Matching, entropy, holes and expansions

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CHAPTER 3

Matching and Ito Tanaka's α-continued fraction expansions

This chapter is joint work with Carlo Carminati and Wolfgang Steiner.

Abstract

Two closely related families of α-continued fractions were introduced in 1981: by Nakada on the one hand, by Ito and Tanaka on the other hand. The entropy and matching for Nakada's family has been studied extensively, whereas the study of Ito Tanaka's family remained on the fringe. This chapter has two parts. In the first part we focus mostly on the similarities; algebraic conditions and monotonicity of the entropy function on matching intervals. The second part focuses mostly on the Ito Tanaka α-continued fraction. We show that the parameter space is almost completely covered by matching intervals. In other words, the set of parameters for which the matching condition does not hold, called the bifurcation set, is a zero measure set (even if it has full Hausdorff dimension). These properties are shared by Nakada's α-continued fractions, though the proof is different. In contrast to Nakada's α-continued fractions, the bifurcation set of Ito Tanaka's α-continued fractions contains several non zero rational values. Moreover, it contains numbers of which the regular continued fraction expansion ends in a sequence that is bounded from below. We give several characterisations of the bifurcation set and have dimensional results for neighbourhoods of the small golden mean and rationals in the bifurcation set.

§3.1 Introduction

Various variants of the regular continued fraction (RCF) have been considered. The most famous ones are the *nearest integer* continued fraction (NICF) and the *backward continued fraction* (BCF). Starting from the 80s, some attention has been devoted to families of continued fraction algorithms; even if different authors have focused on different families one can describe most[1] of these families using the same setting as follows. Let $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ be defined by

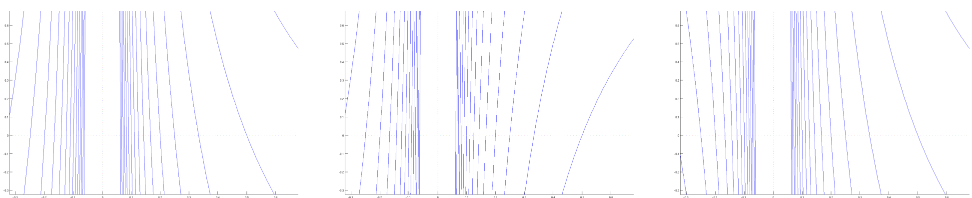
$$T_\alpha(x) = \begin{cases} S(x) - \lfloor S(x) + 1 - \alpha \rfloor & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{3.1.1}$$

Different choices of S in formula (3.1.1) give rise to different generalisations of the classical continued fraction algorithms:

(N) for $S(x) = \frac{1}{|x|}$ one gets the α-continued fractions first studied by Nakada [82],

(KU) for $S(x) = -\frac{1}{x}$ one finds a subfamily of (a, b)-continued fractions (corresponding to the choice $b = \alpha$ and $a = \alpha - 1$), which were first studied by Katok and Ugarcovici [54],

(IT) for $S(x) = \frac{1}{x}$ one gets the α-continued fractions first studied by Ito and Tanaka [103].



(a) the branches of (N) (b) the branches of (KU) (c) the branches of (IT)

Figure 3.1: The different branches for the different transformations.

In Figure 3.1 the different transformations are displayed. In all of the above three cases, for all $\alpha \in (0, 1)$, the dynamical system defined by the map (3.1.1) admits an absolutely continuous invariant probability measure and is ergodic. For the (IT) case this is proven in an unpublished article by Nakada and Steiner. Therefore, we can study the metric entropy $h_{\mu_\alpha}(T_\alpha)$. This determines the speed of convergence of the continued fraction algorithm of typical points (in the same way as in the regular continued fraction case (1.2.1) on page 12). The higher the entropy, the better the convergence. An issue which has been in the spotlight in recent years is the dependence of the entropy on the parameter α. In Figure 3.2 the entropy plotted as a function of α is shown for the Ito Tanaka continued fractions.

[1] Actually some authors, such as the authors of [81], studied the so called *folded algorithms* which are not of the type (3.1.1), however from the metric viewpoint there is hardly any difference between the folded and the unfolded version (see § 3.1 of [9] for a discussion of this issue).

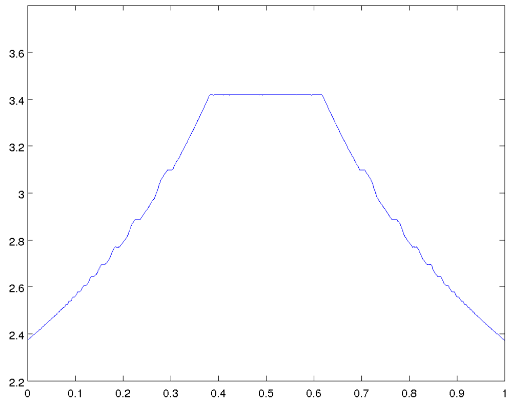


Figure 3.2: The entropy as a function of α for the Ito Tanaka continued fractions.

The behaviour of the entropy is by now quite well understood in case (N), which is by far the most studied [18, 19, 65, 79, 81, 82, 84]. The same is true for the case (KU), which was considered much more recently [17, 54, 56]. However, not much progress has been made in the case (IT) for which there are only partial results dating back to 1981 (see [103]). This chapter studies the similarities and differences between the families, where the results on (IT) are new. As in the cases (N) and (KU), also for Ito Tanaka continued fractions the *matching property* plays a central role; a parameter $\alpha \in [0,1]$ satisfies the *matching condition* with *matching exponents* N, M if

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1). \tag{3.1.2}$$

The peculiar (and somehow surprising) feature of these systems is that a condition like (3.1.2) holds on intervals with non-empty interior; thus what is actually relevant is the definition of a *matching interval.*

Definition 3.1.1 (Matching). *Let $J \subset [0,1]$ be a non-empty open interval. We say that J is a matching interval (with exponents N, M) if $T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1)$ for all $\alpha \in J$, $T_\alpha^{N-1}(\alpha) \neq T_\alpha^{M-1}(\alpha - 1)$ for almost all $\alpha \in J$, and J is not contained in a larger open interval with these properties. The difference $\Delta := M - N$ is called matching index. We call the matching set the union of all matching intervals; its complement will be called the bifurcation set and will be denoted by $\mathcal{E}$.*

Observe that we do not impose conditions on the derivative of T_α^N and T_α^M, as in Definition 1.2.8 on page 13, since these are automatically satisfied whenever matching holds on an open interval (this is proved in Section 3.2). The following lemma shows that two matching intervals cannot overlap (for any choice of $S(x)$ above).

Lemma 3.1.2. *Let M, M', N, N' be such that $M - N \neq M' - N'$. Then there are at most countably many $\alpha \in [0,1]$ such that $T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1)$ and $T_\alpha^{N'}(\alpha) = T_\alpha^{M'}(\alpha - 1)$.*

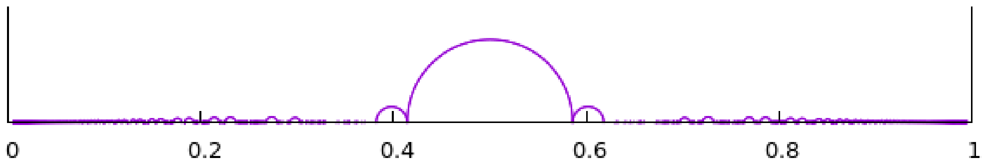


Figure 3.3: Matching intervals, plotted as arcs from a to b for a matching interval (a, b), for the Ito Tanaka continued fractions.

Proof. Assume w.l.o.g. that $N' \geq N$. Then we have $T_\alpha^{M+N'-N}(\alpha - 1) = T_\alpha^{N'}(\alpha) = T_\alpha^{M'}(\alpha - 1)$. Since $M - N \neq M' - N'$, this implies that α is a rational or quadratic number. $\square$

By definition, matching is an open condition. For the α-continued fractions (N) it is conjectured in [84] and shown in [18] that matching holds almost everywhere; the same is true in the case of (KU) (see [17, 54, 56]). In Section 3.3 we show that this is also true for the α-continued fractions of Ito and Tanaka. However, for the bifurcation set the situation is different. Not only does each of the three variants (N), (KU) and (IT) have a different bifurcation set (we denote them by $\mathcal{E}_N, \mathcal{E}_{KU}$ and $\mathcal{E}_{IT}$ respectively) but these bifurcation sets display quite a few differences. For instance, it is not difficult to show that both $\mathcal{E}_N$ and $\mathcal{E}_{KU}$ do not intersect $\mathbb{Q} \cap (0, 1)$ and are made of badly approximable numbers; this is not the case for $\mathcal{E}_{IT}$: not only does it contain infinitely many rational values (such as the values $1/n$ for $n \geq 3$) but it also contains numbers for which the tail of the regular continued fraction expansion has digits bounded from below. In the following subsection, we shall focus on the specific features of the Ito Tanaka case as well as stating the results on the exceptional set $\mathcal{E}_{IT}$. In this section we also state our theorems. In Section 3.2 we show that the entropy formula in terms of q_n is true for all three families as well as the fact that matching implies monotonicity of the entropy. Furthermore, we shed light onto algebraic conditions. Each family comes with different algebraic conditions that hold for $\alpha \in \mathbb{Q} \cap (0, 1)$. They will illustrate the fact that the (IT) case is more complicated than the others. The results displayed in this section in the case of (KU) and (N) are already known but added for comparison. The study of the so called *exceptional set* is specific for every family and is the focus of the second part of this chapter (Section 3.3 and 3.4). In Section 3.3 we prove the results on the exceptional set $\mathcal{E}_{IT}$ as well as the fact that matching holds almost everywhere for which the proof is specific for the (IT) case. Section 3.4 is dedicated to dimensional results for the exceptional set.

§3.1.1 Ito Tanaka continued fractions: old and new results

In this section T_α will always denote the map (3.1.1) for the Ito Tanaka case, i.e., with $S(x) = 1/x$. Let us point out that the dynamical systems of α and $1 - \alpha$ are isomorphic. Indeed, setting $\tau(x) = -x$ gives

$$\tau \circ T_\alpha = T_{1-\alpha} \circ \tau.$$

For this reason, it is enough to study this family for the parameter $\alpha \in [1/2, 1]$. Setting $d_\alpha(x) = \lfloor S(x) + 1 - \alpha \rfloor$, for every $x \in [\alpha - 1, \alpha]$, we use the shorthand $d_{\alpha,n} = d_{\alpha,n}(x) = d_\alpha\left(T_\alpha^n(x)\right)$ to write the continued fraction expansion

$$x = \cfrac{1}{d_{\alpha,1} + \cfrac{1}{d_{\alpha,2} + \cfrac{1}{\ddots}}}.$$

Note that T_1 is the Gauss map and $T_{\frac{1}{2}}$ is the map for Hurwitz continued fraction expansions [48]. Furthermore, $d_{\alpha,n}(x)$ is called the n^{th} digit of x and can be both negative and positive. We define the n^{th} convergent as

$$c_{\alpha,n}(x) = \frac{p_{\alpha,n}(x)}{q_{\alpha,n}(x)} = \cfrac{1}{d_{\alpha,1}(x) + \cfrac{1}{d_{\alpha,2}(x) + \cfrac{1}{\ddots + \cfrac{1}{d_{\alpha,n}(x)}}}}.$$

Let $g = \frac{\sqrt{5}-1}{2}$. For the speed of convergence for any $x \in [\alpha - 1, \alpha]$ we have

$$\left| x - \frac{p_{\alpha,n}}{q_{\alpha,n}} \right| \leq \frac{2}{\sqrt{5}\,|q_{\alpha,n}|^2} \quad \text{for} \quad \frac{1}{2} \leq \alpha \leq g$$

and

$$\left| x - \frac{p_{\alpha,n}}{q_{\alpha,n}} \right| \leq \frac{1}{|q_{\alpha,n}|^2} \qquad \text{for} \quad g < \alpha \leq 1 \tag{3.1.3}$$

with $|q_{\alpha,n}(x)| \geq (g+1)^n$ (see [103]). By symmetry, analogous results could be stated for the convergence of the algorithms when $\alpha \in [0, \frac{1}{2})$. Now let us turn to matching and state our first theorem.

