Matching, entropy, holes and expansions

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Title: Matching, entropy, holes and expansions
Issue Date: 2019-12-10

CHAPTER 2

Natural extensions, entropy and infinite systems

This chapter is based on joint work with Charlene Kalle, Marta Maggioni and Sara Munday.

Abstract

In this chapter we leave the realm of finite ergodic theory and enter the realm of the infinite. We introduce a family of mappings $\{T_\alpha\}$ with $\alpha \in [0,1]$ related to the Gauss map. This family interpolates between the Gauss map and a map isomorphic to the backward continued fraction map. For $\alpha < \frac{1}{2}\sqrt{2}$ we find an explicit expression of an invariant measure that is absolutely continuous with respect to the Lebesgue measure. We do so by means of the natural extension. For $\alpha \leq \frac{\sqrt{5}-1}{2}$ we calculate the Krengel entropy which equals $\frac{\pi^2}{6}$. Furthermore, we show that the maps are so called basic AFN-maps which leads to several nice properties such as the existence of a weak law of large numbers.

§2.1 Introduction

In 1981 Nakada introduced a family of continued fraction transformations, now mainly known as α-continued fraction maps [82]. For each $\alpha \in [0,1]$, one can define the map $S_\alpha : [\alpha-1, \alpha] \to [\alpha-1,\alpha]$ by

$$S_\alpha(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor.$$

Let $x \in [\alpha-1,\alpha]$, define $\varepsilon_0(x) = sign(x)$ and $d_{\alpha,1}(x) = \lfloor |\frac{1}{x}| + 1 - \alpha \rfloor$. By setting $d_{\alpha,n}(x) = d_1(S_\alpha^n(x))$ and $\varepsilon_{\alpha,n}(x) = \varepsilon_0(S_\alpha^n(x))$ we find the following expansion for x

$$x = \cfrac{\varepsilon_0(x)}{d_{\alpha,1}(x) + \cfrac{\varepsilon_1(x)}{d_2(x) + \cfrac{\varepsilon_{\alpha,2}(x)}{d_{\alpha,3}(x) + \ddots}}}.$$

Nakada studied this type of expansions for $\alpha \in \left[\frac{1}{2},1\right]$ and in [80] Marmi, Moussa and Yoccoz extended the study and also included the values $\alpha \in \left[0,\frac{1}{2}\right)$. In [82] Nakada constructed a natural extension for the maps S_α and gave a thorough analysis of the map's metric and ergodic properties for $\alpha \in [\frac{1}{2},1]$. For $\alpha \in [\sqrt{2}-1,\frac{1}{2})$ the natural extension can be found in [79] and for $\alpha \in [\frac{\sqrt{10}-2}{3}, \sqrt{2}-1]$ in [35]. Each of these maps admits a unique absolutely continuous invariant measure ν_α. Nakada already started the study of the dependence on α of the entropy of the map S_α with respect to ν_α for $\alpha \in \left[\frac{1}{2},1\right]$. The function mapping α to the metric entropy of S_α with respect to ν_α, turned out to have a very intricate structure for $\alpha \in [0,\frac{1}{2})$ and has been extensively studied. See for example [18, 65, 79, 84] and the references therein.

It was shown in [27] that the family of folded α-continued fraction maps $\hat{S}_\alpha$ is a particular instance of what the authors called D-continued fraction maps. Folded α-continued fractions are almost the same as α-continued fractions from a metric point of view. The D-continued fraction maps are variants of the classical Gauss map $T : x \mapsto \frac{1}{x} \pmod 1$ where one specifies a region $D \subseteq [0,1]$, such that on $[0,1] \setminus D$ one uses the Gauss map and on D one uses a flipped version of the Gauss map, $F = 1 - T$. It is shown that the folded α-continued fraction maps are obtained by taking $D = \bigcup_{n\geq 2} \left(\frac{1}{n}, \frac{1}{n+\alpha-1}\right]$.

In this chapter, we consider the natural counterparts of the folded α-continued fractions, which we call *flipped α-continued fraction maps.* They are obtained by taking $\alpha \in [0,1]$ and setting

$$D = D_\alpha = \bigcup_{n\geq 1} \left(\frac{1}{n+\alpha}, \frac{1}{n}\right].$$

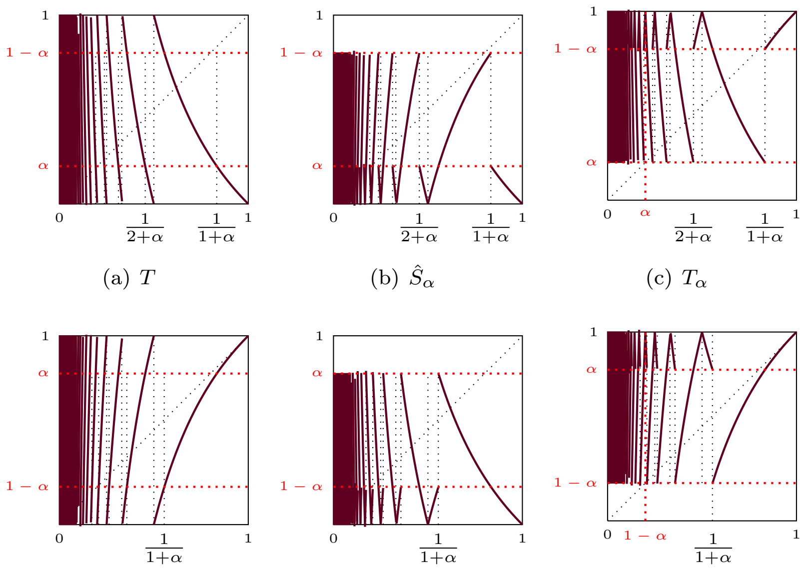
We define $T_\alpha : I_\alpha \to I_\alpha$ as

$$T_\alpha(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \notin D_\alpha, \\[2ex] \left(1 + \left\lfloor \dfrac{1}{x} \right\rfloor\right) - \dfrac{1}{x} & \text{if } x \in D_\alpha \end{cases} \tag{2.1.1}$$

where $I_\alpha := [\min(\alpha, 1-\alpha), 1]$. In terms of the Gauss map T we can define it as follows. Note that $D_\alpha = \{x \in [0,1] : T(x) < \alpha\}$ and

$$T_\alpha(x) = \begin{cases} T(x), & \text{if } x \notin D_\alpha, \\ 1 - T(x), & \text{if } x \in D_\alpha. \end{cases}$$

See Figure 2.1 for a couple of examples.



(a) T (b) $\hat{S}_\alpha$ (c) T_α

(d) $F = 1 - T$ (e) $\hat{S}_\alpha$ (f) T_α

Figure 2.1: The Gauss map T and the flipped map $F = 1 - T$ in (a) and (d). The folded α-continued fraction map $\hat{S}_\alpha$ and the flipped α-continued fraction map T_α for $\alpha < \frac{1}{2}$ in (b) and (c) and for $\alpha > \frac{1}{2}$ in (e) and (f) respectively.

