remark 3.51. Let U_x be a small neighbourhood of a double point, so that $\mathcal{A}_{U_x} = \mathcal{A}_{\mathbb{A}^{l+1}}$. We illustrate in the diagram below the map of Kato cones from $\mathcal{A}_{U_x}$ to $\mathcal{A}_S$ in the case $l = 3$. To this end and to make the diagram more legible, denote the standard generators of $\mathbb{N}^4$ as $\{1, 2, 3, 4\}$, and, for example, the ideal generated by the elements 2 and 3 by 23.

Similarly, the generators of $\mathbb{N}^3$ will be denoted $\{e_1, e_2, e_3\}$. We may assume that the morphism of Kato cones is induced from the morphism of monoids $\mathbb{N}^3 \to \mathbb{N}^4$ that sends e_1 to 1, e_2 to 2, and e_3 to $3 + 4$.

In the diagram, the curved vertical lines represent the map of Kato cones on points. For example, the points $\{(123), (124), (1234)\}$ lie over the closed point of $\mathrm{Spec}(\mathbb{N}^3)$, viz. $(e_1e_2e_3)$. Furthermore, specialisation of points of the spectra of the monoids are indicated by lines going from left to right.



For each geometric point $x \in X$, we now have U_x as described. Set $V_{xy} = U_x \times_X U_y$ for any two geometric points (x, y). The manner in which the U_x were chosen ensures that, whenever x and y are distinct points, $\underline{V}_{xy}$ is necessarily smooth over $\underline{S}$ with no marked points in the image of $\underline{V}_{xy}$ in $\underline{X}$. In particular, its characteristic monoid is always of the form $\overline{\mathcal{M}}_{V_{xy}} = \mathbb{N}^l$, and each V_{xy} has an associated Artin fan $\mathcal{A}_{V_{xy}} = \mathcal{A}_{\mathbb{N}^l}$. Let U be the disjoint union of the U_x and V the disjoint union of the V_{xy}. Recall that $\mathcal{A}_X = \lim \mathcal{A}_V \rightrightarrows \mathcal{A}_U$ in the category of étale spaces over *Log*.

Definition 3.52. Let $\underline{f} : \underline{X} \to \underline{S}$ be a stable log curve of type (g, n) where $\underline{S}$ is the spectrum of an algebraically closed field. The *dual graph* of $\underline{f}$ is the graph with vertex set V, edge set E, and a set of half-edges F where

- we have one vertex v_i for each irreducible component X_i of $\underline{X}$;
- for each non-smooth point of intersection between irreducible components X_i and X_j (where possibly $X_i = X_j$), we have one edge e between v_i and v_j; and
- for each marked point p of $\underline{X}$ on irreducible component X_i, we have

one half-edge f emanating from v_i.

To the dual graph, we associate the following topological space:

Definition 3.53. Let Γ be a graph. Define a topological space Γ_{top} as follows: For every vertex, edge, and half-edge of Γ, we have one point. A point p specialises to a point q if and only if q corresponds to a edge e_q or half edge f_q and p represents a vertex v_p such that v_p is incident with e_q or f_q, respectively.

Remark 3.54. The closed points of Γ_{top} are precisely the points that come from an edge or half-edge of the graph, unless Γ has no edges or half-edges. If p is a point that corresponds to an vertex v_p, then its closure in Γ_{top} is p as well as the closed points corresponding to edges / half-edges with which v_p is incident to.

We are now in a position to prove the last result, which says that the topological space Γ_{top} associated to the dual graph of a stable curve over a field is isomorphic to that of the closed fibre of the morphism of associated Artin fans. This relies on the irreducible components of $\underline{X}$ being smooth over $\underline{S}$.

Theorem 3.55. *Let S be the spectrum of an algebraically closed field. Let $\underline{f} : \underline{X} \to \underline{S}$ be a stable log curve of type (g, n), and suppose the irreducible components of $\underline{X}$ are smooth over $\underline{S}$.*

Let $\varpi : \mathcal{A}_X \to \mathcal{A}_S$ denote the induced morphism of Artin fans from that of X to that of S, and let ζ denote the closed point of $\mathcal{A}_S$. If Γ is the dual graph of X over S, then the underlying topological space of $\varpi^{-1}(\zeta)$ is naturally isomorphic to Γ_{top}.

