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Gravitational Waves from Generalized Newtonian Sources

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I review the elementary theory of gravitational waves on a Minkowski background and the quadrupole approximation. The modified conservation laws for energy and momentum keeping track of the gravitational-wave flux are presented. The theory is applied to two-body systems in bound and scattering states subject to newtonian gravity generalized to include a $1/r^3$ force allowing for orbital precession. The evolution of the orbits is studied in the adiabatic approximation. From these results I derive the conditions for capture of two bodies to form a bound state by the emission of gravitational radiation.

1. Introduction and Overview

The existence of gravitational waves is now well-established from both direct and indirect observations.[1–4] A completely new field of astronomy is opening up which will no doubt have an impact also on other branches of astronomy and astrophysics such as dynamics and evolution of stars and galaxies. The supermassive black holes in the centers of galaxies, and possibly intermediate-mass black holes in stellar clusters, will by the relatively large curvature they create in the surrounding space enhance the emission of gravitational waves from massive objects on trajectories passing close to them, whether these are on bound or open orbits. The emission of gravitational waves can even lead to the capture of objects originally in open orbits to end up in a bound state.

Apart from these radiative phenomena involving very massive black holes, the emission of gravitational waves also affects more common binary star systems like the well-known close binary neutron stars, the recently discovered binary black holes and presumably systems containing white dwarfs.[5] No doubt radiation has an impact on three- and many-body systems, especially on their stability. Detailed investigations of close binary star systems using high-order post-newtonian expansions of the Einstein equations of General Relativity have been carried out with great success; for a review see e.g. [6]. The inspiral and merger of extreme mass-ratio binaries involving a very massive black hole has also been studied directly in the background geometry of the black hole.[7–11] Whenever these theoretical investigations can be compared with data they seem to describe the dynamics of these systems very well, thereby also confirming General Relativity to be the best available theory for gravitational interactions.[12] The study of radiation from two-body scattering has been addressed as well,[13] although no corresponding observations have been announced so far.

Even though they may carry large amounts of energy and momentum, the deformations of space-time created by gravitational waves are extremely small. For example a flux of monochromatic gravitational waves with a frequency of 100 Hz and an extreme intensity of 1 W/m^2 will create spatial deformations of less than 1 part in 10^{19}, the diameter of a proton over a distance of 1 km. This testifies as to the extreme stiffness of space and explains both why it is so difficult to create gravitational waves and to observe them. It also implies that most potential sources of gravitational waves are weak and many move on close-to-stationary almost-newtonian orbits.

This review is devoted to gravitational radiation from such weak or very weak sources. They produce the most abundant, though maybe not the most spectacular, form of gravitational waves in the universe and may eventually become relevant to a wide range of astronomical and astrophysical observations. To lowest order their description and propagation involve straightforward applications of linear field theory in Minkowski space-time. This also provides the starting point for many more elaborate and precise calculations.

We will begin by recapturing in fairly standard fashion the wave equation for gravitational waves, its gauge invariance and its implications for the propagation and polarization states of gravitational waves. We address the quadrupole nature of the waves and the associated sources, and explain how dynamical mass quadrupole motion generates the simplest and most common weak gravitational waves. Next we derive the modification of the conservation laws for energy, momentum and angular momentum by taking account of gravitational radiation. We present equations for the transport of energy and angular momentum by gravitational waves, keeping track of the anisotropic dependence on directions.

This theory is then applied to systems of massive objects moving on generalized newtonian orbits, either in bound states or on open scattering trajectories. The generalization includes the effects of possible $1/r^3$ forces causing orbital precession, which may result e.g. from many-body or post-newtonian interactions. We calculate the evolution of orbital parameters due to emission of gravitational radiation and their relations. We finish by

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establishing which binary scattering orbits are turned into bound states by emission of radiation.

2. The Wave Equation

Weak gravitational waves are dynamical fluctuations of the space-time metric about flat Minkowski geometry.[14–16] Thus we can split the full space-time metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}, \tag{1}$$

where κ is the positive root of

$$\kappa^2 = \frac{8\pi G}{c^4} \simeq 2.1 \times 10^{-41}\,\mathrm{kg}^{-1}\,\mathrm{m}^{-1}\,\mathrm{s}^2, \tag{2}$$

G being the newtonian constant of gravity and c the speed of light in vacuum. This endows $h_{\mu\nu}$ with the standard dimensions of a bosonic tensor field. Up to non-linear corrections the tensor field is postulated to satisfy the field equation

$$\begin{aligned}\Box h_{\mu\nu} &- \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h^\lambda{}_\lambda \\ &- \eta_{\mu\nu}\left(\Box h^\lambda{}_\lambda - \partial^\kappa \partial^\lambda h_{\kappa\lambda}\right) = -\kappa\, T_{\mu\nu},\end{aligned} \tag{3}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian and the inhomogeneous term $T_{\mu\nu}$ on the right-hand side represents the sources of the field. By factoring out the constant κ this tensor has the dimensions of energy per unit of volume or force per unit of area. In this treatise we always use the flat Minkowski metric $\eta_{\mu\nu}$ with signature $(-,+,+,+)$ and its inverse $\eta^{\mu\nu}$ to raise and lower indices on components of mathematical objects like vectors and tensors.

The motivation for postulating this field equation comes from the physical properties of the tensor field $h_{\mu\nu}$ implied by its structure. First note that defining the linear Ricci tensor

$$R_{\mu\nu} = \kappa\left(\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h^\lambda{}_\lambda\right), \tag{4}$$

the trace of which reads

$$R = R^\lambda{}_\lambda = 2\kappa\left(\Box h^\lambda{}_\lambda - \partial^\kappa \partial^\lambda h_{\kappa\lambda}\right), \tag{5}$$

the field equation takes the form

$$R_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,R = -\kappa^2\,T_{\mu\nu}. \tag{6}$$

This is the linearized version of Einstein's gravitational field equation in a flat background. Note also that

$$\partial^\mu R_{\mu\nu} = \frac{1}{2}\,\partial_\nu R, \tag{7}$$

and as a result the inhomogeneous field Equation (6) is seen to imply a conservation law for the source terms:

$$\partial^\mu T_{\mu\nu} = 0. \tag{8}$$

As the energy-momentum tensor of matter and radiation has the required physical dimensions and satisfies the condition (8) in Minkowski space it is the obvious source for the tensor field. As all physical systems possess energy and momemtum this explains the universality of gravity[1].

An observation closely related to (7) is that the linear Ricci tensor is invariant under gauge transformations

$$h_{\mu\nu} \;\to\; h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad R'_{\mu\nu} = R_{\mu\nu}. \tag{9}$$

By such gauge transformations one can straightforwardly eliminate four components of the field to reduce the number of independent components from ten to six. To achieve such a reduction in pratice the standard procedure is to impose the De Donder condition

$$\partial^\mu h_{\mu\nu} = \frac{1}{2}\,\partial_\nu h^\mu{}_\mu. \tag{10}$$

This condition reduces the linear Ricci tensor and its trace to the expressions

$$R_{\mu\nu} = \kappa\,\Box h_{\mu\nu}, \qquad R = \kappa\,\Box h^\lambda{}_\lambda, \tag{11}$$

and therefore the field equation turns into the inhomogeneous wave equation

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu} h^\lambda{}_\lambda\right) = -\kappa\,T_{\mu\nu}. \tag{12}$$

It is then convenient to redefine the field components by

$$\underline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu} h^\lambda{}_\lambda, \tag{13}$$

which transform under gauge transformations as

$$\underline{h}'_{\mu\nu} = \underline{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\,\partial^\lambda \xi_\lambda. \tag{14}$$

After implementing the De Donder condition the field is divergence-free and satisfies the inhomogeneous wave equation:

$$\partial^\mu \underline{h}_{\mu\nu} = 0, \qquad \Box \underline{h}_{\mu\nu} = -\kappa\,T_{\mu\nu}. \tag{15}$$

Finally a second gauge transformation can be made without changing the De Donder condition provided the parameter satisfies itself the homogeneous wave equation:

$$\partial^\mu \underline{h}'_{\mu\nu} = \partial^\mu \underline{h}_{\mu\nu} + \Box\,\xi_\nu = 0 \quad \Leftrightarrow \quad \Box\,\xi_\nu = 0. \tag{16}$$

Such a residual gauge transformation can be made in particular on free fields to remove the trace of the tensor field:

$$\underline{h}'^\lambda{}_\lambda = \underline{h}^\lambda{}_\lambda - 2\,\partial^\lambda \xi_\lambda = 0, \tag{17}$$

in agreement with the Equations (15) provided $\Box\xi_\nu = 0$ and $T^\lambda{}_\lambda = 0$. It follows automatically that the same condition holds for the original tensor field: $h^\lambda{}_\lambda = 0$. Removal of the trace reduces the number of independent components of free fields to five, equal

[1] As is well-known, requiring this universality to encompass the gravitational field itself leads to the non-linear structure of the full theory of General Relativity.

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to the dimension of the irreducible spin-2 representation of the rotation group, However, as dynamical free wave fields propagate on the light cone and have only transverse polarization states, the actual number of independent *dynamical* components of gravitational wave fields is two. This will be discussed in the following.

3. Solutions of the Inhomogeneous Wave Equation

The inhomogeneous linear wave Equation (15) has many solutions: to a given solution one can always add any solution of the homogeneous equation representing free gravitational waves. Free gravitational waves can therefore appear as a background to gravitational wave signals from specific sources.

In the absence of such a background the standard causal solution for sources localized in a finite region of space is the retarded solution

$$\underline{h}_{\mu\nu}(\mathbf{x}, t) = \frac{\kappa}{4\pi} \int_{S_r} d^3x' \, \frac{T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|}, \tag{18}$$

where the integration volume S_r can be taken to be a large sphere of radius $r = |\mathbf{x}|$ containing the finite region of the sources where $T_{\mu\nu} \neq 0$ in its center. To evaluate the field by performing the integration is difficult in practice for any realistic type of sources.