If $\alpha = 0$, we obtain the Gauss map T and if $\alpha = 1$ we obtain the map $1 - T$. Since these maps are well known, we omit them from our analysis. Since the domain of the map T_α is given by $I_\alpha = [\min\{\alpha, 1-\alpha\}, 1]$ it has only finitely many branches. Note that for any $\alpha \in (0,1)$ the map T_α has an indifferent fixed point at 1. This causes the existence of an *infinite* absolutely continuous invariant measure μ_α. This makes the family of maps $\{T_\alpha : I_\alpha \to I_\alpha\}_{\alpha \in (0,1)}$ interesting to study as a natural family of infinite measure systems that do in general not have a Markov partition but have many other nice properties. Let us turn to stating our results. We construct for each

$\alpha \in \big(0, \frac{1}{2}\sqrt{2}\big)$ the natural extension of the map T_α. From the natural extension we obtain the absolutely continuous invariant measures for the maps T_α.

Theorem 2.1.1. *Let $0 \leq \alpha \leq \frac{1}{2}\sqrt{2}$, let $\mathcal{B}_\alpha$ be the Borel σ-algebra on $[\min(\alpha, 1-\alpha), 1]$. The absolutely continuous measure μ_α on $([\min(\alpha, 1-\alpha), 1], \mathcal{B}_\alpha)$ with density*

$$f_\alpha(x) = \begin{cases} \frac{1}{x}\mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha},1]}(x) + \frac{1}{1-x}\mathbf{1}_{[1-\alpha,1]}(x) & \text{for } \alpha \in [0, \frac{1}{2}], \\ \frac{1}{1-x}\mathbf{1}_{[1-\alpha,\alpha]}(x) + \frac{1}{x(1-x)}\mathbf{1}_{[\alpha, \frac{1-\alpha}{\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{1-\alpha}{\alpha},1]}(x) & \text{for } \alpha \in (1/2, g], \\ (\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}})\mathbf{1}_{[1-\alpha, \frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x) + & \\ (\frac{1}{1-x} + \frac{1}{x} - \frac{1}{x+\frac{1}{g}})\mathbf{1}_{[\alpha, \frac{2\alpha-1}{1-\alpha}]}(x) + \frac{x^2+1}{x(1-x^2)}\mathbf{1}_{[\frac{2\alpha-1}{1-\alpha},1]}(x) & \text{for } \alpha \in (g, \frac{2}{3}], \\ (\frac{1}{1-x} + \frac{1}{x+\frac{1}{g-1}})\mathbf{1}_{[1-\alpha, \frac{2\alpha-1}{\alpha}]}(x) + \frac{1}{1-x}\mathbf{1}_{[\frac{2\alpha-1}{\alpha},\alpha]}(x) + & \\ \frac{1+gx^2}{x-(gx)^2-gx^3}\mathbf{1}_{[\alpha, \frac{1-\alpha}{2\alpha-1}]}(x) + & \\ (\frac{2}{1-x^2} - \frac{x+1}{x^2+(2g+1)x+1} + \frac{1}{x})\mathbf{1}_{[\frac{1-\alpha}{2\alpha-1},1]}(x) & \text{for } \alpha \in (\frac{2}{3}, \frac{1}{2}\sqrt{2}], \end{cases}$$

is the unique σ-finite infinite invariant measure for T_α.

For infinite measure systems, there are various notions of entropy generalising the notion of metric entropy for finite measure systems. From these notions Krengel entropy is probably the most used. For $\alpha \in (0, g]$ we are able to calculate this value which brings us to our second result.

Theorem 2.1.2. *For any $\alpha \in (0, g]$ the Krengel entropy $h_{Kr,\mu_\alpha}(T_\alpha)$ for the measure μ_α from Theorem 2.1.1 is equal to $\frac{\pi^2}{6}$.*

Even though we have the invariant measure for $g < \alpha \leq \frac{1}{2}\sqrt{2}$ the exact value of the Krengel entropy eludes us. From evaluating the expressions obtained numerically, one would believe we would also find the value $\frac{\pi^2}{6}$ in that case. Since we have an infinite system we can study the asymptotic behaviour of the maps near the indifferent fixed point 1 and give a finer analysis of their excursion times to the interval $\big(\frac{1}{1+\alpha}, 1\big]$. To be more precise, we study the wandering rates and return sequences. The following proposition summarises the properties that hold for our systems in case $\alpha \in (0, \frac{1}{2})$.

Proposition 2.1.3. *For each $\alpha \in \big(0, \frac{1}{2}\big)$ and each $n \geq 1$ we have $w_n(T) \sim \log n$ for the wandering rate and $a_n(T) \sim \frac{n}{\log n}$ for the return time. Furthermore, a weak law of large numbers holds for T_α:*

$$\frac{\log n}{n} \sum_{k=0}^{n-1} f \circ T_\alpha^k \xrightarrow{\mu_\alpha} \int_{I_\alpha} f \, d\mu_\alpha, \quad \text{for } f \in L_1(\mu_\alpha) \text{ and } \int_{I_\alpha} f \, d\mu_\alpha \neq 0.$$

This chapter is outlined as follows. In Section 2.2 we give some preliminaries. We set up some framework on insertions and singularisations and look at the consequences on the natural extension and entropy. Then we look back to our map and put this into context. In Section 2.3 we define the natural extension of the maps T_α with $\alpha \in (0, \frac{1}{2}\sqrt{2})$ and use this to obtain Theorem 2.1.1. After that we prove Theorem 2.1.2 in Section 2.4. In Section 2.5 several dynamical properties are shown for $\alpha \in (0,1)$. Furthermore we show Proposition 2.1.3. We end this chapter with final observations and remarks.

§2.2 Preliminaries

The properties of various continued fraction expansions are a classical object of study. In 1913 Perron introduced the notion of *semi-regular* continued fraction expansions (see [89]). These continued fractions are a finite or infinite expression for $x \in (0,1]$ of the following form:

$$x = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \cfrac{\varepsilon_2}{d_3 + \ddots}}},$$

where $\varepsilon_n \in \{-1,1\}$, $d_n \in \mathbb{N}$ and $d_n + \varepsilon_n \geq 1$ for each $n \geq 1$. We denote the semi-regular continued fraction expansion of a number x by

$$x = [0; 1/d_1, \varepsilon_1/d_2, \varepsilon_2/d_3, \ldots].$$

In case we have $\varepsilon_n = 1$ for all $n \in \mathbb{N}$ we find a regular continued fraction and simply write $x = [0; d_1, d_2, \ldots]$.

