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(Received 13 November 2018; published 11 April 2019)

Majorana zero modes in a superconductor are midgap states localized in the core of a vortex or bound to the end of a nanowire. They are anyons with non-Abelian braiding statistics, but when they are immobile one cannot demonstrate this by exchanging them in real space and indirect methods are needed. As a real-space alternative, we propose to use the chiral motion along the boundary of the superconductor to braid a mobile vortex in the edge channel with an immobile vortex in the bulk. The measurement scheme is fully electrical and deterministic: edge vortices (π-phase domain walls) are created on demand by a voltage pulse at a Josephson junction and the braiding with a Majorana zero mode in the bulk is detected by the charge produced upon their fusion at a second Josephson junction.

DOI: 10.1103/PhysRevLett.122.146803

Introduction.—Non-Abelian anyons have the property that a pairwise exchange operation may produce a different state, not simply related to the initial state by a phase factor [1]. Because such "braiding" operations are protected from local sources of decoherence they are in demand for the purpose of quantum computations [2]. Charge $e/4$ quasiparticles in the $\nu = 5/2$ quantum Hall effect were the first candidates for non-Abelian statistics [3], followed by vortices in topological superconductors [4,5].

Since experimental evidence for non-Abelian anyons in the quantum Hall effect [6,7] has remained inconclusive, the experimental effort now focuses on the superconducting realizations [8]. While the mathematical description of the braiding operation (the Clifford algebra) is the same in both realizations, the way in which braiding is implemented is altogether different: In the quantum Hall effect one uses the chiral motion along the edge to exchange pairs of non-Abelian anyons and demonstrate non-Abelian statistics [9–11]. In contrast, in a superconductor the non-Abelian anyons are midgap states ("zero modes") bound to a defect (a vortex [12,13] or the end-point of a nanowire [14–16]). Because they are immobile, existing proposals to demonstrate non-Abelian statistics do not actually exchange the zero modes in real space [17–21].

Topological superconductors do have chiral edge modes [4], and recent experimental progress [22] has motivated the search for ways to use the chiral motion for a braiding operation [23]. The obstruction one needs to overcome is that the Majorana fermions which propagate along the edge of a superconductor have conventional *fermionic* exchange statistics. In the quantum Hall effect each charge $e/4$ quasiparticle contains a zero mode and the exchange of two quasiparticles is a non-Abelian operation on a topological qubit encoded in the zero modes. However, Majorana fermions contain no zero mode which might encode a topological qubit, one needs vortices for that.

In this Letter we show how one can exploit the chiral motion along the edge of a topological superconductor to exchange zero modes in real space. The key innovative element of our design, which distinguishes it from Ref. [23], is the use of a biased Josephson junction to *on demand* inject a pair of isolated vortices into chiral edge channels. Previous studies of such "edge vortices" relied on quantum fluctuations of the phase to create a vortex pair in the superconducting condensate [24–27], but here the injection is entirely *deterministic*. When the two mobile edge vortices encircle a localized bulk vortex their fermion parity switches from even to odd, as a demonstration of non-Abelian braiding statistics. The entire operation, injection braiding detection, can be carried out fully electrically, without requiring time-dependent control over Coulomb interactions or tunnel probabilities.

Edge vortex injection.—Figure 1 shows different ways in which the edge vortex can be injected: driven by a flux bias or by a voltage bias over a Josephson junction. We show two possible physical systems that support chiral edge channels moving in the *same* direction on opposite boundaries of the superconductor. Both are hybrid systems, where a topologically trivial superconductor (spin-singlet s-wave pair potential Δ_0) is combined with a topologically nontrivial material: a 2D Chern insulator (quantum anomalous Hall insulator) [22,28] [panel (a)] or a 3D topological insulator gapped on the surface by ferromagnets with opposite magnetization $M_{\uparrow,\downarrow}$ [24,29] [panel (b)].