Theorem 3.1.3. *Matching holds almost everywhere on* $[0, 1]$ *and the only possible indices are* $-2, 0$ *and* 2. *More precisely, the matching indices are* 0 *or* 2 *for* $\alpha \leq 1/2$ *and* 0 *or* -2 *for* $\alpha \geq 1/2$.

Let us recall from [103] that the symmetric parameter interval $(1-g, g)$ is (almost) covered by the three adjacent matching intervals $(1-g, \sqrt{2}-1)$, $(\sqrt{2}-1, 2-\sqrt{2})$ and $(2-\sqrt{2}, g)$ see Figure 3.3; so the interesting part of the bifurcation set is in the ranges of $[0, 1-g]$ and $[g, 1]$. Since the problem is symmetric with respect to $\alpha = 1/2$, we can focus on $\mathcal{E}_{IT} \cap [g, 1]$. We prove the following characterisations of this set.

Theorem 3.1.4. *The bifurcation set on* $[g, 1]$ *is given by*

$$\begin{aligned} &\mathcal{E}_{IT} \cap [g, 1] \\ &\quad = \{\alpha \in [g, 1] : T_\alpha^n(\alpha - 1) \leq \tfrac{1}{\alpha+1} \quad \textit{and } T_\alpha^n(\tfrac{1}{\alpha} - 1) \leq \tfrac{1}{\alpha+1} \quad \textit{for all } n \geq 1\} \qquad (3.1.4) \\ &\quad = \left\{\alpha \in [g, 1] : T_g^n(\alpha - 1) \geq \alpha - 1 \textit{ and } T_g^n(\tfrac{1}{\alpha} - 1) \geq \alpha - 1 \textit{ for all } n \geq 1\right\}. \qquad (3.1.5) \end{aligned}$$

While the characterisation in terms of T_α is natural from the definition of the bifurcation set, the characterisation with a fixed map T_g will be more useful. In particular, from the ergodicity of T_g it easily follows that $\mathcal{E}_{IT}$ is a Lebesgue measure zero set. Note that there is a clear connection with holes namely that $\mathcal{E}_{IT}$ contains those α for which α and $\alpha - 1$ are contained in the survivor set when iterating over T_g with hole $[g-1, \alpha-1)$. Using this characterisation, we retrieve the following dimensional results for $\mathcal{E}_{IT}$.

Theorem 3.1.5. *We have that $\mathcal{E}_{IT}$ is a Lebesgue measure zero set and* $\dim_H(\mathcal{E}_{IT}) = 1$. *Moreover, for all $\delta > 0$ we have* $\dim_H(\mathcal{E}_{IT} \cap (g, g+\delta)) = 1$.

This is similar to the behaviour of Nakada's continued fractions around zero (see [18]). What is different however, is the presence of rationals in the bifurcation set. For those points we have the following theorem.

Theorem 3.1.6. *The bifurcation set $\mathcal{E}_{IT}$ contains infinitely many rational values and the set of rational bifurcation parameters $\mathcal{E}_{IT} \cap \mathbb{Q}$ has no isolated points. Moreover, for all $r \in \mathcal{E}_{IT} \cap \mathbb{Q}$ and for all $\delta > 0$ we have that* $\dim_H(\mathcal{E}_{IT} \cap (r-\delta, r+\delta)) > 1/2$.

Theorem 3.1.3 and 3.1.4 are proved in Section 3.3. In Section 3.4 we prove the theorems on dimensional results (Theorem 3.1.5 and 3.1.6).

§3.2 Algebraic relations, an entropy formula and matching implies monotonicity

Even though the results in this section will be focused on Ito Tanaka α-continued fractions, most of the results also hold for other continued fraction expansion families. Therefore, we will generalise some results to fit a more general framework or refer to other continued fraction expansion families after a proof. We will first prove that for all $\alpha \in (0,1) \cap \mathbb{Q}$ an algebraic condition holds. In the case of Ito Tanaka α-continued fractions this results in 6 different algebraic relations. For KU-continued fractions and Nakada's α-continued fractions the situation greatly simplifies. We find 2 algebraic relations for each family. They are used in the proof of monotonicity on matching intervals later on in this section. But before proving monotonicity we will prove an entropy formula as in (1.2.1) for all three families.

To find the algebraic relations we work with Möbius transformations and matrices.

Definition 3.2.1 (Möbius transformation). *Let* $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ *be a matrix with $a_i \in \mathbb{Z}$. The Möbius transformation induced by A is the map $A : \mathbb{R} \to \mathbb{R}$ given by*

$$A(z) = \frac{a_1 z + a_2}{a_3 z + a_4}.$$

Now let $d \in \mathbb{Z}$. We define the following matrices in $SL_2(\mathbb{Z})$:

$$B_d = \begin{bmatrix} 0 & 1 \\ 1 & d \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that

$$R^d = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

which gives us $B_d = SR^d$. Fix α and $x \in [\alpha-1, \alpha]$ and let $M_{\alpha,x,n} = B_{d_{\alpha,1}(x)} B_{d_{\alpha,2}(x)} B_{d_{\alpha,3}(x)} \cdots B_{d_{\alpha,n}(x)}$. An easy check shows that $M_{\alpha,x,n}(0) = c_{\alpha,n}(x)$.

Lemma 3.2.2 (Recurrence relations). *We have the recurrence relations*

$$\begin{aligned} p_{\alpha,-1} &:= 1; \quad p_{\alpha,0} := 0; \quad p_{\alpha,n}(x) = d_{\alpha,n}(x) p_{\alpha,n-1}(x) + p_{\alpha,n-2}(x), \quad n \geq 1, \\ q_{\alpha,-1} &:= 0; \quad q_{\alpha,0} := 1; \quad q_{\alpha,n}(x) = d_{\alpha,n}(x) q_{\alpha,n-1}(x) + q_{\alpha,n-2}(x), \quad n \geq 1. \end{aligned}$$

The recurrence formulas are also given in [103] however without a proof. We provide a proof using the Möbius transformations. This proof is analogous to the proof for the regular continued fraction given in [29].

Proof. We can obtain the recurrence relations by writing

$$M_{\alpha,x,n} = \begin{bmatrix} r_{\alpha,n}(x) & p_{\alpha,n}(x) \\ s_{\alpha,n}(x) & q_{\alpha,n}(x) \end{bmatrix}.$$

Now

$$\begin{aligned} M_{\alpha,x,n} = M_{\alpha,x,n-1} B_{d_{\alpha,n}(x)} &= \begin{bmatrix} r_{\alpha,n-1}(x) & p_{\alpha,n-1}(x) \\ s_{\alpha,n-1}(x) & q_{\alpha,n-1}(x) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & d_{\alpha,n}(x) \end{bmatrix} \\ &= \begin{bmatrix} p_{\alpha,n-1}(x) & d_{\alpha,n}(x) p_{\alpha,n-1}(x) + r_{\alpha,n-1}(x) \\ q_{\alpha,n-1}(x) & d_{\alpha,n}(x) q_{\alpha,n-1}(x) + s_{\alpha,n-1}(x) \end{bmatrix}. \end{aligned}$$

This gives us $r_{\alpha,n} = p_{\alpha,n-1}$ and $s_{\alpha,n} = q_{\alpha,n-1}$ and the recurrence formulas are found. □

Just as in the classical case we have the following equation

$$p_{\alpha,n-1}(x) q_{\alpha,n}(x) - p_{\alpha,n}(x) q_{\alpha,n-1}(x) = (-1)^n. \tag{3.2.1}$$

Note that this implies that $p_{\alpha,n}(x)$ and $q_{\alpha,n}(x)$ are co-prime for all $n \in \mathbb{N}$ as well as $q_{\alpha,n}(x)$ and $q_{\alpha,n-1}(x)$. The equation is found by looking at the determinant of $M_{\alpha,x,n}$

$$\det(M_{\alpha,x,n}) = \det\left(B_{d_{\alpha,1}(x)} B_{d_{\alpha,2}(x)} \cdots B_{d_{\alpha,n}(x)}\right) = (-1)^n.$$

Also the following equation holds

$$x = \frac{p_{\alpha,n}(x) + p_{\alpha,n-1}(x) T_\alpha^n(x)}{q_{\alpha,n}(x) + q_{\alpha,n-1}(x) T_\alpha^n(x)}. \tag{3.2.2}$$

Note that $T_\alpha(x) = B^{-1}_{d_{\alpha,n}(x)}(x)$ and so $x = B_{d_{\alpha,n}(x)}(T_\alpha(x))$. This gives us

$$x = M_{\alpha,x,n}(T_\alpha^n(x)) = \frac{p_{\alpha,n}(x) + p_{\alpha,n-1}(x) T_\alpha^n(x)}{q_{\alpha,n}(x) + q_{\alpha,n-1}(x) T_\alpha^n(x)}.$$

Let us now turn to the algebraic conditions. For (N) and (KU) continued fractions one can define $M_{\alpha,x,n}$ in the same way as for the (IT) case. The following lemma holds for all three families.

Lemma 3.2.3 (Pre-algebraic condition). *Let $\alpha \in [0,1]$ and suppose matching occurs. Let $b = T_\alpha^N(\alpha) = T_\alpha^M(\alpha-1)$ then the following equation holds*

$$M_{\alpha,\alpha,N}(b) = RM_{\alpha,\alpha-1,M}(b). \tag{3.2.3}$$

Proof. We write

$$\begin{aligned} \alpha &= M_{\alpha,\alpha,N}(b) \\ \alpha - 1 = R^{-1}\alpha &= M_{\alpha,\alpha-1,M}(b) \end{aligned}$$

which gives us (3.2.3). □

From the evaluation in b we get the algebraic conditions (for the (IT) case) that hold for $M_{\alpha,\alpha,N}$ and $M_{\alpha,\alpha-1,M}$.