In order to make progress it makes sense to consider the situation in which the waves are evaluated at large distance from the sources: the radius r of the sphere is taken to be much larger than any typical dimension of the sources. For example we evaluate the waves emitted by a binary star system of orbital extension d at a distance $r \gg d$. Under this assumption one can expand the integral expression on the right-hand side of (18) in inverse powers of r keeping only terms which do not fall off faster than $1/r$. This results in the simpler integral

$$\underline{h}_{\mu\nu}(\mathbf{x}, t) = \frac{\kappa}{4\pi r} \int_{S_r} d^3x' \, T_{\mu\nu}(\mathbf{x}', t - r). \tag{19}$$

Another simplification is possible as it is straightforward to show that for localized sources these solutions have no dynamical time components:

$$\begin{aligned} \partial_0 \underline{h}_{0\mu} &= \frac{\kappa}{4\pi r} \int_{S_r} d^3x' \, \partial_0 T_{0\mu} = \frac{\kappa}{4\pi r} \int_{S_r} d^3x' \, \partial'_i \, T_{i\mu} \\ &= \frac{\kappa}{4\pi r} \oint_{\partial S_r} d^2\sigma \, \hat{r}'_i \, T_{i\mu} = 0. \end{aligned} \tag{20}$$

The second equality on the first line follows from energy-momentum conservation, whilst the last equality uses Gauss' theorem to convert the volume integral to a surface integral over the corresponding normal component of the energy-momentum tensor, $\hat{r}$ being the radial unit vector pointing out of the spherical surface ∂S_r. Finally the localization of the sources in a finite region near the center of the sphere guarantee the vanishing of the energy-momentum tensor on the boundary. We infer that the time components may represent static newtonian fields, but they cannot contribute to the flux of dynamical waves across the boundary of the sphere.

As concerns dynamical fields we are therefore left with the spatial components of the outgoing wave solutions (19):

$$\underline{h}_{ij} = \frac{\kappa}{4\pi r} \int_{S_r} d^3x' \, T_{ij}(\mathbf{x}', t - r). \tag{21}$$

In empty space far from the sources the expression on the right-hand side actually represents an exact formal solution of the wave equation. Now this solution was obtained by imposing the De Donder condition (15); in addition, as argued after (17), in this region one can always find a local gauge transformation of the fields that makes them traceless. For the solution at hand this implies that after such a gauge transformation

$$\partial_i \underline{h}_{ij} = 0 \quad \Rightarrow \quad \hat{r}_i \, \underline{h}_{ij} = 0. \tag{22}$$

and

$$\underline{h}_{jj} = h_{jj} = 0. \tag{23}$$

A detailed discussion of the necessary gauge transformations is presented in appendix A. Tensor fields obeying these conditions of are called transverse and traceless (TT) and satisfy $\underline{h}^{TT}_{ij} = h^{TT}_{ij}$. We will take these properties for granted in what follows and omit the TT in the notation. Combining the above requirements the outgoing wave fields far from the source must then be represented in the TT-gauge by an expression of the form

$$\begin{aligned} h_{ij}(\mathbf{x}, t) &= \underline{h}_{ij}(\mathbf{x}, t) \\ &= \frac{\kappa}{4\pi r} \left(\delta_{ik} - \hat{r}_i \hat{r}_k\right) \left(\delta_{jl} - \hat{r}_j \hat{r}_l\right) \left(I_{kl} + \frac{1}{2} \delta_{kl} \, \hat{r} \cdot I \cdot \hat{r} \right), \end{aligned} \tag{24}$$

where the spatial symmetric 3-tensor $\mathbf{I}$ is traceless: $I_{kk} = 0$. Writing $u \equiv t - r$, agreement of this expression with the result (21) up to gauge transformations is obtained by taking

$$I_{ij}(u) = \int_{S_r} d^3x' \left(T_{ij} - \frac{1}{3} \delta_{ij} T_{kk} \right) (\mathbf{x}', u). \tag{25}$$

With the help of energy-momentum conservation the integral can be rewritten in terms of the quadrupole moment of the total energy density T_{00} of the sources:

$$I_{ij}(u) = \frac{1}{2} \partial_0^2 \int_{S_r} d^3x' \left(x'_i x'_j - \frac{1}{3} \delta_{ij} \, \mathbf{x}'^2 \right) T_{00}(\mathbf{x}', u). \tag{26}$$

The proof is easier in backward fashion; first notice that as $\partial_0 = \partial_u$

$$\partial_0^2 T_{00}(\mathbf{x}', u) = \partial_0 \, \partial'_i \, T_{i0} = \partial'_i \partial'_j \, T_{ij}(\mathbf{x}', u); \tag{27}$$

then perform two partial integrations with respect to $\mathbf{x}'$ to reobtain (25), observing that the full energy-momentum tensor is supposed to vanish at the boundary ∂S_r.

Finally considering non-relativistic sources in the center-of-mass frame, the energy density is dominated by the mass-density

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$\rho(\mathbf{x}, t)$, which allows us to replace the integral in (26) by the components of the mass quadrupole moment and write explicitly:

$$I_{ij} = \frac{1}{2}\frac{d^2 Q_{ij}}{dt^2}, \qquad Q_{ij}(u) = \int_{S_r} d^3x' \left(x'_i x'_j - \frac{1}{3}\delta_{ij}\,\mathbf{x}'^2\right)\rho(\mathbf{x}', u). \tag{28}$$

Thus we get the final expression for the wave field $\underline{h}_{ij}$ for non-relativistic sources in the TT-gauge:

$$h_{ij}(\mathbf{x}, t) = \frac{\kappa}{8\pi \mathrm{r}}\left(\delta_{ik} - \hat{\mathrm{r}}_i\hat{\mathrm{r}}_k\right)\left(\delta_{jl} - \hat{\mathrm{r}}_j\hat{\mathrm{r}}_l\right) \times \frac{d^2}{dt^2}\left(Q_{kl} + \frac{1}{2}\delta_{kl}\,\hat{\mathrm{r}}\cdot Q\cdot\hat{\mathrm{r}}\right)_{u=t-\mathrm{r}}. \tag{29}$$

For the dynamical (non-Newtonian) metric fluctuations $\delta g_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, recalling Equations (1) and (2) this result implies that

$$\delta g_{00} = \delta g_{0i} = 0; \qquad \delta g_{ij} = \frac{2G}{\mathrm{r}}\left(\delta_{ik} - \hat{\mathrm{r}}_i\hat{\mathrm{r}}_k\right)\left(\delta_{jl} - \hat{\mathrm{r}}_j\hat{\mathrm{r}}_l\right) \times \frac{d^2}{dt^2}\left(Q_{kl} + \frac{1}{2}\delta_{kl}\,\hat{\mathrm{r}}\cdot Q\cdot\hat{\mathrm{r}}\right)_{u=t-\mathrm{r}}. \tag{30}$$

4. Conservation Laws and Gravitational-Wave Fluxes

Free radiation fields (always taken in the TT-gauge) define conserved currents of energy, momentum and angular momentum;[15,16] in the conventions of the previous sections

$$\mathscr{E} = \frac{1}{2}\left(\partial_0 h_{ij}\right)^2 + \frac{1}{2}\left(\partial_k h_{ij}\right)^2, \qquad \mathscr{P}_k = \partial_0 h_{ij}\,\partial_k h_{ij}, \qquad \mathscr{M}_k = \partial_0 h_{ij}\left(2\varepsilon_{kmi}h_{mj} - \varepsilon_{kmn}x_m\partial_n h_{ij}\right). \tag{31}$$

Subject to the field equations and gauge conditions these quantities satisfy the continuity equations

$$\frac{\partial\mathscr{E}}{\partial t} = \partial_j\mathscr{P}_j, \qquad \frac{\partial\mathscr{P}_k}{\partial t} = \partial_j\mathscr{S}_{jk}, \qquad \frac{\partial\mathscr{M}_k}{\partial t} = \partial_j\mathscr{J}_{jk}, \tag{32}$$

where

$$\mathscr{S}_{jk} = \partial_j h_{mn}\partial_k h_{mn} + \frac{1}{2}\delta_{jk}\left[(\partial_0 h_{mn})^2 - (\partial_l h_{mn})^2\right], \qquad \mathscr{J}_{jk} = 2\varepsilon_{kmn}h_{ml}\,\partial_j h_{nl} - \frac{1}{2}\varepsilon_{jkl}x_l\left[(\partial_0 h_{mn})^2 - (\partial_l h_{mn})^2\right]. \tag{33}$$

Applying them to the free fields (29) these expressions determine the flux of energy, momentum and angular momentum carried by outgoing gravitational waves far from the source region. First, integration over a large sphere around the center of mass of the source and using Gauss' theorem gives the change in total energy, momentum and angular momentum of gravitational waves in terms of surface integrals

$$\frac{dE}{dt} = \oint_{\partial S_r} d^2\sigma\,\hat{\mathrm{r}}_i\mathscr{P}_i, \qquad \frac{dP_k}{dt} = \oint_{\partial S_r} d^2\sigma\,\hat{\mathrm{r}}_i\mathscr{S}_{ik}, \qquad \frac{dM_k}{dt} = \oint_{\partial S_r} d^2\sigma\,\hat{\mathrm{r}}_i\mathscr{J}_{ik}. \tag{34}$$

Next, on the spherical surface ∂S_r the surface element of integration taken in polar co-ordinates $(\mathrm{r}, \theta, \varphi)$ is

$$d^2\sigma = \mathrm{r}^2\sin\theta\,d\theta d\varphi \equiv \mathrm{r}^2 d^2\Omega. \tag{35}$$

Evaluating the integrands on the right-hand side in Equations (34) while restoring factors of c then results in differential fluxes

$$\frac{dE}{d^2\Omega dt} = -\frac{G}{8\pi c^5}\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{\mathrm{r}}\cdot\dddot{Q}^2\cdot\hat{\mathrm{r}} + \frac{1}{2}(\hat{\mathrm{r}}\cdot\dddot{Q}\cdot\hat{\mathrm{r}})^2\right]_{u=t-\mathrm{r}},$$

$$\frac{dP_k}{d^2\Omega dt} = -\frac{dE}{d^2\Omega\,c dt}\hat{\mathrm{r}}_k = \frac{G}{8\pi c^6}\hat{\mathrm{r}}_k\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{\mathrm{r}}\cdot\dddot{Q}^2\cdot\hat{\mathrm{r}} + \frac{1}{2}(\hat{\mathrm{r}}\cdot\dddot{Q}\cdot\hat{\mathrm{r}})^2\right]_{u=t-\mathrm{r}},$$

$$\frac{dM_k}{d^2\Omega dt} = -\frac{G}{4\pi c^5}\varepsilon_{kij}\left[\left(\ddot{Q}\cdot\dddot{Q}\right)_{ij} - \left(\ddot{Q}\cdot\hat{\mathrm{r}}\right)_i\left(\dddot{Q}\cdot\hat{\mathrm{r}}\right)_j + \hat{\mathrm{r}}_i\left(\ddot{Q}\cdot\dddot{Q}\cdot\hat{\mathrm{r}} - \frac{1}{2}\ddot{Q}\cdot\hat{\mathrm{r}}\,\hat{\mathrm{r}}\cdot\dddot{Q}\cdot\hat{\mathrm{r}}\right)_j\right]_{u=t-\mathrm{r}}. \tag{36}$$

As usual overdots denote derivatives with respect to time t. The integrands themselves represent the anisotropic angular distribution of fluxes. The spherical surface integrals can be performed taking note that the quadrupole moments depend only on retarded time $u = t - r$, and that the angular integrals can be evaluated using the averaging procedure

$$\langle X\rangle \equiv \frac{1}{4\pi}\int d^2\Omega\,X(\theta, \varphi) \Rightarrow \langle\hat{r}_i\rangle = \langle\hat{r}_{i_1}\hat{r}_{i_2}\hat{r}_{i_3}\rangle = \cdots = \langle\hat{r}_{i_1}\cdots\hat{r}_{i_{2n+1}}\rangle = 0, \tag{37}$$

whilst

$$\langle\hat{r}_i\hat{r}_j\rangle = \frac{1}{3}\delta_{ij}, \qquad \langle\hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l\rangle = \frac{1}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{38}$$

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This results in [14–17]

$$\frac{dE}{dt} = -\frac{G}{5c^5}\,\mathrm{Tr}\,\dddot{Q}^2, \qquad \frac{dP_k}{dt} = 0,$$
$$\frac{dM_k}{dt} = -\frac{2G}{5c^5}\,\varepsilon_{kij}\left(\ddot{Q}\cdot\dddot{Q}\right)_{ij}. \tag{39}$$

Note that the total flux of linear momentum vanishes by symmetry (in the present approximation) as it involves only products of odd numbers of $\hat{\mathbf{r}}_i$ integrated over a full spherical surface, whereas the integrands of the energy and angular momentum contain even numbers of outward spherical unit vectors.

