Matching, entropy, holes and expansions

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CHAPTER 4

N-expansions

This chapter is based on joined work with Cor Kraaikamp and has appeared as a paper in Journal of Mathematical Analysis and Applications [64], except for Section 4.3.3 "*why is it so difficult*" where we explain why the methods of Chapter 3 fail to work.

Abstract

As in Chapter 2, we will look at expansions with flips (in the first part) and expansions with digits from a finite alphabet. In this chapter it is combined with N-expansions. By using the natural extension, the density of the invariant measure is obtained in a number of examples. In case this method does not work, a Gauss-Kuzmin-Lévy based approximation method is used. Convergence of this method follows from [99] but the speed of convergence remains unknown. For a lot of known densities the method gives a very good approximation in a low number of iterations. In the second part of this chapter, a subfamily of the N-expansions without flips is studied. In particular, the entropy as a function of a parameter α is estimated for $N = 2$ and $N = 36$. This is done in a similar flavour as Chapter 3. For $N = 2$ we find a matching interval with matching index 0. We show that the entropy is constant on this interval by using the natural extension. We also show that the methods from Chapter 3 to prove that matching is prevalent fail to adapt to this case. This is followed by a numerical exploration. Several conjectures are stated.

§4.1 Introduction

In general, studies on continued fraction expansions focus on expansions for which almost all x have an expansion with digits from an infinite alphabet. A classical example is the regular continued fraction, see [29, 50, 98] and Chapter 1. An example of continued fraction expansions with only finitely many digits has been introduced in [71] by Joe Lehner, where the only possible digits are 1 and 2; see also [28] and, of course, the expansions from Chapter 2. More recently, continued fractions have been investigated for which all x in a certain interval have finitely many possible digits. In [33] the following 4-expansion has been (briefly) studied. Let $T : [1, 2] \to [1, 2]$ be defined as

$$T(x) = \begin{cases} \dfrac{4}{x} - 1 & \text{for} \quad x \in (\frac{4}{3}, 2], \\ \dfrac{4}{x} - 2 & \text{for} \quad x \in [1, \frac{4}{3}] \end{cases}, \tag{4.1.1}$$

see also Figure 4.1.

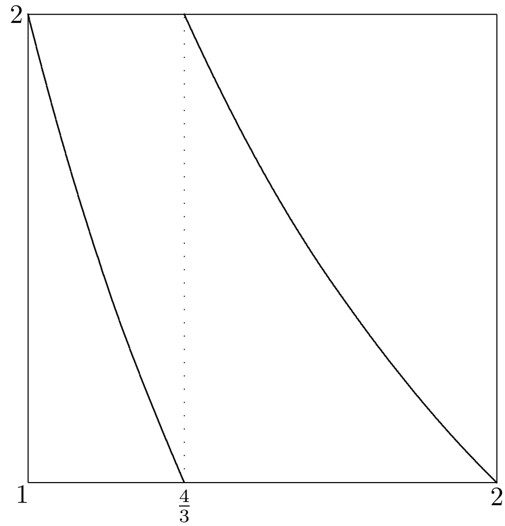

Figure 4.1: The CF-map T from (4.1.1).

By repeatedly using this map we find that every $x \in [1, 2]$ has an infinite continued fraction expansion of the form

$$x = \cfrac{4}{d_1 + \cfrac{4}{d_2 + \ddots}}$$

with $d_n \in \{1, 2\}$ for all $n \geq 1$. The class of continued fractions algorithms that give rise to digits from a finite alphabet is very large. In this chapter we will give examples of such expansions and in Section 4.3 we will take a closer look at an interesting sub-family. Most of the examples will be a particular case of N-expansions (see [3, 15, 33]). Other examples are closely related and can be found by combining the N-expansions with flipped expansions (cf. [68] for 2-expansions; see also [27] for flipped expansions). For all these examples we refer to [27] for ergodicity (which can be obtained in all these cases in a similar way) and existence of an invariant measure. In a number of cases however, it is difficult to find the invariant measure explicitly, while in seemingly closely related cases it is very easy. In case we cannot give an analytic expression for the invariant measure, we will give an approximation using a method that is very suitable (from a computational point of view) for expansions with finitely many different digits. This method is based on the Gauss-Kuzmin-Lévy Theorem. For greedy N-expansions this theorem is proved by Dan Lascu in [69]. The method yields smoother results than by simulating in the classical way (looking at the histogram of the orbit of a typical point as described in Choe's book [22], and used in his papers [21, 23]). We also give an example in which we do know the density and where we use this method to show its strength.

In Section 4.2 we will give the general form of the continued fraction maps we study in this chapter. After that we give several examples of such maps and a way of finding the density of the invariant measure by using the natural extension. In Section 4.2.2 we will see how we simulated the densities in case we were not able to find them explicitly. In the second part of the chapter we will consider a subfamily of the N-expansions which can be parameterized by $\alpha \in (0, \sqrt{N} - 1]$. We study the entropy as function of α. We give some partial results and also show why we cannot adapt the methods from Chapter 3. We proceed by analysing the problem on a numerical basis.

§4.2 The general form of our maps

Within this chapter we will look at continued fraction algorithms of the following form. Fix an integer $N \geq 2$ and let $[a, b]$ be a subinterval of $[0, N]$ with $b - a \geq 1$. Let $T : [a, b] \to [a, b]$ be defined as

$$T(x) = \frac{\varepsilon(x)N}{x} - \varepsilon(x)d(x)$$

where $\varepsilon(x)$ is either -1 or 1 depending on x and $d(x)$ is a positive integer such that $T(x) \in [a, b]$. Note that if $b - a = 1$ then there is exactly one positive integer such that $T(x) \in [a, b)$ if $\varepsilon(x)$ is fixed. For $N = 2$ we find the family that is studied in [68] and for $\varepsilon(x) = 1$ for all x we find the N-expansions from [33]. Whenever $a > 0$ this map can only have finitely many different digits. This family is closely related to the (a, b)-continued fractions introduced and studied by Svetlana Katok and Ilie Ugarcovici in [54, 55, 56]. For (a, b)-continued fractions we have that $\varepsilon(x) = -1$ for all $x \in [a, b]$ and $N = 1$. Also there are restrictions on a, b. These are chosen such that $a \leq 0 \leq b$, $b - a \geq 1$ and $-ab \leq 1$.

Note that this family is rather "large". For the examples in the next section $\varepsilon(x)$ will be plus or minus one on fixed interval(s). In Section 4.3 other restrictions are imposed.

§4.2.1 Two seemingly closely related examples and their natural extension

In [33], using the natural extension the invariant measure of the 4-expansion map T given in (4.1.1) was easily obtained. To (briefly) illustrate the method and the kind of continued fraction algorithms we are interested in we consider a slight variation of this continued fraction. Let $\tilde{T}:[1,2]\to[1,2]$ be defined as

$$\tilde{T}(x)=\begin{cases}\dfrac{4}{x}-1 & \text{for} \quad x\in(\frac{4}{3},2],\\ 5-\dfrac{4}{x} & \text{for} \quad x\in[1,\frac{4}{3}],\end{cases} \tag{4.2.1}$$

i.e. we "flipped" the map T on the interval $[1,\frac{4}{3}]$; see also Figure 4.2.

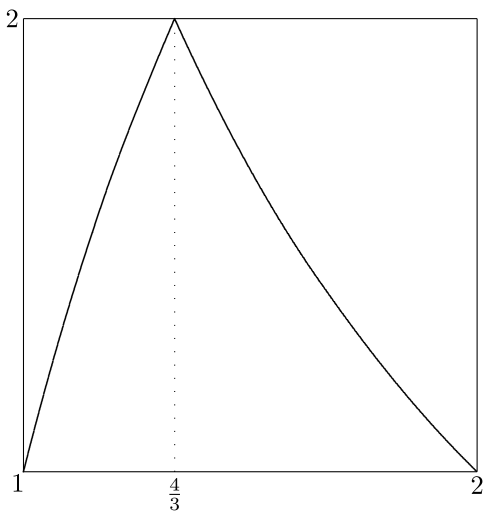

Figure 4.2: The CF map $\tilde{T}$ from (4.2.1).