In general a number has many different semi-regular continued fraction expansions. There are two well studied operations that convert one semi-regular continued fraction expansion of a number to another: singularisation and insertion. Both operations were already introduced in [89] and later appeared in many other places in literature (see for example [63]). *Singularisation* is based on the following equality holding for $a, b \in \mathbb{N}$, $\varepsilon \in \{-1,1\}$ and $\xi \in [0,1]$:

$$a + \cfrac{\varepsilon}{1 + \cfrac{1}{b+\xi}} = a + \varepsilon + \frac{-\varepsilon}{b+1+\xi}.$$

In terms of the semi-regular continued fractions of numbers, it can be written as follows. If

$$x = [0; 1/d_1, \varepsilon_1/d_2, \ldots, \varepsilon_{n-1}/d_n, \varepsilon_n/1, 1/d_{n+2}, \ldots]$$

then

$$x = [0; 1/d_1, \varepsilon_1/d_2, \ldots, \varepsilon_{n-1}/(d_n + \varepsilon_n), -\varepsilon_n/(d_{n+2}+1), \ldots].$$

The inverse operation of singularisation is *insertion*, which is based on the following equality and holds for $a, b \in \mathbb{N}$ with $b \geq 2$ and $\xi \in [0,1]$:

$$a + \frac{1}{b+\xi} = a + 1 + \cfrac{-1}{1 + \cfrac{1}{b-1+\xi}}.$$

In terms of the semi-regular continued fractions of numbers, it gives the following. If

$$x = [0; 1/d_1, \varepsilon_1/d_2, \ldots, \varepsilon_{n-1}/d_n, 1/d_{n+1}, \varepsilon_{n+1}/d_{n+2}, \ldots],$$

with $d_{n+1} \geq 2$, then

$$x = [0; 1/d_1, \varepsilon_1/d_2, \ldots, \varepsilon_{n-1}/(d_n+1), -1/1, 1/(d_{n+1}-1), \varepsilon_{n+1}/d_{n+2}, \ldots].$$

Singularisations and insertions are directly related to D-continued fraction expansions. In [27] it is explained that if $x \in \cup_{n=1}^{\infty}(\frac{2}{2n+1}, \frac{1}{n}]$ then the map $1-T$ acts as an insertion on x. If $x \notin \cup_{n=1}^{\infty}(\frac{2}{2n+1}, \frac{1}{n}]$ then the map $1-T$ acts as a singularisation. Furthermore, by the action of singularisation one removes a convergent in the sequence $(\frac{p_n}{q_n})_{n\geq 1}$. On the other hand, by insertion one adds a mediant of two consecutive convergents (see [63]).

In the next section we will see how insertion can affect the natural extension and the entropy. This is in essence the opposite of what singularisation would do which is explained in [29].

§2.2.1 Insertions and the natural extension

Let $D \subset [0,1] \times \mathbb{R}$ and define $\mathcal{T}_D : \Omega_D \to \Omega_D$ by

$$\mathcal{T}_D(x,y) := \left(T_D(x), \frac{\varepsilon_D(x,y)}{d_1(x)+y} \right), \quad (x,y) \in \Omega_D$$

where $\varepsilon_D(x,y) = -1$ if $(x,y) \in D$ and 1 otherwise. Note that Ω_D is not yet specified. The game is to find a suitable Ω_D such that $\mathcal{T}_D$ is bijective almost everywhere because of the following proposition.

Proposition 2.2.1. *Let $\Omega_D \subset [0,1] \times \mathbb{R}$ such that $\mathcal{T}_D$ is bijective almost everywhere. Then*

$$\mu_D(A) = \int_A \frac{1}{(1+xy)^2}\, d\lambda(x)$$

is an invariant measure for $\mathcal{T}_D$.

The proof is essentially the same as for [82]. We now show that for a certain class of subsets of $D \subset [0,1] \times [0,1]$ we can easily find the natural extension. From the natural extension we can show that there is a decrease in entropy when comparing it with the regular continued fraction. For this subset, the corresponding continued fraction algorithm only uses insertions and no singularisations.

For suitable *insertion sets* D we show that $h(\mathcal{T}_D) = \frac{h(\mathcal{T})}{1+\bar{\mu}(D)}$ with h the (metric) entropy and $\bar{\mu}$ the 2-dimensional Gauss measure from page 8. Let

$$D^* = \bigcup_{n=1}^{\infty} \left(\frac{2}{2n+1}, \frac{1}{n} \right) \times [0,1]$$

and pick $D \subset D^*$ such that $\mathcal{T}(D) \cap D = \emptyset$ with $\mathcal{T}$ the natural extension map of the regular continued fraction (see Theorem 1.1.12 from page 8). Note that this does not include the case where the measure of the system of $\mathcal{T}_D$ is infinite. Neither the case where we do "insertion on the newly added regions". We will first find the natural extension domain of $\mathcal{T}_D$ by building it from the natural extension domain of $\mathcal{T}$. Note that $\mathcal{T}(x,y) = \mathcal{T}_D(x,y)$ for $(x,y) \in D^c$. The new natural domain will be $(([0,1] \times [0,1]) \backslash \mathcal{T}(D)) \cup \mathcal{T}_D(D) \cup \mathcal{T}_D^2(D)$. We first show that there is no overlap.
Note that $\mathcal{T}_D(x,y) \notin [0,1] \times [0,1]$ since $\frac{-1}{n+1+y} < 0$. We also have that $\mathcal{T}_D^2(x,y) \notin [0,1] \times [0,1]$ because $\cfrac{1}{1+\cfrac{-1}{n+1+y}} = 1 + \frac{1}{n+y} > 1$. For the same reason we find $\mathcal{T}_D(D) \cap \mathcal{T}_D^2(D) = \emptyset$.

Now we show that $\mathcal{T}^2(D) = \mathcal{T}_D^3(D)$ which gives that "no holes appear" in the natural domain of $\mathcal{T}_D$ besides $\mathcal{T}(D)$, see also Figure 2.2. Let $(x,y) \in D$ then we can write $x = \frac{1}{n+\frac{1}{k+z}}$ with $k \in \mathbb{N}$ and $z \in [0,1]$ since $(x,y) \in D^*$. We find $\mathcal{T}^2(x,y) = \left(z, \frac{1}{k+\frac{1}{n+y}} \right)$,

$$\mathcal{T}_D(x,y) = \left(\cfrac{1}{1+\cfrac{1}{k-1+z}}, \frac{-1}{n+1+y} \right), \quad \mathcal{T}_D^2(x,y) = \left(\frac{1}{k-1+z}, \cfrac{1}{1+\cfrac{-1}{n+1+y}} \right)$$

and

$$\mathcal{T}_D^3(x,y) = \left(z, \cfrac{1}{k-1+\cfrac{1}{1+\cfrac{-1}{n+1+y}}} \right).$$