Theorem 3.2.4 (Algebraic conditions). *Let $\alpha = \frac{p}{q} \in \mathbb{Q} \cap (0,1)$ with $T_\alpha^N(\alpha) = T_\alpha^M(\alpha-1) = 0$ and N, M minimal. Then one of the following algebraic conditions holds*

(a) $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}$	*(b)* $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}SRS$
(c) $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}SR^{-1}S$	*(d)* $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}VSRS$
(e) $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}VSR^{-1}S$	*(f)* $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}V$

with $V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Proof. Let $\alpha = \frac{p}{q} \in \mathbb{Q} \cap (0,1)$ with $T_\alpha^N(\alpha) = T_\alpha^M(\alpha-1) = 0$ and N, M minimal. Now (3.2.3) gives us

$$M_{\alpha,\alpha,N} = \begin{bmatrix} a_1 & p \\ a_2 & q \end{bmatrix} \quad RM_{\alpha,\alpha-1,M} = \begin{bmatrix} b_1 & p \\ b_2 & q \end{bmatrix}$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}\backslash\{0\}$. We have that $a_2 = q_{\alpha,N-1}(\alpha)$ and $b_2 = q_{\alpha,M-1}(\alpha-1)$. We prove that $|q_{\alpha,N-1}(\alpha)| < |q_{\alpha,N}(\alpha)|$ and $|q_{\alpha,M-1}(\alpha-1)| < |q_{\alpha,M}(\alpha-1)|$ which gives us

$$0 < |a_2| < q, \quad 0 < |b_2| < q. \tag{3.2.4}$$

This is used in all 6 cases. We have

$$\begin{aligned} \left|\frac{p_{\alpha,N}(\alpha)}{q_{\alpha,N}(\alpha)} - \frac{p_{\alpha,N-1}(\alpha)}{q_{\alpha,N-1}(\alpha)}\right| &= \left|\frac{p_{\alpha,N}(\alpha)q_{\alpha,N-1}(\alpha) - p_{\alpha,N-1}(\alpha)q_{\alpha,N}(\alpha)}{q_{\alpha,N}(\alpha)q_{\alpha,N-1}(\alpha)}\right| \\ &= \left|\frac{(-1)^N}{q_{\alpha,N}(\alpha)q_{\alpha,N-1}(\alpha)}\right| \leq \frac{1}{q_{\alpha,N}(\alpha)^2} \end{aligned}$$

from (3.2.1) and the speed of convergence (Equation (3.1.3) on page 43) for (IT) . This gives us $|q_{\alpha,N-1}(\alpha)| \leq |q_{\alpha,N}(\alpha)|$. Suppose that equality holds. From the recurrence formulas we find

$$\pm q_{\alpha,N-1}(\alpha) = q_{\alpha,N}(\alpha) = d_{\alpha,N}(\alpha)q_{\alpha,N-1}(\alpha) + q_{\alpha,N-2}(\alpha)$$

which implies $(\pm 1 - d_{\alpha,N}(\alpha))\, q_{\alpha,N-1}(\alpha) = q_{\alpha,N-2}(\alpha)$. This contradicts with $q_{\alpha,N-1}(\alpha)$ and $q_{\alpha,N-2}(\alpha)$ being co-prime. Therefore we find

$$|q_{\alpha,N-1}(\alpha)| < |q_{\alpha,N}(\alpha)| .$$

Now

$$\det(M_{\alpha,\alpha,N}) = (-1)^N \text{ and } \det(RM_{\alpha,\alpha-1,M}) = (-1)^M .$$

Whenever $N - M$ is odd we find $\det(M_{\alpha,\alpha,N}) = -\det(RM_{\alpha,\alpha-1,M})$ and if $N - M$ is even we find $\det(M_{\alpha,\alpha,N}) = \det(RM_{\alpha,\alpha-1,M})$. Furthermore, we either have $a_2b_2 > 0$ or $a_2b_2 < 0$. These different cases lead to different algebraic conditions. Table 3.1 shows which algebraic condition we find in which case. Left to prove is that this table holds.

	$a_2b_2 > 0$	$a_2b_2 < 0$
$\det(M_{\alpha,\alpha,N}) = \det(RM_{\alpha,\alpha-1,M})$	(a)	(b,c)
$\det(M_{\alpha,\alpha,N}) = -\det(RM_{\alpha,\alpha-1,M})$	(d,e)	(f)

Table 3.1: The different cases.

When $\det(M_{\alpha,\alpha,N}) = \det(RM_{\alpha,\alpha-1,M})$ we find

$$(a_1 - b_1)q = (a_2 - b_2)p \tag{3.2.5}$$

by writing out the determinants. Since p and q are co-prime, $a_2 - b_2$ is a multiple of q. Together with (3.2.4) we get that $a_2 - b_2 \in \{-q, 0, q\}$. If $a_2b_2 > 0$, then $a_2 - b_2 = 0$ and so $a_2 = b_2$. Note that this also gives us $a_1 = b_1$ using (3.2.5). This results in

$$M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}$$

which is condition (a). Now suppose $a_2b_2 < 0$. We find that $(a_2 - b_2) = \pm q$. In case $a_2 - b_2 = q$ we have $a_1 - b_1 = p$ by (3.2.5) which gives us

$$M_{\alpha,\alpha,N} = \begin{bmatrix} b_1 + p & p \\ b_2 + q & q \end{bmatrix}$$

and so

$$M_{\alpha,\alpha,N} = \begin{bmatrix} b_1 & p \\ b_2 & q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We find

$$M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M}SRS$$

which is case (b). In case $a_2 - b_2 = -q$ we have $a_1 - b_1 = -p$ which gives

$$M_{\alpha,\alpha,N} = \begin{bmatrix} b_1 - p & p \\ b_2 - q & q \end{bmatrix}$$

and so

$$M_{\alpha,\alpha,N} = \begin{bmatrix} b_1 & p \\ b_2 & q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

We find

$$M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M} SR^{-1} S$$

which is case (c). When $\det(M_{\alpha,\alpha,N}) = -\det(RM_{\alpha,\alpha-1,M})$ we get

$$(a_1 + b_1)q = (a_2 + b_2)p.$$

This time $a_2 + b_2$ is a multiple of q and together with (3.2.4) this gives $a_2 + b_2 \in \{-q, 0, q\}$. Now assume that $a_2 b_2 > 0$. We find $a_2 + b_2 = \pm q$. In case $a_2 + b_2 = q$ we have $a_1 + b_1 = p$ which gives

$$M_{\alpha,\alpha,N} = \begin{bmatrix} p - b_1 & p \\ q - b_2 & q \end{bmatrix}$$

and so

$$M_{\alpha,\alpha,N} = \begin{bmatrix} b_1 & p \\ b_2 & q \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This results in

$$M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M} VSRS$$

which is case (d).
Suppose $a_2 + b_2 = -q$. Then $a_1 + b_1 = -p$ which gives

$$M_{\alpha,\alpha,N} = \begin{bmatrix} -p - b_1 & p \\ -q - b_2 & q \end{bmatrix}$$

and so

$$M_{\alpha,\alpha,N} = \begin{bmatrix} b_1 & p \\ b_2 & q \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

This results in

$$M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M} VSR^{-1} S$$

which is case (e). If $a_2 b_2 < 0$ then $a_2 + b_2 = 0$ and so $a_1 + b_1 = 0$ which gives us

$$M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M} V$$

which is case (f). □

For (N) continued fractions we know that $q_n(x) > 0$ for all choices of α and x. With the same reasoning as above we find that $a_2 b_2 > 0$. Furthermore, $a_2 + b_2 = -q$ is excluded. The two algebraic relations that remain are (a) and (d). For details see the appendix of [18] and [84].
For KU-continued fractions we have that $\det(M_{\alpha,x,k}) = 1$ for any allowed triple (α, x, k). With the above reasoning we can find that either (a),(b) or (c) holds. In [17] it is shown that (b) holds for a special class of rationals $Q \subset \mathbb{Q} \cap (0,1)$. For other rationals (a) holds. Beware that the matrix S is defined slightly differently in the (KU)-case since $S(x) = -\frac{1}{x}$.

Theorem 3.2.4 results in the following corollary.

Corollary 3.2.5. *Let $\alpha \in \mathbb{Q} \cap (0,1)$ with $T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1) = 0$ and N, M minimal and x in the neighbourhood of α. Then*

if (a) holds, $T_x^N(x) = T_x^M(x-1)$	*if (b) holds,* $T_x^{N+1}(x) = T_x^{M+1}(x-1)$
if (c) holds, $T_x^{N+1}(x) = T_x^{M+1}(x-1)$	*if (d) holds,* $T_x^{N+1}(x) = -T_x^{M+1}(x-1)$
if (e) holds, $T_x^{N+1}(x) = -T_x^{M+1}(x-1)$	*if (f) holds,* $T_x^N(x) = -T_x^M(x-1)$.

Proof. Fix $\alpha \in \mathbb{Q} \cap (0,1)$. We first prove that there is a neighbourhood of α such that for every x in this neighbourhood we have

$$M_{x,x,N} = M_{\alpha,\alpha,N} \text{ and } M_{x,x-1,M} = M_{\alpha,\alpha-1,M}. \tag{3.2.6}$$

In other words, the functions $T_z^N(z)$ and $T_z^M(z-1)$ are continuous in $z = \alpha$. Suppose that $T_z^N(z)$ is not continuous in $z = \alpha$ then there exists a $k \leq N$ such that $T_\alpha^k(\alpha) = \alpha - 1$. This gives

$$\alpha = \cfrac{1}{d_{\alpha,1} + \cfrac{1}{\ddots + \cfrac{1}{d_{\alpha,k} + \alpha - 1}}}$$

which is an infinite (periodic) expansion of α and so α is irrational. In the same way we find a contradiction for $T_z^M(z-1)$.
Now pick x in the neighbourhood of α so that (3.2.6) holds. We write

$$x = M_{x,x,N}(T_x^N(x)) \tag{3.2.7}$$

and

$$x = RM_{x,x-1,M}(T_x^M(x-1)). \tag{3.2.8}$$

Since x is in the neighbourhood of α we have that

$$M_{x,x,N} = M_{\alpha,\alpha,N} \text{ and } M_{x,x-1,M} = M_{\alpha,\alpha-1,M}.$$

If condition (a) holds, we find

$$M_{x,x,N} = M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-1,M} = RM_{x,x-1,M}.$$

This gives us, together with (3.2.7) and (3.2.8), that

$$T_x^N(x) = T_x^M(x-1).$$

In the second case we get from condition (b) and (3.2.7), (3.2.8) that

$$SRST_x^N(x) = T_x^M(x-1)$$

and so

$$\frac{T_x^N(x)}{T_x^N(x)+1} = T_x^M(x-1)$$

which implies

$$T_x^N(x) = \frac{T_x^M(x-1)}{1 - T_x^M(x-1)}. \tag{3.2.9}$$

Now

$$\begin{aligned} T_x^{N+1}(x) &= \frac{1}{T_x^M(x-1)} - 1 - d_{x,N+1}(x), \\ T_x^{M+1}(x-1) &= \frac{1}{T_x^N(x)} + 1 - d_{x,M+1}(x-1). \end{aligned}$$

This gives

$$T_x^{N+1}(x) - T_x^{M+1}(x-1) = \frac{1}{T_x^M(x-1)} - \frac{1}{T_x^N(x)} - 2 - d_{x,N+1}(x) + d_{x,M+1}(x-1).$$

Using (3.2.9) we find

$$T_x^{N+1}(x) - T_x^{M+1}(x-1) = \frac{1}{T_x^M(x-1)} - \frac{1}{T_x^M(x-1)} + 1 - d_{x,N+1}(x) + d_{x,M+1}(x-1).$$

so

$$T_x^{N+1}(x) - T_x^{M+1}(x-1) = r$$

for some $r \in \mathbb{Z}$. Since $T_x^{N+1}(x), T_x^{M+1}(x-1) \in [x-1, x)$ we find $r = 0$. Case (c),(d) and (e) can be found in a similar way as case (b). Case (f) is similar to case (a). ☐

Note that from (3.2.9) it follows that $T_x^N(x) \neq T_x^M(x-1)$ whenever (b) holds. In the same manner we find that whenever (c) holds $T_x^N(x) \neq T_x^M(x-1)$. For (d), (e) and (f) we also find $T_x^N(x) \neq T_x^M(x-1)$. We can conclude that on a matching interval (a) must hold, otherwise not all points in that matching interval have the same matching exponents. Simulations suggest that (b) only holds for $\alpha = \frac{1}{2}$ and (c) only holds for $\alpha \in \{\frac{2}{5}, \frac{3}{5}\}$.
We now prove the fact that if $\det(M_{\alpha,\alpha,N}) = \det(M_{\alpha,\alpha-1,M})$ for $\alpha \in (0,1) \cap \mathbb{Q}$, then the condition on the derivatives, as in Definition 1.2.8 on page 13, of $T_\alpha^N(\alpha)$ and $T_\alpha^M(\alpha-1)$ are satisfied. This lemma holds for all three families.

