Absolutely continuous invariant measures for non-autonomous dynamical systems

Góra, P.; Boyarsky, A.; Keefe, C.

ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS.

PAWEŁ GÓRA, ABRAHAM BOYARSKY, AND CHRISTOPHER KEEFE

Abstract. We consider the non autonomous dynamical system $\{\tau_n\}$, where τ_n is a continuous map $X \to X$, and X is a compact metric space. We assume that $\{\tau_n\}$ converges uniformly to τ. The inheritance of chaotic properties as well as topological entropy by τ from the sequence $\{\tau_n\}$ has been studied in [4, 5, 10, 13, 17]. In [16] the generalization of SRB measures to non-autonomous systems has been considered. In this paper we study absolutely continuouus invariant measures (acim) for non autonomous systems. After generalizing the Krylov-Bogoliubov Theorem [7] and Straube's Theorem [14] to the non autonomous setting, we prove that under certain conditions the limit map τ of a non autonomous sequence of maps $\{\tau_n\}$ with acims has an acim.

1. Introduction

Autonomous systems are rare in nature. A more realistic approach to modeling real life processes is to consider non autonomous models. In this note we consider a sequence of maps $\{\tau_n\}$ on a compact metric space $X \to X$. We assume that $\{\tau_n\}$ converges uniformly to τ. Let $\tau_{(0,n)} = \tau_n \circ \tau_{n-2} \circ \cdots \circ \tau_1 \circ \tau_0$. For an initial measure η we consider the sequence $\mu_n = (\tau_{(0,n)})_* \eta$. Since X is compact the space of probability measures on X is $*$-weakly compact and hence we can assume that $\{\mu_n\}$ converges to a measure μ. In this note we study conditions under which the limit map τ preserves μ. In particular we are interested in the situation when μ_n and μ are absolutely continuous.

The behaviour of non autonomous sequences of piecewise expanding maps was studied before. In the paper [12] the authors consider a family $\mathcal{E}$ of exact piecewise expanding maps with uniform expanding properties and show that for any two initial densities f_1, f_2 the iterates $P_{\tau_{(0,n)}} f_1$ and $P_{\tau_{(0,n)}} f_2$ get closer to each other with exponential speed. Using the notation of Section 2:

$$\int |P_{\tau_{(0,n)}} f_1 - P_{\tau_{(0,n)}} f_2| dm \le C(f_1, f_2)\Lambda^n, \quad n \ge 1,$$

for some constants $C(f_1, f_2) > 0$, $0 < \Lambda < 1$ and any sequence of maps $\tau_n \in \mathcal{E}$. In this situation, in general, there is no limit map and the densities $P_{\tau_{(0,n)}} f$ do not converge. In this note we assume the uniform convergence $\tau_n \rightrightarrows \tau$. This allows us to prove that, under some assumptions, the densities $P_{\tau_{(0,n)}} f$ converge to a τ-invariant density.

Date: August 30, 2019.
2000 *Mathematics Subject Classification.* 37A05, 37E05.
Key words and phrases. absolutely continuous invariant measures, non-autonomous systems.
The research of the authors was supported by NSERC grants.

Another approach to dealing with compositions of different maps is to consider a random map. Maps from a family $\mathcal{E}=\{\tau_a\}_{a\in\mathcal{A}}$ are applied randomly according to a probability on $\mathcal{A}$, which might depend on the current position of the process. The literature on random maps is quite rich. A recent article is [1]. The authors study, in particular, random maps based on the set $\mathcal{E}$ of the Liverani-Saussol-Vaienti maps

$$\tau_a(x)=\begin{cases} x(1+2^a x^a), & x\in[0,1/2],\\ 2x-1, & x\in(1/2,1],\end{cases}$$

with parameters in $[a_0,a_1]\subset(0,1)$ chosen independently with respect to a distribution ν on $[a_0,a_1]$. These maps have indifferent fixed points which makes them non-exponentially mixing. The authors study the fibre-wise (quenched) dynamics of the system. For this point of view a skew-product approach is convenient.

Let $(\mathcal{A},\mathcal{F},p)$ be a Borel probability space, let $\Omega=\mathcal{A}^{\mathbb{Z}}$ be equipped with the product measure $P:=p^{\mathbb{Z}}$ and let $\sigma:\Omega\to\Omega$ denote the P-preserving two-sided shift map. Let $(X,\mathcal{B})$ be a measurable space. Suppose that $\tau_a:X\to X$ is a family of measurable maps defined for p-almost every $a\in\mathcal{A}$ such that the skew product

$$T:X\times\Omega\to X\times\Omega,\ T(x,\omega)=(\tau_{[\omega]_0},\sigma\omega),$$

is measurable with respect to $\mathcal{B}\times\mathcal{F}$. If $X_\omega=X\times\{\omega\}$ denotes the fiber over ω and