5. Generalized Newtonian 2-Body Forces

In the following we will apply the results to systems of masses moving under the influence of mutual newtonian forces, considering two-body systems interacting via a central potential. The classical description of such systems simplifies greatly, first as one can effectively reduce it to a single-body system by separating off the center-of-mass (CM) motion; second as angular momentum conservation implies the relative motion to be confined to a two-dimensional plane. Of course, the emission of gravitational radiation introduces limitations to these simplifications, but as long as the rate of energy and angular-momentum loss by the system is small the orbits will change only gradually and one can evaluate the effect of gravitational-wave emission in terms of adiabatic changes in the orbital parameters. In this section we first discuss non-disspiative motion; the effects of gravitational wave emission will be analysed afterwards.

Let the bodies have masses m_1 and m_2 and positions $\mathbf{r}_1$ and $\mathbf{r}_2$. To make maximal use of the simplifications we work in the CM frame in which

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0.$$

In terms of the relative separation vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ the positions w.r.t. the CM are

$$\mathbf{r}_1 = -\frac{m_2}{M}\,\mathbf{r}, \qquad \mathbf{r}_2 = \frac{m_1}{M}\,\mathbf{r},$$

and Newton's third law of motion implies that

$$m_1\ddot{\mathbf{r}}_1 = -m_2\ddot{\mathbf{r}}_2 = \mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}}, \tag{40}$$

where μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

and $F(r)$ is the magnitude of the central force acting on the masses. As usual r and $\hat{\mathbf{r}}$ represent the modulus and unit direction vector of the separation. In the absence of dissipation the energy and angular momentum of the system are conserved. In the CM frame these quantities can be written as

$$E = \frac{1}{2}\,\mu\dot{\mathbf{r}}^2 + V(r), \quad \text{such that} \quad F(r) = -\frac{dV}{dr}, \tag{41}$$

and

$$\mathbf{L} = \mu\mathbf{r}\times\dot{\mathbf{r}}. \tag{42}$$

Angular momentum being a conserved vector, the relative motion takes place in the plane perpendicular to $\mathbf{L}$, which we take to be the equatorial plane $\theta = \pi/2$. Then

$$\mathbf{r} = r\hat{\mathbf{r}} = r\,(\cos\varphi, \sin\varphi, 0)\,, \tag{43}$$

and

$$\mathbf{L} = (0, 0, \mu\ell)\,, \qquad \ell = r^2\dot{\varphi}. \tag{44}$$

In the following we will always orient the orbit such that the motion is counter-clockwise and therefore $\ell \geq 0$. The orbit is represented by the parametrized curve $r(\varphi)$ such that

$$\dot{r} = r'\dot{\varphi} = \frac{\ell r'}{r^2}, \tag{45}$$

the prime denoting a derivative w.r.t. φ. Newton's law of central force (40) then takes the form

$$F(r) = \frac{\mu\ell^2}{r^3}\left(\frac{r''}{r} - \frac{2r'^2}{r^2} - 1\right) = -\frac{\mu\ell^2}{r^2}\left[\left(\frac{1}{r}\right)'' + \frac{1}{r}\right]. \tag{46}$$

This result is tailored to suit Newton's original program of finding the law of force corresponding to a given orbit.[18] We will demonstrate it for the particular case of precessing conic sections: ellipses, parabolae and hyperbolae; these orbits are parametrized by

$$r = \frac{\rho}{1 - e\cos n\varphi}. \tag{47}$$

Here ρ is known as the semi-latus rectum; e is the eccentricity: $e = 0$ for circles, $0 < e < 1$ for precessing ellipses, $e = 1$ for similar parabolae and $e > 1$ for hyperbolae. Finally the number n determines the rate of precession. For circles this is of course irrelevant. For precessing ellipses the apastra occur for

$$\varphi = \frac{2\pi k}{n}, \tag{48}$$

where k is an integer; thus the apastron shift is $\Delta\varphi = 2\pi(1 - n)/n$ per turn. For precessing parabolae n determines the angle over which the directrix turns during the passage of the two bodies, i.e. the asymptotic scattering angle due to precession, also measuring

$$\Delta\varphi = \frac{2\pi(1-n)}{n}. \tag{49}$$

Similarly for hyperbolae it determines the angle between the incoming and outgoing asymptotes:

$$\Delta\varphi = \varphi_{out} - \varphi_{in} = \frac{2}{n}\left(\pi - \arccos\frac{1}{e}\right). \tag{50}$$

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Substitution of the expression (47) into Equation (46) leads to the result

$$F(r) = -\frac{\mu n^2 \ell^2}{\rho}\frac{1}{r^2} - \mu(1-n^2)\ell^2\frac{1}{r^3}, \tag{51}$$

the sum of an inverse square and an inverse cube force. Identifying the inverse square term with newtonian gravity and introducing an inverse cubic force with strength $\beta\mu$:

$$F(r) = -\frac{GM\mu}{r^2} - \frac{\beta\mu}{r^3}, \tag{52}$$

we find

$$n^2\ell^2 = GM\rho, \qquad n^2 = \frac{GM\rho}{GM\rho+\beta}. \tag{53}$$

with $M = m_1 + m_2$ the total mass of the two-body system. Such a force follows from a potential

$$V(r) = -\frac{G\mu M}{r} - \frac{\beta\mu}{2r^2}. \tag{54}$$

The eccentricity is determined by the radial velocity when the system is at the semi-latus rectum $\varphi = \pi/2n$, $r = \rho$:

$$\dot{r}|_{\varphi=\pi/2n} = -\frac{en\ell}{\rho} = -e\sqrt{\frac{GM}{\rho}}. \tag{55}$$

Evaluating the total energy at the semi-latus rectum and observing it is a constant of motion then tells us that

$$E = \frac{GM\mu}{2\rho}\left(e^2-1\right). \tag{56}$$

This confirms that for $e^2 < 1$ the orbits are bound, whilst for $e^2 \geq 1$ the orbits are open. Obviously the total angular momentum is by definition

$$L_z = \mu\ell = \mu\sqrt{GM\rho+\beta}. \tag{57}$$

Note that taking the first-order result for relativistic precession in Schwarzschild space-time with innermost circular orbit $R_{isco} = 6GM/c^2$ one gets

$$n^2 \simeq 1 - \frac{6GM}{c^2\rho} \quad \Rightarrow \quad \beta = \frac{6G^2M^2}{c^2} = GMR_{isco}. \tag{58}$$

6. Gravitational Waves from Two-Body Systems

In this section and the following we address the emission of gravitational radiation by the two-body systems described in section 5. As announced we treat this as a form of adiabatic dissipation changing the orbital parameters (ρ, e, n) of the system. This applies only to systems in which no head-on collisions or mergers involving strong gravity effects take place; these require more powerful methods of computation.[6]

To compute the amplitude h_{ij} from Equation (29) for point masses on the quasi-newtonian orbits (47) we must first determine the components of the quadrupole moment and their derivatives. For a two-body system in the CM frame they read

$$\begin{aligned} Q_{ij} &= m_1\left(r_{1i}r_{1j} - \frac{1}{3}\delta_{ij}r_1^2\right) + m_2\left(r_{2i}r_{2j} - \frac{1}{3}\delta_{ij}r_2^2\right) \\ &= \mu r^2\left(\hat{r}_i\hat{r}_j - \frac{1}{3}\delta_{ij}\right) \equiv \mu r^2\hat{R}_{ij}, \end{aligned} \tag{59}$$

where $\hat{\mathbf{r}}$ is the orbital unit vector in the equatorial plane defined in (43). We explicitly factor out the three-tensor array $\hat{\mathbf{R}}$ with components $\hat{R}_{ij}$ describing the angular dependence of the orbits used in computing the quadrupole moments:

$$\hat{\mathbf{R}} = \frac{1}{2}\begin{bmatrix} \cos 2\varphi + \frac{1}{3} & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}. \tag{60}$$

Next we want to compute the time derivatives of the quadrupole moment $\mathbf{Q}$. For ease of computation it is convenient to introduce a set of basic three-tensors in which all our results can be expressed:

$$\mathbf{M} = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -\sin 2\varphi & \cos 2\varphi & 0 \\ \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{61}$$

and

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}. \tag{62}$$

They have simple algebraic properties

$$\begin{aligned} &\mathbf{E}^2 = \frac{2}{9}\mathbf{I} - \frac{1}{3}\mathbf{E}, \quad \mathbf{M}^2 = \mathbf{N}^2 = -\mathbf{J}^2 = \frac{2}{3}\mathbf{I} + \mathbf{E}, \\ &\mathbf{E}\cdot\mathbf{M} = \mathbf{M}\cdot\mathbf{E} = \frac{1}{3}\mathbf{M}, \quad \mathbf{E}\cdot\mathbf{N} = \mathbf{N}\cdot\mathbf{E} = \frac{1}{3}\mathbf{N}, \\ &\mathbf{M}\cdot\mathbf{N} = -\mathbf{N}\cdot\mathbf{M} = \mathbf{J}. \end{aligned} \tag{63}$$