The superconducting phase difference $\phi(t)$ across the Josephson junction is incremented with 2π by application of a voltage pulse $V(t)$ (with $\int V(t)dt = h/2e$), or by an $h/2e$ increase of the flux $\Phi(t)$ through an external

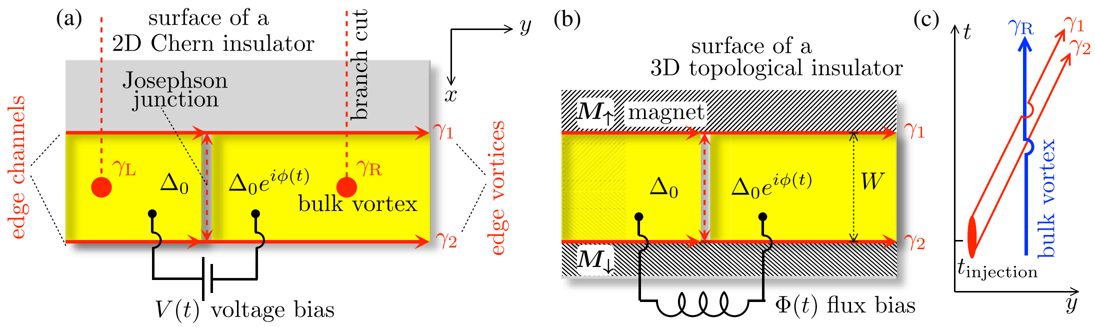

FIG. 1. Panels (a) and (b): Josephson junction geometries to deterministically inject a pair of edge vortices in chiral edge channels at opposite boundaries of a superconductor (yellow). The injection happens in response to a 2π increment in the superconducting phase difference $\phi(t)$, driven by a time-dependent voltage $V(t)$ or flux $\Phi(t)$. In panel (a) edge vortex 1 crosses the 2π branch cut of bulk vortex R, resulting in a fermion parity switch. Panel (c) shows the corresponding braiding of world lines in space-time: an overpass indicates that the vortex crosses a branch cut. Two crossings jointly switch the fermion parity of the edge vortices and of the bulk vortices, so that overall fermion parity is conserved. The orientation of the branch cuts can be varied by a gauge transformation and the measurable fusion outcome does not depend on it.

superconducting loop. If the width W of the superconductor is large compared to the coherence length $\xi_0 = \hbar v/\Delta_0$, the edge channels at $x = \pm W/2$ are not coupled by the Josephson junction—except when ϕ is near π, as follows from the junction Hamiltonian [13,29]

$$H_J = v p_x \sigma_z + \Delta_0 \sigma_y \cos(\phi/2). \tag{1}$$

The Pauli matrices act on excitations moving in the $\pm x$ direction with velocity v, in a single mode for ξ_0 large compared to the thickness of the junction in the y direction.

At $\phi = \pi$ a Josephson vortex passes through the superconductor [30,31]. A Josephson vortex is a 2π phase winding for the pair potential, so a π phase shift for an unpaired fermion. As explained in Ref. [32], the passage of the Josephson vortex leaves behind a pair of edge vortices: a phase boundary $\sigma(y)$ on each edge, at which the phase of the Majorana fermion wave function $\psi(y)$ jumps by π. Because of the reality constraint on ψ, a π phase jump (a minus sign) is stable: it can only be removed by merging with another π phase jump. And because the phase boundary is tied to the fermion wave function, it shares the same chiral motion, $\sigma(y,t) = \sigma(y - vt)$.

Braiding of an edge vortex with a bulk vortex.—Two vortices may be in a state of odd or even fermion parity, meaning that when they fuse they may or may not leave behind an unpaired electron. The fermion parity of vortices σ_1 and σ_2 is encoded in the ± 1 eigenvalue of the parity operator $P_{12} = i\gamma_1\gamma_2$, where γ_n is the Majorana operator associated with the zero mode in vortex n [33]. The two edge vortices are created at the Josephson junction in a state of even fermion parity, $P_{12} = +1$, but as illustrated in Fig. 1(a) that may change as they move away from the junction. If one of the edge vortices, say σ_1, crosses the branch cut of the phase winding around a bulk vortex, γ_1 picks up a minus sign and the fermion parity $P_{12} \mapsto -1$ switches from even to odd [5]. This is the essence of the non-Abelian braiding statistics of vortices. Overall fermion parity is conserved, because a second branch cut crossing [see Fig. 1(c)] also switches the fermion parity of the bulk vortices.