Setting

$$\varepsilon_1(x)=\begin{cases}1 & \text{for } x\in(\frac{4}{3},2]\\ -1 & \text{for } x\in[1,\frac{4}{3}]\end{cases} \quad \text{and} \quad d_1(x)=\begin{cases}1 & \text{for } x\in(\frac{4}{3},2]\\ 5 & \text{for } x\in[1,\frac{4}{3}],\end{cases}$$

we define $\varepsilon_n(x)=\varepsilon_1\left(\tilde{T}^{n-1}(x)\right)$ and $d_n(x)=d_1\left(\tilde{T}^{n-1}(x)\right)$.
From $\tilde{T}(x)=\varepsilon_1\cdot\left(\frac{4}{x}-d_1\right)$, it follows that

$$x=\frac{4}{d_1+\varepsilon_1\tilde{T}(x)}=\ldots=\cfrac{4}{d_1+\cfrac{4\,\varepsilon_1}{d_2+\ddots+\cfrac{4\,\varepsilon_{n-1}}{d_n+\varepsilon_n\tilde{T}^n(x)}}}\,.$$

Taking finite truncations, we find the so called convergents

$$c_n = \frac{p_n}{q_n} = \cfrac{4}{d_1 + \cfrac{4\,\varepsilon_1}{d_2 + \ddots + \cfrac{4\,\varepsilon_{n-1}}{d_n}}}$$

of x. One can show that $\lim_{n\to\infty} c_n = x$; see [68] for further details. Therefore we write

$$x = \cfrac{4}{d_1 + \cfrac{4\,\varepsilon_1}{d_2 + \ddots}}\,, \tag{4.2.2}$$

or in short hand notation $x = [4/d_1, 4\varepsilon_1/d_2, \ldots]$ or $x = [d_1, \varepsilon_1/d_2, \ldots]_4$.

Using the natural extension to find the invariant measure

As in Chapter 2, we will use the method of natural extensions to find the density for some of our dynamical systems. We briefly recall this procedure using $\tilde{T}$ from (4.2.1). The idea is to build a two-dimensional system (the natural extension) $(\Omega = [1,2] \times [A,B], \mathcal{T})$ which is almost surely invertible and contains $([1,2], \tilde{T})$ as a factor (see Definition 1.1.11 on page 8). In [33] it was shown that a suitable candidate for the natural extension map $\mathcal{T}$ is given by

$$\mathcal{T}(x,y) = \left(\tilde{T}(x), \frac{4\varepsilon_1(x)}{d_1(x) + y}\right).$$

Now we choose A and B in such a way that the system is indeed (almost surely)

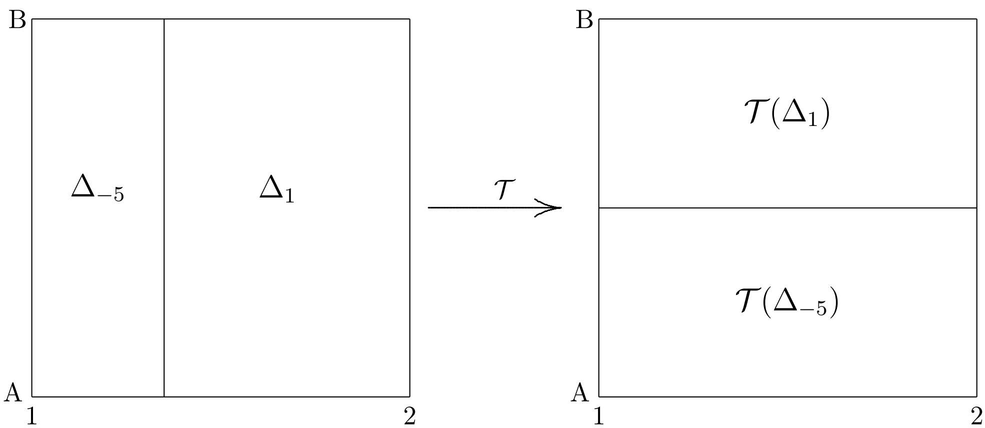

Figure 4.3: The suitable domain for $\mathcal{T}$.

invertible. We define fundamental intervals $\Delta_n = \{(x,y) \in \Omega : d_1(x) = n\}$ if $\varepsilon = 1$ and $\Delta_{-n} = \{(x,y) \in \Omega : d_1(x) = n\}$ if $\varepsilon = -1$. When the fundamental intervals fit exactly under the action of $\mathcal{T}$, the system is almost surely invertible; see Figure 4.3.

An easy calculation shows that $A = -1$ and $B = \infty$ is the right choice here. It is shown in [33], that the density of the invariant measure (for the 2-dimensional system) is given by

$$f(x, y) = C\frac{4}{(4 + xy)^2}, \quad \text{for } (x, y) \in \Omega,$$

where C is a normalising constant (which is $\frac{1}{\log(3)}$ in this example). Projecting on the first coordinate yields the invariant measure for the 1-dimensional system $([1, 2], \tilde{T})$, with density

$$\frac{1}{\log(3)}\left(\frac{1}{x} + \frac{1}{4 - x}\right), \quad \text{for } x \in [1, 2].$$

Note that if we would consider the map

$$\hat{T}(x) = \begin{cases} 4 - \dfrac{4}{x} & \text{for } x \in (\frac{4}{3}, 2], \\ \dfrac{4}{x} - 2 & \text{for } x \in [1, \frac{4}{3}], \end{cases} \tag{4.2.3}$$

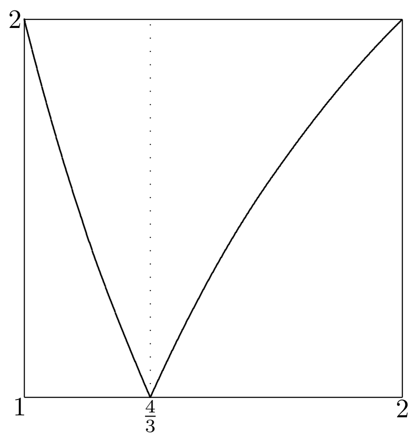

Figure 4.4: The CF map $\hat{T}$ from (4.2.3).

which is a "flipped version" of the map T from (4.1.1) where we flipped the branch on the interval $(\frac{4}{3}, 2]$, we get another continued fraction of the form (4.2.2) but now with digits $d_n \in \{2, 4\}$; see Figure 4.4. Our approach now gives $A = -2$ and $B = \infty$ which shows that the underlying dynamical system has a σ-finite infinite measure with "density" $f(x)$, given by

$$f(x) = \frac{1}{x} + \frac{1}{2 - x}, \quad \text{for } x \in [1, 2].$$

The method from [33] we just used does not always "work". As an example we will use an expansion given in [68]. Let $\bar{T}(x)$ be defined as

$$\bar{T}(x) = \begin{cases} \dfrac{2}{x} - 3 & \text{for} \quad x \in (\frac{1}{2}, \frac{4}{7}], \\ 4 - \dfrac{2}{x} & \text{for} \quad x \in (\frac{4}{7}, \frac{2}{3}], \\ \dfrac{2}{x} - 2 & \text{for} \quad x \in (\frac{2}{3}, \frac{4}{5}], \\ 3 - \dfrac{2}{x} & \text{for} \quad x \in [\frac{4}{5}, 1], \end{cases} \tag{4.2.4}$$

see Figure 4.5.

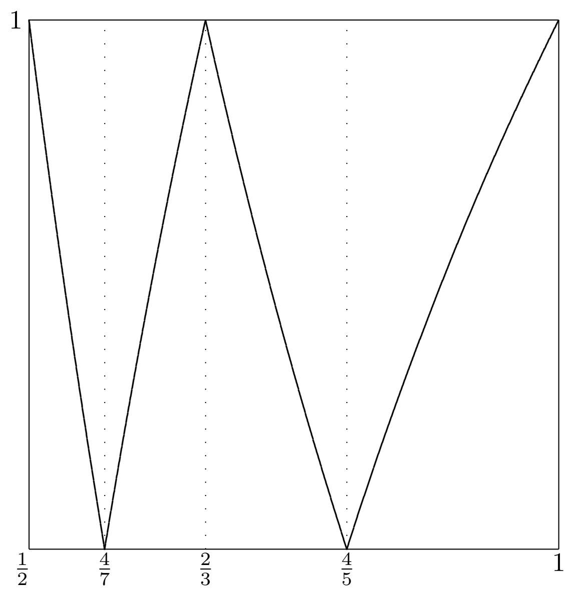

Figure 4.5: An expansion map on $[\frac{1}{2}, 1]$.

When trying to construct the domain of the natural extension one quickly notices that "holes" appear. This is not an entirely new phenomenon in continued fractions, it also appears in constructing the natural extension of Nakada's α-expansions when $\alpha \in (0, \sqrt{2} - 1)$; see [79]. One might hope that there are finitely many holes, but a simulation of the domain indicates otherwise; see Figure 4.6.