$$\tau_\omega^n=\tau_{\sigma^{n-1}\omega}\circ\cdots\circ\tau_\omega:X_\omega\to X_{\sigma^n\omega},$$

we have $T^n(x,\omega)=(\tau_\omega^n(x),\sigma^n\omega$. If a probability measure μ is T-invariant and $\pi_*\mu=P$ (π is the projection onto Ω), then there exists a family of probability fiber measures μ_ω on X_ω such that $\mu(A)=\int\mu_\omega(A)dP(\omega)$ for any $A\in\mathcal{B}\times\mathcal{F}$. Since μ is T-invariant the measures $\{\mu_\omega\}$ form an equivariant family, i.e., $(\tau_\omega)_*\mu_\omega=\mu_{\sigma\omega}$ for almost all ω.

The authors study future and past quenched correlations: given $\phi,\psi:X\times\Omega\to\mathbb{R}$ the future and past fibre-wise correlations are defined as

$$Cor^{(f)}_{n,\omega}=\int(\phi\circ\tau^n_\omega)\psi d\mu_\omega-\int\phi d\mu_{\sigma^n\omega}\int\psi d\mu_\omega,$$

$$Cor^{(p)}_{n,\omega}=\int(\phi\circ\tau^n_{\sigma^{-n}\omega})\psi d\mu_{\sigma^{-n}\omega}-\int\phi d\mu_\omega\int\psi d\mu_{\sigma^{-n}\omega}.$$

They prove that for the random map based on family $\mathcal{E}$ there exists an equivariant family of measures μ_ω which are absolutely continuous P-a.e., characterize their densities and show that both future and past quenched correlations are of order $\mathcal{O}(n^{1-1/a_0}+\delta)$ for bounded ϕ and Hölder continuous ψ and arbitrary $\delta>0$. The system (T,μ) is mixing.

In this note we assume that $\tau_n\rightrightarrows\tau$ and consider the compositions $\tau_{(0,n)}=\tau_n\circ\tau_{n-2}\circ\cdots\circ\tau_1\circ\tau_0$, so we can say that we study one fixed fiber under very special assumptions.

In Section 2 we give the definitions and introduce the notation. In Section 3 we generalize the Krylov-Bogoliubov Theorem [7] and Straube's Theorem [14] to the non autonomous setting. Section 4 is independent of the previous section. We make stronger assumptions on the τ_n's and establish the existence of an acim for the limit map τ and show that any convergent subsequence of $\{P_{\tau_{(0,n)}}f\}_{n\geq1}$ converges to an invariant density of the limit map, where $P_{\tau_{(0,n)}}$ is the Frobenius-Perron operator induced by $\tau_{(0,n)}$ and f is a density.

2. Notation and Definitions

Let (X,ρ) be a compact metric space. Let $\{\tau_n\}$ be a sequence of maps $\tau_n : X \to X$ which converges uniformly to a continuous map τ. We shall consider the non-autonomous dynamical system defined by

$$x_{m+1} = \tau_m(x_m), \quad m = 0, 1, 2, \dots$$

where we assume that τ_0 is the identity and $x_0 \in I$.

We write

$$\tau_{(m,n)} = \tau_n \circ \tau_{n-2} \circ \cdots \circ \tau_{m+1} \circ \tau_m, \quad n > m.$$

In particular,

$$\tau_{(0,n)} = \tau_n \circ \tau_{n-2} \circ \cdots \circ \tau_1 \circ \tau_0.$$

Let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X.

For a map $\tau : X \to X$ we define an operator on measures on $\mathcal{B}(X)$:

$$\tau_* \mu(A) = \mu(\tau^{-1}A),$$

for any measurable set A.

3. Generalization of the Krylov-Bogoliubov Theorem and Straube's Theorem

We will now prove a generalization of the Krylov-Bogoliubov Theorem:

Theorem 1. *Let $\{\tau_n\}$ be a sequence of transformations defining a nonautonomous dynamical system on the metric compact space X with a continuous limit τ. We assume that the τ_n's converge uniformly to τ. Let η be a fixed probability measure on X. Define the measures $\mu_n = \frac{1}{n}\sum_{i=1}^n \nu_i$, where $\nu_i = \left(\tau_{(0,i)}\right)_*(\eta)$. Let μ be a $*$-weak limit point of the sequence $\{\mu_n\}_{n\geq 1}$. Then μ is a τ-invariant measure, i.e., $\tau_*\mu = \mu$.*

Proof. We follow the proof of the original Krylov-Bogoliubov Theorem. Let η be a probability measure X. Then the sequence $\mu_n = \frac{1}{n}\sum_{i=1}^n \nu_i$, where $\nu_i = \left(\tau_{(0,i)}\right)_*(\eta)$ is a sequence of probability measures and contains a convergent subsequence μ_{n_k}. Let $\mu = \lim_{k\to\infty}\mu_{n_k}$. We will prove that $\tau_*\mu = \mu$. To this end it is enough to show that for any $g \in C^0(X)$, $\mu(g) = \tau_*\mu(g) = \mu(g\circ\tau)$.