Note that

$$\cfrac{1}{k-1+\cfrac{1}{1+\cfrac{-1}{n+1+y}}} = \cfrac{1}{k+\cfrac{1}{n+y}}.$$

We will derive a formula for the entropy of the new system. Let $A := ([0,1] \times [0,1]) \backslash \mathcal{T}(D)$ and denote by $\hat{\mathcal{T}} : A \to A$ the induced transformation for $\mathcal{T}$ on A and $\hat{\mathcal{T}}_D : A \to A$ the induced transformation for $\mathcal{T}_D$ on A, see page 9 for the definition. Abramov's entropy formula give us:

$$h(\hat{\mathcal{T}}) = \frac{h(\mathcal{T})}{\bar{\mu}(A)} \tag{2.2.1}$$

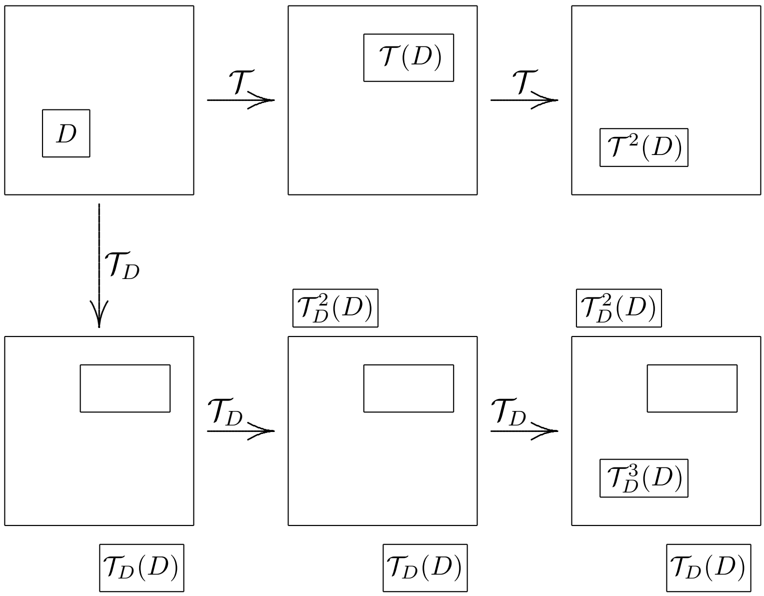


Figure 2.2: A diagram of the construction of the natural extension domain.

and

$$h(\hat{\mathcal{T}}_D) = \frac{h(\mathcal{T}_D)}{\nu(A)}, \tag{2.2.2}$$

where $\bar{\mu}$ is the invariant measure for the original system and ν for the new system i.e. $\bar{\mu}(A) = C \int_A f(x,y)\lambda \times \lambda(x,y)$ and $\nu(A) = C' \int_A f(x,y)\lambda \times \lambda(x,y)$ with $f(x,y) = \frac{1}{(1+xy)^2}$. We will now find an expression for C'. We have

$$\begin{aligned}
(C')^{-1} &= \int_{\Omega_D} f(x,y)\lambda \times \lambda(x,y) \\
&= \int_{\Omega} f(x,y)\lambda \times \lambda(x,y) - \int_{\mathcal{T}(D)} f(x,y)\lambda \times \lambda(x,y) \\
&\quad + \int_{\mathcal{T}_D(D)} f(x,y)\lambda \times \lambda(x,y) + \int_{\mathcal{T}_D^2(D)} f(x,y)\lambda \times \lambda(x,y) \\
&= \int_{\Omega} f(x,y)\lambda \times \lambda(x,y) + \int_{D} f(x,y)\lambda \times \lambda(x,y) \\
&= \frac{1}{C} + \int_{D} f(x,y)\lambda \times \lambda(x,y) \\
&= \frac{1}{C}(1 + \bar{\mu}(D)).
\end{aligned}$$

This gives us $C' = \frac{C}{1+\bar{\mu}(D)}$ which results in

$$\bar{\mu}(A) = (1 + \bar{\mu}(D))\nu(A). \tag{2.2.3}$$

Since $\mathcal{T}^2(D) = \mathcal{T}_D^3(D)$ we have that $\hat{\mathcal{T}} = \hat{\mathcal{T}}_D$. Using (2.2.1) and (2.2.2) we now find

$$\frac{h(\mathcal{T})}{\bar{\mu}(A)} = \frac{h(\mathcal{T}_D)}{\nu(A)},$$

which together with (2.2.3) gives

$$\frac{h(\mathcal{T})}{1 + \bar{\mu}(D)} = h(\mathcal{T}_D).$$

Given this result one might expect that, in our setting, we would have a decrease in entropy whenever we add insertions. Though, Theorem 2.1.1 shows this is not the case (even though for $\alpha \leq \frac{1}{2}$ there are only insertions). This illustrates well that there are differences between similar systems when one system is an infinite system and the other is a finite one. Another observation is that for Nakada's α-continued fractions we have only singularisations for $\alpha \in (g, 1]$. In a similar way as for insertions, one finds that the entropy as a function of α is in this case decreasing on $(g, 1]$ since the measure of the singularisation region decreases as α goes to one.

§2.2.2 Back to our map

The map T_α generates semi-regular continued fraction expansions of real numbers. For any $\alpha \in (0, 1)$ and any $x \in I_\alpha$, the map T_α from (2.1.1) defines a continued fraction expansion for any $x \in I_\alpha$ in the following way. Define the partial quotients $d_k = d_k(x)$ and the signs $\varepsilon_k = \varepsilon_k(x)$ by $d_k(x) := d_1(T_\alpha^{k-1}(x))$, where

$$d_1(x) := \begin{cases} \lfloor \frac{1}{x} \rfloor, & \text{if } x \notin D_\alpha, \\ \lfloor \frac{1}{x} \rfloor + 1, & \text{otherwise;} \end{cases}$$

and by $\varepsilon_k(x) := \varepsilon_1(T_\alpha^{k-1}(x))$, where

$$\varepsilon_1(x) := \begin{cases} 1, & \text{if } x \notin D_\alpha, \\ -1, & \text{otherwise.} \end{cases}$$

With this notation the map T_α can be written as

$$T_\alpha(x) = \varepsilon_1(x)\left(\frac{1}{x} - d_1(x)\right)$$

and so

$$x = \frac{1}{d_1 + \varepsilon_1 T_\alpha(x)} = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \ddots + \cfrac{\varepsilon_{n-1}}{d_n + \varepsilon_n T_\alpha^n(x)}}}. \tag{2.2.4}$$

Denote by $(p_k/q_k)_{k\geq 1}$ the sequence of convergents of such an expansion, i.e., we write

$$\frac{p_k}{q_k} = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \ddots + \cfrac{\varepsilon_{k-1}}{d_k}}}.$$

Since we obtained T_α from the Gauss map by flipping on the domain D_α, it follows immediately from [27, Theorem 1] that for any $x \in I_\alpha$ the expression from (2.2.4) converges to a continued fraction expansion of x: $\lim_{k\to\infty} \frac{p_k}{q_k} = x$. Therefore, we can write

$$x = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \ddots + \cfrac{\varepsilon_{k-1}}{d_k + \ddots}}} =: [0; 1/d_1, \varepsilon_1/d_2, \varepsilon_2/d_3, \ldots]_\alpha,$$

which we call the *flipped α-continued fraction expansion* of x. For $x \in [0,1]$ it is well known that the regular continued fraction is finite if and only if $x \in \mathbb{Q}$. In our case 0 is not in the domain of T_α. Therefore, we cannot find finite expansions. Instead of a finite continued fraction all rational numbers will end in 1 where $1 = [0; 1/2, -1/2, -1/2, \ldots]_\alpha$ for all $\alpha \in (0,1)$.