Although the method might still work in this case, it does not really seem to help us to find a description of the invariant density. In order to get an idea of the density, we will use two different approaches. One will be based on the Gauss-Kuzmin-Lévy Theorem. The other will be a more classical approach based on Choe's book [22].

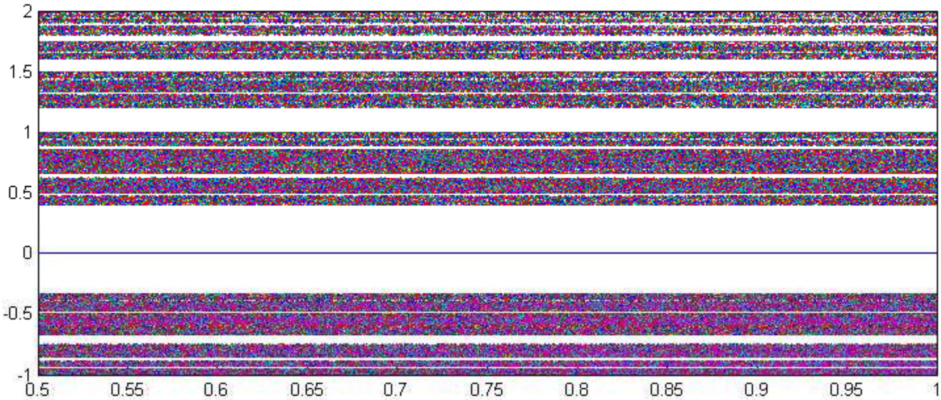

Figure 4.6: A simulation of the domain of the natural extension for the map $\bar{T}$ from (4.2.4).

§4.2.2 Two different methods for approximating the density

The first way is based on the Gauss-Kuzmin-Lévy Theorem. This theorem states that for the regular continued fraction the Lebesgue measure of the pre-images of a measurable set A will converge to the Gauss measure.

$$\lambda\left(T^{-n}(A)\right) \to \mu(A) \quad \text{as } n \to \infty.$$

There are many proofs of this theorem and refinements on the speed of convergence; see e.g. Khinchine's book [58], or [50] for such refinements.
The idea for our method is to look at the pre-images of $[\frac{1}{2}, z]$ for our map $\bar{T}$ from (4.2.4) and take the Lebesgue measure of the intervals found. Note that the number of intervals doubles every iteration. Also the size of the intervals shrink relatively fast. Fortunately it seems that a low number of iterates (around 10) is already enough to give a good approximation; see Figure 4.8 where the theoretical density and its approximation are displayed and Figure 4.7, where both methods of approximating are compared for a density we do not know the theoretical density of.

The other way of finding an approximation is by iterating points and looking at the histogram of the orbits. The way we iterated is that we used a lot of points and iterated them just a few times. To be more precise we iterated 2500 uniformly sampled points 20 times, repeated this process 400 times and took the average density of all points. Then we redid the process but instead of sampling uniformly we sampled from the previously found density (see also [32]). In Figure 4.7 we see both methods applied to our example.
The two methods give results that are relatively close but the approximation found with the Gauss-Kuzmin-Lévy method is far more smooth. Since we do not know the density we cannot compare the theoretical density with the approximation and since the Gauss-Kuzmin-Lévy method is the new method we will look at how well this method performs in an example in which we know the invariant density explicitly. For the map T from (4.1.1) we know the density which was given in [33].

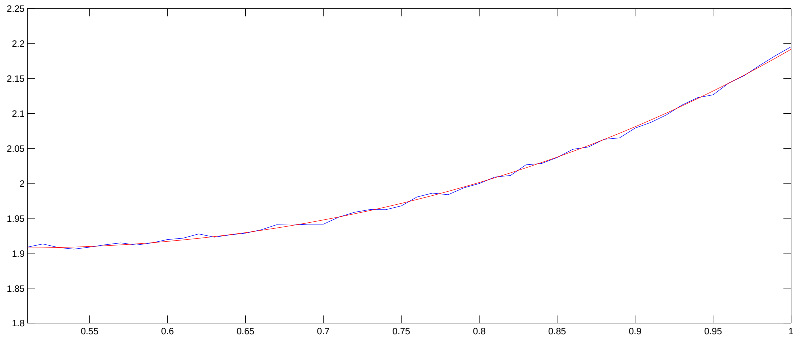

Figure 4.7: Approximations of the density of the invariant measure of $\bar{T}(x)$ using the Gauss-Kuzmin-Lévy method (red) and the classical way (blue).

In Figure 4.8 we see a plot of both the theoretical density and the approximation found by the Gauss-Kuzmin-Lévy method.

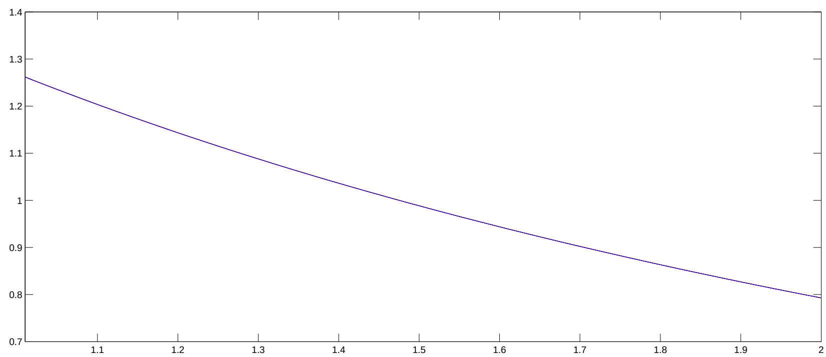

Figure 4.8: An approximation of the true density for the CF map T from (4.1.1) and the true T-invariant density.

The difference can barely be seen by the naked eye. If we look at the difference in 2-norm we get

$$\left(\int_1^2 (f(x) - \hat{f}(x))^2\, dx\right)^{\frac{1}{2}} = 1.1235 * 10^{-5}$$

where $f(x)$ is the true density and $\hat{f}(x)$ the approximation.

§4.3 A sub-family of the N-expansions

In this section we study a subfamily of the N-expansions (so $\varepsilon(x) = 1$ for all x in the domain) with digits from a finite alphabet and an interval $[\alpha, \beta]$ as domain. For our subfamily we want that it has finitely many digits. Furthermore we would like that there is a unique digit such that $T(x) \in [\alpha, \beta)$. This results in $\alpha > 0$ and $\beta - \alpha = 1$. In this way the map is uniquely determined by the domain. With these restrictions we define for $\alpha \in (0, \sqrt{N} - 1]$ the map $T_{\alpha,N} : [\alpha, \alpha + 1] \to [\alpha, \alpha + 1]$ as

$$T_{\alpha,N} = \frac{N}{x} - \left\lfloor \frac{N}{x} - \alpha \right\rfloor .$$

We call the associated continued fractions N_α-expansions. Note that for all these expansions we have a finite number of digits since $\alpha > 0$. Also note that this is the largest range in which we can choose α because for $\alpha > \sqrt{N} - 1$ the digit would be 0 or less (see Section 4.3.2 for a calculation). Simulations show that a lot of maps have an attractor smaller than $[\alpha, \alpha + 1]$. When $N \geq 9$ we find that if $\alpha = \sqrt{N} - 1$ there is always an interval $[c, d] \subsetneq [\alpha, \alpha + 1]$ for which the $T_{\alpha,N}$-invariant measure of $[c, d]$ is zero. Whenever $N > 4$ we have that $T_{\alpha,N}$ always has 2 branches for $\alpha = \sqrt{N} - 1$. Calculations of these observations are given in Section 4.3.2 in which we take a closer look at which sequences are admissible for a given N and α. In Section 4.3.3 we study the behaviour of the entropy as a function of α for a fixed N.

The examples in [33] with "fixed range" are all member of this kind of sub-family of the N-expansions. Though, these examples are cases for which all the branches of the mapping are full. In such case the natural extension can be easily build using the method described previously. If not all branches are full we can still make the natural extension in some cases. We will start this section with such a case.