We estimate the difference

(1)

$$\begin{aligned}
|\mu_n(g) - \mu_n(g\circ\tau)| &= \frac{1}{n}\left|\sum_{i=1}^n \nu_i(g) - \sum_{i=1}^n \nu_i(g\circ\tau)\right| \\
&= \frac{1}{n}\big|\eta(g\circ\tau_{(0,1)}) + \eta(g\circ\tau_{(0,2)}) + \cdots + \eta(g\circ\tau_{(0,n-1)}) + \eta(g\circ\tau_{(0,n)}) \\
&\qquad -\eta(g\circ\tau\circ\tau_{(0,1)}) - \eta(g\circ\tau\circ\tau_{(0,2)}) - \cdots - \eta(g\circ\tau\circ\tau_{(0,n-1)}) - \eta(g\circ\tau\circ\tau_{(0,n)})\big| \\
&= \frac{1}{n}\left|\eta(g\circ\tau_{(0,1)}) + \sum_{i=2}^n\left(\eta(g\circ\tau_{(0,i)}) - \eta(g\circ\tau\circ\tau_{(0,i-1)})\right) - \eta(g\circ\tau\circ\tau_{(0,n)})\right|.
\end{aligned}$$

Let ω_g be the modulus of continuity of g, i.e.,

$$\omega_g(\delta) = \sup_{\rho(x,y)<\delta}|g(x) - g(y)|.$$

For an arbitrary $\varepsilon > 0$ we can find a $\delta > 0$ such that $\omega_g(\delta) < \varepsilon$. Since $\tau_n \to \tau$ uniformly for this δ we can find an $N \geq 1$ such that $\sup_{x\in X} \rho(\tau_n(x), \tau(x)) < \delta$ for all $n > N$.

For $i > N$, we have

$$\begin{aligned}&\left|\eta(g \circ \tau_{(0,i)}) - \eta(g \circ \tau \circ \tau_{(0,i-1)})\right| = \left|\eta(g \circ \tau_i \circ \tau_{(0,i-1)} - g \circ \tau \circ \tau_{(0,i-1)})\right| \\ &= \left|\eta((g \circ \tau_i - g \circ \tau)(\tau_{(0,i-1)}))\right| \leq \omega_g(\delta) < \varepsilon.\end{aligned}$$

Thus, for $n > N$, we have

$$|\mu_n(g) - \mu_n(g \circ \tau)| \leq \frac{1}{n} \left(N \cdot 2 \cdot \sup |g| + (n - N)\varepsilon\right),$$

which becomes arbitrarily close to ε as $n \to \infty$. This shows that $\mu_{n_k}(g) - \mu_{n_k}(g \circ \tau) \to 0$ as $k \to \infty$.

We have $\mu_{n_k}(g) \to \mu(g)$ and since τ is continuous $\mu_{n_k}(g\circ\tau) \to \mu(g\circ\tau) = \tau_*\mu(g)$. Thus, μ is a τ-invariant measure. □

Remark: The only place where we needed the continuity of τ is the last line of the proof: since τ is continuous $g \circ \tau$ is continuous for any continuous g and then the $*$-weak convergence of μ_{n_k} implies $\mu_{n_k}(g \circ \tau) \to \mu(g \circ \tau)$.

Theorem 1 does not yield any more information about the τ-invariant measure μ. The next result is a generalization of a theorem by Straube [14], which provides a sufficient condition for μ to be absolutely continuous.

Theorem 2. *Let $(X, \mathcal{B}, \nu)$ be a normalized measure space and let $\{\tau_n\}$ be a sequence of non-singular transformations defining a non-autonomous dynamical system on X. We do not assume that the limit τ is continuous. Assume there exists $\delta > 0$ and $0 < \alpha < 1$ such that*

$$\nu(E) < \delta \implies \sup_{k\geq 1} \nu\left(\tau^{-1}_{(0,k)}(E)\right) < \alpha,$$

for all $E \in \mathcal{B}$. Then there exists a τ-invariant normalized measure μ which is absolutely continuous with respect to ν.

(The proof uses a number of facts from the theory of finitely additive measures which are collected in the Appendix. The proof is similar to the proof in [14] but is modified to allow the use of the estimates from the proof of Theorem 1.)