In addition their derivatives are

$$\begin{aligned} &\frac{d\mathbf{M}}{dt} = \frac{2\ell}{r^2}\mathbf{N}, \qquad \frac{d\mathbf{N}}{dt} = -\frac{2\ell}{r^2}\mathbf{M}, \\ &\frac{d\mathbf{E}}{dt} = \frac{d\mathbf{I}}{dt} = \frac{d\mathbf{J}}{dt} = 0. \end{aligned} \tag{64}$$

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It follows that

$$\hat{\mathbf{R}} = \frac{1}{2}\left(\mathbf{E}+\mathbf{M}\right). \tag{65}$$

Using these results and the ones in appendix B it is now straightforward to establish expressions for the quadrupole moment and its derivatives:

$$\begin{aligned}
\mathbf{Q} &= \frac{\mu r^2}{2}\left(\mathbf{E}+\mathbf{M}\right), \qquad \dot{\mathbf{Q}} = \mu\ell\left(\frac{r'}{r}\mathbf{E}+\frac{r'}{r}\mathbf{M}+\mathbf{N}\right),\\
\ddot{\mathbf{Q}} &= \frac{\mu\ell^2}{r^2}\left[\left(\frac{r''}{r}-\frac{r'^2}{r^2}\right)\mathbf{E}+\left(\frac{r''}{r}-\frac{r'^2}{r^2}-2\right)\mathbf{M}+\frac{2r'}{r}\mathbf{N}\right],\\
\dddot{\mathbf{Q}} &= \frac{\mu\ell^3}{r^4}\left[\left(\frac{r'''}{r}-\frac{5r''r'}{r^2}+\frac{4r'^3}{r^3}\right)\left(\mathbf{E}+\mathbf{M}\right)\right.\\
&\quad \left.+4\left(\frac{r''}{r}-\frac{2r'^2}{r^2}-1\right)\mathbf{N}\right].
\end{aligned} \tag{66}$$

More generally we can write for the n-th derivative

$$\mathbf{Q}^{(n)} = \frac{\mu\ell^n}{r^{2(n-1)}}\left(Q_E^{(n)}\mathbf{E}+Q_M^{(n)}\mathbf{M}+Q_N^{(n)}\mathbf{N}\right), \quad n=0,1,2,3,\ldots, \tag{67}$$

where the coefficients $Q_{E,M,N}^{(n)}$ can be read off from the expressions (66) or computed by taking still higher derivatives. These results can now be used to evaluate the amplitude $h_{ij}(\mathbf{x},t)$; the expression (29) for the amplitude is equivalent to

$$\begin{aligned}
h_{ij}(\mathbf{x},t) &= \frac{\kappa}{8\pi r}\left[\ddot{Q}_{ij}-\hat{r}_i(\ddot{\mathbf{Q}}\cdot\hat{\mathbf{r}})_j-\hat{r}_j(\ddot{\mathbf{Q}}\cdot\hat{\mathbf{r}})_i\right.\\
&\quad \left.+\frac{1}{2}\left(\delta_{ij}+\hat{r}_i\hat{r}_j\right)\hat{\mathbf{r}}\cdot\ddot{\mathbf{Q}}\cdot\hat{\mathbf{r}}\right]_{u=t-r}.
\end{aligned} \tag{68}$$

Note that the direction of the observer is given by the polar unit vector

$$\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{69}$$

which is distinct from the orbital unit vector $\hat{\mathbf{r}}$; then the amplitude in three-tensor notation takes the form

$$\begin{aligned}
\mathbf{h} &= \frac{\kappa}{8\pi r}\frac{\mu\ell^2}{r^2}\left[Q_E^{(2)}\mathbf{E}+Q_M^{(2)}\mathbf{M}+Q_N^{(2)}\mathbf{N}\right.\\
&\quad -\hat{\mathbf{r}}\left(Q_E^{(2)}\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(2)}\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(2)}\mathbf{N}\cdot\hat{\mathbf{r}}\right)^T\\
&\quad -\left(Q_E^{(2)}\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(2)}\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(2)}\mathbf{N}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}^T\\
&\quad \left.+\frac{1}{2}\left(\mathbf{I}+\hat{\mathbf{r}}\hat{\mathbf{r}}^T\right)\left(Q_E^{(2)}\hat{\mathbf{r}}\cdot\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(2)}\hat{\mathbf{r}}\cdot\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(2)}\hat{\mathbf{r}}\cdot\mathbf{N}\cdot\hat{\mathbf{r}}\right)\right].
\end{aligned} \tag{70}$$

To evaluate this expression use

$$\begin{aligned}
\mathbf{E}\cdot\hat{\mathbf{r}} &= \frac{1}{3}\left(\sin\theta\cos\phi, \sin\theta\sin\phi, -2\cos\theta\right),\\
\mathbf{M}\cdot\hat{\mathbf{r}} &= \sin\theta\left(\cos(2\varphi-\phi), \sin(2\varphi-\phi), 0\right),\\
\mathbf{N}\cdot\hat{\mathbf{r}} &= \sin\theta\left(-\sin(2\varphi-\phi), \cos(2\varphi-\phi), 0\right),
\end{aligned} \tag{71}$$

and

$$\begin{aligned}
&\hat{\mathbf{r}}\cdot\mathbf{E}\cdot\hat{\mathbf{r}} = \sin^2\theta-\frac{2}{3}, \quad \hat{\mathbf{r}}\cdot\mathbf{M}\cdot\hat{\mathbf{r}} = \sin^2\theta\cos 2(\phi-\varphi),\\
&\hat{\mathbf{r}}\cdot\mathbf{N}\cdot\hat{\mathbf{r}} = \sin^2\theta\sin 2(\phi-\varphi).
\end{aligned} \tag{72}$$

The simplest case is that of circular orbits with $r'=0$ and $\ell = \omega r^2$, where ω is the constant angular velocity such that $\varphi(t)=\omega t$. Then

$$Q_E^{(2)} = Q_N^{(2)} = 0, \quad Q_M^{(2)} = -2, \tag{73}$$

and

$$\begin{aligned}
\mathbf{h} &= \frac{\kappa\mu\,\omega^2 r^2}{8\pi r}\left[-2\mathbf{M}+2\hat{\mathbf{r}}\left(\mathbf{M}\cdot\hat{\mathbf{r}}\right)^T+2(\mathbf{M}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}^T\right.\\
&\quad \left.-\hat{\mathbf{r}}\cdot\mathbf{M}\cdot\hat{\mathbf{r}}\left(\mathbf{I}+\hat{\mathbf{r}}\hat{\mathbf{r}}^T\right)\right].
\end{aligned} \tag{74}$$

In particular in the equatorial plane $\theta=\pi/2$ and

$$\mathbf{h} = \frac{\kappa\mu\,\omega^2 r^2}{16\pi r}\cos 2(\phi-\omega t)\begin{pmatrix}1-\cos 2\phi & -\sin 2\phi & 0\\ -\sin 2\phi & 1+\cos 2\phi & 0\\ 0 & 0 & -2\end{pmatrix}. \tag{75}$$

whilst along the axis perpendicular to the equatorial plane $\theta=0$ and

$$\begin{aligned}
\mathbf{h} &= -\frac{\kappa\mu\,\omega^2 r^2}{4\pi r}\mathbf{M}\\
&= \frac{\kappa\mu\,\omega^2 r^2}{4\pi r}\left[\sin 2(\phi-\omega t)\begin{pmatrix}\sin 2\phi & \cos 2\phi & 0\\ \cos 2\phi & -\sin 2\phi & 0\\ 0 & 0 & 0\end{pmatrix}\right.\\
&\quad \left.-\cos 2(\phi-\omega t)\begin{pmatrix}\cos 2\phi & \sin 2\phi & 0\\ \sin 2\phi & -\cos 2\phi & 0\\ 0 & 0 & 0\end{pmatrix}\right].
\end{aligned} \tag{76}$$

Note that the frequency of the gravitational waves is twice that of the orbital motion, which is a direct consequence of their quadrupole nature.

7. Radiative Energy Loss

The first Equation (36) describes the energy flux of gravitational waves per unit of spherical angle as a function of the direction specified by the unit vector $\hat{\mathbf{r}}$. Equations (66) specify the quadrupole moments and their derivatives for two-body systems

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in generalized newtonian orbits (47). To evaluate the differential energy flux these quadrupole moments are to be substituted into the energy flux equation. First we compute

$$\left[\mathbf{Q}^{(3)}\right]^2 = \frac{\mu^2\ell^6}{r^8}\left[\frac{2}{3}\left(\frac{1}{3}\,Q_E^{(3)\,2} + Q_M^{(3)\,2} + Q_N^{(3)\,2}\right)\mathbf{I} + \left(-\frac{1}{3}\,Q_E^{(3)\,2} + Q_M^{(3)\,2} + Q_N^{(3)\,2}\right)\mathbf{E} + \frac{2}{3}\,Q_E^{(3)}\,Q_M^{(3)}\,\mathbf{M} + \frac{2}{3}\,Q_E^{(3)}\,Q_N^{(3)}\,\mathbf{N}\right]. \tag{77}$$

It follows that

$$\mathrm{Tr}\left[\mathbf{Q}^{(3)}\right]^2 = \frac{2\mu^2\ell^6}{r^8}\left(\frac{1}{3}\,Q_E^{(3)\,2} + Q_M^{(3)\,2} + Q_N^{(3)\,2}\right), \tag{78}$$

and

$$\hat{\mathbf{r}}\cdot\left[\mathbf{Q}^{(3)}\right]^2\cdot\hat{\mathbf{r}} = \frac{\mu^2\ell^6}{r^8}\left[\frac{4}{9}\,Q_E^{(3)\,2} + \sin^2\theta\left(-\frac{1}{3}\,Q_E^{(3)\,2} + Q_M^{(3)\,2} + Q_N^{(3)\,2} + \frac{2}{3}\cos 2(\phi-\varphi)\,Q_E^{(3)}\,Q_M^{(3)} + \frac{2}{3}\sin 2(\phi-\varphi)\,Q_E^{(3)}\,Q_N^{(3)}\right)\right]. \tag{79}$$

Finally

$$\hat{\mathbf{r}}\cdot\mathbf{Q}^{(3)}\cdot\hat{\mathbf{r}} = \frac{\mu\ell^3}{r^4}\left[-\frac{2}{3}\,Q_E^{(3)} + \sin^2\theta\left(Q_E^{(3)} + \cos 2(\phi-\varphi)\,Q_M^{(3)} + \sin 2(\phi-\varphi)\,Q_N^{(3)}\right)\right]. \tag{80}$$

Inserting the coefficients taken from Equation (66):

$$Q_E^{(3)} = Q_M^{(3)} = \left(\frac{r'''}{r} - \frac{5r''r'}{r^2} + \frac{4r'^3}{r^3}\right) \equiv A,$$
$$Q_N^{(3)} = 4\left(\frac{r''}{r} - \frac{2r'^2}{r^2} - 1\right) \equiv B, \tag{81}$$

the general result is

$$\frac{dE}{d^2\Omega dt} = -\frac{G\mu^2\ell^6}{8\pi c^5 r^8}\left[2\left(A^2+B^2\right)\cos^2\theta - 2\,A^2\sin^2\theta\cos 2(\phi-\varphi) - 2\,AB\sin^2\theta\sin 2(\phi-\varphi) + \frac{1}{2}\,\sin^4\theta\left(A^2+B^2+2\,A^2\cos 2(\phi-\varphi) + 2\,AB\sin 2(\phi-\varphi) + \left(A^2-B^2\right)\cos^2 2(\phi-\varphi + 2\,AB\sin 2(\phi-\varphi)\cos 2(\phi-\varphi)\right)\right]. \tag{82}$$

For purely Keplerian orbits this result was derived in [20]. Using the results from appendix B for the generalized newtonian orbits (47) the expressions for the quantities A and B take the form

$$A = \frac{n^3 r}{\rho}\sqrt{(e^2-1)\,\frac{r^2}{\rho^2} + \frac{2r}{\rho} - 1},$$
$$B = -\frac{4n^2 r}{\rho} + 4\left(n^2-1\right). \tag{83}$$

The intensity distribution of gravitation radiation emitted by a bound binary system in elliptical orbit, precessing and non-precessing, is illustrated for a particular choice of parameters in appendix C.