§4.3.1 A 2-expansion with $\alpha = \sqrt{2} - 1$

Let $T(x) : [\sqrt{2} - 1, \sqrt{2}] \to [\sqrt{2} - 1, \sqrt{2}]$ be defined by

$$T(x) = \begin{cases} \dfrac{2}{x} - 1 & \text{for} \quad 2(\sqrt{2} - 1) < x \leq \sqrt{2}, \\ \dfrac{2}{x} - 2 & \text{for} \quad 2 - \sqrt{2} < x \leq 2(\sqrt{2} - 1), \\ \dfrac{2}{x} - 3 & \text{for} \quad \frac{1}{7}(6 - 2\sqrt{2}) < x \leq 2 - \sqrt{2}, \\ \dfrac{2}{x} - 4 & \text{for} \quad \sqrt{2} - 1 \leq x \leq \frac{1}{7}(6 - 2\sqrt{2}). \end{cases}$$

A graph of this map is shown in Figure 4.9. We can find the invariant measure for this map by using the method as in Section 4.2.1 though we now need to determine 3 "heights" in order to make the mapping of the natural extension almost surely bijective on the domain (see Figure 4.10). We get the following equations for the heights A, B and C:

$$A = \frac{2}{4 + C}\ , \quad B = \frac{2}{3 + C} \quad \text{and } C = \frac{2}{1 + B}.$$

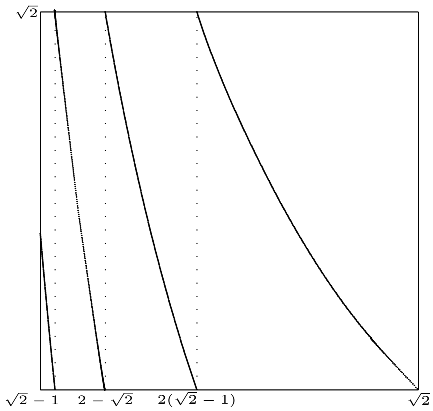

Figure 4.9: A 2-expansion on the interval $[\sqrt{2}-1, \sqrt{2}]$.

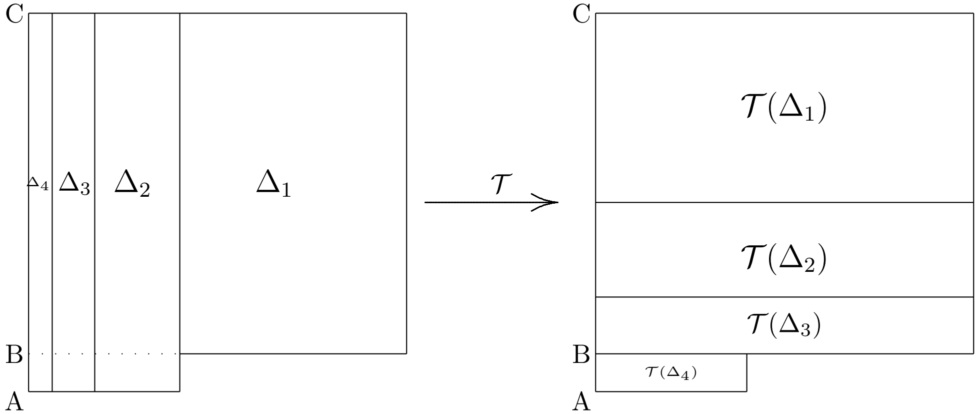

Figure 4.10: Ω *and* $\mathcal{T}(\Omega)$.

This results in $A = \frac{1}{2}(\sqrt{33}-5)$, $B = \frac{1}{6}(\sqrt{33}-3)$ and $C = \frac{1}{2}(\sqrt{33}-3)$. We find the following invariant density up to a normalising constant (which is given in Theorem 4.3.3)

$$f(x) = \begin{cases} \frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-5}{4+(\sqrt{33}-5)x} & \text{for } \sqrt{2}-1 < x \leq 2(\sqrt{2}-1), \\ \frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-3}{12+(\sqrt{33}-3)x} & \text{for } 2(\sqrt{2}-1) < x \leq \sqrt{2}. \end{cases}$$

The graph of the density is given in Figure 4.11. In this case we were lucky. But in general it seems to be very hard to construct the natural extension explicitly. Still we can simulate the densities and calculate the entropy for a given α. Also for the 2-expansions, we can extend the above result to all $\alpha \in [\frac{\sqrt{33}-5}{2}, \sqrt{2}-1]$; see Theorem 4.3.2.

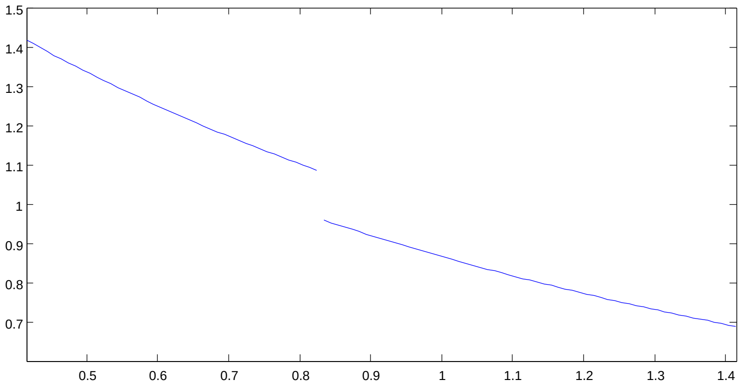

Figure 4.11: The density of the invariant measure for the 2-expansion on $[\sqrt{2}-1, \sqrt{2}]$.

§4.3.2 Admissibility

In this section we look at how the alphabet is determined by α for a fixed N. It turns out that not all different sequences of such an alphabet will occur in a continued fraction expansion (or only finitely many times). This is a consequence of some cylinders having zero mass. The range of the first digits of a continued fraction for given α and N are easily described since the smallest digit will be attained by the right end point of the domain and the largest digits will be attained by the left end point of the domain. Let

$$n_{min} = \left\lfloor \frac{N}{\alpha+1} - \alpha \right\rfloor \quad \text{and} \quad n_{max} = \left\lfloor \frac{N}{\alpha} - \alpha \right\rfloor .$$

Note that $n_{min} \leq 0$ when $\alpha > \sqrt{N}-1$ and therefore $\alpha = \sqrt{N}-1$ is the largest value for which we have positive digits. Furthermore the alphabet is given by $\{n_{min}, \dots, n_{max}\}$. Now to see for which N we have that for $\alpha = \sqrt{N}-1$ there are two branches we must check that $n_{max} = 2$. This happens when $\frac{N}{\sqrt{N}-1} - 2 \in [\sqrt{N}-1, \sqrt{N}]$. We have 2 inequalities

$$\frac{N}{\sqrt{N}-1} - 2 \leq \sqrt{N} \tag{4.3.1}$$

and

$$\sqrt{N} - 1 \leq \frac{N}{\sqrt{N}-1} - 2. \tag{4.3.2}$$

Inequality (4.3.1) gives $4 \leq N$ and inequality (4.3.2) gives $N-1 \leq N$. We find that for all $N \geq 4$ we have two branches for $\alpha = \sqrt{N}-1$. If $N \geq 9$ we also have an attractor which is strictly smaller than the entire interval for $\alpha = \sqrt{N}-1$ as the following calculation shows

$$\begin{aligned} T_{\alpha,N}\left([\alpha, \frac{N}{\alpha+2}]\right) &= \left[\alpha, \frac{N}{\alpha}-2\right], \\ T_{\alpha,N}\left([\alpha, \frac{N}{\alpha}-2]\right) &= \left[\alpha, \frac{N}{\alpha}-2\right] \cup \left[\frac{N\alpha}{N-2\alpha}-1, \alpha+1\right], \\ T_{\alpha,N}\left([\frac{N\alpha}{N-2\alpha}-1, \alpha+1]\right) &= \left[\alpha, \frac{N^2-(1-3\alpha)N-2\alpha}{(\alpha-1)N+2\alpha}\right]. \end{aligned}$$

If we substitute α with $\sqrt{N}-1$ and the following two inequalities hold we find an attractor strictly smaller than the interval $[\alpha, \alpha+1]$;

$$\frac{N}{\sqrt{N}-1} - 2 < \frac{N(\sqrt{N}-1)}{N-2(\sqrt{N}-1)} - 1,$$

$$\frac{N^2-(4-3\sqrt{N})N-2(\sqrt{N}-1)}{(\sqrt{N}-2)N+2(\sqrt{N}-1)} < \frac{N}{\sqrt{N-1}} - 2,$$

yielding that $N \geq 9$. We take a closer look at $N = 9$ and $\alpha = \sqrt{9}-1 = 2$. This example is briefly discussed in [33] where it is stated that computer experiments suggest that the orbit of 2 never becomes periodic and therefore it is hard to find the natural extension explicitly. However, when simulating the natural extension, it seems that there are finitely many discontinuities; see Figure 4.12.

