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## Matching, entropy, holes and expansions

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# Matching, entropy, holes and expansions

Niels Langeveld



# Matching, entropy, holes and expansions

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