After integrating the result (82) over all angles the standard result (39) for the total energy loss becomes

$$\frac{dE}{dt} = -\frac{2G\mu^2\ell^6}{15c^5r^8}\left(4A^2+3B^2\right). \tag{84}$$

Substitution of the expressions (83) then results in

$$\frac{dE}{dt} = -\frac{8G^4M^3\mu^2}{15c^5n^6\rho^5}\left[n^6\left(e^2-1\right)\frac{\rho^4}{r^4} + 2n^6\,\frac{\rho^5}{r^5} - n^4\left(n^2-12\right)\frac{\rho^6}{r^6} - 24n^2\left(n^2-1\right)\frac{\rho^7}{r^7} + 12(n^2-1)^2\,\frac{\rho^8}{r^8}\right]. \tag{85}$$

In the simplest case, that of a circular orbit with $e=0$, $n=1$, $r=\rho$ and with angular velocity given by

$$\ell^2 = r^4\omega^2 = GM\rho, \tag{86}$$

this result reduces to the well-known expression

$$\frac{dE}{dt} = -\frac{32G^4M^3\mu^2}{5c^5\rho^5} = -\frac{2}{5}\left(\frac{2GM}{c^2\rho}\right)^4\frac{\mu^2c^3}{M\rho}. \tag{87}$$

The last result has been cast in terms of the dimensionless compactness parameter $2GM/c^2\rho$, defined as the ratio of the Schwarzschild radius for the combined system and the actual orbital scale characterized by ρ. For non-precessing orbits for which $n=1$, $\ell^2 = GM\rho$, the rate of energy loss is

$$\frac{dE}{dt} = -\frac{1}{30}\left(\frac{2GM}{c^2\rho}\right)^4\frac{\mu^2c^3}{M\rho}\left[\left(e^2-1\right)\frac{\rho^4}{r^4} + 2\,\frac{\rho^5}{r^5} + 11\,\frac{\rho^6}{r^6}\right]. \tag{88}$$

The expression (85) can also be used to compute the total energy lost by the two-body system in a definite period between times t_1 and t_2, e.g. between two periastra for bound orbits, or during the total passage of two objects in an open orbit:

$$\Delta E = \int_{t_1}^{t_2} dt\,\frac{dE}{dt} = \frac{\rho^2}{\ell}\int_{\varphi_1}^{\varphi_2} d\varphi\,\frac{r^2}{\rho^2}\,\frac{dE}{dt} = \frac{\rho^2}{n\ell}\int_{\psi_1}^{\psi_2} d\psi\,\frac{r^2}{\rho^2}\,\frac{dE}{dt}, \tag{89}$$

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where we have introduced the integration variable $\psi = n\varphi$. Now substitute (84) for the energy change and use

$$\frac{\rho}{r} = 1 - e\cos\psi.$$

Recalling that $n^2\ell^2 = GM\rho$ and expanding the integrand transforms the expression to

$$\begin{aligned}\Delta E = -\frac{\sqrt{2}}{30n^6}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\int_{\psi_1}^{\psi_2} d\psi\,\big[&12 + n^6e^2\\ &+ e\cos\psi\left(24n^2 - 72 - 2n^6e^2\right)\\ &+ e^2\cos^2\psi\left(-n^6 + 12n^4 - 120n^2 + 180 + n^6e^2\right)\\ &+ e^3\cos^3\psi\left(2n^6 - 48n^4 + 240n^2 - 240\right)\\ &+ e^4\cos^4\psi\left(-n^6 + 72n^4 - 240n^2 + 180\right)\\ &+ e^5\cos^5\psi\left(-48n^4 + 120n^2 - 72\right)\\ &+ 12(n^2-1)^2e^6\cos^6\psi\big].\end{aligned} \tag{90}$$

The adiabatic approximation implies that we treat the parameters e and n in this interval as constants; then it is straightforward to perform the integrations. For a bound orbit with succesive periastra at $\psi_1 = 0$ and $\psi_2 = 2\pi$ the total energy lost per period to gravitational waves is

$$\begin{aligned}\Delta E = -\frac{4\pi\sqrt{2}}{5n^6}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}&\\ \Bigg[1 + \frac{e^2}{24}\left(n^6 + 12n^4 - 120n^2 + 180\right)&\\ + \frac{e^4}{96}\left(n^6 + 216n^4 - 720n^2 + 540\right) + \frac{5e^6}{16}\left(n^2-1\right)^2\Bigg].&\end{aligned} \tag{91}$$

In particular for non-precessing orbits with $n = 1$:

$$\Delta E = -\frac{4\pi\sqrt{2}}{5}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{92}$$

For the simplest case, a circular orbit with $e = 0$:

$$\Delta E = -\frac{4\pi\sqrt{2}}{5}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}. \tag{93}$$

On the other hand, for open orbits with $e \geq 1$ and asymptotic values of the azimuth (ψ_1, ψ_2) satisfying

$$\begin{aligned}&\cos\psi_1 = \cos n\varphi_1 = \frac{1}{e}, \quad \sin\psi_1 = \frac{1}{e}\sqrt{e^2-1},\\ &\psi_2 = 2\pi - \psi_1,\end{aligned} \tag{94}$$

the result of the integral (90) in a somewhat hybrid notation is

$$\Delta E = -\frac{\sqrt{2}}{15n^6}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\sum_{k=0}^{6} I_k(n,\psi_1)\,e^k, \tag{95}$$

with coefficients

$$\begin{aligned}&I_0 = 12\left(\pi - \psi_1\right), \qquad I_1 = \left(-24n^2 + 72\right)\sin\psi_1,\\ &I_2 = \frac{1}{2}\left(3n^6 + 12n^4 - 120n^2 + 180\right)\left(\pi - \psi_1\right)\\ &\quad + \frac{1}{2}\left(n^6 - 12n^4 + 120n^2 - 180\right)\sin\psi_1\cos\psi_1,\\ &I_3 = \left(48n^4 - 240n^2 + 240\right)\sin\psi_1\\ &\quad + \frac{1}{3}\left(2n^6 - 48n^4 + 240n^2 - 240\right)\sin^3\psi_1,\\ &I_4 = \frac{1}{8}\left(n^6 + 216n^4 - 720n^2 + 540\right)\left(\pi - \psi_1\right)\\ &\quad + \frac{1}{8}\left(n^6 - 360n^4 + 1200n^2 - 900\right)\sin\psi_1\cos\psi_1\\ &\quad - \frac{1}{4}\left(n^6 - 72n^4 + 240n^2 - 180\right)\sin^3\psi_1\cos\psi_1,\\ &I_5 = \left(48n^4 - 120n^2 + 72\right)\left(\sin\psi_1 - \frac{2}{3}\sin^3\psi_1 + \frac{1}{5}\sin^5\psi_1\right),\\ &I_6 = 12\left(n^2-1\right)^2\Bigg[\frac{5}{16}\left(\pi - \psi_1\right)\\ &\quad - \cos\psi_1\left(\frac{11}{16}\sin\psi_1 - \frac{13}{24}\sin^3\psi_1 + \frac{1}{6}\sin^5\psi_1\right)\Bigg].\end{aligned} \tag{96}$$

For non-precessing orbits with $n = 1$ the expression simplifies as $I_5 = I_6 = 0$. The simplest case is the parabolic orbit with $e = 1$, $n = 1$ and $\psi_1 = 0$, resulting in

$$\Delta E = -\frac{433\pi\sqrt{2}}{120}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}. \tag{97}$$

These results are based on the generalized newtonian approximation. Results for scattering in the Effective One-Body formalism to all orders in v/c have been obtained in ref. [19].

8. Radiative Loss of Angular Momentum

The gravitational waves emitted by a system of masses in motion not only carry away energy, they also change the system's angular momentum. The last Equation (36) quantifies the directional angular momentum loss per unit of time of a non-relativistic system in terms of the change in the mass quadrupole. In this section we compute the angular momentum lost by a quasi-newtonian two-body system as we did for the energy in the previous section.