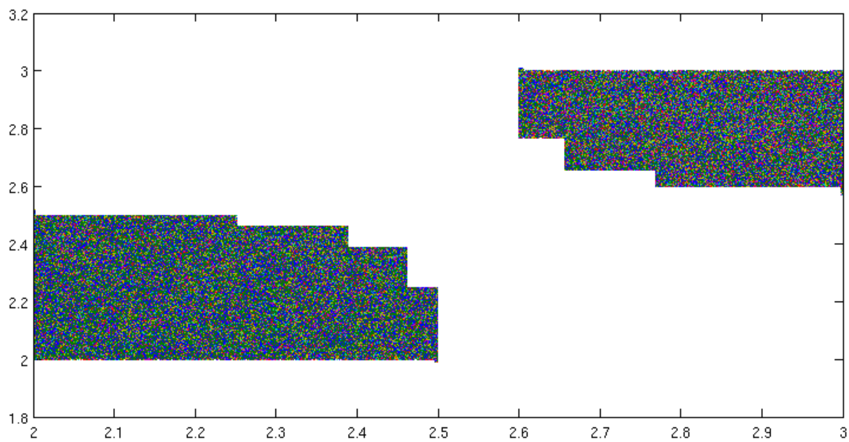

Figure 4.12: A simulation of the natural extension for $N = 9$ and $\alpha = 2$.

We can also simulate the density of the invariant measure; see Figure 4.13.
Remark that cylinders with zero mass tells us which sequences are not apparent in any continued fraction of numbers outside the attractor and for those numbers not in the attractor these sequences only appear in the start of the continued fraction. We

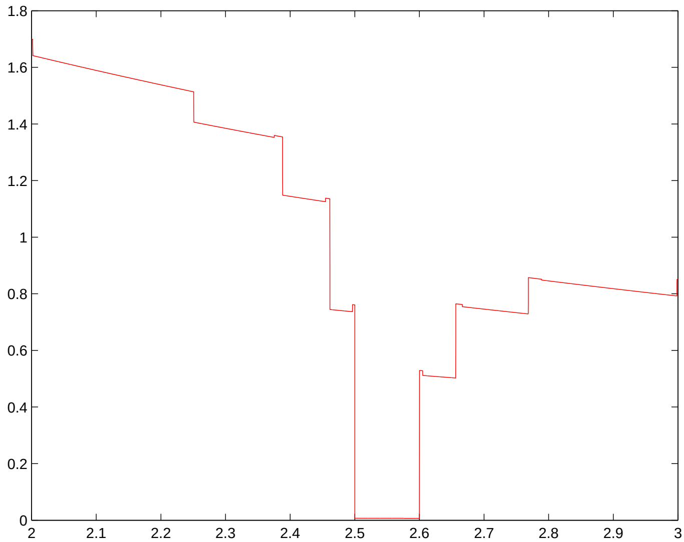

Figure 4.13: A Simulation of the density for $N = 9$ and $\alpha = 2$ using the Gauss-Kuzmin-Lévy method.

can describe which cylinders these are. The hole is given by $[2.5, 2.6]$.
Now $2.5 = [1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, \ldots]_9$
and $2.6 = [1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, \ldots]_9$. The boundary of a cylinder $\Delta(a_1, \ldots, a_n)$ is given by $[a_1, \ldots, a_n, 1, r]_9$ and $[a_1, \ldots, a_n, 2, r]_9$ where r is the expansion of 2. Now a cylinder is contained in $[2.5, 2.6]$ if $2.5 < [a_1, \ldots, a_n, 1, r]_9 < 2.6$ and $2.5 < [a_1, \ldots, a_n, 2, r]_9 < 2.6$. Note that here we can have a clear description of the attractor and therefore for the admissible sequences. Simulation shows us that there are a lot of different settings in which you find an attractor strictly smaller than the interval. In Figure 4.14 simulations for several values of N are shown. On the y-axis α is given and on the x-axis the attractor is plotted. For example, for $N = 9$ we see that for $\alpha = 1$ there is no attractor strictly smaller than the interval. There is an attractor for $\alpha = 1.8$ and also for example for $\alpha = 2$. The pattern seems to be rather regular. Moreover, more “holes” seem to appear for large N and large α[1].

§4.3.3 Entropy and matching

In this section we look at entropy as a function of $\alpha \in (0, \sqrt{N} - 1]$ for a fixed N and the relation with matching. We will use the following definition.

Definition 4.3.1 (Matching). *We say matching holds for α if there are $K, M \in \mathbb{N}$ such that $T^K_{\alpha,N}(\alpha + 1) = T^M_{\alpha,N}(\alpha)$. The numbers K and M are called the matching*

[1]Recently Jaap de Jonge, Cor Kraaikamp and Hitoshi Nakada studied these holes. The overall structure seems rather complicated. See the dissertation of Jaap de Jonge for more details [34].

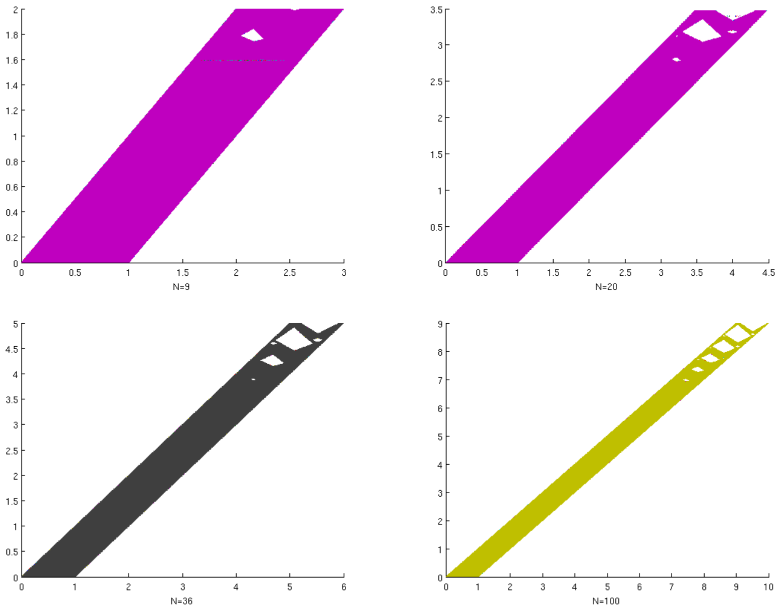

Figure 4.14: Attractors plotted for several values of N.

exponents, $K - M$ is called the matching index and an interval (c, d) such that for all $\alpha \in (c, d)$ the matching exponents are the same is called a matching interval.

For our family we do not know whether matching holds almost everywhere. In fact, it is not even clear whether for all $N \in \mathbb{N}$ we can find a matching interval. Also, whether matching implies monotonicity is not clear. Certain conditions used to prove it as in Chapter 3 are not met.

Why is it so difficult?

Note that for the maps studied in Chapter 3 all rationals have a finite expansion for any choice of α. Therefore, all $\alpha \in (0, 1) \cap \mathbb{Q}$ match in 0 or before. For any rational in the domain a matching interval is found. Since for all $\alpha \in (0, \sqrt{N} - 1)$ the expansion of any number in the interval is infinite, there are no values for which we find matching "trivially". Another obstruction is the fact that a necessary condition for a matching interval is that the derivatives should match (see also Definition 1.2.8 on page 13). To fix ideas let us fix $N = 2$. Suppose we want to prove that the derivatives match for $\alpha \in (0, \sqrt{2} - 1) \cap \mathbb{Q}$. For simplicity assume that there exist a K and an L such that $T^K_{2,\alpha}(\alpha) = 1 = T^L_{2,\alpha}(\alpha + 1)$. Following the lines in the proof of Lemma 3.2.6 on page 50 we find that we need $(a - c)^2(\frac{1}{2})^{2K-1} = (e - g)^2(\frac{1}{2})^{2L-1}$ in order to find matching derivatives. One can check that this equation holds for $\alpha = \frac{1}{n}$ with $n \in \mathbb{N}$. On the other hand, there is no simple way of proving this and no reason to believe it holds in general.

If we look back at Lemma 3.2.7 on page 51 it was necessary to have $|x - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$. For N, α-expansions we do not have such a good approximation. The sharpest bound known is $|x - \frac{p_n}{q_n}| < \frac{N^n}{q_n^2}$, therefore we cannot mimic the proof. Lemma 3.2.7 is needed to prove the monotonicity on a matching interval. The only reason to believe matching can help us to prove monotonicity is the fact that it works for related families. In fact, for specific choices of N and α we can actually find matching intervals on which the function is monotonic. If we want to mimic the proof of Theorem 3.1.3 which states that matching holds almost everywhere, we find that we start in a state $(x+1)(y+1) = 2$ whenever we take $N = 2$. This implies that for the iterates we have $x + y = 1$ or $x + y = 2$. From there, at least 12 other states can be reached. Computer simulation showed that for arbitrary α we cannot find iterates of α and $\alpha + 1$ that return to these states.
We will now discuss the 2_α-expansions in more detail. Moreover, we prove that the entropy is constant for $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ when $N = 2$.