Proof. If X is smooth, then clearly $\varpi^{-1}(\zeta)$ is a point, which is trivially isomorphic to Γ_{top}. Assume now that X is not smooth.

Let x and y denote two smooth points of X. If they lie in distinct irreducible components, then clearly $V_{xy} = \emptyset$, so suppose they lie in the same irreducible component. Then $\overline{\mathcal{M}}_{V_{xy}}$ maps to both $\overline{\mathcal{M}}_{U_x}$ and $\overline{\mathcal{M}}_{U_y}$ via the identity, and the points of $\mathcal{A}_{U_x}$ and $\mathcal{A}_{U_y}$ lying over ζ glue to the same point in $\mathcal{A}_X$.

Let x be a marked point of X and let y be a smooth point such that $V_{xy} \neq \emptyset$. The closed point of $\mathcal{A}_{V_{xy}} = \mathcal{A}_{U_y}$ is identified with the open point in the fibre over ζ of $\mathcal{A}_{U_x}$ over $\mathcal{A}_S$ as $\overline{\mathcal{M}}_{U_x} = \overline{\mathcal{M}}_{U_y} \bigoplus \mathbb{N}$.

Let x be a double point of X and let y be a smooth point such that $V_{xy} \neq \emptyset$. Note that $\underline{U_x}$ is étale over $\mathrm{Spec}(k[t_1, t_2]/(t_1t_2))$ by how the U_x were chosen. The irreducible components of $\mathrm{Spec}(k[t_1, t_2]/(t_1t_2))$ cut out by (t_1) and (t_2) are geometrically irreducible as k is algebraically closed, and after shrinking $\underline{U_x}$ as needed, we may assume that the fibre of $\underline{U_x}$ lying over the points $x \in X$ and $(t_1, t_2) \in \mathrm{Spec}(k[t_1, t_2]/(t_1t_2))$ is a single point and hence that there is a unique point $u_1 \in \underline{U_x}$ lying over (t_1) and u_2 lying over (t_2).

Let us assume that y lies in the irreducible component of X whose generic point is mapped to by $u_1 \in U_x$. Note that as X has smooth irreducible components both u_1 and u_2 lie over distinct irreducible components of X. The logarithmic structure on U_x is defined by the chart $\overline{\pi^*\mathcal{M}_S} \bigoplus \mathbb{N}\alpha \bigoplus \mathbb{N}\beta/(\alpha + \beta = e_i)$, where without loss of generality α maps to the generator of u_1 and β maps to the generator of u_2. In particular, when restricted to the component of U_x defined by u_1, β maps to a unit and so is 0 in $\overline{\mathcal{M}}_{U_x}$. We conclude that the closed point of $\mathcal{A}_{V_{xy}}$ maps to the closed point of $\mathcal{A}_{U_y}$ and the point of $\mathcal{A}_{U_x}$ defined by $(e_1, \ldots, 0, \ldots, e_l) \bigoplus (1, 0)$, where the j-th coordinate is 0.

As every irreducible component has some double point by the fact X is connected and not smooth, the only non-closed points in the closed fibre of $\mathcal{A}_X$ over $\mathcal{A}_S$ correspond to vertices. Two points have an edge between them in the dual graph if and only if there is a double point x lying in their intersection if and only if the points in the closed fibre corresponding to the irreducible components have a shared specialisation. Finally, a vertex has a half-edge if and only if the corresponding irreducible component contains a marked point, if and only the corresponding point of the closed fibre of $\mathcal{A}_X$ over $\mathcal{A}_S$ contains a specialisation in the fibre shared by no other points. □

Remark 3.56. In Theorem 3.55 it was assumed that the irreducible components of X were smooth over S. This was to avoid ambiguity in distinguishing the marked points of X with the loop edges in the dual graph, as either would be represented by an open point specialising to a closed point in Γ_{top}. Instead, one could assume that the logarithmic structure of X is *vertical*, i.e. with no marked points. In this case Theorem 3.55 would hold while allowing for non-smooth irreducible components of X.