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After substitution of Equations (66), (67) in the expression (36) for the differential flux of angular momentum we get

$$\begin{aligned}\frac{dM_k}{d^2\Omega dt} &= -\frac{G}{4\pi c^5}\frac{\mu^2\ell^5}{r^6}\varepsilon_{kij}\\ &\times\Big[\Big(Q_E^{(2)}\mathbf{E}+Q_M^{(2)}\mathbf{M}+Q_N^{(2)}\mathbf{N}\Big)\cdot\Big(Q_E^{(3)}\mathbf{E}+Q_M^{(3)}\mathbf{M}+Q_N^{(3)}\mathbf{N}\Big)_{ij}\\ &-\Big(Q_E^{(2)}\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(2)}\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(2)}\mathbf{N}\cdot\hat{\mathbf{r}}\Big)_i\\ &\times\Big(Q_E^{(3)}\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(3)}\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(3)}\mathbf{N}\cdot\hat{\mathbf{r}}\Big)_j\\ &+\hat{\mathbf{r}}_i\Big(Q_E^{(2)}\mathbf{E}+Q_M^{(2)}\mathbf{M}+Q_N^{(2)}\mathbf{N}\Big)_{jl}\\ &\times\Big(Q_E^{(3)}\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(3)}\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(3)}\mathbf{N}\cdot\hat{\mathbf{r}}\Big)_l\\ &-\frac{1}{2}\hat{\mathbf{r}}_i\Big(Q_E^{(2)}\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(2)}\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(2)}\mathbf{N}\cdot\hat{\mathbf{r}}\Big)_j\\ &\times\Big(Q_E^{(3)}\hat{\mathbf{r}}\cdot\mathbf{E}\cdot\hat{\mathbf{r}}+Q_M^{(3)}\hat{\mathbf{r}}\cdot\mathbf{M}\cdot\hat{\mathbf{r}}+Q_N^{(3)}\hat{\mathbf{r}}\cdot\mathbf{N}\cdot\hat{\mathbf{r}}\Big)\Big]\end{aligned} \tag{98}$$

The total loss of angular momentum obtained by integration over all angles as given by the result (39) is

$$\frac{dM_k}{dt} = -\frac{2G}{5c^5}\varepsilon_{kij}\,[\mathbf{Q}^{(2)}\cdot\mathbf{Q}^{(3)}]_{ij}.$$

According to the expansion (67) and the multiplication rules (63) the only antisymmetric contribution to the product of $\mathbf{Q}^{(2)}$ and $\mathbf{Q}^{(3)}$ comes from

$$\mathbf{M}\cdot\mathbf{N} = -\mathbf{N}\cdot\mathbf{M} = \mathbf{J},$$

which has only a non-vanishing $J_{xy} = -J_{yx} = 1$ component. As the only non-trivial component of orbital angular momentum is M_z this is as expected. Using the results of appendix B it follows that

$$\begin{aligned}\frac{dM_z}{dt} &= -\frac{4G\mu^2\ell^5}{5c^5r^6}\Big(Q_M^{(2)}Q_N^{(3)}-Q_N^{(2)}Q_M^{(3)}\Big)\\ &= -\frac{8G\mu^2\ell^5}{5c^5r^6}\Big[n^4(1-e^2)\frac{r^3}{\rho^3}\\ &\quad -2n^2(n^2-1)(1-e^2)\frac{r^2}{\rho^2}+n^2(n^2+2)\frac{r}{\rho}-4(n^2-1)\Big].\end{aligned} \tag{99}$$

For circular orbits with $r = \rho$, $e = 0$ and $n = 1$ this reduces to

$$\begin{aligned}\frac{dM_z}{dt} &= -\frac{32G^3\mu^2M^2}{5c^5\rho^3}\sqrt{\frac{GM}{\rho}}\\ &= -\frac{2\sqrt{2}}{5}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M},\end{aligned} \tag{100}$$

and for other non-precessing orbits

$$\frac{dM_z}{dt} = -\frac{\sqrt{2}}{10}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\left[(1-e^2)\frac{\rho^3}{r^3}+3\frac{\rho^5}{r^5}\right]. \tag{101}$$

Following a procedure similar to the treatment of energy we can compute the change in angular momentum in a fixed period of time between precessing angles $\psi_{1,2}$:

$$\begin{aligned}\Delta M_z &= \frac{\rho^2}{n\ell}\int_{\psi_1}^{\psi_2}d\psi\,\frac{r^2}{\rho^2}\frac{dM_z}{dt}\\ &= -\frac{1}{5n^5}\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\int_{\psi_1}^{\psi_2}d\psi\,\big[4+e^2n^2(n^2-2)\\ &\quad +e\cos\psi\,\big(6n^2-16-e^2n^2(3n^2-4)\big)\\ &\quad +e^2\cos^2\psi\,\big(n^4-16n^2+24+2e^2n^2(n^2-1)\big)\\ &\quad +e^3\cos^3\psi\,\big(-n^4+14n^2-16\big)-4(n^2-1)e^4\cos^4\psi\big].\end{aligned} \tag{102}$$

It follows that for a bound state the angular momentum lost per period between successive periastra $\psi_1 = 0$ and $\psi_2 = 2\pi$ is

$$\begin{aligned}\Delta M_z &= -\frac{8\pi}{5n^5}\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\left[1+\frac{e^2}{8}\big(3n^4-20n^2+24\big)\right.\\ &\quad \left.+\frac{e^4}{8}\big(2n^2-3\big)\big(n^2-1\big)\right].\end{aligned} \tag{103}$$

For $n = 1$ this becomes:

$$\Delta M_z = -\frac{8\pi}{5n^5}\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\left[1+\frac{7e^2}{8}\right]; \tag{104}$$

for circular motion just take $e = 0$. Next considering open orbits with asymptotic directions as in (94) Equation (102) takes the form

$$\Delta M_z = -\frac{2}{5n^5}\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\sum_{k=0}^{4}m_k(n,\psi_1)\,e^k, \tag{105}$$

with coefficients

$$\begin{aligned}m_0 &= 4(\pi-\psi_1), \qquad m_1 = \big(-6n^2+16\big)\sin\psi_1,\\ m_2 &= \left(\frac{3}{2}n^4-10n^2+12\right)(\pi-\psi_1)\\ &\quad -\left(\frac{1}{2}n^4-8n^2+12\right)\sin\psi_1\cos\psi_1,\\ m_3 &= \big(4n^4-18n^2+16\big)\sin\psi_1\\ &\quad -\frac{1}{3}\big(n^4-14n^2+16\big)\sin^3\psi_1,\\ m_4 &= \big(n^2-1\big)\left[\left(n^2-\frac{3}{2}\right)(\pi-\psi_1)\right.\end{aligned} \tag{106}$$

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$$-\left(n^2-\frac{5}{2}\right)\sin\psi_1\cos\psi_1-\sin^3\psi_1\cos\psi_1\Bigg].$$

In particular for *parabolic* orbits with $e=n=1$ and $\psi_1=0$:

$$\Delta M_z=-3\pi\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}. \tag{107}$$

In ref. [13] a similar result was derived for small-angle scattering in purely newtonian gravity with $\beta=0$.

9. Evolution of Orbits

The flux of energy and angular momentum carried by gravitational waves as expressed by Equations (34) can be determined only if all components of the wave signal are known. With present interferometric detectors this is barely possible by combining the signals received by at least three instruments at different locations. However, the loss of energy and angular momentum by sources such as binary star systems is observable and allows the gravitational-wave flux to be reconstructed as in the well-known case of the binary pulsar systems. Therefore it is of some practical use to evaluate the orbital changes due to the emission of gravitational radiation by such systems. Here as in the previous sections we consider non-relativistic two-body systems, either in bound orbit or on scattering trajectories.

In the adiabatic approximation on which our calculations are based the orbits of two-body systems in the CM frame are parametrized by the expression (47). We take the orbital parameters (ρ, e, n) to be slowly changing functions of time; they would be constant in the absence of gravitational radiation. According to Equations (56) and (57) the orbital energy and angular momentum are expressed in terms of these parameters by

$$E=\frac{GM\mu}{2\rho}\left(e^2-1\right),\qquad L_z=\mu\sqrt{GM\rho+\beta}. \tag{108}$$

For comparison with observational data of bound orbits it is sometimes convenient to consider the (possibly precessing) semi-major axis of the orbit related to the semi-latus rectum by

$$a=\frac{\rho}{1-e^2}\quad\Rightarrow\quad E=-\frac{GM\mu}{2a}. \tag{109}$$

This quantity is also related to the precession parameter by

$$\frac{1}{n^2}=1+\frac{\beta}{GM\rho}\quad\Rightarrow\quad L_z=\frac{\mu}{n}\sqrt{GM\rho}. \tag{110}$$

It follows that for bound orbits the orbital parameter changes are related to change in orbital energy and angular momentum by

$$\frac{dE}{dt}=\frac{GM\mu}{2a^2}\frac{da}{dt},\qquad \frac{dL_z}{dt}=\frac{n\mu}{2}\sqrt{\frac{GM}{\rho}}\frac{d\rho}{dt}. \tag{111}$$

As these parameters are related by (109) the changes in ρ and in eccentricy e are related as well:

$$\frac{1}{\rho}\frac{d\rho}{dt}=\frac{1}{a}\frac{da}{dt}-\frac{1}{1-e^2}\frac{de^2}{dt}. \tag{112}$$

Also for constant β:

$$\frac{1}{\rho}\frac{d\rho}{dt}=\frac{2}{n(1-n^2)}\frac{dn}{dt}. \tag{113}$$

Now by equating the change in energy and orbital angular momentum to the amount of energy ΔE and angular momentum ΔM_z carried away by gravitational waves we can relate the change in orbital parameters to these parameters themselves. In particular according to Equations (91) and (103) during a period between to succesive periastra the orbital parameters change by

$$\begin{aligned}\frac{\Delta a}{a}&=-\frac{\Delta E}{E}\\&=-\frac{16\pi\sqrt{2}}{5n^6}\frac{\mu}{M}\left(\frac{2GM}{c^2\rho}\right)^{5/2}\frac{1}{1-e^2}\\&\quad\times\Bigg[1+\frac{e^2}{24}\left(n^6+12n^4-120n^2+180\right)\\&\quad+\frac{e^4}{96}\left(n^6+216n^4-720n^2+540\right)+\frac{5e^6}{16}\left(n^2-1\right)^2\Bigg],\\\frac{\Delta\rho}{\rho}&=\frac{2}{n\mu\sqrt{GM\rho}}\Delta M_z\\&=-\frac{16\pi\sqrt{2}}{5n^6}\frac{\mu}{M}\left(\frac{2GM}{c^2\rho}\right)^{5/2}\\&\quad\times\left[1+\frac{e^2}{8}\left(3n^4-20n^2+24\right)+\frac{e^4}{8}\left(2n^2-3\right)\left(n^2-1\right)\right],\end{aligned} \tag{114}$$

Furthermore from these results we can determine the period of the orbit between periastra and its evolution. The period itself is

$$\begin{aligned}T&=\int_0^{2\pi/n}d\varphi\,\frac{dt}{d\varphi}=\frac{\rho^2}{n\ell}\int_0^{2\pi}d\psi\,\frac{1}{(1-e\cos\psi)^2}\\&=\frac{2\pi}{(1-e^2)^{3/2}}\frac{\rho^2}{n\ell}=\frac{2\pi a^{3/2}}{\sqrt{GM}}.\end{aligned} \tag{115}$$

This is the appropriate generalization of Kepler's third law for precessing orbits, which holds provided the period T is taken to be that between two periastra. From this it follows that the rate of change of the period is

$$\frac{dT}{dt}=3\pi\sqrt{\frac{a}{GM}}\frac{da}{dt}, \tag{116}$$

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and the relative change per turn is

$$\frac{\Delta T}{T} = \frac{3}{2}\frac{\Delta a}{a}. \tag{117}$$

This amounts to a generalization of the Peter-Matthews Equation[20]

$$\frac{dT}{dt} \simeq \frac{\Delta T}{T} = -\frac{192\pi}{5c^5}\frac{G^{5/3}M^{2/3}\mu}{(1-e^2)^{7/2}}\left(\frac{T}{2\pi}\right)^{-5/3} \times\left[\frac{1}{n^6}+\frac{e^2}{24}\left(1+\frac{12}{n^2}-\frac{120}{n^4}+\frac{180}{n^6}\right) + \frac{e^4}{96}\left(1+\frac{216}{n^2}-\frac{720}{n^4}+\frac{540}{n^6}\right)+\frac{5e^6}{16n^6}\left(n^2-1\right)^2\right]. \tag{118}$$