The entropy of 2_α-expansions

We start with an example for which there is no α such that we have an attractor strictly smaller than the interval. Also, simulation indicates that there seems to be a plateau in the neighborhood of $\sqrt{2}-1$. For this value we can calculate the entropy since we have the density for this specific case of α; see Section 4.3.1, also see Figure 4.15 for a plot of the entropy function. When taking a closer look at this plateau we found

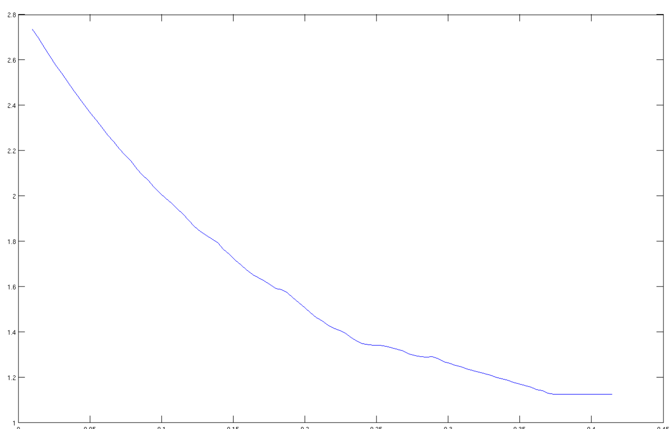

Figure 4.15: Entropy as function of α for $N = 2$.

that on the interval $[\frac{\sqrt{33}-5}{2}, \sqrt{2}-1]$ the entropy is constant. The point $\frac{\sqrt{33}-5}{2}$ is the point so that for all smaller α there are always 5 or more branches and for all larger α there are always 4 branches. If we look at a simulation of the natural extension it seems that for these values of α we can construct a natural extension. Indeed this turned out to be the case (see Theorem 4.3.3). For $\frac{\sqrt{33}-5}{2}$ we find matching exponents

$(0, 2)$ and for $\sqrt{2}-1$ we find matching exponents $(0, 1)$. Inside the interval itself we find $(3, 3)$. These values were first found by simulation, in Theorem 4.3.2 we give a proof of this.

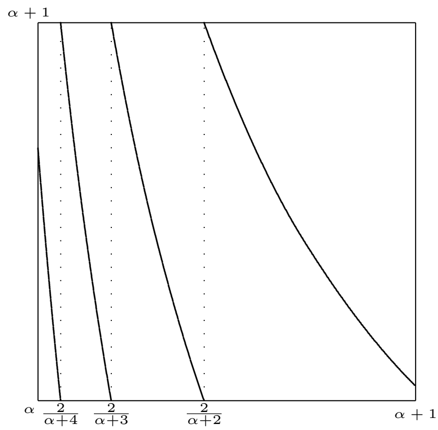

Figure 4.16: The map $T_{0.395,2}$.

Theorem 4.3.2. *Let $N = 2$ and let $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$. Furthermore, denote $T_{\alpha,2}$ by T. Then $T^3(\alpha) = T^3(\alpha + 1)$.*

Proof. Note that the interval $(\alpha, \alpha+1)$ has as natural partition $\{I_1, I_2, I_3, I_4\}$, where

$$I_1 = \left(\frac{2}{\alpha+2}, \alpha+1\right), \quad I_2 = \left(\frac{2}{\alpha+3}, \frac{2}{\alpha+2}\right], \quad I_3 = \left(\frac{2}{\alpha+4}, \frac{2}{\alpha+3}\right],$$

and

$$I_4 = \left[\alpha, \frac{2}{\alpha+4}\right],$$

where

$$T(x) = \frac{2}{x} - d, \quad \text{if } x \in I_d \text{ for } d = 1, 2, 3, 4.$$

An easy calculation shows that

$$T(\alpha) = \frac{2-4\alpha}{\alpha} \in I_1,$$

(and $T(\alpha) = \frac{2}{\alpha+2}$ when $\alpha = \sqrt{2}-1$, and $T(\alpha) = \alpha + 1$ when $\alpha = \frac{\sqrt{33}-5}{2}$), so that

$$T^2(\alpha) = \frac{3\alpha - 1}{1 - 2\alpha}.$$

Furthermore, we have that

$$T(\alpha+1) = \frac{1-\alpha}{\alpha+1} \in I_4,$$

(and $T(\alpha+1)=\sqrt{2}-1$ when $\alpha=\sqrt{2}-1$; $T(\alpha+1)=\frac{2}{\alpha+4}$ when $\alpha=\frac{\sqrt{33}-5}{2}$), so that

$$T^2(\alpha+1)=\frac{6\alpha-2}{1-\alpha}.$$

Now let

$$K_1=\left(\frac{\sqrt{33}-5}{2},\frac{\sqrt{51}-6}{3}\right],\quad K_2=\left(\frac{\sqrt{51}-6}{3},\frac{\sqrt{129}-9}{6}\right]$$

and

$$K_3=\left(\frac{\sqrt{129}-9}{6},\sqrt{2}-1\right).$$

For $\alpha\in K_1$ we have that $T^2(\alpha)\in I_3$ and so

$$T^3(\alpha)=\frac{5-13\alpha}{3\alpha-1}$$

and $T^2(\alpha+1)\in I_4$ which results in

$$T^3(\alpha+1)=\frac{5-13\alpha}{3\alpha-1}=T^3(\alpha).$$

For $\alpha\in K_2$ we have that $T^2(\alpha)\in I_2$ and so

$$T^3(\alpha)=\frac{4-10\alpha}{3\alpha-1}$$

and $T^2(\alpha+1)\in I_3$ which results in

$$T^3(\alpha+1)=\frac{4-10\alpha}{3\alpha-1}=T^3(\alpha).$$

For $\alpha\in K_3$ we have that $T^2(\alpha)\in I_1$ and so

$$T^3(\alpha)=\frac{3-7\alpha}{3\alpha-1}$$

and $T^2(\alpha+1)\in I_2$ which results in

$$T^3(\alpha+1)=\frac{3-7\alpha}{3\alpha-1}=T^3(\alpha).$$

□

Earlier we thought we were just lucky finding the natural extension in case $N=2$ and $\alpha=\sqrt{2}-1$. Note that from this natural extension we immediately also have the case $N=2$, $\alpha=\frac{\sqrt{33}-5}{2}$; just "invert" the time and exchange the two coordinates in the natural extension we found for $N=2$ and $\alpha=\sqrt{2}-1$. However, from Theorem 4.3.2 it immediately follows that we can also "build" the natural extension for every $\alpha\in\left(\frac{\sqrt{33}-5}{2},\sqrt{2}-1\right)$. Clearly, from Theorem 4.3.2 we see that we have three different cases.

Theorem 4.3.3. *For* $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ *the natural extension can be build as in Figure 4.17. Moreover the invariant density is given by*

$$
\begin{aligned}
f(x) &= H\Big(\frac{D}{2+Dx}\mathbf{1}_{(\alpha,T(\alpha+1))} + \frac{E}{2+Ex}\mathbf{1}_{(T(\alpha+1),T^2(\alpha))} + \frac{F}{2+Fx}\mathbf{1}_{(T^2(\alpha),\alpha+1)} \\
&\quad - \frac{A}{2+Ax}\mathbf{1}_{(\alpha,T^2(\alpha+1))} - \frac{B}{2+Bx}\mathbf{1}_{(T^2(\alpha+1),T(\alpha))} - \frac{C}{2+Cx}\mathbf{1}_{(T(\alpha),\alpha+1)}\Big)
\end{aligned}
$$

with $A = \frac{\sqrt{33}-5}{2}, B = \sqrt{2}-1, C = \frac{\sqrt{33}-3}{6}, D = 2\sqrt{2}-2, E = \frac{\sqrt{33}-3}{2}, F = \sqrt{2}$ *and* $H^{-1} = \log\left(\frac{1}{32}(3+2\sqrt{2})(7+\sqrt{33})(\sqrt{33}-5)^2\right) \approx 0.25$ *the normalising constant.*

Proof. We guessed the shape of the domain of natural extension by studying a simulation. For the map on this domain we used $\mathcal{T}(x,y) = \left(T(x), \frac{2}{d_1(x)+y}\right)$.