Proposition 2.2.2. *Let $\alpha \in (0,1)$ and $x \in I_\alpha$ be given. Then $x \in \mathbb{Q}$ if and only if there is an $N \geq 0$ such that $T_\alpha^N(x) = 1$.*

Proof. If there is an $N \geq 0$ such that $T_\alpha^N(x) = 1$, then it follows immediately that $x \in \mathbb{Q}$. Suppose $x \in \mathbb{Q}$. Note that $T_\alpha^n(x) \in \mathbb{Q} \cap I_\alpha$ for all $n \geq 0$ and write $T_\alpha^n(x) = \frac{s_n}{t_n}$ with $s_n, t_n \in \mathbb{N}$ and t_n as small as possible. Assume for a contradiction that $T_\alpha^n(x) \neq 1$ for all $n \geq 1$. Then $s_n < t_n$ and since either $T_\alpha^{n+1}(x) = \frac{t_n - ks_n}{s_n}$ or $T_\alpha^{n+1}(x) = \frac{(k+1)s_n - t_n}{s_n}$, we get $0 < t_{n+1} < t_n$. This gives a contradiction. □

§2.3 Natural extensions for our maps

To find the invariant density of the absolutely continuous invariant measure of T_α, we construct a natural extension domain such that $\mathcal{T}_\alpha$ is almost bijective and minimal from a measure theoretic point of view. In that case Proposition 2.2.1 gives us the wanted result. We are able to construct the domain for $\alpha \in (0, \frac{1}{2}\sqrt{2}]$. We will go through a subset of the parameter space to show the method. Invariant densities of other values are found in the same way but with different computations. Let $\alpha \in (0, \frac{1}{2})$ such that $\frac{1}{n+\alpha} < \alpha < \frac{1}{n}$ with $n \in \mathbb{N}_{\geq 2}$. We define $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ as

$$(x,y) \mapsto \left(T_\alpha(x), \frac{\varepsilon_1(x)}{d_1(x) + y} \right)$$

where

$$\Omega_\alpha := \left[\alpha, \frac{\alpha}{1-\alpha}\right] \times \left[0, \infty\right) \cup \left(\frac{\alpha}{1-\alpha}, 1-\alpha\right] \times \left[0,1\right] \cup \left(1-\alpha, 1\right] \times \left[-1, 1\right].$$

	Ω_α	$\mathcal{T}_\alpha(\Omega_\alpha)$	
a	$\left[\alpha, \frac{1}{n}\right] \times \left[0, \infty\right)$	$\left[\frac{\alpha-1+n\alpha}{\alpha}, 1\right] \times \left[-\frac{1}{n+1}, 0\right]$	
b	$\left[\frac{1}{n}, \frac{1}{n-1+\alpha}\right] \times \left[0, \infty\right)$	$\left[\alpha, 1\right] \times \left[0, \frac{1}{n-1}\right]$	
c	$\left[\frac{1}{n-1+\alpha}, \frac{\alpha}{1-\alpha}\right] \times \left[0, \infty\right)$	$\left[1-\alpha, \frac{\alpha-1+n\alpha}{\alpha}\right] \times \left[-\frac{1}{n}, 0\right]$	
d	$\left[\frac{\alpha}{1-\alpha}, \frac{1}{n-1}\right] \times \left[0, 1\right]$	$\left[\frac{\alpha-1+n\alpha}{\alpha}, 1\right] \times \left[-\frac{1}{n}, -\frac{1}{n+1}\right]$	
e	$\left[\frac{1}{k+1}, \frac{1}{k+\alpha}\right] \times \left[0, 1\right]$	$\left[\alpha, 1\right] \times \left[\frac{1}{k+1}, \frac{1}{k}\right]$	for $k \in \mathbb{N}_{\leq n-2}$
f	$\left[\frac{1}{k+1+\alpha}, \frac{1}{k+1}\right] \times \left[0, 1\right]$	$\left[1-\alpha, 1\right] \times \left[-\frac{1}{k+2}, -\frac{1}{k+3}\right]$	for $k \in \mathbb{N}_{\leq n-3}$
g	$\left[1-\alpha, \frac{1}{1+\alpha}\right] \times \left[-1, 0\right]$	$\left[\alpha, \frac{\alpha}{1-\alpha}\right] \times \left[1, \infty\right)$	
h	$\left[\frac{1}{1+\alpha}, 1\right] \times \left[-1, 1\right]$	$\left[1-\alpha, 1\right] \times \left[-1, -\frac{1}{3}\right]$	

Table 2.1: Ω_α split up in disjoint pieces in the left column and their image under $\mathcal{T}_\alpha$ in the right column.

Table 2.1 shows that $\mathcal{T}_\alpha$ is bijective almost everywhere on Ω_α. See Figure 2.3 for a visualisation of the map.

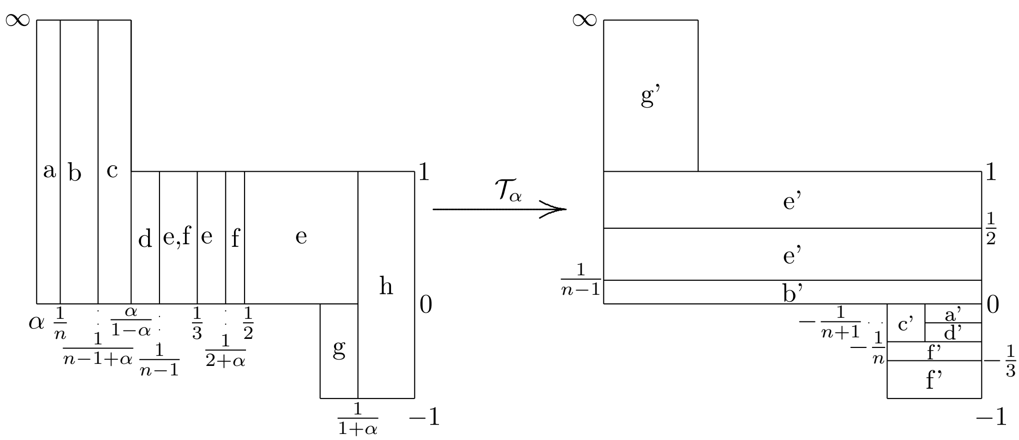


Figure 2.3: The natural extension domain for $\mathcal{T}_\alpha$ where $\mathcal{T}_\alpha(a) = a'$, $\mathcal{T}_\alpha(b) = b'$ etc..