Proof. Let us define the measures

$$\nu_n(E) = \frac{1}{n} \sum_{k=0}^{n-1} \nu(\tau^{-1}_{(0,k)}(E)) \ , \ E \in \mathcal{B}.$$

Then, for all n,

(a) $\nu_n(X) = 1$;
(b) $\nu_n \ll \nu$ (τ_n is non-singular for every n);
(c) $\nu_n(\cdot) \geq 0$.

Thus, $\{\nu_n\}$ is a sequence of positive, normalized, absolutely continuous measures and can be treated as a sequence in the unit ball of $L^*_\infty(X)$ with the $*$-weak topology. Thus, it contains a convergent subsequence $\nu_{n_k} \to z$ and z can be identified with a finitely additive measure on X. The measure z is finitely additive, positive, normalized and absolutely continuous with respect to ν.

By Lemma 7 in the Appendix we can uniquely decompose z into

$$z = z_c + z_p,$$

where z_c is countably additive and z_p is purely finitely additive. Now, we claim that $z_c \neq 0$. Otherwise, by Lemma 6, there exists a decreasing sequence $\{E_n\} \subset \mathcal{B}$ such that $\lim_{n\to\infty} \nu(E_n) = 0$ and $z(E_n) = z(X) = 1$ for all $n \geq 1$. Since $\nu(E_n) \to 0$, for any $\delta > 0$, there exists an n_0 such that $n > n_0 \implies \nu(E_n) < \delta$. Now, by our assumptions, there is an $\alpha < 1$ such that,

$$\sup_k \nu(\tau_{(0,k)}^{-1}(E_n)) < \alpha < 1.$$

Thus, $\nu(\tau_{(0,k)}^{-1}(E_n) < \alpha$ for all k. So,

$$z(E_n) < \alpha < 1,$$

which is a contradiction. We have demonstrated that $z_c \neq 0$.

Now we will prove that z_c is τ-invariant. Consider the finitely additive measure

$$\kappa = z - z \circ \tau^{-1} = z_c - z_c \circ \tau^{-1} + z_p - z_p \circ \tau^{-1}.$$

In the proof of Theorem 1 we showed that for any continuous function g on X we have

$$\mu_{n_k}(g) - \mu_{n_k}(\tau^{-1}(g)) \to 0 \ , \ k \to \infty.$$

This means that for any continuous function g (which is bounded since X is compact) we have

$$\kappa(g) = z(g) - z \circ \tau^{-1}(g) = 0.$$

We do not need continuity of τ here as $\mu_{n_k}(h) \to z(h)$ for all bounded h. By Lemma 9 in the Appendix the countably additive component of κ is 0, which means

$$z_c - z_c \circ \tau^{-1} = 0,$$

or that z_c is τ-invariant. □

In the following example we show that, unlike in the case of one transformation, the converse implication in Theorem 2 may not hold. We will construct a sequence of maps $\tau_n \to \tau$, such that τ admits an acim and

$$\forall_{\ \delta>0} \ \exists_{\ E\in\mathcal{B}} \ \sup_{k\geq 1} \ \nu\left(\tau_{(2,k)}^{-1}(E)\right) = 1. \tag{2}$$

Example 3. *Let us consider maps $\tau_n : [0,1] \to [0,1]$, $n = 2, 3, \ldots$, defined as follows*

$$\tau_n(x) = \begin{cases} (1 - \frac{1}{n})x, & \text{for } x \in [0, \frac{1}{2}); \\ 2x - 1, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

The limit map $\tau(x) = x\chi_{[0,\frac{1}{2})}(x) + (2x+1)\chi_{[\frac{1}{2},1]}(x)$ admits an acim and condition (2) holds.

Proof. Let $\rho_n = \tau_{n|[0,\frac{1}{2})}$ be the first branch of τ_n. The slope of $\rho_n = \frac{n-1}{n}$ so the slope of $\rho_{m,n} = \rho_n \circ \rho_{n-1} \circ \rho_{n-2} \circ \cdots \circ \rho_m$, $n > m$, is $\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{m-1}{m} = \frac{m}{n} < 1$. Then, the interval $\rho_{m,n}^{-1}([0,\delta])$ is the interval from 0 to the minimum of $\delta \cdot \frac{n}{m}$ and $\frac{1}{2}$. Note, that for any k, we have

$$\rho_k^{-1}([0, \frac{1}{2}]) = [0, \frac{1}{2}]. \tag{3}$$

Letting $\varrho = \varrho_n = \tau_{n|[\frac{1}{2},1]}$ be the second branch of τ_n, we have

$$\varrho^{-1}\left(\left[0,\frac{1}{2}\right]\right) = \left[\frac{1}{2},\frac{1}{2}+\frac{1}{4}\right];$$
$$\varrho^{-1}\left(\left[\frac{1}{2},\frac{1}{2}+\frac{1}{4}\right]\right) = \left[\frac{1}{2}+\frac{1}{4},\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right];$$
$$\vdots$$
$$\varrho^{-1}\left(\left[\sum_{i=1}^{k}\frac{1}{2^i},\sum_{i=1}^{k+1}\frac{1}{2^i}\right]\right) = \left[\sum_{i=1}^{k+1}\frac{1}{2^i},\sum_{i=1}^{k+2}\frac{1}{2^i}\right].$$