Detection of the fermion-parity switch.—Figure 2 shows the voltage-biased layout for a fully electrical measurement. The fermion parity of the edge vortices cannot be detected if they remain separated on opposite edges, so we first fuse them at a second Josephson junction. The characteristic time scale of the injection process [29] is the time $t_{\rm inj} = (\xi_0/W)(d\phi/dt)^{-1}$ when $\phi(t)$ is within ξ_0/W from π, and if the distance L between the two Josephson junctions is less than $v t_{\rm inj}$ we can neglect the time delay between the injection at the first junction J_1 and the fusion at the second junction J_2. This is convenient, because then the whole process can be driven by a single voltage pulse $V(t)$ applied to the region $|y| < L/2$ between the two junctions, relative to the grounded regions $y < -L/2$ and $y > L/2$ outside.

Both these grounded regions are connected to normal metal electrodes N_1 and N_2 and the electrical current $I(t)$ between them is measured. As we will now show, the transferred charge $Q = \int I(t)dt$ is quantized at unit electron charge if the region between the Josephson junctions contains a bulk vortex, while $Q = 0$ if it does not.

Mapping onto a scattering problem.—Tunneling of edge vortices driven by quantum fluctuations of the phase is a many-body problem of some complexity [32]. We avoid this because we rely on an external bias to inject the edge vortices; hence the phase $\phi(t)$ can be treated as a classical variable with a given time dependence.

The dynamics of the Majorana fermions remains fully quantum mechanical, governed by the Hamiltonian

$$H = i\begin{pmatrix} -v\partial/\partial y & -\mu[y,\phi(t)] \\ \mu[y,\phi(t)] & -v\partial/\partial y \end{pmatrix} \equiv v p_y \sigma_0 + \mu \sigma_y. \tag{2}$$

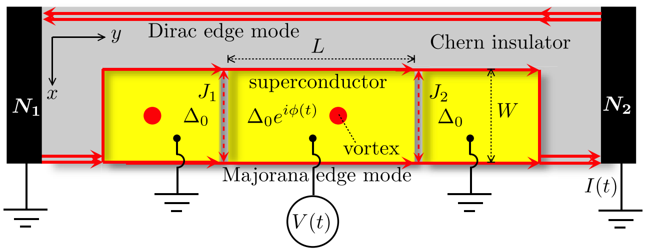

FIG. 2. Starting from the layout of Fig. 1(a), we have inserted a second Josephson junction (J_2) and we have added normal metal contacts (N_1, N_2) to measure the current $I(t)$ carried by the edge modes in response to the voltage $V(t)$ applied to the superconductor. A unit charge per 2π increment of ϕ is transferred from the superconductor into the normal metal contact. The counterpropagating Dirac edge mode along the upper edge of the Chern insulator is decoupled from the superconductor and plays no role in the analysis.

(We set $\hbar = 1$.) The 2×2 Hermitian matrix H acts on the Majorana fermion wave functions $\Psi = (\psi_1, \psi_2)$ at opposite edges of the superconductor, both propagating in the $+y$ direction (hence the unit matrix σ_0) The interedge coupling μ multiplies the σ_y Pauli matrix to ensure that H is purely imaginary and the wave equation $\partial\Psi/\partial t = -iH\Psi$ is purely real (as it should be for a Majorana fermion).

For low-energy, long-wavelength wave packets the y dependence of the interedge coupling may be replaced by a delta function, $\mu[y, \phi(t)] = v\delta(y)\eta(t)$. This "instantaneous scattering approximation" [34] is valid if the transit time $t_{\text{transit}} \simeq L/v$ of the wave packet through the system is short compared to the characteristic time scale t_{inj} of the vortex injection, hence if $d\phi/dt \ll v\xi_0/A_{\text{junction}}$, where $A_{\text{junction}} = WL$ is the area of the region between J_1 and J_2. In this regime there is no need to explicitly consider the vortex dynamics in between the Josephson junctions, instead we can treat this as a scattering problem "from the outside."

Incoming and outgoing states are related by

$$\Psi_{\text{out}}(E) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\omega)\Psi_{\text{in}}(E-\omega), \tag{3}$$

where $S(\omega)$ is the Fourier transform of the adiabatic (or "frozen") scattering matrix $S(t)$,

$$S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} S(t), \qquad S(t) = \exp(-i\eta(t)\sigma_y), \tag{4}$$

describing the scattering at $E = 0$ for a fixed $\phi(t)$. Note that $S(t)$ is unitary but $S(\omega)$ is not.