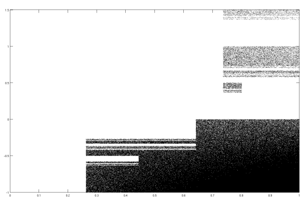
§2.3.1 From natural extension to invariant measure

To find the invariant measure for the original system (I_α, T_α) one simply projects onto the first coordinate. For $\alpha \in (0, \frac{1}{2})$ such that $\frac{1}{n+\alpha} < \alpha < \frac{1}{n}$ with $n \in \mathbb{N}_{\geq 2}$ we find invariant density

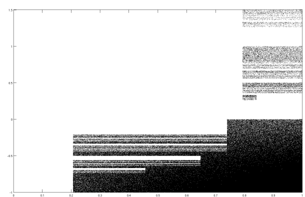
$$
\begin{aligned}
f_\alpha(x) &= \int_0^\infty \frac{1}{(1+xy)^2} dy \mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \int_0^1 \frac{1}{(1+xy)^2} dy \mathbf{1}_{[\frac{\alpha}{1-\alpha}, 1]}(x) \\
&\quad + \int_{-1}^0 \frac{1}{(1+xy)^2} dy \mathbf{1}_{[1-\alpha, 1]}(x) \\
&= \frac{1}{x}\mathbf{1}_{[\alpha, \frac{\alpha}{1-\alpha}]}(x) + \frac{1}{1+x}\mathbf{1}_{[\frac{\alpha}{1-\alpha}, 1]}(x) + \frac{1}{1-x}\mathbf{1}_{[1-\alpha, 1]}(x).
\end{aligned}
$$

Proof of Theorem 2.1.1. By the same method as explained in this section one can find all the densities given in the theorem. Since only the calculations are different from the case explained, we omit them here. □

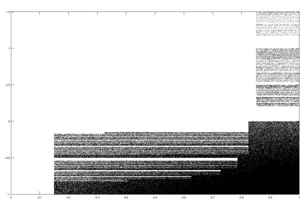
For $\alpha \in (\frac{1}{2}\sqrt{2}, 1)$ the structure of the domain Ω_α of natural extension becomes more complicated as Figure 2.4 shows.



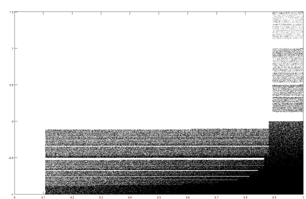
(a) $\alpha \approx 0.73694949$



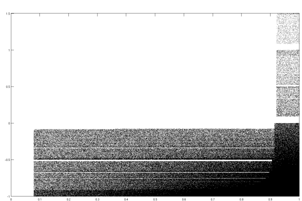
(b) $\alpha \approx 0.79347519$



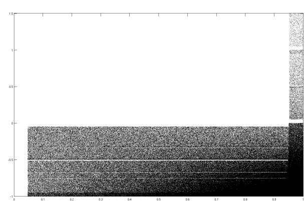
(c) $\alpha \approx 0.85019348$



(d) $\alpha \approx 0.89348572$



(e) $\alpha \approx 0.92087668$



(f) $\alpha \approx 0.95234649$

Figure 2.4: Several numerical simulations of the natural extension domain for $\alpha \in (\frac{1}{2}\sqrt{2}, 1)$.

§2.4 Entropy

To be able to calculate the entropy we first have to do some preliminary work. We show the following proposition.

Proposition 2.4.1. *Let $\alpha \in (0,1)$. The system $(I_\alpha, \mathcal{B}, \mu_\alpha, T_\alpha)$ is a basic AFN map: a conservative system, a piecewise monotonic system (there exists a partition $\mathcal{P} = \{I_i\}$ such that T_α restricted to each element of $\mathcal{P}$ is continuous, strictly monotonic and twice differentiable) and a system with at least one indifferent fixed point such that the following conditions hold.*

- ***(A)*** *Adler's condition: $\frac{T''_\alpha}{(T'_\alpha)^2}$ is bounded on $\cup_i I_i$,*
- ***(F)*** *Finite image condition: $T_\alpha(\mathcal{P}) := \{T_\alpha(I_i) : I_i \in \mathcal{P}\}$ is finite,*
- ***(N)*** *Indifferent fixed point condition: there exists a finite set $\mathcal{Z} \subseteq \mathcal{P}$, such that each $Z_i \in \mathcal{Z}$ has an indifferent fixed point x_{Z_i}, i.e.*

$$\lim_{x \to x_{Z_i}, x \in Z_i} T_\alpha(x) = x_{Z_i} \quad \text{and} \quad \lim_{x \to x_{Z_i}, x \in Z_i} T'_\alpha(x) = 1$$

  and T'_α decreases on $(-\infty, x_{Z_i}) \cap Z_i$ respectively increases on $(x_{Z_i}, \infty) \cap Z_i$. Last, T is uniformly expanding on sets bounded away from $\{x_{Z_i} : Z_i \in \mathcal{Z}\}$.

Proof. First, recall that a system T_α is said to be conservative if every wandering set (a set for which all the pre-images under the map are pairwise disjoint) for T_α is a set of null measure. Maharam's Recurrence Theorem (see [57, Theorem 2.2.14]) ensures the conservativity through the existence of a sweep-out set (a positive but finite measure set for which the set of all pre-images covers almost everything). It is easy to see that any subinterval of I_α is a sweep-out set for the map T_α so that the system is conservative.
For each $\alpha \in (0,1)$, let $k(\alpha) \in \mathbb{N}$ be such that $\min(\alpha, 1-\alpha) \in \left(\frac{1}{k(\alpha)+1}, \frac{1}{k(\alpha)}\right]$ and let

$$W_\alpha := \begin{cases} \left[\min(\alpha, 1-\alpha), \frac{1}{k(\alpha)}\right] & \text{if } \frac{1}{k(\alpha)+\alpha} < \alpha < \frac{1}{k(\alpha)}, \\ \left[\min(\alpha, 1-\alpha), \frac{1}{k(\alpha)+\alpha}\right], \left(\frac{1}{k(\alpha)+\alpha}, \frac{1}{k(\alpha)}\right], & \text{if } \frac{1}{k(\alpha)+1} < \alpha < \frac{1}{k(\alpha)+\alpha}. \end{cases}$$

A finite partition $\mathcal{P}$ can be given by

$$\left\{W_\alpha, \left(\frac{1}{n+1}, \frac{1}{n+\alpha}\right], \left(\frac{1}{n+\alpha}, \frac{1}{n}\right], \text{ for } n = 1, 2, ..., k(\alpha) - 1\right\}.$$

On each of these subintervals the map is continuous, strictly monotonic and twice differentiable. Furthermore we see that the conditions **(A)**,**(F)**,**(N)** hold:

- **(A)** $|2x| \leq 2$ on I_α,
- **(F)** $T_\alpha(\mathcal{P})$ consists at most of three subintervals depending on α,
- **(N)** $x_Z = 1$ is the only indifferent fixed point for $Z = (\frac{1}{1+\alpha}, 1]$, and $T'_\alpha(x) = 1/x^2$ decreases on Z and it is strictly greater than 1 on sets bounded away from x_Z.