This and (3) imply that

$$\tau^{-1}_{(2,m-1)}([0,\frac{1}{2}]) = \left[0,\sum_{i=1}^{m-1}\frac{1}{2^i}\right].$$

Let $\varepsilon > 0$ and m such that $1-\sum_{i=1}^{m-1}\frac{1}{2^i} < \varepsilon$. Let n satisfy $\delta\cdot\frac{n}{m} > \frac{1}{2}$. Then the Lebesgue measure of $\tau^{-1}_{2,n}([0,\delta])$ is larger than $1-\varepsilon$. □

4. Existence of an absolutely continuous invariant measure for the limit map

In this section we will assume that all the maps τ_n are piecewise expanding maps of an interval. For the general theory of such maps we refer the reader to [3] or [8].

Let $I = [0,1]$. The map $\tau : I \to I$ is called piecewise expanding iff there exists a partition $\mathcal{P} = \{I_i := [a_{i-1},a_i], i = 1,\dots,q\}$ of I such that $\tau : I \to I$ satisfies the following conditions:

(i) τ is monotonic on each interval I_i;

(ii) $\tau_i := \tau|_{I_i}$ is C^2, i.e., C^2 in the interior and the one-sided limits of the derivatives are finite at endpoints;

(iii) $|\tau_i'(x)| \ge s_i \ge s > 1$ for any i and for all $x \in (a_{i-1},a_i)$.

The following Frobenius-Perron operator $P_\tau : L^1(I,m) \to L^1(I,m)$, where m is Lebesgue measure, is a basic tool in the theory of piecewise expanding maps. For a general non-singular map τ $\left[m(A)=0 \implies m(\tau^{-1}(A)=0\right]$, we define $P_\tau f$ as a Radon-Nikodym derivative $\frac{d(\tau_* m)}{dm}$. For piecewise expanding maps the operator can be written explicitly [3]:

$$P_\tau f(x) = \sum_{i=1}^{q}\frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}.$$

In particular $P_\tau f = f$ iff $f\cdot m$ is an acim of τ. Piecewise expanding maps of the interval satisfy the following Lasota-Yorke inequality [9]. For any bounded variation function $f \in BV(I)$ the variation $V(P_\tau f)$ satisfies

$$V(P_\tau f) \le AV(f) + B\int_I |f|dm,$$

where the constants $A = \frac{2}{s}$, $B = \frac{\max|\tau''|}{s} + \frac{2}{h}$ and $h = \min_i\{m(I_i)\}$. In particular, we can assume that $A < 1$, considering an iterate τ^k, if necessary. We always assume that bounded variation functions are modified to satisfy $f(x_0) = \limsup_{x\to x_0} f(x)$ for all $x_0 \in I$.

We will prove the following:

Theorem 4. *Assume that τ_n, $n = 1, 2, \dots$ are piecewise expanding maps of an interval and satisfy the Lasota-Yorke inequality with common constants $A < 1$ and B. Then, for any density $f \in BV(I)$, the sequence $f_n = \frac{1}{n}\sum_{i=1}^{n} P_{\tau_{(1,i)}} f$ forms a precompact set in L^1 and any convergent subsequence converges to a density of an acim of the limit map τ.*

Remark: We do not assume that the maps τ_n are defined on a common partition. We assume that they all satisfy Lasota-Yorke inequality with the same constant B. In the following lemma we show that this implies that the limit map τ is defined on a finite partition and the partitions for maps τ_n are "asymptotically" the same as the partition for τ.

Lemma 5. *Under the assumptions of Theorem 4 the limit map τ is piecewise monotonic and there exists a constant K such that for any interval J we have $m(\tau^{-1}(J)) \le K m(J)$. In particular, it follows that the limit map τ is non-singular.*

Proof. Since the constant B depends on the reciprocal of h, there is a universal bound q_u on the number of elements of the partition $\mathcal{P}$ for τ_n. This places a restriction on the number k of iterates we can use to make $A < 1$. Thus, there exists a universal lower bound s_u for the modulus of the derivative τ_n'.

Now, we prove that τ is piecewise monotonic. Assume that the graph of τ contains p points forming a "zigzag", i.e., there exist $x_1 < x_2 < x_3 < \cdots < x_{p-1} < x_p$ such that $\tau(x_i) < \tau(x_{i+1})$ for odd i and $\tau(x_i) > \tau(x_{i+1})$ for even i (or other way around). Then, $p \le 2q_u$. If not, then since $\tau_n \rightrightarrows \tau$ uniformly, for large n the graph of τ_n also contains a zigzag of length p. This is impossible as τ_n has at most q_u branches of monotonicity. Thus, τ is piecewise monotonic with at most q_u branches of monotonicity.