Lemma 3.2.6. *Fix $\alpha \in (0,1)$ and let N, M be minimal such that $T_\alpha^N(\alpha) = T_\alpha^M(\alpha-1) = 0$ with $\det(M_{\alpha,\alpha,N}) = \det(M_{\alpha,\alpha-1,M}) = t$ with $t \in \{-1,1\}$. Then $(T_\alpha^N)'(\alpha) = (T_\alpha^M)'(\alpha-1)$.*

Proof. We know that there are $a, b, \ldots, f \in \mathbb{Z}$ such that

$$T_\alpha^N(x) = \frac{ax+b}{cx+d}\ , \quad T_\alpha^M(x) = \frac{ex+f}{gx+h}.$$

We have that $T_\alpha^N(\alpha) = 0$ gives $\alpha = -\frac{b}{a}$ and $T_\alpha^M(\alpha-1) = 0$ gives $\alpha - 1 = -\frac{f}{e}$. For any choice of S we have that $ad - bc = t$ and $eh - fg = t$. This gives us

$$(T_\alpha^N)'(x) = \frac{t}{(cx+d)^2}\ , \quad (T_\alpha^M)'(x) = \frac{t}{(gx+h)^2}.$$

Filling in $\alpha = -\frac{b}{a}$ and $\alpha - 1 = -\frac{d}{e}$ respectively gives

$$(T_\alpha^N)'(\alpha) = \frac{t}{(\frac{-cb}{a} + d)^2} = \frac{ta^2}{(ad - cb)^2} = ta^2$$

and

$$(T_\alpha^M)'(\alpha - 1) = \frac{t}{(\frac{-fg}{e} + h)^2} = \frac{te^2}{(eh - fg)^2} = te^2.$$

Furthermore, note that a and b are co-prime and f and e are co-prime. Since $\alpha - 1 = \frac{-b-a}{a} = -\frac{f}{e}$ we find $a = \pm e$ so that $a^2 = e^2$. This finalises the proof. □

Note that on a matching interval the determinants are equal (since condition (a) holds). Let us now turn to the entropy formula. We prove it for the (IT) and (N) case and show where the proof fails to work for the (KU) case.

Lemma 3.2.7. *Let T_α be as in (3.1.1) with $S(x) = \frac{1}{x}$ or $S(x) = \frac{1}{|x|}$. For almost every $x \in [\alpha - 1, \alpha]$ we have that*

$$h(\alpha) := h(T_\alpha) = 2 \lim_{n\to\infty} \frac{1}{n} \left|\log(q_{\alpha,n}(x))\right|. \tag{3.2.10}$$

where $q_{\alpha,n}(x)$ is the denominator associated to the n^{th} convergent of x for the corresponding map S.

The proof of Lemma 3.2.7 is very similar to the proof in the classical case (see [29]).

Proof of Lemma 3.2.7. Let T be T_α for some choice of S and $\alpha \in (0,1)$ and let x be a typical point. For all three cases one has recurrence relations for the convergents of the following form

$$\begin{array}{llll} p_{\alpha,-1} := 1; & p_{\alpha,0} := 0; & p_{\alpha,n}(x) = d_{\alpha,n}(x)p_{\alpha,n-1}(x) + \varepsilon_{n-1}(x)p_{\alpha,n-2}(x), & n \geq 1, \\ q_{\alpha,-1} := 0; & q_{\alpha,0} := 1; & q_{\alpha,n}(x) = d_{\alpha,n}(x)q_{\alpha,n-1}(x) + \varepsilon_{n-1}(x)q_{\alpha,n-2}(x), & n \geq 1. \end{array}$$

Here $d_{\alpha,n}(x) = d_{\alpha,1}(T^{n-1}(x))$ and $\varepsilon_{\alpha,n}(x) = \varepsilon_{\alpha,1}(T^{n-1}(x))$ where $d_{\alpha,1}(x)$ and $\varepsilon_{\alpha,1}(x)$ depend on the choice of S and $\varepsilon_0 := 1$. In the proof we will omit the dependence of α in our notation. First we show that for all $n \in \mathbb{N}$ we have

$$p_n(x) = q_{n-1}(T(x)) \tag{3.2.11}$$

by using induction. For $n = 0$ we find $p_0(x) = 0 = q_{-1}(T(x))$, for $n = 1$ we find $p_1(x) = \varepsilon_0 = 1 = q_0(T(x))$. We assume $p_n(x) = q_{n-1}(T(x))$ and $p_{n-1}(x) = q_{n-2}(T(x))$ to find

$$\begin{aligned} p_{n+1}(x) &= d_{n+1}(x)p_n(x) + \varepsilon_n(x)p_{n-1}(x) \\ &= d_{n+1}(x)q_{n-1}(T(x)) + \varepsilon_n(x)q_{n-2}(T(x)) \\ &= d_n(T(x))q_{n-1}(T(x)) + \varepsilon_{n-1}(T(x))q_{n-2}(T(x)) \\ &= q_n(T(x)) \end{aligned}$$

which finalises the induction. Using (3.2.11) we write

$$\begin{aligned}\frac{1}{q_n(x)} &= \frac{1}{q_n(x)}\frac{p_n(x)}{q_{n-1}(T(x))}\frac{p_{n-1}(T(x))}{q_{n-2}(T^2(x))}\cdots\frac{p_1(T^{n-1}(x))}{q_0(T^n(x))}\\ &= \frac{p_n(x)}{q_n(x)}\frac{p_{n-1}(T(x))}{q_{n-1}(T(x))}\cdots\frac{p_1(T^{n-1}(x))}{q_1(T^{n-1}(x))}.\end{aligned}$$

Taking the absolute value and the logarithm on both sides we find

$$-\log|q_n(x)| = \log\left|\frac{p_n(x)}{q_n(x)}\right| + \log\left|\frac{p_{n-1}(T(x))}{q_{n-1}(T(x))}\right| + \ldots + \log\left|\frac{p_1(T^{n-1}(x))}{q_1(T^{n-1}(x))}\right|. \tag{3.2.12}$$

Now we write

$$-\log|q_n(x)| = \log|x| + \log|T(x)| + \cdots + \log|T^{n-1}(x)| + E(n,x) \tag{3.2.13}$$

and determine the error term $E(n,x)$ by substituting the right hand side of (3.2.12) for $-\log|q_n(x)|$ in (3.2.13) and rewriting the equation. We get

$$E(n,x) = \log\left|\frac{p_n(x)}{q_n(x)}\right| - \log|x| + \cdots + \log\left|\frac{p_1(T^{n-1}(x))}{q_1(T^{n-1}(x))}\right| - \log|T^{n-1}(x)|. \tag{3.2.14}$$

Now we prove that for any $y \in [\alpha-1,\alpha]\backslash\mathbb{Q}$ we have

$$\left|\log\left|\frac{p_n(y)}{q_n(y)}\right| - \log|y|\right| \le \frac{1}{|q_n(y)|}. \tag{3.2.15}$$

First we prove that if we write $|x| = |\frac{d}{q_n}|$ then $d > 1$ for $n \ge 2$. We have

$$\left||x| - |\frac{p_n}{q_n}|\right| \le \left|\frac{d-p_n}{q_n}\right| \le \frac{1}{q_n^2}. \tag{3.2.16}$$

For the (IT) case this follows from 3.1.3 and for the (N) case this estimate can be found in [81]. We do not have this estimate for the (KU) case where only $|x - \frac{p_n}{q_n}| < \frac{1}{q_n}$ is proven in [54]. Now (3.2.16) gives $|d-p_n||q_n| \le 1$. Now suppose $|\frac{p_n}{q_n}| \le |x|$. Using the Mean Value Theorem with $f(x) = \log|x|$ on $[|\frac{p_n}{q_n}|, |x|]$ we find

$$0 \le \left|\log|x| - \log|\frac{p_n}{q_n}|\right| = \left||x| - |\frac{p_n}{q_n}|\right|\frac{1}{c} \le \left|x - \frac{p_n}{q_n}\right|\frac{1}{c} \le \frac{1}{q_n^2}\frac{1}{c} \le \frac{1}{q_n^2}|\frac{q_n}{p_n}| \le \frac{1}{|q_n|}$$

for some $c \in [|\frac{p_n}{q_n}|, |x|]$. Suppose $|x| \le |\frac{p_n}{q_n}|$. Using the Mean Value Theorem with $f(x) = \log|x|$ on $[|x|, |\frac{p_n}{q_n}|]$ we find

$$0 \le \left|\log|x| - \log|\frac{p_n}{q_n}|\right| = \left||x| - |\frac{p_n}{q_n}|\right|\frac{1}{c} \le \left|x - \frac{p_n}{q_n}\right|\frac{1}{c} \le \frac{1}{q_n^2}\frac{1}{c} \le \frac{1}{q_n^2}|\frac{q_n}{d}| \le \frac{1}{|q_n|}.$$

In both cases we find that (3.2.15) holds. Using this estimate in (3.2.14) we find

$$|E(n,x)| \le \frac{1}{|q_n(x)|} + \cdots + \frac{1}{|q_1(T_\alpha^{n-1}(x))|}.$$

Since for all choices of S and $\alpha \in (0,1]$ we have that the sequence $|q_n(x)|$ grows exponentially fast (see [54, 80, 103][2]) there is a $b \in \mathbb{R}_{>1}$ such that $|q_n| > b^{n-1}$ for $n > 1$. Furthermore $q_1 \geq 1$ and so we get

$$|E(n,x)| \leq 1 + \frac{1}{b} + \cdots + \frac{1}{b^{n-1}} < \sum_{k=0}^{\infty} \frac{1}{b^k} = \frac{b}{b-1}. \tag{3.2.17}$$

Using Rohlin's formula and Birkhoff's formula we find

$$h(\alpha) = \int \log |T'(x)| d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |T'(T^i(x))|.$$

With (3.2.13) this gives us

$$h(\alpha) = -2 \lim_{n\to\infty} \frac{1}{n} \left(-\log |q_n(x)| - E(n,x)\right).$$

Since (3.2.17) holds we can now conclude

$$h(\alpha) = 2 \lim_{n\to\infty} \frac{1}{n} \log |q_n(x)|$$

which is equation (3.2.10).

□

Now we will prove that on a matching interval the entropy is monotonic. The general idea is the same as for the α-continued fractions (see [84]).