□

For infinite measure-preserving and conservative systems $(X, \mathcal{B}, \mu, T)$ there exists an extension of the notion of entropy (w.r.t μ) due to Krengel [66]:

$$h_{\mathrm{Kr},\mu}(T) = h(T_A, \mu\mid_A),$$

for A a sweep-out set for T, T_A the induced transformation T on A and $\mu\mid_A$ the measure μ restricted to the set A. A result of Zweimüller tells us that we can use Rohlin's formula to calculate it.

Theorem 2.4.2 (Zweimüller [106]). *Let $(I, \mathcal{B}, \mu, T)$ be a basic AFN map with μ_α an (absolutely continuous) invariant measure, then*

$$h_{Kr,\mu}(T) = \int_X \log(|T'(x)|)d\mu.$$

We are now in the position of proving Theorem 2.1.2.

Proof of Theorem 2.1.2. From Proposition 2.4.1 and Theorem 2.4.2 it follows we can calculate the Krengel entropy by using Rohlin's formula. We use some properties of dilogarithm functions (see also [73]). We have

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad \text{for } |x| < 1$$

and

- $\mathrm{Li}_2(0) = 0$,
- $\mathrm{Li}_2(-1) = -\pi^2/12$,
- $\mathrm{Li}_2(x) + \mathrm{Li}_2(-\frac{x}{1-x}) = -\frac{1}{2}\log^2(1-x)$,
- $\frac{d}{dx}\mathrm{Li}_2(x) = \frac{-\log(1-x)}{x}$.

We compute the entropy for $\alpha \in (0, \frac{1}{2})$. The computation for $\alpha \in [\frac{1}{2}, g)$ works in a similar way.

$$\begin{aligned}
\int_{[\alpha,1]} \log(|T'_\alpha(x)|)d\mu_\alpha &= -2\Big[\int_\alpha^{\alpha/(1-\alpha)} \frac{\log x}{x}dx + \int_{\alpha/(1-\alpha)}^{1-\alpha} \frac{\log x}{1+x}dx + \int_{1-\alpha}^{1} \frac{\log x}{1-x^2}dx\Big] \\
&= -\log^2 x \,|_\alpha^{\alpha/(1-\alpha)} -2[\mathrm{Li}_2(-x) + \log x \log(x+1)] \,|_{\alpha/(1-\alpha)}^{1-\alpha} \\
&\quad - 2[\mathrm{Li}_2(1-x) + \mathrm{Li}_2(-x) + \log x \log(x+1)] \,|_{1-\alpha}^{1} \\
&= -\log^2 \big(\frac{\alpha}{1-\alpha}\big) + \log^2(\alpha) + 2\,\mathrm{Li}_2\,\big(\frac{-\alpha}{1-\alpha}\big) \\
&\quad + 2\log\big(\frac{\alpha}{1-\alpha}\big)\log\big(\frac{1}{1-\alpha}\big) - 2\,\mathrm{Li}_2(0) - 2\,\mathrm{Li}_2(-1) \\
&\quad - 2\log(0)\log(2) + 2\,\mathrm{Li}_2(\alpha) \\
&= \log^2(\alpha) - \log^2\big(\frac{\alpha}{1-\alpha}\big) + 2[\mathrm{Li}_2\,\big(\frac{-\alpha}{1-\alpha}\big) + \mathrm{Li}_2(\alpha)]+ \\
&\quad - 2\log\big(\frac{\alpha}{1-\alpha}\big)\log(1-\alpha) + \frac{\pi^2}{6} \\
&= \log^2(\alpha) - \log^2\big(\frac{\alpha}{1-\alpha}\big) - \log^2(1-\alpha) \\
&\quad - 2\log\big(\frac{\alpha}{1-\alpha}\big)\log(1-\alpha) + \frac{\pi^2}{6} \\
&= 2\log(1-\alpha)[\log(\alpha) - \log(1-\alpha) - \log(\alpha) + \log(1-\alpha)] + \frac{\pi^2}{6} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

□

§2.5 Return sequences and wandering rates

Other ergodic properties can be obtained from the asymptotic type of the maps, which is the asymptotic proportionality class of any return sequence of the map. Let P_α denote the transfer operator of the map T_α, defined by the equation

$$\int_A P_\alpha f\, d\mu_\alpha = \int_{T_\alpha^{-1}} f\, d\mu_\alpha \quad \text{for } f \in L^1(I_\alpha, \mathcal{B}_\alpha, \mu_\alpha) \text{ and } A \in \mathcal{B}_\alpha.$$

The *return sequence* for T_α is the sequence $(a_n(T_\alpha))_{n\geq 1} \subseteq (0,\infty)$ satisfying

$$\lim_{n\to\infty} \frac{1}{a_n(T_\alpha)} \sum_{k=0}^{n-1} P_\alpha^k f = \int_{I_\alpha} f\, d\mu_\alpha.$$

The result from [106, Theorem 1] implies that each map T_α is pointwise dual ergodic. This ensures that such a sequence, which is unique up to asymptotic equivalence, exists. The *asymptotic type* of any map T_α is the asymptotic proportionality class of T_α, containing all sequences that are asymptotically equivalent to some positive multiple of $(a_n(T_\alpha))_{n\geq 1}$.

The return sequence of a system is related to its wandering rate, which quantifies how big the system is in relation to its subsets of finite measure. To be more precise, if $(X, \mathcal{B}, \mu, T)$ is a conservative, ergodic, measure preserving system and $A \in \mathcal{B}$ a set of finite positive measure, then the wandering rate of A with respect to T is the sequence given by $(w_n(A))_{n \geq 1}$ for

$$w_n(A) := \mu\Big(\bigcup_{k=0}^{n-1} T^{-k} A \Big).$$

It follows from [106, Theorem 2] that for each of the maps T_α there is a sequence $(w_n(T_\alpha)) \subseteq (0, \infty)$ such that $w_n(T_\alpha) \uparrow \infty$ and $w_n(T_\alpha) \sim w_n(A)$ as $n \to \infty$ for all sets $A \in \mathcal{B}$ such that $0 < \mu(A) < \infty$ and that are bounded away from one. The asymptotic equivalence class of $(w_n(T_\alpha))$ is called the *wandering rate* of T_α. Using the machinery from [106] we can prove Proposition 2.1.3.