Let $[a, b] \subset I$ be an interval. Each line $y = a$, $y = b$ intersects the graph of τ in at most q_u points. Let points (x_1, a), (x_2, b) be the points of intersection of these lines with one monotonic, say increasing, branch of τ. Then,

$$b - a = \lim_{n\to\infty} \tau_n(x_2) - \tau_n(x_1) \ge \lim_{n\to\infty} s_u \cdot (x_2 - x_1) = s_u \cdot (x_2 - x_1).$$

If one (or two) of the intersections is empty, we replace appropriate x_i by the endpoint of the interval of monotonicity. Thus, for any interval J we have

$$m(\tau^{-1}(J)) \le \frac{q_u}{s_u} m(J). \tag{4}$$

□

We can now prove Theorem 4.

Proof of Theorem 4. Since f is a density and the Frobenius-Perron operator preserves the integral of positive functions, we have $\int |P_{\tau_n} f| dm = 1$ for all $n \ge 1$. Since $P_{\tau_{(1,i)}} = P_{\tau_i} \circ P_{\tau_{i-1}} \circ \cdots \circ P_{\tau_2} \circ P_{\tau_1}$, we can apply the Lasota-Yorke inequality consecutively and obtain

$$V(P_{\tau_{(1,i)}} f) \le A^i V(f) + B(A^{i-1} + A^{i-2} + \cdots + A^2 + A + 1) \le A^i V(f) + \frac{B}{1 - A}\ , \ i \ge 1.$$

Thus, the functions $P_{\tau_{(1,i)}} f$ and also the functions f_n, $i, n \ge 1$, have uniformly bounded variation. Since for a bounded variation density f, $\sup_{x\in I} f(x) \le 1 + V(f)$, these functions are also uniformly bounded. The sequence $\{f_n\}_{n\ge 1}$, being both uniformly bounded and of uniformly bounded variation contains a subsequence $\{f_{n_k}\}_{k\geq 1}$ convergent almost everywhere to a function f^* of bounded variation by Helly's Theorem [11]. Additionally, by the Lebesgue Dominated Convergence Theorem, $\int_I f^* dm = 1$. This means that, by Scheffe's Theorem [2], $f_{n_k} \to f^*$ in the L^1-norm. Thus, the sequence $\{f_n\}_{n\geq 1}$ forms a pre-compact set in L^1 and in particular, contains a subsequence convergent in L^1 to a function of bounded variation.

Now, we will prove that for any density F, $(P_{\tau_n}F - P_\tau F) \to 0$ weakly in L^1, as $n \to \infty$. Let $g \in L^\infty(I, m)$ be an arbitrary bounded function and let us fix an $\varepsilon > 0$. By Lusin's Theorem [6, Th. 7.10] for any $\eta > 0$ there exists an open set $U \subset I$, $m(U) < \eta$, and a continuous function $G \in C^0(I)$ such that $g = G$ on $I \setminus U$ and $\sup |G| \leq \|g\|_\infty$. The Frobenius-Perron operator is a conjugate of the Koopman operator, that is for any $f \in L^1$ and any $g \in L^\infty$, we have $\int_I P_\tau f \cdot g \, dm = \int_I f \cdot g \circ \tau \, dm$. Therefore, we can write

$$\begin{aligned}
&\left| \int_I (P_\tau F \cdot g - P_{\tau_n} F \cdot g) \, dm \right| \leq \int_I F \left| g \circ \tau - g \circ \tau_n \right| \, dm \\
&= \int_I F \left| g \circ \tau - G \circ \tau + G \circ \tau - G \circ \tau_n + G \circ \tau_n - g \circ \tau_n \right| \, dm \\
&\leq \int_{\tau^{-1}(U)} F \left| g \circ \tau - G \circ \tau \right| \, dm + \int_I F \left| G \circ \tau_n + G \circ \tau_n \right| \, dm + \int_{\tau_n^{-1}(U)} F \left| g \circ \tau_n - G \circ \tau_n \right| \, dm.
\end{aligned}$$