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Summary

> There was no God in his heart, he knew; his ideas were still in riot; there was ever the pain of memory; the regret for his lost youth - yet the waters of disillusion had left a deposit on his soul, responsibility and a love of life, the faint stirring of old ambitions and unrealized dreams...
>
> F. Scott Fitzgerald, *This Side of Paradise*

This thesis looks at a special class of mathematical objects called *semistable varieties*. These are meant to generalise a well-studied one-dimensional case, namely semistable curves. A *curve* is a one-dimensional object defined by an polynomial equation. For example, $y = x$ gives a straight line, whereas $y = x^2$ gives a parabola. These are both curves, but you can imagine that they can become very complicated.

Curves can also have multiple irreducible components. There's a way to make the word *irreducible* precise, but, for example, one can take the curve with irreducible components $y = x$ *and* $y = x^2$. Now you have a parabola *and* a line. We regard this as *one* curve with *two* irreducible components. The two components intersect one another at the points $(0, 0)$ and $(1, 1)$. Another curve with two components is the union of the $x-$ and $y-$ axis, given by the equation $0 = xy$. It has two irreducible components, both of which are lines, that meet at the point $(0, 0)$.

A *semistable curve* is a curve such that, if you look at all the points of intersection of its irreducible components, the intersections look locally like the union of the coordinate axis. Locally, in this sense, has the intuitive meaning of being able to draw a small circle around the point of intersection and looking at the result, but the technical term is *étale locally*. There are some other technical conditions we place on being a semistable curve, such as "proper" and being "geometrically connected", but these are unimportant for the intuition.

To a semistable curve one associates a graph, called the *dual graph*. Graphs,

in this sense, refer to a collection of points ("vertices") and edges between them. For each irreducible component of a semistable curve, we'll have one vertex. For every point of intersection between the components, we'll have one edge between the corresponding vertices. For example, the dual graph of the semistable curve $0 = xy$ is just two vertices with one edge between them. Admittedly, not the most exciting example, but one that illustrates well what is going on. We care about the dual graph because it's often easier to envision than the actual curve, but more importantly we can often say a lot about the curve by looking at the dual graph. An important, movitating example was work by Holmes in [Hol16], which said that a certain condition on the dual graph of a semistable curve called *alignment* implied the existence of an important mathematical object called a Néron model associated to the curve.

A semistable variety takes the one-dimensional case of a curve and tries to generalise it to higher dimensions. If we work in three dimensions, for example, the local description of a semistable variety should look like the intersection of the xy, xz, and yz planes. What is the dual object in this case? This turned out to be a bit more tricky to answer, and each chapter tries to address this.

In the first chapter, we claim that the dual complex should still be a graph. Each vertex will correspond to an irreducible component, but now an edge will correspond to a connected component of the intersection of different irreducible components. Think of the intersection of the xy, xz, and yz planes: the result is the union of the x, y, and z axis, and they all meet in the origin. In this case we only have one connected component, and the resulting dual graph consist of 3 vertices with one edge between each of them. This turns out to be reasonable insofar as one can derive results from it. The big result of the first chapter says that, with this definition of the dual complex of a semistable variety, an associated Néron model exists as in the one-dimensional case if the dual graph is aligned. What appears to be a more complicated problem in higher dimensions can be reduced to looking at a one-dimensional object, viz. the dual graph!

In the second chapter we are motivated by an area of mathematics called tropical geometry. By the by, the "tropical" part refers to where this branch of mathematics was developed and less so to the actual mathematics. A *tropical curve* is a graph, but where every edge has a number associated to it, referred to as the labelling. If one starts with a semistable curve that is defined over a special base (a *discrete valuation ring*, for the technically-inclined), one can associate a tropical curve whose underlying graph is the usual dual graph, but whose labels on edges represent the algebraic expression of the point of intersection associated to the edge.

In this chapter we attempt to generalise this to higher dimensions. Rather than work with a dual graph, we instead begin working with *simplicial complexes*. Returning briefly to the example of the intersection of the xy, xz, and yz planes, we will still have 3 vertices and one edge between each of them. The dual simplicial complex, however, will also include the "filled-in" portion of the triangle formed by the vertices and edges. This should better represent the combinatorics of the intersection, as intuitively the origin will look different than the coordinate axis. This definition turns out to be too restrictive for a number of reasons, so we eventually turn to looking an *generalised cone complexes*. It turns out that to every simplicial complex there's a unique generalised cone complex, so we don't lose any information in this more general setting. After working out what the labels should be we spend this chapter proving that such dual complexes share important properties in common with dual tropical curves of semistable curves.