Proof of Proposition 2.1.3. The wandering rate for AFN-maps is given in [106, Theorem 3] and the return sequence in [106, Theorem 4]. Using the Taylor expansion of the maps T_α one sees that for $x \to 1$ we have $T_\alpha(x) = x - (x-1)^2 + o((x-1)^2)$. Hence, T_α admits what is called *nice expansions* in [106]. Secondly, on the right most interval $\big(\frac{1}{1+\alpha}, 1\big]$ the density f_α for $\alpha \in (0, \frac{1}{2})$ is given by $f_\alpha(x) = \frac{2}{1-x^2}$. This can be written as $f_\alpha(x) = G(x)H(x)$, where $G(x) = \frac{x-2}{x-1}$ and $H(x) = \frac{2}{(1+x)(2-x)}$. As a consequence, at the indifferent fixed point it holds that $H(1) = 1$. It then follows from [106, Theorems 3 and 4], that the wandering rate is

$$w_n(T) \sim \log n$$

and the return sequence is

$$a_n(T) \sim \frac{n}{\log n}.$$

In our setting [106, Theorem 5] translates to

$$\frac{\log n}{n} \sum_{k=0}^{n-1} f \circ T_\alpha^k \xrightarrow{\mu_\alpha} \int_{I_\alpha} f \, d\mu_\alpha, \qquad \text{for } f \in L_1(\mu_\alpha) \text{ and } \int_{I_\alpha} f \, d\mu_\alpha \neq 0,$$

i.e., a weak law of large numbers holds for T_α. □

§2.5.1 Isomorphic?

When considering a family of transformations with similar dynamical properties, a natural question to ask is whether the maps in question are isomorphic. Since the maps T_α all have an infinite invariant measure, these measures cannot be normalised and the appropriate notion to consider is that of c-isomorphism, which is defined as follows (see for example [1]): Two measure preserving dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ on σ-finite measure spaces are called *c-isomorphic* for $c \in \mathbb{R}_{>0} \cup \{\infty\}$ if there are sets $N \in \mathcal{B}$, $M \in \mathcal{C}$ with $\mu(N) = 0 = \nu(M)$ and $T(X \setminus N) \subseteq X \setminus N$ and $S(Y \setminus M) \subseteq Y \setminus M$ and if there is a map $\phi : X \setminus N \to Y \setminus M$ that it invertible, bi-measurable and satisfies $\phi \circ T = S \circ \phi$ and $\mu \circ \phi^{-1} = c \cdot \nu$. Well known invariants for c-isomorphism are the Krengel entropy and the asymptotic proportionality classes of the return sequence and the wandering rate.

We have seen that the Krengel entropy for $\alpha \in (0, g]$ is constant and not depending on α. Also the wandering rate as well as the return sequence do not display any dependence on α, so that also these invariants do not give us information on the existence (or non-existence) of isomorphisms between the maps T_α either. Using the technique from [53] we can show that in general it is not true that for any α, α' there is a $c \in \mathbb{R}_{>0} \cup \{\infty\}$ such that T_α and $T_{\alpha'}$ are c-isomorphic. Consider for example any $\alpha \in \left(\sqrt{2}-1, \frac{1}{2}\right)$, so that $\alpha \in \left(\frac{1}{2+\alpha}, \frac{1}{2}\right)$ and any $\alpha' \in \left(\frac{1}{3}, \frac{3-\sqrt{5}}{2}\right)$, so that $T_{\alpha'}(\alpha') > 1-\alpha'$, see Figure 2.5. For a contradiction, suppose that there is a c-isomorphism $\phi : I_\alpha \to I_{\alpha'}$ for some $c \in \mathbb{R}_{>0} \cup \{\infty\}$. Let $J = [\alpha, \min\{T_\alpha(\alpha), 1-\alpha\}]$ and note that any $x \in J$ has precisely one pre-image. Since $\phi \circ T_\alpha = T_{\alpha'} \circ \phi$ and ϕ is invertible, any element of the set $\phi(J)$ must also have precisely one pre-image. Since $T_{\alpha'}(\alpha') > 1 - \alpha'$, there are no such points, so $\mu_{\alpha'}(\phi(J)) = 0$. On the other hand, since J is bounded away from 1, it follows that $0 < \mu_\alpha(J) < \infty$. Hence, there can be no c, such that $\mu_{\alpha'} \circ \phi^{-1} = c \cdot \mu_\alpha$. Obviously a similar argument holds for many other combinations of α and α', even for $\alpha > \frac{1}{2}$, and in case the argument does not work for T_α and $T_{\alpha'}$, one can also consider iterates of the transformation. Hence, even though the above discussed isomorphism invariants are equal for all $\alpha \in \left(0, \frac{1}{2}\right)$, in general one cannot conclude that any two of the maps T_α are c-isomorphic.

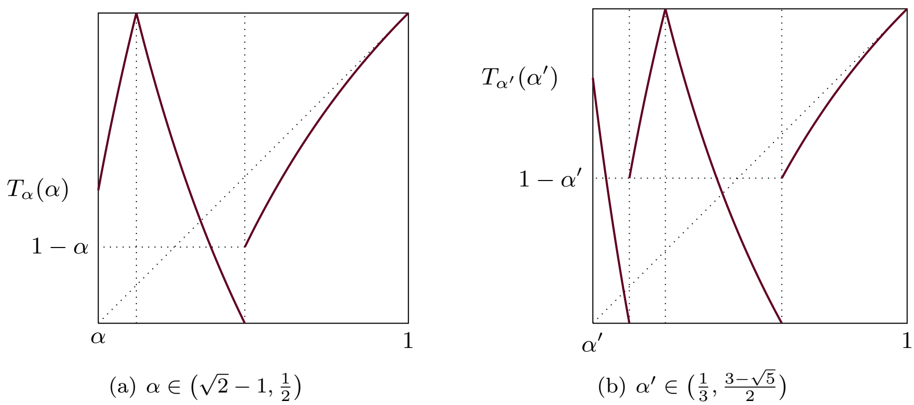


Figure 2.5: Maps T_α and $T_{\alpha'}$ that are not c-isomorphic for any $c \in \mathbb{R}_{>0} \cup \{\infty\}$.

§2.5.2 Final observations and remarks

We have seen that the natural extension is a powerful tool to find invariant measures for families of continued fractions. Though, for $\alpha > \frac{1}{2}\sqrt{2}$ the domain becomes more complicated. In other families of continued fractions (α-continued fractions or Ito

Tanaka's α-continued fractions studied in Chapter 3) similar behaviour is seen for certain values of the parameter space.

What one can do with the natural extensions that we found, is the study of Diophantine approximation. Using for example tools from [29] one can study the quality of convergence for typical points. One can show that for any $\alpha \in (0, 1)$ the convergence of a typical point in I_α is not exponential. Though, maybe more can be said about the quality.

Something we did not study in this chapter is matching. Though, matching can be easily found. For example on the interval $(0, \frac{1}{2})$ matching holds with exponents $(1, 3)$. In the study of other families, matching often has implications for the entropy whereas for our family we did not observe a relation between the Krengel entropy and matching. Maybe there is another observable which is related to matching in our case.

In an upcoming paper we show a very close connection between matching intervals for our family and for α-continued fractions. By using this relation we can prove that matching holds almost everywhere, even though it is unclear to us how these families are exactly related.