Let $\sup G \leq \|g\|_\infty = M_g$. Let $I_F(t) = \sup_{\{A: m(A) < t\}} \int_A |F| \, dm$. It is known that $I_F(t) \to 0$ as $t \to 0$. Let ω_G be the modulus of continuity of G: $\omega_G(t) = \sup_{|x-y|\leq t} |G(x) - G(y)|$. Again, $\omega_G(t) \to 0$ as $t \to 0$. Using estimate (4) we obtain

$$\begin{aligned}
&\left| \int_I (P_\tau F \cdot g - P_{\tau_n} F \cdot g) \, dm \right| \\
&\leq 2M_g I_F \left(\frac{q_u}{s_u} \eta \right) + \omega_G(\sup |\tau_n - \tau|) + 2M_g I_F \left(\frac{q_u}{s_u} \eta \right) \\
&= \omega_G(\|\tau_n - \tau\|_\infty) + 4M_g I_F \left(\frac{q_u}{s_u} \eta \right).
\end{aligned} \tag{5}$$

Let us fix an $\varepsilon > 0$. Since $\|\tau_n - \tau\|_\infty \to 0$, as $n \to \infty$ we can find $N \geq 1$ such that for all $n \geq N$ we have $\omega_G(\|\tau_n - \tau\|_\infty) < \varepsilon$. We can also find an $\eta > 0$ such that $4M_g I_F \left(\frac{q_u}{s_u} \eta \right) < \varepsilon$. This shows that $(P_{\tau_n}F - P_\tau F) \to 0$ weakly in L^1, as $n \to \infty$. Note, that this convergence is uniform over precompact subsets of L^1, since the estimate (5) can be made common for all F in such a set (the functions in a precompact set are uniformly integrable).

Let $\{f_{n_k}\}_{k\geq 1}$ be a subsequence of $\{f_n\}_{n\geq 1}$ convergent in L^1 to f^*. To simplify the notation we will skip the subindex k. We will show that f^* is the density of an acim of τ, i.e., $P_\tau f^* = f^*$. We have

$$P_\tau f^* = P_\tau (\lim_{n\to\infty} f_n) = \lim_{n\to\infty} P_\tau f_n.$$

We will show that $P_\tau f_n - f_n$ converges weakly in L^1 to 0. Let $\phi_i = P_{\tau_{(1,i)}} f$, $i = 1, 2, \dots$. Then, $f_n = \frac{1}{n}(\phi_1 + \phi_2 + \cdots + \phi_{n-1} + \phi_n)$. We can write

$$P_\tau f_n - f_n = \frac{1}{n}(P_\tau\phi_1 + P_\tau\phi_2 + P_\tau \cdots + P_\tau\phi_{n-1} + P_\tau\phi_n) - \frac{1}{n}(\phi_1 + \phi_2 + \cdots + \phi_{n-1} + \phi_n)$$

$$= \frac{1}{n}(P_\tau\phi_n - \phi_1) + \frac{1}{n}\sum_{i=1}^{n-1}(P_\tau\phi_i - \phi_{i+1}) = \frac{1}{n}(P_\tau\phi_n - \phi_1) + \frac{1}{n}\sum_{i=1}^{n-1}\left(P_\tau\phi_i - P_{\tau_{i+1}}\phi_i\right).$$

Let I_Φ be a common I_F function for all ϕ_i's. Let N and η be chosen as above. Let $n \ge N + 2$. Then, using estimate (5), we have

$$\left|\int_I (P_\tau f_n - f_n)\, g\, dm\right|$$
$$\le \frac{1}{n}\int_I |(P_\tau\phi_n - \phi_1)\, g|\, dm + \frac{1}{n}\sum_{i=1}^{N}\int_I \left|\left(P_\tau\phi_i - P_{\tau_{i+1}}\phi_i\right) g\right|\, dm$$
$$+ \frac{1}{n}\sum_{i=N+1}^{n-1}\int_I \left|\left(P_\tau\phi_i - P_{\tau_{i+1}}\phi_i\right) g\right|\, dm$$
$$\le \frac{2}{n}M_g + \frac{2}{n}NM_g + \frac{n-1-N}{n}(2\varepsilon).$$

As $n \to \infty$ the right hand side becomes smaller than say 3ε. Since $\varepsilon > 0$ is arbitrary this proves that $P_\tau f_n - f_n$ converges weakly in L^1 to 0 and $P_\tau f^* = f^*$. □

5. Appendix

Here we collect the results about finitely additive measures necessary for the proof of Theorem 2

Lemma 6. [Theorem 1.22 of [15]] *Let $(X, \mathcal{B})$ be a compact measure space. Let the measure η be purely finitely additive and $\eta \ge 0$. Let κ be a countably additive measure defined on $(X, \mathcal{B})$ such that $\kappa \ge 0$. Then, there exists a decreasing sequence $\{E_n\} \subset \mathcal{B}$ such that $\lim_{n\to\infty}\kappa(E_n) = 0$ and $\eta(E_n) = \eta(X)$ for all $n \ge 1$. Conversely, if kappa is a measure and the above conditions hold for all countably additive κ, then η is purely finitely additive.*