Theorem 3.2.8. *Let $A \subset [0,1]$ be a matching interval for (N),(KU) or (IT). Then the entropy is monotonic on A. Furthermore, if the matching index is positive $h(\alpha)$ is increasing, if the matching index is zero $h(\alpha)$ is constant and if the matching index is negative $h(\alpha)$ is decreasing.*

Proof. Fix $s \in \mathbb{Q} \backslash \mathcal{E}$, where $\mathcal{E} \in \{\mathcal{E}_N, \mathcal{E}_{KU}, \mathcal{E}_{IT}\}$ depends on the choice of S, with $T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1) = 0$ and N, M minimal and let $(l(s), r(s))$ be such that $s \in (l(s), r(s))$ where $l(s)$ and $r(s)$ are chosen in such a way that $M_{\alpha,\alpha,N} = M_{s,s,N}$ and $M_{\alpha,\alpha-1,M} = M_{s,s-1,M}$ for all $\alpha \in (l(s), r(s))$. Note that this implies that $(l(s), r(s))$ is contained in a matching interval. Let $\alpha, \beta \in (l(s), r(s))$ and define $T_\alpha^m(\alpha) = \alpha_m$ and $T_\beta^m(\beta) = \beta_m$. We prove the following equation holds for $m \leq M$

$$|\alpha_m - \beta_m| < \frac{|\alpha - \beta|}{|p_{\alpha,m-1}^2 - 2p_{\alpha,m-1}q_{\alpha,m-1}|}. \tag{3.2.18}$$

From (3.2.2) and the fact that α_m and β_m have the same partial quotients we can get

$$\alpha_m = \hat{\varepsilon} \frac{\alpha q_{\alpha,m} - p_{\alpha,m}}{p_{\alpha,m-1} - q_{\alpha,m-1}\alpha}, \quad \beta_m = \hat{\varepsilon} \frac{\beta q_{\alpha,m} - p_{\alpha,m}}{p_{\alpha,m-1} - q_{\alpha,m-1}\beta},$$

[2]Actually, in [54, 80] linear growth is proven, but the proof can be adjusted easily to get exponential growth since, in the sequence of digits for any $x \in [\alpha-1, \alpha]$, a sequence of consecutive 2's or -2's is uniformly bounded for a fixed $\alpha < 1$. All proofs are based on the recurrence relations.

where $\hat{\varepsilon} \in \{-1,1\}$ depending on the family (always 1 for (IT), always -1 for (KU) and $\varepsilon_{\alpha,m}(\alpha)$ for (N)). This gives us by (3.2.1)

$$\begin{aligned}
|\alpha_m - \beta_m| &= \left|\frac{\alpha q_{\alpha,m} - p_{\alpha,m}}{p_{\alpha,m-1} - q_{\alpha,m-1}\alpha} - \frac{\beta q_{\alpha,m} - p_{\alpha,m}}{p_{\alpha,m-1} - q_{\alpha,m-1}\beta}\right| \\
&= \left|\frac{(\alpha q_{\alpha,m} - p_{\alpha,m})(p_{\alpha,m-1} - q_{\alpha,m-1}\beta) - (\beta q_{\alpha,m} - p_{\alpha,m})(p_{\alpha,m-1} - q_{\alpha,m-1}\alpha)}{(p_{\alpha,m-1} - q_{\alpha,m-1}\alpha)(p_{\alpha,m-1} - q_{\alpha,m-1}\beta)}\right| \\
&= \left|\frac{(q_{\alpha,m}p_{\alpha,m-1} - q_{\alpha,m-1}p_{\alpha,m})\alpha + (q_{\alpha,m-1}p_{\alpha,m} - q_{\alpha,m}p_{\alpha,m-1})\beta}{p^2_{\alpha,m-1} - p_{\alpha,m-1}q_{\alpha,m-1}(\alpha+\beta) + q^2_{\alpha,m-1}\alpha\beta}\right| \\
&= \left|\frac{(-1)^m\alpha - (-1)^m\beta}{p^2_{\alpha,m-1} - p_{\alpha,m-1}q_{\alpha,m-1}(\alpha+\beta) + q^2_{\alpha,m-1}\alpha\beta}\right| \\
&< \left|\frac{\alpha - \beta}{p^2_{\alpha,m-1} - 2p_{\alpha,m-1}q_{\alpha,m-1}}\right|.
\end{aligned}$$

Fix α and let us define the set $L(\alpha) = \cup_{n=1}^{N} T_\alpha^n(\alpha) \cup \cup_{n=1}^{M} T_\alpha^n(\alpha-1)$. We show that there is an $\varepsilon > 0$ such that for all $\beta \in (\alpha-\varepsilon, \alpha)$ we have that $L(\alpha) \subset (\alpha-1,\beta)$ and $L(\beta) \subset (\alpha-1,\beta)$. Let $\varepsilon' > 0$ such that the minimum of $L(\beta)$ is attained after the same amount of iterations for all $\beta \in (\alpha-\varepsilon',\alpha)$ so that when $T_\alpha^m(\alpha) = \alpha_m = \min(L(\alpha))$ then $T_\beta^m(\beta) = \beta_m = \min(L(\beta))$ for $\beta \in (\alpha-\varepsilon',\alpha)$. This can be done since the maps $T_z^n(z)$ and $T_z^{n'}(z-1)$ are continuous in $z=\alpha$ for $n \le N$ and $n' \le M$. If the minimum is attained in a point of the orbit of $\alpha-1$ and $\beta-1$ the proof works the same.
We now find an $\varepsilon_1 > 0$ such that $|\alpha_m - \beta_m| < |\alpha - 1 - \alpha_m|$ for all $\beta \in (\alpha-\varepsilon_1,\alpha) \cap (\alpha - \varepsilon',\alpha) \cap (l(s), r(s))$ which implies $L(\beta) \subset (\alpha-1,\beta)$. Let $c = \frac{1}{|p^2_{\alpha,m-1} - 2p_{\alpha,m-1}q_{\alpha,m-1}|}$ and set $\varepsilon_1 := \frac{|\alpha-1-\alpha_m|}{c}$. We find for $\beta \in (\alpha-\varepsilon_1,\alpha)$ and from equation (3.2.18) that

$$|\alpha_m - \beta_m| < c|\alpha-\beta| < c\varepsilon_1 = |\alpha - 1 - \alpha_m|.$$

Now let $\varepsilon_2 = \alpha - \max(L(\alpha))$, then $L(\alpha) \subset (\alpha-1,\beta)$ for all $\beta \in (\alpha-\varepsilon_2,\alpha)$. Let $\varepsilon = \min(\varepsilon_1,\varepsilon_2,\varepsilon')$ then we have $L(\alpha) \subset (\alpha-1,\beta)$ and $L(\beta) \subset (\alpha-1,\beta)$ for all $\beta \in (\alpha-\varepsilon,\alpha)$.
Fix $\beta \in (\alpha-\varepsilon,\alpha) \cap (l(s),r(s))$ and pick $x \in (\beta,\alpha)$ such that x is a typical point for the system $(T_\alpha, (\alpha-1,\alpha), \mu_\alpha)$ and $x-1$ is a typical point for the system $(T_\beta, (\beta-1,\beta), \mu_\beta)$. By typical we mean that $\lim_{n\to\infty} \frac{1}{n}\#\{i < n : T_\alpha^i(x) \in (\beta,\alpha)\} = \mu_\alpha((\beta,\alpha))$. We iterate x over T_α and $x' = x-1$ over T_β. Let n_k be the k^{th} return time of x to (β,α) and m_k the k^{th} return time of x' to $(\beta-1,\alpha-1)$. We show that $n_k - m_k = (N-M)k$ and $q_{\alpha,n_k-1,}(x) = q_{\beta,m_k-1}(x')$.

Because $L(\alpha), L(\beta) \subset (\alpha-1,\beta)$ we have that x will not return to (β,α) before N iterations of T_α and x' will not return to $(\beta-1,\alpha-1)$ before M iterations. On the interval $(\alpha-1,\beta)$ we have that $T_\alpha(x) = T_\beta(x)$ whenever $T_\alpha(x) \in (\alpha-1,\beta)$. This gives us that $T_\alpha^N(x) = T_\beta^M(x')$ and $T_\alpha^{n_1-1}(x) = T_\beta^{m_1-1}(x')$ and we find $n_1 - m_1 = N - M$. Furthermore, since x is contained in the same matching interval as s we have that condition (a) holds and so

$$M_{x,x,N} = RM_{x,x-1,M}$$

which gives us $q_{x,N}(x) = q_{x,M}(x')$ and so $q_{\alpha,N}(x) = q_{\beta,M}(x')$. Since the orbits of x and x' coincide after N and M iterations respectively we have that also the fractional transformations coincide. This results in $q_{\alpha,n_1-1}(x) = q_{\beta,m_1-1}(x')$. Now $T_\beta^{m_1}(x')+1 = T_\alpha^{n_1}(x)$ and $T_\beta^{m_1}(x') \in (\beta-1,\alpha-1)$ is a typical point for $(T_\beta,(\beta-1,\beta),\mu_\beta)$ and $T_\alpha^{n_1}(x) \in (\beta,\alpha)$ is a typical point for $(T_\alpha,(\alpha-1,\alpha),\mu_\alpha)$. This means we are in the same situation as we started and so we can repeat this process and find $n_k - m_k = (N-M)k$ and $q_{\alpha,n_k-1,}(x) = q_{\beta,m_k-1}(x')$. We will now prove

$$h(T_\alpha) = (1+(M-N)\mu_\alpha((\beta,\alpha)))h(T_\beta).$$

It follows from Birkhoff's Theorem that for typical x we have

$$\lim_{n\to\infty} \frac{1}{n}\#\{i<n : T_\alpha^i(x) \in (\beta,\alpha)\} = \mu_\alpha((\beta,\alpha)).$$

This gives us

$$\lim_{k\to\infty} \frac{k}{n_k} = \mu_\alpha((\beta,\alpha)).$$

We find the following limit:

$$\begin{aligned}\lim_{k\to\infty}\frac{m_k}{n_k} &= \lim_{k\to\infty}\left(1+\frac{m_k-n_k}{n_k}\right)\\ &= \lim_{k\to\infty}\left(1+\frac{(M-N)k}{n_k}\right)\\ &= 1+(M-N)\mu_\alpha((\beta,\alpha)).\end{aligned}$$

We will now use Lemma 3.2.7 to find the wanted result

$$\begin{aligned}h(T_\alpha) &= 2\lim_{n_k\to\infty}\frac{1}{n_k-1}|\log(q_{\alpha,n_k-1}(x))|\\ &= \lim_{k\to\infty}\frac{m_k-1}{n_k-1}\frac{1}{m_k-1}|\log(q_{\beta,m_k-1}(x'))|\\ &= (1+(M-N)\mu_\alpha(\beta,\alpha))\,h(T_\beta).\end{aligned}$$

This finalises the proof. □

In the next section we primarily focus on the (IT) case. Most techniques used cannot be mimicked to prove statements for the other two families.

§3.3 Matching almost everywhere and characterisations of the bifurcation set

The main tool that lies at the basis of the results in this section is the following technical lemma. It can be used both to compare α-continued fractions of two numbers (in particular of $\alpha-1$ and $T_\alpha(\alpha) = \frac{1}{\alpha}-1$) as well as to translate an α-continued fraction into a β-continued fraction. Recall that $g = \frac{\sqrt{5}-1}{2}$.