Finally, in the last chapter, we turn our attention to *logarithmic geometry*. Logarithmic geometry sits somewhere between tropical geometry and algebraic geometry in its approach to studying algebraic curves. Rather than dive into the technical details, let us say that logarithmic geometry treats semistable curves as algebraic geometry treats smooth curves. To simplify matters, we restrict ourselves to the one-dimensional case of looking at curves, and restrict ourselves furthermore to looking at *stable curves*. These curves are a subset of semistable curves, but satisfy some additional conditions to make them "better-behaved".

The main problem this chapter tries to address is how to recover the dual graph of a stable curve from an associated object called the Artin fan of the curve. Artin fans are a new object used to study logarithmic geometry. Amongst other good things, Artin fans capture the combinatorics of a logarithmic variety in a way that the dual graph does of a semistable curve. We make this result precise. It should be noted that this is likely already known by the experts in the area, but we could not find it explicitly written down and hope as well that the intermediate results will be of interest.

Nederlandse samenvatting

Het is zo lang geleden
Dat het vergeten had moeten zijn,
Het is zo vers
Als een voetstap in het gras,
Als rook die wegtrekt uit een open raam,
Dauw die druppelt langs gewas
Door aarde en stof,
Een gedachte die er niet meer was.

Jan Wolkers, *De Herinnering*

Dit proefschrift poogt om eendimensionale problemenen naar hogere dimensies te generaliseren. In algebraische meetkunde is het primaire object van studie in één dimensie een *kromme.* De vergelijking $y = x$ geeft bijvoorbeeld een lijn, maar er zijn heel veel andere soorten krommen zoals een parabool (i.e. $y = x^2$). Beschouw het volgende voorbeeld: de vergelijking $0 = xy$. Je kunt snel zien dat dit bestaat uit twee delen, namelijk de x-as en de y-as. In algebraische meetkunde zeggen we dat dit één kromme is, maar een met twee *irreducibele componenten.*

Een belangrijke klasse van krommen is de klasse van *semi-stabiele krommen.* Zo'n kromme is er een waarin het snijpunt van elk paar irreducibele componenten lokaal lijkt op de doorsnijding van de x- en y-as. Lokaal betekent hier dat je een kleine cirkel rond het snijpunt kunt tekenen, waarbinnen de doorsnijding lijkt op de twee assen. (Eigenlijk is de juiste term "lokaal voor de étale topologie", maar dit jargon helpt niet voor de intuiïtie! Algebraische meetkunde zit vol met dit soort vervelende termen - bah!)

Krommen zijn best wel moeilijk om te bestuderen. Wat vaak makkelijker te doen is, is om naar de *duale graaf* te kijken. Een graaf bestaat uit knopen met takken tussen de knopen. Één knoop is een graaf, net als twee knopen met drie takken ertussen. Je kunt ook een tak met hetzelfde begin- en eindpunt hebben - dit noemen we een *lus.*

Stel nu dat je een semi-stabiele kromme hebt die bestaat uit irreducibele componenten met snijpunten. De duale graaf is de graaf die een knoop heeft voor elke irreducibele component, en een tak tussen twee knopen als de overeenkomende componenten elkaar snijden. Een lus komt voor als een component zichzelf doorsnijdt. De duale graaf is heel nuttig! Vaak kun je iets over de kromme zeggen als je iets over de graaf weet. Uit het in de artikel [Hol16] blijkt bijvoorbeeld dat een belangrijk wiskundig object geassocieerd met een kromme (een "Néron model") bestaat als de duale graaf voldoet aan een voorwaarde die "uitgelijnd" wordt genoemd.

Dit proefschrift vraagt of semistabiliteit zinvol is in hogere dimensies, en zo ja, wat de duale graaf in hogere dimensies dan is. Het antwoord op de eerste vraag is dat we in hogere dimensies *semi-stabiele variëteiten* kunnen overwegen. Deze zien er lokaal uit als de doorsnijding van hypervlakken. In drie dimensies hebben we de doorsnijding van het xy-, xz-, en yz- vlak. Elk paar vlakken snijdt elkaar in één lijn, en bovendien ligt de oorsprong in de doorsnijding van alle de drie vlakken.