Next we consider open orbits. These we will characterize in terms of ρ and e directly with rates of change determined by (108) and (111)

$$\frac{1}{e^2-1}\frac{de^2}{dt} = \frac{1}{\rho}\frac{d\rho}{dt}+\frac{1}{E}\frac{dE}{dt}. \tag{119}$$

This results in

$$\frac{d\rho}{dt} = -\frac{2}{5n^6}\frac{\mu c}{M}\left(\frac{2GM}{c^2\rho}\right)^3\left[n^4(1-e^2)\frac{\rho^3}{r^3} -2n^2(n^2-1)(1-e^2)\frac{\rho^4}{r^4}+n^2(n^2+2)\frac{\rho^5}{r^5}-4(n^2-1)\frac{\rho^6}{r^6}\right], \tag{120}$$

$$\begin{aligned}\frac{de^2}{dt} = \frac{1}{60n^6}\frac{\mu c}{M\rho}\left(\frac{2GM}{c^2\rho}\right)^3\Bigg[&24n^4\left(e^2-1\right)^2\frac{\rho^3}{r^3}\\ &-n^2(e^2-1)\left(n^4+48(n^2-1)(e^2-1)\right)\frac{\rho^4}{r^4}\\ &-2n^2\left(n^4+12(n^2+2)(e^2-1)\right)\frac{\rho^5}{r^5}\\ &+\left(n^2(n^2-12)+96(n^2-1)(e^2-1)\right)\frac{\rho^6}{r^6}\\ &+24n^2\left(n^2-1\right)\frac{\rho^7}{r^7}-12\left(n^2-1\right)^2\frac{\rho^8}{r^8}\Bigg].\end{aligned} \tag{121}$$

The corresponding changes over the complete orbit are

$$\frac{\Delta\rho}{\rho} = -\frac{4\sqrt{2}}{5n^6}\frac{\mu}{M}\left(\frac{2GM}{c^2\rho}\right)^{5/2}\sum_{k=0}^{4} m_k(n,\psi_1)e^k, \tag{122}$$

and

$$\Delta e^2 = \left(e^2-1\right)\frac{\Delta\rho}{\rho}-\frac{4\sqrt{2}}{15n^6}\frac{\mu}{M}\left(\frac{2GM}{c^2\rho}\right)^{5/2}\sum_{k=0}^{6} I_k(n,\psi_1)e^k. \tag{123}$$

The total energy change in such an open orbit is given by

$$\frac{\Delta E}{E} = -\frac{4\sqrt{2}}{15n^6}\frac{\mu}{M}\left(\frac{2GM}{c^2\rho}\right)^{5/2}\frac{\sum_{k=0}^{6}(I_k e^k)}{e^2-1}. \tag{124}$$

Finally one can determine for which open orbits the loss of energy by gravitational radiation results in a bound orbit, at least in lowest-order approximation. Such a capture process happens when the initial energy is positive and the final energy is negative: $|\Delta E| > E$. From (124) this requires

$$\frac{4\sqrt{2}}{15n^6(e^2-1)}\frac{\mu}{M}\sum_{k=0}^{6} I_k(n,\psi_1)e^k > \left(\frac{c^2\rho}{2GM}\right)^{5/2}.$$

As the semi-latus rectum ρ must be greater than the Schwarzschild radius of the system, the quantity on the left-hand side must be definitely larger than one, and as $\mu < M$ it follows that e^2-1 must be small, i.e. the orbit must be close to parabolic.

Appendix A: The Transverse Traceless Gauge

In this appendix we explain in more detail how starting from an arbitrary solution of the field Equations (3) for the massless tensor field one can reach the TT-gauge (24) in the far-field region. We will do this in the hamiltonian formulation in which space- and time components of the fields are considered separately. In this formulation the space-components h_{ij} and their conjugate momentum fields π_{ij} satisfy field equations which are first-order in time derivatives. In contrast the time components represent auxiliary fields $N = -h_{00}$ and $N_i = h_{0i}$ acting as Lagrage multipliers to impose constraints: time-independent field equations restricting the allowed field configurations of the space components. The full set of dynamical equations for these fields read

$$\begin{aligned}\pi_{ij} &= \dot{h}_{ij}-\delta_{ij}\dot{h}_{kk}+2\delta_{ij}\partial_k N_k-\partial_i N_j-\partial_j N_i,\\ \dot{\pi}_{ij} &= \Delta h_{ij}-\partial_i\partial_k h_{kj}-\partial_j\partial_k h_{ki}+\partial_i\partial_j h_{kk}\\ &\quad -\delta_{ij}\left(\Delta h_{kk}-\partial_k\partial_l h_{kl}\right)-\delta_{ij}\Delta N+\partial_i\partial_j N+\kappa T_{ij}.\end{aligned} \tag{125}$$

The constraints imposed by the auxiliary fields are

$$\Delta h_{jj}-\partial_i\partial_j h_{ij} = -\kappa T_{00}, \qquad \partial_j\pi_{ji} = \kappa T_{i0}. \tag{126}$$

Together these equations are fully equivalent to the covariant field Equations (3). Our analysis will show that the split in dynamical space- and non-dynamical time components is in full agreement with the properties of the causal solutions (18)–(21).

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As expected the full set of Equations (125), (126) is invariant under local gauge transformations which in this formulation take the form

$$h'_{ij} = h_{ij} + \partial_i \xi_j + \partial_j \xi_i, \qquad N'_i = N_i + \dot{\xi}_i + \partial_i \xi,$$
$$\pi'_{ij} = \pi_{ij} + 2\delta_{ij}\Delta\xi - 2\,\partial_i\partial_j\xi, \;\; N' = N - 2\,\dot{\xi}, \tag{127}$$

Observe that h_{ij} changes only by terms depending on ξ_i, whilst the change of π_{ij} is determined only by ξ. Clearly the transformations of the auxiliary fields (N, N_i) suffice to remove these components by taking

$$\dot{\xi} = \frac{1}{2}\,N, \qquad \dot{\xi}_i = N_i - \partial_i \xi. \tag{128}$$

This results in $N' = N'_i = 0$ and

$$\pi'_{ij} = \dot{h}'_{ij} - \delta_{ij}\dot{h}'_{kk},$$
$$\dot{\pi}'_{ij} = \Delta h'_{ij} - \partial_i\partial_k h'_{kj} - \partial_j\partial_k h'_{ki} + \partial_i\partial_j h'_{kk} - \delta_{ij}\left(\Delta h'_{kk} - \partial_k\partial_l h'_{kl}\right) + \kappa\, T_{ij}, \tag{129}$$

constrained by

$$\Delta h'_{jj} - \partial_i\partial_j h'_{ij} = -\kappa\, T_{00}, \qquad \partial_j \pi'_{ji} = \kappa\, T_{i0} \tag{130}$$

Now note that the choice of gauge parameters (128) does not fix these transformations completely: one can still make residual gauge transformations with parameters (ξ', ξ'_i) subject to the conditions

$$\dot{\xi}' = 0, \qquad \dot{\xi}'_i = -\partial_i \xi', \qquad \ddot{\xi}'_i = 0. \tag{131}$$

To see how these can be used, first note that combining the second field Equation (129) with the first constraint (130) results in

$$\dot{\pi}'_{jj} = \kappa\left(T_{jj} + T_{00}\right). \tag{132}$$

This condition is invariant under the residual gauge transformations, and therefore *in empty space* where $T_{jj} = T_{00} = 0$ the trace π'_{jj} is seen to be constant in time and can be removed by a time-independent gauge transformation:

$$\Delta\xi' = -\frac{1}{4}\left(\pi'_{jj}\right)_{t=0} \quad \Rightarrow \quad \pi''_{jj} = \pi'_{jj} + 4\Delta\xi' = 0. \tag{133}$$

In view of the first Equation (129) this also implies that at all times $\dot{h}''_{jj} = 0$ and therefore h''_{jj} is time-independent. In empty space the first constraint (130) then asserts that also $\partial_i\partial_j h''_{ij}$ is time-independent. Next the residual gauge parameters ξ'_i can be used to restrict the field combination

$$\partial_j h''_{ji} - \frac{1}{2}\,\partial_i h''_{jj} = \partial_j h'_{ji} - \frac{1}{2}\,\partial_i h'_{jj} + \Delta\xi'_i. \tag{134}$$

First it can be removed from the initial configuration by taking

$$\Delta\xi'_i = -\left(\partial_j h'_{ji} - \frac{1}{2}\,\partial_i h'_{jj}\right)_{t=0}$$
$$\Rightarrow \left(\partial_j h''_{ji} - \frac{1}{2}\,\partial_i h''_{jj}\right)_{t=0} = 0. \tag{135}$$

In combination with the first constraint (130), and knowing that h''_{jj} and $\partial_i\partial_j h''_{ij}$ themselves are constant in time, this implies that in empty space

$$\left(\Delta h''_{jj}\right)_{t=0} = \left(\partial_i\partial_j h''_{ij}\right)_{t=0} = 0 \;\; \Rightarrow \;\; \Delta h''_{jj} = \partial_i\partial_j h''_{ij} = 0 \tag{136}$$

at all times. Finally one can still make one more residual gauge transformation, with harmonic parameters (ξ'', ξ''_i) satisfying

$$\Delta\xi''_i = 0, \qquad \Delta\xi'' = -\partial_i\dot{\xi}''_i = 0. \tag{137}$$

These transformations can be used to remove the trace of the field at $t = 0$ and therefore at all times:

$$\partial_i\xi''_i = -\frac{1}{2}\left(h''_{jj}\right)_{t=0}$$
$$\Rightarrow h'''_{jj} = \left(h'''_{jj}\right)_{t=0} = \left(h''_{jj} + 2\partial_i\xi''_i\right)_{t=0} = 0. \tag{138}$$

Finally as the second constraint (130) in empty space requires

$$\partial_j\dot{h}'''_{ji} = 0, \tag{139}$$

we also find that by combining with (135) and (138)

$$\partial_j h'''_{ji} = \left(\partial_j h'''_{ji}\right)_{t=0} = 0. \tag{140}$$

In conclusion, we have proved that we can find local gauge transformations such that in empty space any solution of the field equation can be transformed to the TT-gauge

$$\partial_j h'''_{ji} = h'''_{jj} = 0,$$

by the gauge transformations specified in (128), (133), (135) and (138). The vanishing of the trace also implies that in the TT-gauge $h'''_{ij} = \underline{h}'''_{ij}$.