Lemma 3.3.1. *Let $g \le \alpha \le \beta \le 1$, $x \in [\alpha-1,\alpha)$, $y \in [\beta-1,\beta)$.*

(i) If $x = y$, then $T_\beta(y) - T_\alpha(x) \in \{0,1\}$.

(ii) If $y - x = 1$, then $(x+1)(T_\beta(y)+1) = 1$.

(iii) If $(x+1)(y+1) = 1$ or $x + y = 0$, then $T_\alpha(x) + T_\beta(y) \in \{0,1\}$.

(iv) If $x + y = 1$, then

$$\begin{cases} T_\beta(y) - T_\alpha^2(x) \in \{0,1\} & \text{if } x > \frac{1}{\alpha+1}, \\ T_\beta^2(y) - T_\alpha(x) \in \{0,1\} & \text{if } y > \frac{1}{\beta+1}, \\ \big(T_\alpha(x)+1\big)\big(T_\beta(y)+1\big) = 1 & \text{otherwise.} \end{cases}$$

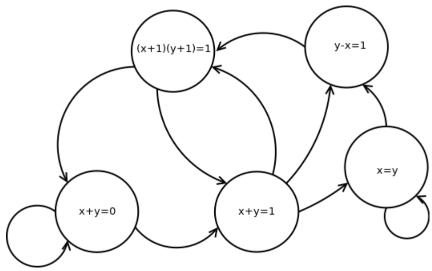


Figure 3.4: A diagram for Lemma 3.3.1.

In Figure 3.4 one can see which condition can imply which other condition.

Proof. **Case (i).** We have $T_\beta(y) - T_\alpha(x) \in \mathbb{Z} \cap (\beta - 1 - \alpha, \beta - \alpha + 1) = \{0,1\}$.
Case (ii). Since $x \geq \alpha - 1$, we have $y \geq \alpha$, thus $(x+1)(T_\beta(y)+1) = \frac{x+1}{y} = 1$.
Case (iii). Dividing the equations by xy gives us $\frac{1}{x} + \frac{1}{y} = -1$ and $\frac{1}{x} + \frac{1}{y} = 0$ respectively. This implies that $T_\alpha(x) + T_\beta(y) \in \mathbb{Z} \cap [\alpha + \beta - 2, \alpha + \beta) = \{0,1\}$.
Case (iv). If $x > \frac{1}{\alpha+1}$, then $\frac{1}{T_\alpha(x)} = \frac{1}{\frac{1}{x}-1} = \frac{x}{1-x} = \frac{1-y}{y} = \frac{1}{y} - 1$, thus $\frac{1}{y} - \frac{1}{T_\alpha(x)} = 1$ and so $T_\beta(y) - T_\alpha^2(x) \in \mathbb{Z} \cap (\beta - 1 - \alpha, \beta - \alpha + 1) = \{0,1\}$. Similarly, $y > \frac{1}{\beta+1}$ implies that $T_\beta^2(y) - T_\alpha(x) \in \{0,1\}$.
If $x \leq \frac{1}{\alpha+1}$ and $y \leq \frac{1}{\beta+1}$, then $x = 1 - y \geq \frac{\beta}{\beta+1} \geq \frac{g}{g+1} = \frac{1}{g+2} \geq \frac{1}{\alpha+2}$ and $y = 1 - x \geq \frac{\alpha}{\alpha+1} \geq \frac{1}{g+2} \geq \frac{1}{\beta+2}$. We cannot have $x = \frac{1}{\alpha+2}$ because this would imply that $\alpha = g = \beta = y$, contradicting that $y < \beta$. Similarly, we cannot have $y = \frac{1}{\beta+2}$. From $x \in (\frac{1}{\alpha+2}, \frac{1}{\alpha+1}]$ and $y \in (\frac{1}{\beta+2}, \frac{1}{\beta+1}]$, we infer that $(T_\alpha(x)+1)(T_\beta(y)+1) = (\frac{1}{x} - 1)(\frac{1}{y} - 1) = 1$. $\square$

Lemma 3.3.1 greatly simplifies when taking $\alpha = \beta$ and only looking at the orbits of $\alpha - 1$ and $\frac{1}{\alpha} - 1$ before exceeding $\frac{1}{\alpha+1}$. We use the notation

$$x_n := T_\alpha^n(\alpha - 1), \qquad y_n := T_\alpha^n(\tfrac{1}{\alpha} - 1).$$

Lemma 3.3.2. *Let $\alpha \in (g, 1]$ and $m \in \mathbb{N}$ be such that*

$$x_n \leq \tfrac{1}{\alpha+1} \quad \text{and} \quad y_n \leq \tfrac{1}{\alpha+1} \quad \text{for all } 0 \leq n < m. \tag{3.3.1}$$

Then for any $0 \leq n \leq m$ the pair (x_n, y_n) satisfies one of the following relations:

$$\begin{array}{ll} \text{(A)} & (x_n+1)(y_n+1) = 1, \\ \text{(B)} & \qquad x_n + y_n = 0, \\ \text{(C)} & \qquad x_n + y_n = 1. \end{array}$$

If $x_m > \frac{1}{\alpha+1}$ or $y_m > \frac{1}{\alpha+1}$, then $x_m + y_m = 1$.

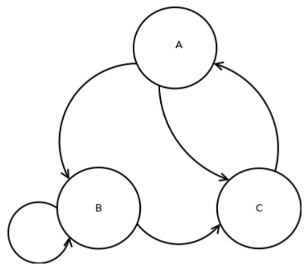


Figure 3.5: A diagram for Lemma 3.3.2.

In Figure 3.5 one can see from which state to which state you can get.

Proof. The proof is a straightforward application of Lemma 3.3.1. The pair (x_0, y_0) satisfies (A), condition (i) in Lemma 3.3.1. Let $0 \leq n < m$ then $y_n - x_n = 1$ is impossible since $x_n, y_n \in [\alpha - 1, \alpha)$. Also $x_n = y_n$ is impossible since we have $x_n \leq \frac{1}{\alpha+1}$ and $y_n \leq \frac{1}{\alpha+1}$ which implies that (x_n, y_n) always are in state (A), (B) or (C). We find that if (x_n, y_n) satisfies (A) or (B), then (x_{n+1}, y_{n+1}) satisfies (B) or (C). If (x_n, y_n) satisfies (C), then (A) holds for (x_{n+1}, y_{n+1}).
Now suppose that $x_m > \frac{1}{\alpha+1}$ and (B) holds. Then $y_m < -\frac{1}{\alpha+1} < \alpha - 1$ which contradicts with $y_m \in [\alpha-1, \alpha)$. If $x_m > \frac{1}{\alpha+1}$ and (A) holds we find $y_m = \frac{1}{x_m+1} - 1 < \frac{1}{\frac{1}{\alpha+1}+1} - 1 = -\frac{1}{2+\alpha} < \alpha - 1$ since $\alpha > g$ which also contradicts with $y_m \in [\alpha - 1, \alpha)$. Note that the role of x_m and y_m are interchangeable. We find that if $x_m > \frac{1}{\alpha+1}$ or $y_m > \frac{1}{\alpha+1}$, then $x_m + y_m = 1$. ☐

We focus now on the complement of the set

$$\tilde{\mathcal{E}} = \{\alpha \in [g, 1] : x_n \leq \tfrac{1}{\alpha+1} \text{ and } y_n \leq \tfrac{1}{\alpha+1} \text{ for all } n \geq 1\}$$

and show that it belongs to the matching set ($\tilde{\mathcal{E}}$ is the set in (3.1.4)).

Proposition 3.3.3. *Let $\alpha \in (g, 1]$ with $m \in \mathbb{N}$ such that (3.3.1) holds and $\epsilon \in \{-1, 1\}$. If $T_\alpha^m(\alpha^\epsilon - 1) > \frac{1}{\alpha+1}$, then α belongs to a matching interval J with exponents $M = m + 2 - \frac{1-\epsilon}{2}$, $N = m + 2 + \frac{1-\epsilon}{2}$. Furthermore, let $f(z) = T_z^m(z^\epsilon - 1)$. The boundaries of J satisfy $f(z) = \frac{1}{z+1}$ and $f(z) = z$ respectively.*

See Figure 3.6 for an example. For the proof of the proposition, we use the following lemma.

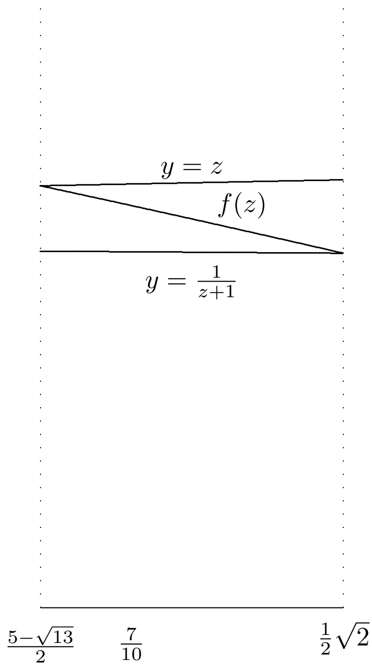


Figure 3.6: An example of Proposition 3.3.3 for $\alpha = \frac{7}{10}$ with $m = 1$, $\epsilon = 1$ and matching exponents $(3,3)$ where $f(z) = \frac{1}{z-1} + 4$.

Lemma 3.3.4. *Let $\alpha \in (g, 1]$, $m \in \mathbb{N}$ such that (3.3.1) holds, and $x_m > \frac{1}{\alpha+1}$ or $y_m > \frac{1}{\alpha+1}$. Then the maps $T_z^m(z-1)$ and $T_z^m(\frac{1}{z}-1)$ are continuous at $z = \alpha$.*

Proof. The maps $T_z^n(z-1)$ and $T_z^n(\frac{1}{z}-1)$ are continuous at $z = \alpha$ for all $1 \le n \le m$ if and only if $x_n \neq x_0$ and $y_n \neq x_0$ for all $1 \le n \le m$. Suppose that $x_n = x_0$ or $y_n = x_0$ for some $1 \le n \le m$. If (x_n, y_n) satisfies (C) and $x_n = x_0$ then $x_n + y_n = \alpha - 1 + y_n = 1$ and so $y_n = 2 - \alpha > \alpha$. Since we can use the same reasoning for $y_n = x_0$ we find that (x_n, y_n) satisfies (A) or (B). This gives $n < m$ and $x_{n+1} + y_{n+1} \in \{0, 1\}$, $x_1 + y_1 \in \{0, 1\}$. We find $x_{n+1} + y_{n+1} - (x_1 + y_1) \in \{-1, 0, 1\}$. If $x_n = x_0$, then we have $x_{n+1} = x_1$ and $x_{n+1} + y_{n+1} - (x_1 + y_1) = y_{n+1} - y_1 \in \{-1, 0, 1\}$ where we can exclude ± 1 since $y_{n+1}, y_1 \in [\alpha - 1, \alpha)$ and thus $y_{n+1} = y_1$; if $y_n = x_0$, then we have $y_{n+1} = x_1$ and thus $x_{n+1} = y_1$. We get that $\{x_{m-n}, y_{m-n}\} = \{x_m, y_m\}$, contradicting (3.3.1). □