In het eerste hoofdstuk beschouwen wij de duale graaf in hogere dimensies als een geschikt object analoog aan de duale graaf van een kromme. De duale graaf bestaat weer uit een knoop voor elke irreducibele component, maar nu er is een tak tussen twee knopen voor elk samenhangende component van de doorsnijding van de twee corresponderende irreducibele componenten. In ons vorige voorbeeld met de doorsnijding van het xy-, xz-, en yz- vlak was er maar één samenhangende component in de doorsnijding van de vlakken. De duale graaf bestaat in dit geval uit drie knopen (overeenkomend met de drie vlakken) en één tak tussen elk paar knopen. Wij beweren dat dit een goede definitie voor het duale object in hogere dimensies is, want als de duale graaf uitgelijnd is, dan bestaat er een Néron model geassocieerd met de semi-stabiele variéteit. Wat betreft de resultaten over de variëteit die de duale graaf impliceert is dit dus bijna hetzelfde als in het eendimensionele geval.

In hoofdstuk twee richten we ons op *tropische meetkunde*. Dit gebied van de wiskunde is zo genoemd omdat het in Brazilïe is ontwikkeld. Er zijn helaas nog maar weinig originele woordspelingen met deze naam over. Een *tropische kromme* is een graaf waarin elke tak een label heeft. Normaal gesproken zijn de labels cijfers. Als je een semi-stabiele kromme hebt kun je een tropische kromme bouwen. De graaf is dezelfde als de duale graaf van de kromme, en de labels zijn cijfers met informatie over de algebraische uitdrukking van de doorsnijding van de overeenkomende irreducibele componenten.

Wij proberen dit naar hogere dimensies te generaliseren, i.e. wat is het juiste object geassocieerd met een semi-stabiele variëteit dat analoog is aan een semistable variëteit? We beginnen met simpliciale complexen. In het voorbeeld met de doorsnijding van de xy-, xz-, en yz- vlakken bestaat het duale simplex uit de drie knopen met een tak tussen elk paar knopen, maar nu heeft het ook een inwendige driehoek. Dit is informatie die de duale graaf niet heeft. Het bleek dat simpliciale complexen te eenvoudig waren voor meer complexe zaken, en daarom gebruiken wij ook *gegeneraliseerde kegel complexen*. Het primaire resultaat in dit hoofdstuk is dat er een gegeneraliseerde kegel complex geassocieerd met een semistable variëteit is dat dezelfde eigenschappen als een tropische kromme heeft.

In het laatse hoofdstuk werken we in één dimensie, en bovendien met een specifiek type van semi-stabiele krommen die *stabiele krommen* worden genoemd. Dit hoofdstuk gebruikt resultaten uit een gebied van wiskunde dat "logaritmische meetkunde" wordt genoemd. Dit is nodig voor sommige resultaten - de aanvullende eigenschappen van stabiele krommen maken het noodzakelijk om bepaalde resultaten uit de logaritmische meetkunde te gebruiken. Elke stabiele kromme is ook een *basislogaritmische kromme*, en een aan elke dergelijke kromme is een *Artin waaier* verbonden. Deze objecten zijn heel technisch, maar het zijn objecten die informatie over de structuur van de logaritmische kromme geven.

De bedoeling van dit hoofdstuk is dus om de duale graaf van een stable kromme de bouwen als wij met de Artin waaier beginnen. Eigenlijk is dit resultaat niet helemaal nieuw, maar we konden het niet expliciet opgeschreven, en wij hopen ook dat de tussenresultaten interessant zijn.

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> All this, all of this love we're talking about, it would just be a memory. Maybe not even a memory. Am I wrong? Am I way off base? Because I want you to set me straight if you think I'm wrong. I want to know. I mean, I don't know anything, and I'm the first one to admit it.
>
> Raymond Carver, *What We Talk About When We Talk About Love*

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Curriculum Vitae

> The man who wrote it, I suppose, was some wretched fellow who writes these things for a drink.
>
> James Joyce, *Dubliners*

Garnet was born August 5, 1992 in Mississauga, Ontario. His pre-university education lasted from 2006 to 2010. He graduated with a Bachelor of Mathematics in 2014 from the University of Waterloo, and with a Masters of Mathematics in 2016 from the University of Waterloo under the supervision of David McKinnon. He began his PhD at Universiteit Leiden in 2016 under the supervision of David Holmes. Beginning in September of 2019, Garnet has begun working at Ortec Finance.