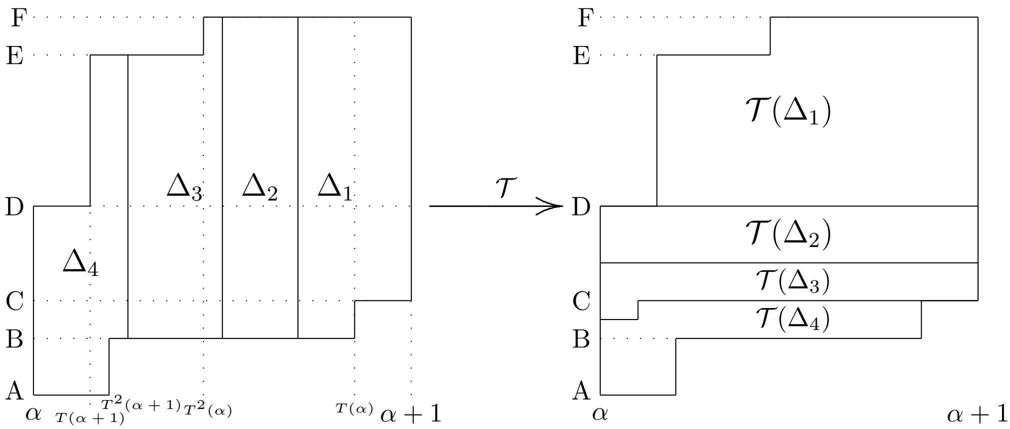

Figure 4.17: Ω *and* $\mathcal{T}(\Omega)$ *with* $\alpha \in K_1$.

For $\alpha \in K_1$, we find the following equations:

$$
\begin{array}{ll}
A = \frac{2}{4+E} & \\
B = \frac{2}{4+D} & \\
C = \frac{2}{3+E} & \\
D = \frac{2}{2+A} & \text{implying that} \\
E = \frac{2}{1+C} & \\
F = \frac{2}{1+B} &
\end{array}
\qquad
\begin{array}{l}
A = \frac{\sqrt{33}-5}{2} \\
B = \sqrt{2}-1 \\
C = \frac{\sqrt{33}-3}{6} \\
D = 2\sqrt{2}-2 \\
E = \frac{\sqrt{33}-3}{2} \\
F = \sqrt{2}.
\end{array}
$$

A similar picture emerges for $\alpha \in K_2$ and $\alpha \in K_3$. Moreover, you will find the same set of equations and thus the same heights! Note that for $\alpha < \frac{2}{5}$ we have $T^2(\alpha) < T(\alpha)$, for $\alpha = \frac{2}{5}$ we have $T^2(\alpha) = T(\alpha)$ and for $\alpha > \frac{2}{5}$ we have $T^2(\alpha) > T(\alpha)$. When you integrate over the second coordinate you find the density given in the statement of the theorem. For the normalising constant we have the following integral

$$\begin{aligned} H &= \int_{\alpha}^{\alpha+1} \frac{D}{2+Dx}\mathbf{1}_{(\alpha,T(\alpha+1))} \cdots - \frac{C}{2+Cx}\mathbf{1}_{(T(\alpha),\alpha+1)}\,dx \\ &= \log\left(\frac{2+DT(\alpha+1)}{2+D\alpha}\right) + \ldots + \log\left(\frac{2+CT(\alpha)}{2+C(\alpha+1)}\right). \end{aligned}$$

It seems that H depends on α but this is not the case as the following calculation shows

$$\begin{aligned} H &= -\log\left(\frac{2+A\frac{6\alpha-2}{1-\alpha}}{2+A\alpha}\right) - \log\left(\frac{2+B\frac{2-4\alpha}{\alpha}}{2+B\frac{6\alpha-2}{1-\alpha}}\right) - \log\left(\frac{2+C(\alpha+1)}{2+C\frac{2-4\alpha}{\alpha}}\right) \\ &\quad +\log\left(\frac{2+D\frac{1-\alpha}{\alpha+1}}{2+D\alpha}\right) + \log\left(\frac{2+E\frac{3\alpha-1}{1-2\alpha}}{2+E\frac{1-\alpha}{\alpha+1}}\right) + \log\left(\frac{2+F(\alpha+1)}{2+F\frac{3\alpha-1}{1-2\alpha}}\right) \\ &= -\log\left(\frac{2-2\alpha+A(6\alpha-2)}{2+A\alpha}\right) + \log(1-\alpha) \\ &\quad -\log\left(\frac{2\alpha+B(2-4\alpha)}{2-2\alpha+B(6\alpha-2)}\right) + \log(\alpha) - \log(1-\alpha) \\ &\quad -\log\left(\frac{2+C(\alpha+1)}{2\alpha+C(2-4\alpha)}\right) - \log(\alpha) \\ &\quad +\log\left(\frac{2\alpha+2+D(1-\alpha)}{2+D\alpha}\right) - \log(\alpha+1) \\ &\quad +\log\left(\frac{2-4\alpha+E(3\alpha-1)}{2\alpha+2+E(1-\alpha)}\right) + \log(\alpha+1) - \log(1-2\alpha) \\ &\quad +\log\left(\frac{2+F(\alpha+1)}{2-4\alpha+F(3\alpha-1)}\right) + \log(1-2\alpha) \\ &= -\log\left(\frac{2-2\alpha+\frac{\sqrt{33}-5}{2}(6\alpha-2)}{2+\frac{\sqrt{33}-5}{2}\alpha}\right) - \log\left(\frac{2\alpha+(\sqrt{2}-1)(2-4\alpha)}{2-2\alpha+(\sqrt{2}-1)(6\alpha-2)}\right) \\ &\quad -\log\left(\frac{2+\frac{\sqrt{33}-3}{6}(\alpha+1)}{2\alpha+\frac{\sqrt{33}-3}{6}(2-4\alpha)}\right) + \log\left(\frac{2\alpha+2+(2\sqrt{2}-2)(1-\alpha)}{2+(2\sqrt{2}-2)\alpha}\right) \\ &\quad +\log\left(\frac{2-4\alpha+(\frac{\sqrt{33}-3}{2})(3\alpha-1)}{2\alpha+2+(\frac{\sqrt{33}-3}{2})(1-\alpha)}\right) + \log\left(\frac{2+\sqrt{2}(\alpha+1)}{2-4\alpha+\sqrt{2}(3\alpha-1)}\right) \\ &= -\log(\frac{7-\sqrt{33}}{2}) - \log(\frac{1}{\sqrt{2}}) - \log(\frac{5+\sqrt{33}}{4}) + \log(\sqrt{2}) + \log(\frac{\sqrt{33}-5}{4}) \\ &\quad +\log(3+2\sqrt{2}). \end{aligned}$$

□

One might hope that when calculating the entropy using Rohlin's formula, terms will cancel as well. These integrals result in Li_2 functions depending on α and things are not so easy anymore. We provide a more elegant proof to show that the entropy is constant on $\left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ and calculate the entropy for $\alpha = \sqrt{2}-1$ afterwards. We will use quilting introduced in [65]. Proposition 1 in [65] can be formulated (specific to our case) in the following way:

Proposition 4.3.4 ([65], Proposition 1). *Let $(\mathcal{T}_\alpha, \Omega_\alpha, \mathcal{B}_\alpha, \mu)$ and $(\mathcal{T}_\beta, \Omega_\beta, \mathcal{B}_\beta, \mu)$ be two dynamical systems as in our setting. Furthermore let $D_1 = \Omega_\alpha \backslash \Omega_\beta$ and $A_1 = \Omega_\beta \backslash \Omega_\alpha$. If there is a $k \in \mathbb{N}$ such that $\mathcal{T}_\alpha^k(D_1) = \mathcal{T}_\beta^k(A_1)$ then the dynamical systems are isomorphic.*

Since isomorphic systems have the same entropy it will give us the following corollary.

Corollary 4.3.5. *For $N = 2$ the entropy function is constant on $\left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ and the value is approximately* 1.14.