Proof of Proposition 3.3.3. By Lemma 3.3.2, we have $x_m + y_m = 1$. Then Lemma 3.3.1 gives that $x_{m+2} = y_{m+1}$, i.e., $T_\alpha^{m+2}(\alpha - 1) = T_\alpha^{m+2}(\alpha)$, if $x_m > \frac{1}{\alpha+1}$, and that $x_{m+1} = y_{m+2}$, i.e., $T_\alpha^{m+1}(\alpha - 1) = T_\alpha^{m+3}(\alpha)$, if $y_m > \frac{1}{\alpha+1}$.
Let f be the linear fractional transformation satisfying $f(z) = T_z^m(z^\epsilon - 1)$ around $z = \alpha$, which exists by Lemma 3.3.4. By Lemma 3.3.4, we also get that $T_z^m(z-1)$ and $T_z^m(\frac{1}{z} - 1)$ are continuous at all points z with $\frac{1}{z+1} < f(z) < z$. Note that (3.3.1) holds for these points because $T_z^n(z^{\pm 1} - 1) = \frac{1}{z+1}$ implies that $T_z^{n+1}(z^{\pm 1} - 1)$ is not continuous. Since the maps $z \mapsto T_z^{n+1}(z^{\pm 1} - 1)$ are continuous at all points $\bar{z}$ for

which $\frac{1}{\bar{z}+1} < T_z^{n+1}(\bar{z}^{\pm 1} - 1) < \bar{z}$ holds. Therefore, f is expanding at these points and we have some z_-, z_+ with $f(z_-) = \frac{1}{z_-+1}$ and $f(z_+) = z_+$; let J be the open interval with boundaries z_-, z_+. Since (3.3.1) holds for all points in J, the interval J has matching exponents $N = m + 2 + \frac{1-\epsilon}{2}$, $M = m + 2 - \frac{1-\epsilon}{2}$.

Arbitrarily close to z_- and z_+, we can find points z where the minimal n such that $T_z^n(z-1) \geq \frac{1}{\alpha+1}$ or $T_z^n(\frac{1}{z}-1) \geq \frac{1}{\alpha+1}$ is different from m. Therefore, these points are in matching intervals with different matching exponents than J. Hence, by Lemma 3.1.2, they are not in J, and J is a matching interval. □

Proposition 3.3.3 shows that $\mathcal{E}_{IT} \cap [g, 1] \subset \tilde{\mathcal{E}}$.

Lemma 3.3.5. *Let $\alpha \in (g, 1]$, $z \in [\alpha - 1, g)$. The following conditions are equivalent:*

(i) $T_\alpha^n(z) = T_g^n(z)$ *for all* $n \in \mathbb{N}$.

(ii) $T_g^n(z) \geq \alpha - 1$ *for all* $n \in \mathbb{N}$.

(iii) $T_\alpha^n(z) < g$ *for all* $n \in \mathbb{N}$.

(iv) $T_\alpha^n(z) \leq \frac{1}{\alpha+1}$ *for all* $n \in \mathbb{N}$.

In particular, we have

$$\tilde{\mathcal{E}} = \left\{\alpha \in [g, 1] : T_g^n(\alpha - 1) \geq \alpha - 1 \textit{ and } T_g^n(\tfrac{1}{\alpha} - 1) \geq \alpha - 1 \textit{ for all } n \geq 1\right\}.$$

Proof. The equivalences (ii) ⇔ (i) ⇔ (iii) are direct consequences of the definition of T_α. Since $\frac{1}{1+\alpha} < g$, we have (iv) ⇒ (iii). For the converse, suppose that $T_\alpha^n(z) > \frac{1}{\alpha+1}$ for some n. Then we have $T_\alpha^{n+1}(z) = \frac{1}{T_\alpha^n(z)} - 1$, thus $T_\alpha^n(z) \geq g$ or $T_\alpha^{n+1}(z) > \frac{1}{g} - 1 = g$, hence (iii) does not hold. □

Now we prove that matching is prevalent and the only indices are $-2, 0, 2$.

Proof of Theorem 3.1.3. We have

$$\begin{aligned} \mathcal{E}_{IT} \subset \tilde{\mathcal{E}} \quad &\subset \quad \{\alpha \in (g, 1] : T_g^n(\alpha - 1) \geq \alpha - 1 \text{ for all } n \geq 1\} \\ &\subset \quad \bigcup_{k=1}^{\infty} \{\alpha \in (g, 1] : T_g^n(\alpha - 1) \geq g - 1 + \tfrac{1}{k} \text{ for all } n \geq 1\}. \end{aligned}$$

Since T_g is ergodic (with respect to an invariant measure that is equivalent to the Lebesgue measure), all the sets in this union have Lebesgue measure zero. Therefore, by Proposition 3.3.3 and Lemma 3.3.5, the matching set has full measure on $[g, 1]$. Since matching is an open condition, Lemma 3.1.2 tells us that Proposition 3.3.3 gives all matching exponents on $[g, 1]$, hence the only possible indices are $0, -2$. Recalling that for almost all matching parameters in $(1 - g, g)$ we have matching index 0, we can exploit the symmetry to conclude the proof of the theorem. □

Next we prove Theorem 3.1.4.

Proof of Theorem 3.1.4. Proposition 3.3.3 gives us $\mathcal{E}_{IT} \cap [g, 1] \subset \tilde{\mathcal{E}}$ and the set $\tilde{\mathcal{E}}$ is the set in (3.1.4) so left to show is $\tilde{\mathcal{E}} \subset \mathcal{E}_{IT}$. Let $x \in \tilde{\mathcal{E}}$ and suppose $x \notin \mathcal{E}_{IT}$ then $x \in (a, b)$ for some matching interval (a, b). From the proof of Theorem 3.1.3 we find that the complement of $\tilde{\mathcal{E}}$ covers almost everything. Furthermore, the complement is the union of matching intervals. We find that $(a, b) \subset [0, 1] \setminus \tilde{\mathcal{E}}$ and in particular $x \in [0, 1] \setminus \tilde{\mathcal{E}}$ which gives a contradiction. We find $\tilde{\mathcal{E}} \subset \mathcal{E}_{IT}$. Lemma 3.3.5 gives the second characterisation (3.1.5). □

§3.4 Dimensional results for $\mathcal{E}_{IT}$

Now that we established several characterisations of $\mathcal{E}_{IT}$ we will focus on dimensional results of $\mathcal{E}_{IT}$ in this section. We make use of two sets and the following proposition:

Proposition 3.4.1. *Let us consider the sets*

$$F_n = \{x \in [0, 1] : x = [0; a_1, a_2, \ldots] \text{ such that } a_j \geq n \text{ for all } j \in \mathbb{N}\},$$
$$C_n = \{x \in [0, 1] : x = [0; a_1, a_2, \ldots] \text{ and } a_j, \ldots, a_{j+2n-1} \neq 1^{2n} \text{ for all } j \in \mathbb{N}\}.$$

where $[0; a_1, a_2, \ldots]$ *denotes the regular continued fraction. For these sets we have* $\dim_H(F_n) > \frac{1}{2}$ *and* $\lim_{n\to+\infty} \dim_H(C_n) = 1$.

Proof. In [45] it is shown that $\dim_H(F_n) > \frac{1}{2} + \frac{1}{2\log(n+2)}$ for $n > 20$. Since $F_{n+1} \subset F_n$ we find that $\dim_H(F_n) > \frac{1}{2}$ for all $n \in \mathbb{N}$.
Let $BAD(g) = \{x \in [0, 1] : g \notin \overline{\{T^n(x) : n \in \mathbb{N}\}}\}$ where T denotes the Gauss map. Then [47] gives us that $BAD(g)$ is α-winning and therefore it has Hausdorff dimension 1. On the other hand, it is not difficult to check that $BAD(g) = \cup_n C_n$ and since C_n is an increasing sequence of sets we get

$$1 = \sup_n \dim_H(C_n) = \lim_{n\to+\infty} \dim_H(C_n).$$

□

We will now give a lemma to prove Theorem 3.1.5.

Lemma 3.4.2. *Let* $x \in [g - 1, g)$ *have the RCF expansion* $x = [a_0; a_1, a_2, a_3, ...]$ *with* $a_0 \in \{0, -1\}$, $a_j \in \mathbb{N}\ \forall j \geq 1$. *Furthermore, let* $\{x\} = x$ *for* $x \geq 0$ *and* $\{x\} = x + 1$ *for* $x < 0$. *Then there is*

- *a sequence* $j_k \to +\infty$ *such that* $0 \leq j_k - j_{k-1} \leq 2$,
- *a sequence of prefixes* $P_k \in \{\emptyset, (1), (a), (1, a)\}$,

such that $\{T_g^k(x)\} = [0; P_k, a_{j_k}, a_{1+j_k}, a_{2+j_k}, ...]$ *for all* k.

Proof. Let us set $x_k := T_g^k(x)$ and $j_{-1} = 0$ and proceed by induction. It is clear that the statement holds for $k = 0$. Now suppose the statement holds for x_k then $\{x_k\} = [0; P_k, a_{j_k}, a_{1+j_k}, a_{2+j_k}, ...]$. We treat the following cases:

(a) if $x_k > 0$ then $\{T_g(x_k)\} = \{T(x_k)\} = T(x_k)$ and $x_k = \{x_k\}$ so we find the desired form for $\{x_k\}$ with $j_k = j_{k-1} + 1$.

(b) if $x_k \in (g-1, -\frac{1}{3})$ then $\{x_k\} \in (g, \frac{2}{3})$ and so we can write $\{x_k\} = [0; 1^{2i+1}, a, X]$ with $i \geq 1$. This implies that $T_g(x_k) = \frac{1}{x_k} + 3 = [0; a+1, X]$ in case $i = 1$ and $T_g(x_k) = \frac{1}{x_k} + 3 = [0; 2, 1^{2i-3}, a, X]$ otherwise. We find $\{x_{k+1}\} = \{T_g(x_k)\} = T_g(x_k)$ so it has the desired form with $j_k = j_{k-1} + 2$.

(c) if $x_k \in (-\frac{1}{3}, 0)$ then $\{x_k\}$ is of the form $\{x_k\} = [0; 1, a, X]$ which gives us $x_k = -[0; a+1, X]$ and so $\{T_g(x_k)\} = 1 - [0; X]$. Using the relation $1 - [0; c_1 + 1, c_2, c_3, ...] = [0; 1, c_1, c_2, c_3, ...]$ we find that $\{x_{k+1}\}$ has the desired form with $j_{k+1} = j_k + 1$ if $c_1 = 0$ and $j_{k+1} = j_k$ otherwise.

So in any case the RCF expansion of x_{k+1} is a short prefix (possibly empty) followed by the tail of the RCF expansion of x. □

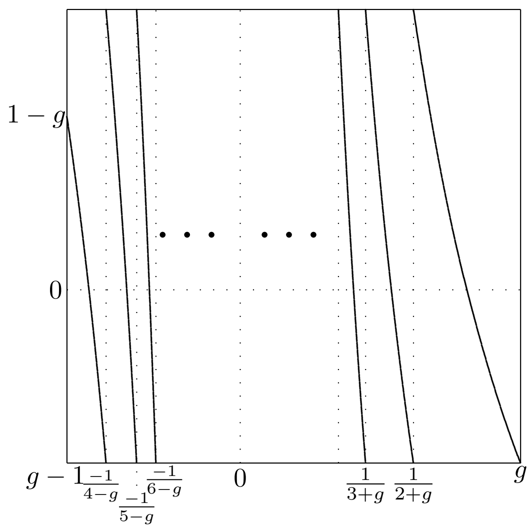


Figure 3.7: The map T_g.