Lemma 7. [Theorems 1.23 and 1.24 of [15]] *Let η be a measure such that $\eta \ge 0$. Then there exist unique measures η_p and η_c such that $\eta_p \ge 0$, $\eta_c \ge 0$, η_p is purely finitely additive, η_c is countably additive and*

$$\eta = \eta_p + \eta_c.$$

Lemma 8. [Contained in the proof of Theorem 1.23 of [15]] *Let η be a measure decomposed as $\eta = \eta_p + \eta_c$.. Then, η_c is the greatest of the measures κ, such that $0 \le \kappa \le \eta$.*

Lemma 9. *If η is a non-negative finitely additive measure and*

$$\int_X g d\eta = 0,$$

for any continuous function on X, then η is purely finitely additive measure.

Proof. According to the Definition 1.13 of [15] we have to show that any countably additive measure κ satisfying

$$0 \leq \kappa \leq \eta \tag{6}$$

is a zero measure. Let κ satisfy (6). Then for any continuous function g, we have

$$0 \leq \kappa(g) \leq \eta(g) = 0.$$

Therefore $\kappa(g) = 0$ for all continuous functions g. Since κ is a countably additive measure, $\kappa = 0$. □

Acknowledgments: The authors are grateful to the anonymous reviewer for his very detailed comments which helped to improve the paper.

References

[1] Bahsoun, Wael; Bose, Christopher; Ruziboev, Marks, *Quenched decay of correlations for slowly mixing systems*, arXiv:1706.04158v3 [math.DS].

[2] Billingsley, Patrick, *Convergence of probability measures*, Second edition, Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[3] Boyarsky, Abraham; Góra, Paweł; *Laws of chaos. Invariant measures and dynamical systems in one dimension*, Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

[4] [1] MR2783362 Cánovas, Jose S., *Li-Yorke chaos in a class of nonautonomous discrete systems*, J. Difference Equ. Appl. **17** (2011), no. 4, 479–486.

[5] [5] MR3111777 Cánovas, Jose S., *On entropy of non-autonomous discrete systems*, Progress and challenges in dynamical systems, 143–159, Springer Proc. Math. Stat., **54**, Springer, Heidelberg, 2013.

[6] Folland, Gerald B., *Real analysis. Modern techniques and their applications*, Second edition, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[7] MR1503326 Kryloff, Nicolas; Bogoliouboff, Nicolas, *La théorie générale de la mesure dans son application l'étude des systèmes dynamiques de la mécanique non linéaire*, (French) Ann. of Math. (2) **38** (1937), no. 1, 65–113.

[8] Lasota, Andrzej, Mackey, Michael C., *Chaos, fractals, and noise. Stochastic aspects of dynamics*, Second edition, Applied Mathematical Sciences, 97, Springer-Verlag, New York, 1994.

[9] Lasota, A., Yorke, James A., *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488 (1974).

[10] [4] MR3430474 Li, Zhiming, *Remarks on topological entropy of nonautonomous dynamical systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **25** (2015), no. 12, 1550158, 5 pp.

[11] Natanson, I. P., *Theory of functions of a real variable*, Translated by Leo F. Boron with the collaboration of Edwin Hewitt, Frederick Ungar Publishing Co., New York, 1955.

[12] MR2511626 Ott, William; Stenlund, Mikko; Young, Lai-Sang(1-NY-X), *Memory loss for time-dependent dynamical systems*, Math. Res. Lett. **16** (2009), no. 3, 463–475.

[13] [3] MR3485444 Štefánková, Marta, *Inheriting of chaos in uniformly convergent nonautonomous dynamical systems on the interval*, Discrete Contin. Dyn. Syst. **36** (2016), no. 6, 3435–3443. arXiv. 1311.4083v1. Nov. 16 2013.

[14] E. Straube, *On the existence of invariant absolutely continuous measures*, Commun. Math. Phys. **81**, (1982), 27–30.

[15] MR0045194 Yosida, Kôsaku, Hewitt, Edwin, *Finitely additive measures*, Trans. Amer. Math. Soc. **72**, (1952), 46–66.

[16] [6] MR3607578 Young, Lai-Sang, *Generalizations of SRB measures to nonautonomous, random, and infinite dimensional systems*, J. Stat. Phys. **166** (2017), no. 3–4, 494–515.

[17] [2] Zhu, Hao; Shi, Yuming; Shao, Hua, *Devaney chaos in nonautonomous discrete systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **26** (2016), no. 11, 1650190, 10 pp.

(P. Góra) Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada
E-mail address, P. Góra: pawel.gora@concordia.ca

(A. Boyarsky) Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada
E-mail address, A. Boyarsky: abraham.boyarsky@concordia.ca

(Ch. Keefe) Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada
E-mail address, Ch. Keefe: chriskeefe3.14159@gmail.com