As we shall see in a moment, the transferred charge is independent of how $\eta(t) = \eta[\phi(t)]$ is varied as a function of time, only the net increment $\delta\eta = \eta(t \to \infty) - \eta(t \to -\infty)$ matters. When there is no vortex in the region between the two Josephson junctions J_1 and J_2 there is no difference between $\phi = 0$ and $\phi = 2\pi$, hence $\delta\eta = 0$. On the contrary, when there is a bulk vortex in this region we find [35]

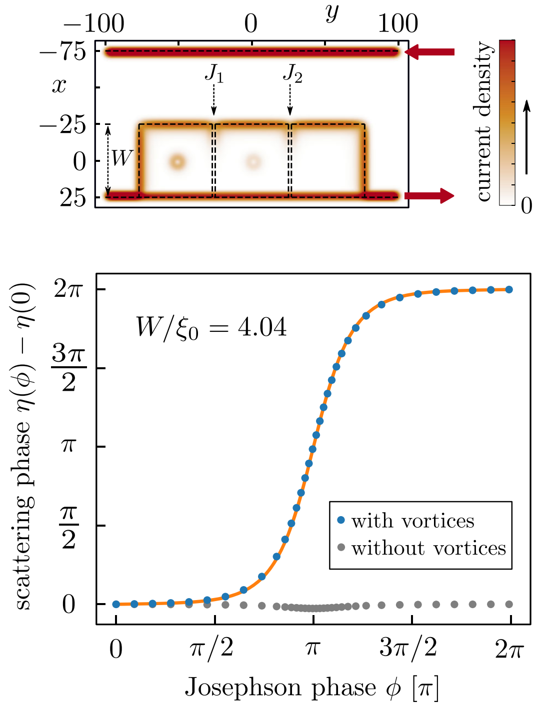

FIG. 3. Bottom panel: Scattering phase $\eta(\phi) - \eta(0)$ according to Eq. (5) (solid curve) and as obtained numerically (blue data points) from a lattice model [28] of the system shown in Fig. 2. There are no fit parameters in the comparison, the ratio $W/\xi_0 = 4.04$ was obtained directly from the simulation [35,37]. The grey data points show the result without vortices, when there is no net increment as ϕ advances from 0 to 2π. Top panel: Current density in the lattice model. The two vortices are faintly visible.

$$\eta = 2 \arccos\left(\frac{\cos(\phi/2) + \tanh\beta}{1 + \cos(\phi/2)\tanh\beta}\right), \qquad \beta = \frac{W}{\xi_0}\cos\frac{\phi}{2}, \tag{5}$$

hence $\delta\eta = 2\pi$. More generally, when there are N_{vortex} vortices between J_1 and J_2 the phase increment is

$$\delta\eta = \pi[1 - (-1)^{N_{\text{vortex}}}]. \tag{6}$$

In Fig. 3 we show that the analytical result Eq. (5) agrees well with a computer simulation (using Kwant [35,36]) of a lattice model of a quantum anomalous Hall insulator with induced s-wave superconductivity [28].

Transferred charge.—The expectation value of the transferred charge [38],

$$Q = e\int_0^{\infty} \frac{dE}{2\pi} \langle \Psi_{\text{out}}^{\dagger}(E)\sigma_y\Psi_{\text{out}}(E)\rangle, \tag{7}$$

is given at zero temperature, when

$$\langle \Psi_{\text{in},n}(E)\Psi_{\text{in},m}(E')\rangle = \delta_{nm}\delta(E-E')\theta(-E), \tag{8}$$

by an integral over positive excitation energies,

$$Q = \frac{e}{4\pi^2}\int_{0^+}^{\infty} d\omega\, \omega \mathrm{Tr} S^\dagger(\omega)\sigma_y S(\omega). \tag{9}$$

(The factor $\omega = \int_0^\infty dE\theta(\omega - E)$ appears from the integration over the step function.) Because $S(-\omega) = S^*(\omega)$ the integrand in Eq. (9) is an even function of ω and the integral can be extended to negative ω,

$$\begin{aligned} Q &= \frac{e}{8\pi^2}\int_{-\infty}^{\infty} d\omega\, \omega \mathrm{Tr} S^\dagger(\omega)\sigma_y S(\omega) \\ &= \frac{ie}{4\pi}\int_{-\infty}^{\infty} dt\, \mathrm{Tr} S^\dagger(t)\sigma_y \frac{\partial}{\partial t} S(t). \end{aligned} \tag{10}$$

This is the superconducting analogue of Brouwer's charge-pumping formula [39] (see Ref. [40] for an alternative derivation).