Proof of Theorem 3.1.5. Let $f_a : [0,1] \to [0,1]$ be defined as $f_a(x) = \frac{1}{a+x}$ with $a \in \mathbb{N}$ and let $\hat{C}_n := f_1^{2n+5} \circ f_2(C_n)$. We first prove that $\hat{C}_n \subset \mathcal{E}_{IT}$. Let $\alpha \in \hat{C}_n$ then we can write $\alpha = [0; 1^{2n+5}, 2, X]$ for some string X without any subsequence 1^{2n}. From Lemma 3.3.5 we get that if $\{T_g^k(\frac{1}{\alpha} - 1)\} \notin (g, \alpha)$ and $\{T_g^k(\alpha - 1)\} \notin (g, \alpha)$ for all k then $\alpha \in \mathcal{E}_{IT}$. Suppose there is a k such that $\{T_g^k(\frac{1}{\alpha} - 1)\} \in (g, \alpha)$. Then $k > 2$ since $T_g(\frac{1}{\alpha} - 1) = -[0; 2, 1^{2n+1}]$ and $T_g^2(\frac{1}{\alpha} - 1) = [0; 2, 1^{2n-2}]$. Furthermore, we can write $\{T_g^k(\frac{1}{\alpha} - 1)\} = [0; 1^{2n+5}, Y]$ for some string Y. From Lemma 3.4.2 we find $\{T_g^k(\frac{1}{\alpha} - 1)\} = [0; P_{k+1}, a_{j_{k+1}}, a_{1+j_{k+1}}, a_{2+j_{k+1}}, \ldots]$.
We find that $a_{j_{k+1}}, a_{1+j_{k+1}}, a_{2+j_{k+1}}, \ldots, a_{2n-1+j_{k+1}} = 1^{2n}$ which is not a part of the initial string. This contradicts with $\alpha \in \hat{C}_n$. Since $\alpha - 1 = -[0; 2, 1^{2n+2}, X]$ we can find the same contradiction for $\alpha - 1$. Together with Lemma 3.3.5 and the proof of Theorem 3.1.4 we can conclude $\alpha \in \mathcal{E}_{IT}$. Of course, if $\hat{C}_n \subset \mathcal{E}_{IT}$ then $\cup \hat{C}_n \subset \mathcal{E}_{IT}$.

Since f_a is bi-Lipschitz for all $a \in \mathbb{N}$, the same is true for any finite composition of these maps. Since bi-Lipschitz maps preserve the Hausdorff dimension we find

$$\dim_H(C_n) = \dim_H(\hat{C}_n). \tag{3.4.1}$$

From (3.4.1) and Proposition 3.4.1 it follows that

$$\dim_H\left(\cup_{n>20}\hat{C}_n\right) = \sup_{n>20} \dim_H(\hat{C}_n) = \sup_{n>20} \dim_H(C_n) = 1.$$

Since $\hat{C}_n \subset \mathcal{E}_{IT}$ we find $\dim_H(\mathcal{E}_{IT}) = 1$. Now let $\delta > 0$. For sufficiently large N we have that $\hat{C}_n \subset (g, g+\delta]$ for all $n \geq N$ and so $\dim_H((g, g+\delta) \cap \mathcal{E}_{IT}) = 1$ which finishes the proof.

□

To get results on the Hausdorff dimension around a point $b \in \mathcal{E}_{IT} \cap \mathbb{Q}$ we need more insight in the behaviour around such a point. We establish this with the following lemma.

Lemma 3.4.3. *If $\alpha_0 \in \mathcal{E}_{IT} \cap \mathbb{Q} \cap (g, 1]$ has RCF expansion $\alpha_0 = [0; a_1, a_2, ..., a_k]$ then there is a $c \in \mathbb{N}$ such that*

$$E_{\alpha_0} := \{\alpha \in [g, 1] : \ \alpha = [0; a_1, a_2, ..., a_k, c, c_1, c_2, ...] \quad \text{with } c_j > a_2 + 1 \ \ \forall j\} \subset \mathcal{E}_{IT}$$

with $c_1, c_2, \ldots$ either a finite (possibly empty) or an infinite sequence. Furthermore, we have that matching condition (3.1.2) holds for α_0 with $N - M = 1$.

Proof. Let $\alpha_0 \in \mathcal{E}_{IT} \cap \mathbb{Q} \cap (g, 1]$ and define $x_n = T^n_{\alpha_0}(\alpha_0 - 1)$ and $y_n = T^n_{\alpha_0}\left(\frac{1}{\alpha_0} - 1\right) = T^{n+1}_{\alpha_0}(\alpha_0)$. Since $\alpha_0 \in \mathbb{Q}$, both the T_{α_0}-orbit of $\alpha_0 - 1$ and the T_{α_0}-orbit of $1/\alpha_0 - 1$ will eventually reach zero, and since $\alpha_0 \in \mathcal{E}_{IT}$ this will happen in one of the states (A), (B) or (C) from Lemma 3.3.2. Let m be minimal such that $x_m = 0$. From the equations for (A), (B), (C) we have that (C) cannot happen and that $y_m = 0$. This gives us that $T^m_{\alpha_0}(\alpha_0 - 1) = T^{m+1}_{\alpha_0}(\alpha_0)$ and matching condition (3.1.2) holds with $N - M = 1$.

Now observe that Lemma 3.3.5 (ii) gives us that $\{T^n_g(\alpha_0)\} \notin [g, \alpha_0]$ and $\{T^n_g(\alpha_0 - 1)\} \notin [g, \alpha_0]$ for all $n \in \mathbb{N}$. This implies that there is a $\delta > 0$ such that for all $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$ we have $\{T^n_g(\alpha)\} \notin [g, \alpha]$ for $0 \leq n \leq m' + 1$ and $\{T^j_g(\alpha - 1)\} \notin [g, \alpha]$ for $0 \leq j \leq m'$ with m' minimal such that $T^{m'}_g(\frac{1}{\alpha_0} - 1) = 0$. Pick $c \in \mathbb{N}$ such that $E_{\alpha_0} \subset (\alpha_0 - \delta, \alpha_0 + \delta)$ and α_0 and α have the same partial quotients in their g-expansion up to m' for all $\alpha \in E_{\alpha_0}$. Let $\alpha \in E_{\alpha_0}$. From Lemma 3.4.2 we find $\{T^{m'}_g(\frac{1}{\alpha} - 1)\} = [0; P_{m'}, c, c_1, \ldots]$ or $\{T^{m'}_g(\frac{1}{\alpha} - 1)\} = [0; P_{m'}, c_1, \ldots]$. In the first case we find $T^{m'}_g(\frac{1}{\alpha} - 1) > 0$ and $P_{m'} = \emptyset$. Since $\{T^j_g(\alpha - 1)\} \notin [g, \alpha]$ for $0 \leq j \leq m'$ we find $T^j_g(\frac{1}{\alpha} - 1) \notin [g - 1, \alpha - 1]$ and so $T^j_g(\frac{1}{\alpha} - 1) = T^j_\alpha(\frac{1}{\alpha} - 1)$ for $0 \leq j \leq m'$. This gives us $T^{m'}_\alpha(\frac{1}{\alpha} - 1) = [0; c, c_1, c_2 \ldots]$. It follows that $T^{m'+j}_\alpha(\frac{1}{\alpha} - 1) = [0; c_j, c_{j+1}, \ldots]$ for all $j \in \mathbb{N}$. Note that matching did not happen before $T^{m'}_\alpha(\frac{1}{\alpha} - 1)$ so that $(T^{m'}_\alpha(\frac{1}{\alpha} - 1), T^{m'}_\alpha(\alpha - 1))$ is in one of the states $(A), (B), (C)$ from Lemma 3.3.2. State (C) would imply that $T^{m'}_\alpha(\alpha - 1) > \alpha$ so we can exclude it. If we are in state (A) we find $T_\alpha^{m'}(\alpha-1) = -[0; c, c_1, c_2 \ldots]$ and so $T_\alpha^{m'+j}(\alpha-1) = -[0; c, c_1, c_2 \ldots]$. We find $\alpha \in \mathcal{E}_{IT}$. If we are in state (B) we have $T_\alpha^{m'}(\alpha-1) = -[0; c+1, c_1, c_2 \ldots]$ and we can draw the same conclusion. In the second case we find $P_{m'} = c+1$ and no difference to the proof of the first case.

□

Now Theorem 3.1.6 follows almost directly.

Proof of Theorem 3.1.6. The fact that there are infinitely many rationals in $\mathcal{E}_{IT}$ is given by the fact that $\frac{n-1}{n} \in \mathcal{E}_{IT}$ for all $n \in \mathbb{N}_{\geq 3}$. Furthermore, $\mathcal{E}_{IT} \cap \mathbb{Q} \cap (g, 1]$ does not have isolated points since in Lemma 3.4.3 one can take a string c_1 with c_1 arbitrarily high. For the dimensional result we reason as follows. The composition $f_{a_1} \circ \ldots \circ f_{a_k}$ is bi-Lipschitz. Furthermore, from Lemma 3.4.3 it follows that $f_{a_1} \circ \ldots \circ f_{a_k}(F_n) \subset \mathcal{E}_{IT}$ for all $n > N$ for some N. Using Proposition 3.4.1 and symmetry the theorem now follows. □

§3.5 Final observations and remarks

In the first part of this chapter, we have seen that a lot of machinery works for all three families. In the second part we have seen some differences. Since for the (KU) and (N) case the set of possible matching indices is $\mathbb{Z}$ we cannot expect that we can obtain a tool like Lemma 3.3.1 for these families.

Note that this chapter was concerned mostly with matching and the non-matching set rather than the entropy as a function of α. We do know that the set for which the entropy as a function of α is not locally monotonic is a subset of $\mathcal{E}_{IT}$ however we do not know whether equality holds. Furthermore, we did not prove the fact that the entropy function is continuous. For the matching set this should follow from the fact that we have

$$h(T_\alpha) = (1 + (M-N)\mu_\alpha((\beta, \alpha)))h(T_\beta)$$

for $\beta < \alpha$ on the same matching interval and the fact that $\mu_\alpha((\beta, \alpha))$ is continuous in β. To prove continuity on the non-matching set might be more challenging.

Worth mentioning is that Wolfgang Steiner and Hitoshi Nakada have (unpublished) results on the natural extension for Ito Tanaka's continued fractions. In particular they can show that for every $\alpha \in [0,1]$ there is a solid rectangle $[\alpha-1, \alpha] \times [A, B]$ that is fully contained in the domain of the natural extension. This implies that the invariant measure has full support.

Now it is proven that for all three families matching holds almost everywhere, one can take the challenge of mixing the maps. When, instead of iterating over one fixed map, you flip a coin to decide whether you pick $S(x) = \frac{1}{x}$ or $S(x) = -\frac{1}{x}$ the orbit of α and $\alpha - 1$ become random. Can we prove that for almost every $\alpha \in [0,1]$ we have matching almost surely? And what does matching imply in this case? A different toolbox would be needed to tackle this problem.