Proof. We show that for $k = 3$ we satisfy the condition in Proposition 4.3.4. Define $D_i = \mathcal{T}_\alpha^{i-1}(D_1)$ and $A_i = \mathcal{T}_\beta^{i-1}(A_1)$ for $i = 1, 2, 3, 4$. We find the following regions (see Figure 4.18):

$$\begin{aligned}
D_1 &= [\alpha, \beta] \times [A, D], \\
D_2 &= [T_{\alpha,2}(\beta), T_{\alpha,2}(\alpha)] \times [B, C], \\
D_3 &= [T_{\alpha,2}^2(\beta), T_{\alpha,2}^2(\alpha)] \times [E, F], \\
D_4 &= [T_{\alpha,2}^3(\beta), T_{\alpha,2}^3(\alpha)] \times \left[\frac{2}{3+F}, \frac{2}{3+E}\right], \\
A_1 &= [\alpha+1, \beta+1] \times [C, F], \\
A_2 &= [T_{\beta,2}(\beta+1), T_{\beta,2}(\alpha+1)] \times [D, E], \\
A_3 &= [T_{\beta,2}^2(\beta+1), T_{\beta,2}^2(\alpha+1)] \times [A, B], \\
A_4 &= [T_{\beta,2}^3(\beta+1), T_{\beta,2}^3(\alpha+1)] \times \left[\frac{2}{4+B}, \frac{2}{4+A}\right].
\end{aligned}$$

Note that since we have matching $[T_{\beta,2}^3(\beta), T_{\alpha,2}^3(\alpha)] = [T_{\beta,2}^3(\beta+1), T_{\alpha,2}^3(\alpha+1)]$. Now

$$\frac{2}{3+F} = \frac{2}{3+\sqrt{2}} = \frac{2}{4+B} = \frac{2}{4+\sqrt{2}-1},$$

$$\frac{2}{3+E} = \frac{2}{3+\frac{\sqrt{33}-3}{2}} = \frac{2}{4+A} = \frac{2}{4+\frac{\sqrt{33}-5}{2}}$$

and so we find $D_4 = A_4$.

CHAPTER 4

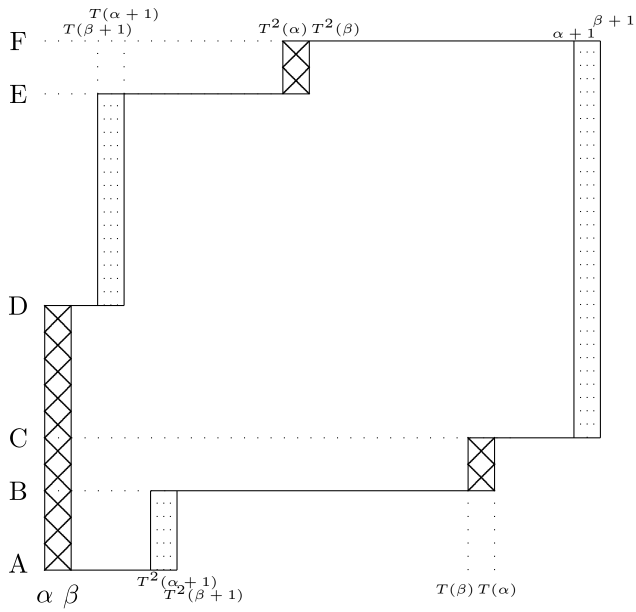

Figure 4.18: Illustration of the quilting.

For the value of the entropy we use Rohlin's formula for $\alpha = \sqrt{2}-1$ (see [29, 98]);

$$
\begin{aligned}
h(T_{\sqrt{2}-1,2}) &= \int_{\sqrt{2}-1}^{\sqrt{2}} \log |T'_{\sqrt{2}-1,2}(x)| f(x)\,dx \\
&= H\int_{\sqrt{2}-1}^{\sqrt{2}} (\log(2) - 2\log(x))\, f(x)\,dx \\
&= \log(2) - 2H\int_{\sqrt{2}-1}^{\sqrt{2}} \log(x) f(x)\,dx \\
&= \log(2) - 2H\int_{\sqrt{2}-1}^{\sqrt{2}} \log(x)\Bigg((\frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-5}{4+(\sqrt{33}-5)x})\mathbf{1}_{\sqrt{2}-1,2(\sqrt{2}-1)} \\
&\quad +(\frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-3}{12+(\sqrt{33}-3)x})\mathbf{1}_{2(\sqrt{2}-1),\sqrt{2}}\Bigg)\,dx \\
&= \log(2) - 2H\Bigg((Li_2(-\frac{x(\sqrt{33}-3)}{4}) + \log(\frac{x(\sqrt{33}-3)}{4}+1) \\
&\quad -Li_2(-\frac{x(\sqrt{33}-5)}{4}) + \log(\frac{x(\sqrt{33}-5)}{4}+1))|_{\sqrt{2}-1}^{2(\sqrt{2}-1)} \\
&\quad +(Li_2(-\frac{x(\sqrt{33}-3)}{4}) + \log(\frac{x(\sqrt{33}-3)}{4}+1) \\
&\quad -Li_2(-\frac{x(\sqrt{33}-5)}{12}) + \log(\frac{x(\sqrt{33}-5)}{12}+1))|_{2(\sqrt{2}-1)}^{\sqrt{2}}\Bigg) \\
&\approx 1.14.
\end{aligned}
$$

□

By looking at the graph displayed in Figure 4.15 we cannot find other matching exponents easily. To check for other matching exponents we can do the following. Suppose we are interested in finding a matching interval with exponents (n_1, n_2). We select a large number of random points (say 10 000) from $(0, \sqrt{N}-1)$. Then we look at $T^{n_1}_{\alpha,2}(\alpha) - T^{n_2}_{\alpha,2}(\alpha+1)$ for these random points and we check whether it is very close to 0. Note that if an interval was found this way with matching exponents (n_1, n_2) then we also find the same interval for (n_1+1, n_2+1). Table 4.1 shows which matching exponents we found. This is very different from Nakada's α-continued fractions where

M\K	1	2	3	4	5	6	7	8	9	10
1	0	0	1	0	1	0	0	0	0	0
2	0	0	0	1	0	1	0	0	0	0
3	0	0	1	0	1	0	1	0	1	0
4	0	0	0	1	0	1	0	1	0	1
5	0	0	0	0	1	0	1	0	1	0
6	0	0	0	0	0	1	0	1	0	1
7	0	0	0	0	0	0	1	0	1	0
8	0	0	0	0	0	0	0	1	0	1
9	0	0	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	1	0	1

Table 4.1: observed matching exponents for $N = 2$: 1 if seen, 0 if not.

you can find all possible matching exponents. The fact that we did not observe them does not mean they are not there. Maybe they are too small to observe using this method.

The entropy of 36_α-expansions

For $N \geq 9$ we expect different behaviour because we know that for some α there is at least one subinterval on which the invariant measure is zero. If we pick $N = 36$ we have a map with only full branches for $\alpha = 1, 2, 3$. Figure 4.19 shows the entropy as function of α. The stars indicate those values which we could calculate theoretically.

Clearly, we can observe plateaus however, if we look at the matching exponents we can observe that for all $M, K \leq 10$ the only matching exponents we find are (n, n) with $n \in \{3, 4, \ldots, 10\}$.

§4.4 Conclusion

We have seen that the general form of the examples given yields a rather large family. In some examples we were able to construct the natural extension and therefore to find the invariant measure. In other examples this was not the case. There does not seem to be an easy rule which tells us when the method works and when it does not.

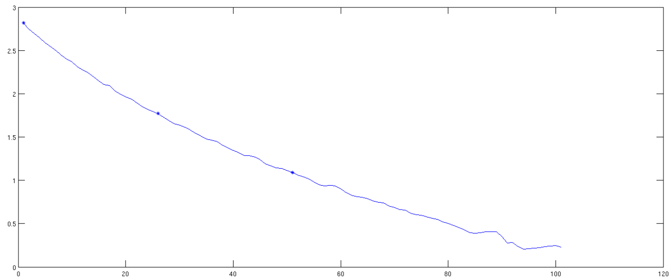

Figure 4.19: Entropy as function of α for $N = 36$.

The subfamily of the N-expansions we studied is not new, but it has not been studied in this detail with finitely many digits. Note that having the Gauss-Kuzmin-Lévy method for approximating the densities allowed us to study the entropy much easier due to much shorter computation time. We have seen that matching is helpful to prove monotonicity even though we did not mimic the proof for α-expansions. Motivated by similar results in the case of Nakada's α-expansions the following questions about entropy arise:

- For every $N \in \mathbb{N}_{\geq 2}$ is there an interval in $(0, \sqrt{N} - 1)$ for which the entropy function is constant?
- For a fixed $N \in \mathbb{N}_{\geq 2}$ for which $\alpha \in (0, \sqrt{N} - 1)$ do we have matching?
- Does matching hold on an open dense set? Does matching hold almost everywhere?
- What is the influence of an attractor strictly smaller than the interval $[\alpha, \alpha + 1]$ on the entropy?