Substitution of $S(t) = \exp(-i\eta(t)\sigma_y)$ results in

$$Q = (e/2\pi)\delta\eta = e \tag{11}$$

if N_{vortex} is odd, while $Q = 0$ if N_{vortex} is even.

Transferred particle number.—This quantized transfer of one electron charge may be accompanied by the non-quantized transfer of neutral electron-hole pairs. To assess this we calculate the expectation value of the transferred particle number, given by Eq. (9) upon substitution of the charge operator $e\sigma_y$ by unity:

$$N_{\mathrm{particles}} = \frac{1}{4\pi^2}\int_{0^+}^{\infty} d\omega\, \omega \mathrm{Tr} S^\dagger(\omega) S(\omega). \tag{12}$$

This integrand is an odd function of ω, so we cannot easily transform it to the time domain.

We proceed instead by calculating $S(\omega)$ from Eq. (4), in the approximation $\eta(t) \approx 2\arccos[-\tanh(t/2t_{\mathrm{inj}})]$, accurate when $W/\xi_0 \gg 1$. The result is

$$S(\omega) = -\frac{8\pi\omega t_{\mathrm{inj}}^2 \sigma_0}{\sinh(\pi\omega t_{\mathrm{inj}})} - \frac{8\pi\omega t_{\mathrm{inj}}^2 \sigma_y}{\cosh(\pi\omega t_{\mathrm{inj}})} - 2\pi\delta(\omega)$$
$$\Rightarrow N_{\mathrm{particles}} = (84/\pi^4)\zeta(3) = 1.037. \tag{13}$$

One can construct a special t-dependent phase variation [41] that makes $N_{\mathrm{particles}}$ exactly equal to unity, by analogy with the "leviton" [34,42], but even without any fine tuning the charge transfer is nearly noiseless.

Discussion.—We have shown how the chiral motion of edge modes in a topological superconductor can be harnessed to braid a pair of non-Abelian anyons: one immobile in a bulk vortex, the other mobile in an edge vortex. The experimental layout of Fig. 2 is directly applicable to the recently reported chiral Majorana fermion modes in quantum anomalous Hall insulator-superconductor structures [22,43]. The fermion parity switch can be measured fully electrically: a constant voltage V applied to the region between the Josephson junctions transfers charge into the grounded normal metal contact N_2, in a series of narrow current pulses $I(t)$ of quantized area e, spacing $\Delta t = h/2eV$, and width $t_{\mathrm{inj}} = (\xi_0/W)(\Delta t/2\pi) \ll \Delta t$.

While the presence of a bulk vortex and the crossing of its branch cut is essential for the charge transfer, it is of the essence for braiding that no tunnel coupling or Coulomb coupling to the edge vortices is needed. This distinguishes the braiding experiment proposed here to tunnel probes of Majorana zero modes that can also produce a quantized charge transfer [40]. In the quantum Hall effect, attempts to use edge modes for braiding [7] have been inconclusive because of Coulomb coupling with bulk quasiparticles [44]. The superconductor offers a large gap, to suppress tunnel coupling, and a large capacitance, to suppress Coulomb coupling, which could make the edge mode approach to braiding a viable alternative to existing approaches using zero modes bound to superconducting nanowires [17–21].

In the quantum Hall effect there is a drive to use quasiparticles in edge modes as "flying qubits" for quantum information processing [45]. Edge vortices in a topological superconductor could play the same role for topological quantum computation. The pair of edge vortices in the geometry of Fig. 1(a) carries a topologically protected qubit encoded in the fermion parity. The deterministic voltage-driven injection of edge vortices that we have proposed here could become a key building block for such applications.

We have benefited from discussions with N. V. Gnezdilov. This research was supported by the Netherlands Organization for Scientific Research (NWO/OCW) and by the European Research Council.

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