We close this section by noting that the hamiltonian field Equations (125), (126) follow directly from the action

$$S = \int d^4x\left(\dot{h}_{ij}\pi_{ij} - \mathscr{H}\right), \tag{141}$$

with hamiltonian density

$$\mathscr{H} = \frac{1}{2}\,\pi_{ij}^2 - \frac{1}{4}\,\pi_{jj}^2 + \frac{1}{2}\left(\partial_k h_{ij}\right)^2 - \left(\partial_j h_{ji} - \frac{1}{2}\partial_i h_{jj}\right)^2$$
$$- \frac{1}{4}\left(\partial_i h_{jj}\right)^2 - \kappa\, h_{ij}\, T_{ij} - 2\,N_i\left(\partial_j\pi_{ji} - \kappa\, T_{i0}\right)$$
$$+ N\left(\Delta h_{jj} - \partial_i\partial_j h_{ij} + \kappa\, T_{00}\right). \tag{142}$$

As is to be expected, in the TT-gauge this hamiltonian reduces to the energy density (31).

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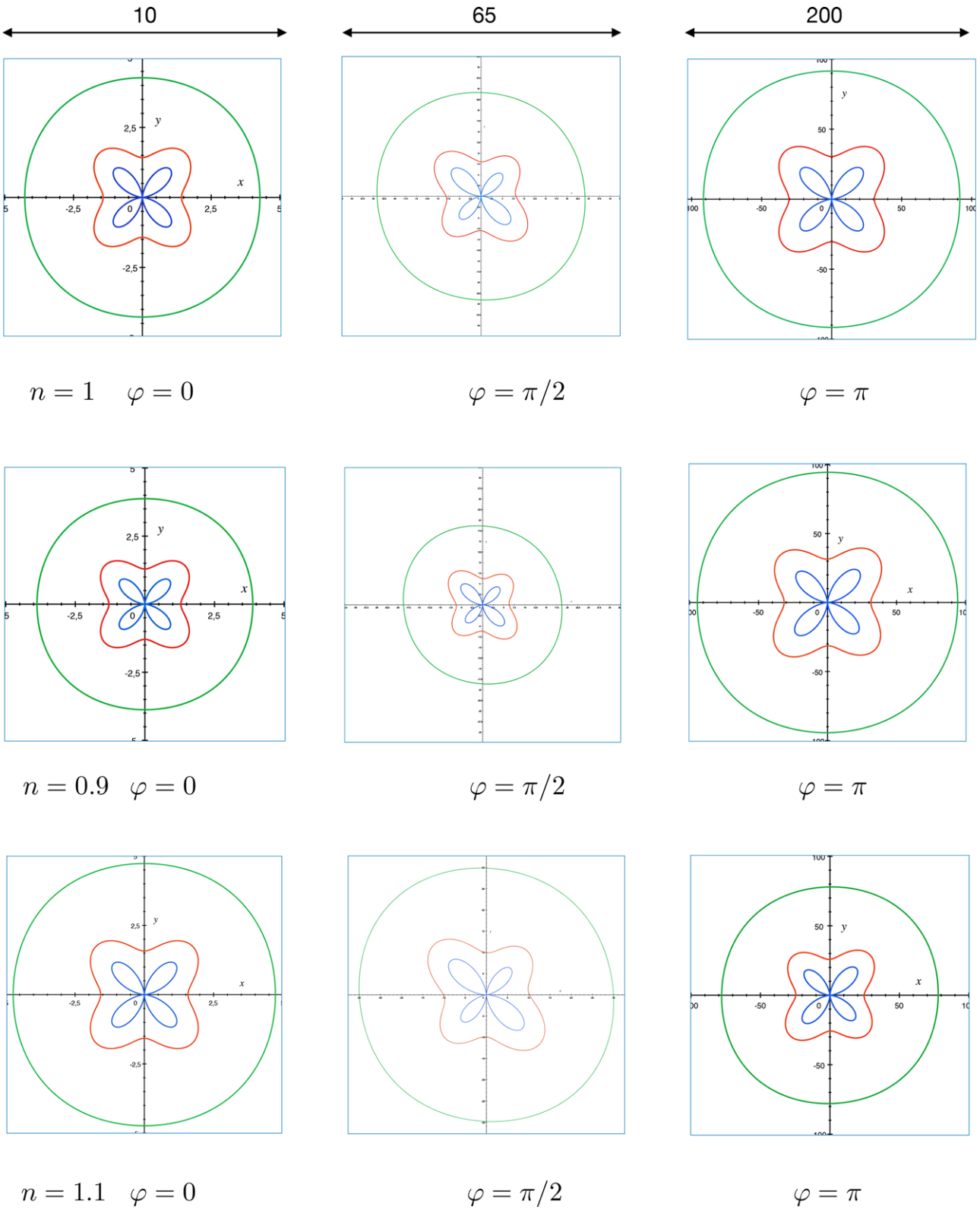

Figure B1. Intensity patterns of gravitational radiation emitted by a binary system in (quasi-)elliptical orbits (characterized by the value of n) with eccentricity $e = 0.25$ at three different points in the orbit at orientations $\varphi = (0, \pi/2, \pi)$, and as emitted in three different directions w.r.t. the polar axis: $\theta = 90^\circ$ (blue inner contour), $\theta = 60^\circ$ (red middle contour) and $\theta = 30^\circ$ (green outer contour). Note that the scales agree in vertical columns, but differ from left to right in proportion 10 : 65 : 200.

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Appendix B: Generalized Newtonian Orbits

The generalized newtonian orbits (47) are parametrized by

$$r = \frac{\rho}{1 - e\cos n\varphi}.$$

In our computations we also need the derivatives of this expression, up to the third derivative. Taking anti-clockwise motion they read

$$\frac{r'}{r} = -n\sqrt{(e^2-1)\frac{r^2}{\rho^2} + \frac{2r}{\rho} - 1},$$
$$\frac{r''}{r} = n^2\left[2\left(e^2-1\right)\frac{r^2}{\rho^2} + \frac{3r}{\rho} - 1\right],$$
$$\frac{r'''}{r} = -n^3\left[6\left(e^2-1\right)\frac{r^2}{\rho^2} + \frac{6r}{\rho} - 1\right]\sqrt{(e^2-1)\frac{r^2}{\rho^2} + \frac{2r}{\rho} - 1}. \tag{143}$$

Appendix C: Intensity of Emission from a Binary System

In this appendix we show an example of the intensity distribution of gravitational-wave emission in various directions produced by generalized newtonian binary systems in elliptic orbit with eccentricity $e = 0.25$ and precession rates $n = 1$ (newtonian, non-precessing), $n = 0.9$ (prograde precession) and $n = 1.1$ (retrograde precession). The intensity distribution is represented by the dimensionless quantity

$$\begin{aligned} Y(\theta,\phi) &= -128\pi n^6 \frac{M^2}{\mu^2}\left(\frac{c^2\rho}{2GM}\right)^4 \frac{\rho\, d(E/Mc^2)}{c\,dt\, d^2\Omega} \\ &= \frac{\rho^8}{r^8}\Big[2\left(A^2+B^2\right)\cos^2\theta - 2A^2\sin^2\theta\cos 2(\phi-\varphi) \\ &\quad - 2AB\sin^2\theta\sin 2(\phi-\varphi) + \frac{1}{2}\sin^4\theta\Big(A^2+B^2 \\ &\quad + 2A^2\cos 2(\phi-\varphi) + 2AB\sin 2(\phi-\varphi) \\ &\quad + \left(A^2-B^2\right)\cos^2 2(\phi-\varphi) \\ &\quad + 2AB\sin 2(\phi-\varphi)\cos 2(\phi-\varphi)\Big)\Big]. \end{aligned} \tag{144}$$

It is plotted as a function of azimuth ϕ for three different polar angles θ: in the equatorial plane $\theta = 90^\circ$, and in the directions $\theta = 60^\circ$ and $\theta = 30^\circ$ with respect to the axis of angular momentum, at three different instants during the orbit where the relative orientation of the two masses is $\varphi = 0$, $\varphi = 90^\circ$ and $\varphi = 180^\circ$ corresponding in the non-precessing case with $n = 1$ to apastron, semi-latus rectum and periastron. The same distributions for the same polar angles are also plotted for the case of prograde precession with $n = 0.9$, and for retrograde precession with $n = 1.1$.

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Conflict of Interest

The author has declared no conflict of interest.

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gravitational capture, gravitational waves, precessing orbits

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[1] J. H. Taylor, J. M. Weisberg, *Astrophys. J.* **1989**, *345*, 434.
[2] B. P. Abbott et al. *Phys. Rev. Lett.* **2016**, *116*, 061102.
[3] B. P. Abbott et al. *Phys. Rev. Lett.* **2017**, *119*, 141101.
[4] B. P. Abbott et al. *Phys. Rev. Lett.* **2017**, *119*, 161101.
[5] G. Nelemans, arXiv:1807.01060 [astro-ph.SR].
[6] L. Blanchet, *Living Rev. Relativity* **2014**, *17*, 2.
[7] C. O. Lousto, R. H. Price, *Phys. Rev. D* **1997**, *55*, 2124.
[8] K. Martel, E. Poisson, *Phys. Rev. D* **2005**, *71*, 084001.
[9] R. Fujita, W. Hikida, H. Tagoshi, *Prog. Theor Phys.* **2009**, *121*, 843.
[10] G. Koekoek, J. W. van Holten, *Class. Quantum Grav.* **2011**, *28*, 225022.
[11] G. d'Ambrosi, J. W. van Holten, *Class. Quantum Grav.* **2015**, *32*, 015012.
[12] B. P. Abbott et al. *Phys. Rev. Lett.* **2016**, *116*, 221101.
[13] T. Damour, N. Deruelle, *Phys. Lett. A* **1981**, *87*, 81.
[14] A. Einstein, *Sitzungsber. K. Preuss. Akad. Wiss.* **1918**, *I*, 154.
[15] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, Freeman, San Francisco, **1970**.
[16] M. Maggiore, *Gravitational Waves*, Oxford Univ. Press, **2008**.
[17] P. C. Peters, *Phys. Rev. B* **1964**, *136*, 1224.
[18] I. Newton, *Principia Mathematica*, Royal Society, London, **1687**.
[19] T. Damour, *Phys. Rev. D* **2016**, *94*, 104015.
[20] P. C. Peters, J. Mathews, *Phys. Rev.* **1963**, *131*, 435.

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