Arithmetic of affine del Pezzo surfaces

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Summary

This thesis concerns the mathematical area of arithmetic geometry. I will first give a general introduction to this field before touching upon the more specific topics and results discussed in this thesis.

What is arithmetic geometry?

In *number theory* it is common to study rational solutions to equations such as $x^2 + y^3 = 5$. This means that one is interested in pairs of rational numbers x and y for which this equation is true. Some of these solutions can be found by simply trying some small values for x and y. For example, the solutions $x = 2$ and $y = 1$, and $x = -2$ and $y = 1$ could be found by simple inspection.

In *arithmetic geometry* one approaches such questions using geometric techniques and terminology. For example, one can draw all the real points on the xy-plane for which the equation $x^2 + y^3 = 5$ is satisfied. We recover the curved line as shown in the diagram on the next page and we see that the solution $x = 2$ and $y = 1$ corresponds to the point $(2, 1)$ on this geometric object. This interpretation shows why a solution to an equation is often called a *point*; it is a point on the geometric object defined by the equation. We will use the terms point and solution interchangeably.

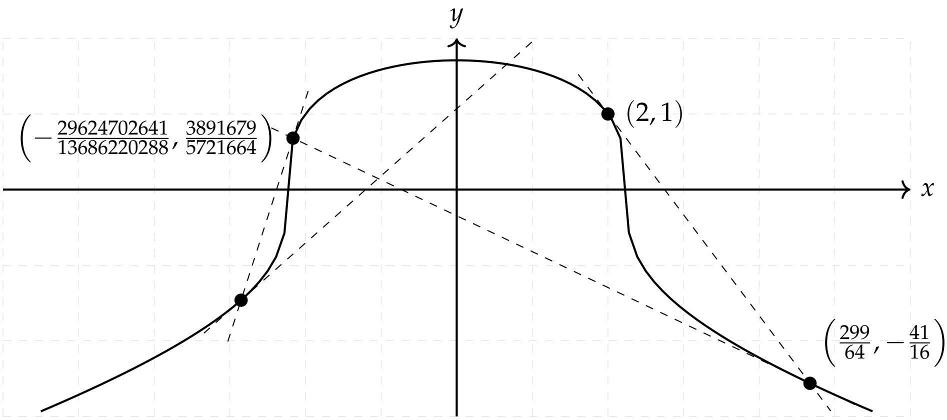
We will now show that the equation $x^2 + y^3 = 5$ has infinitely many solutions over the rational numbers. It has been shown that if (x, y) is a rational solution, then another solution is given by

$$\left(\frac{y^3 + 10}{2x} - \frac{3y^3}{8x}\frac{y^3 + 40}{y^3 - 5}, \frac{y}{4}\frac{y^3 + 40}{y^3 - 5}\right).$$

If we start with the point $(x, y) = (2, 1)$ we can iterate this procedure to find the following points

$$(2, 1) \mapsto \left(\frac{299}{64}, -\frac{41}{16}\right) \mapsto \left(-\frac{29624702641}{13686220288}, \frac{3891679}{5721664}\right) \mapsto \dots$$

A suspicious reader is invited to check that these pairs are really solutions to the equation $x^2 + y^3 = 5$.



These formulas for determining new solutions are as ugly as they are mysterious, but this procedure has a very nice geometric interpretation: consider the geometric object defined by $x^2 + y^3 = 5$. Then draw the tangent line in the point $(2, 1)$ and intersect it again with the curve. This new intersection point is $\left(\frac{299}{64}, -\frac{41}{16}\right)$; precisely the point obtained from applying the procedure described above. To compute the next point one draws the tangent line in this new point $\left(\frac{299}{64}, -\frac{41}{16}\right)$ and again intersects it with the curve. We can repeat this procedure indefinitely to find infinitely many rational points.

This is a first example of how geometric techniques are used to study equations. We also use geometric properties to classify equations. The equation above describes a 'bent line', which is called a *curve*. This thesis however is about *surfaces* which look more like 'bent, twisted or curved planes' such as on the cover of this thesis.

Surfaces

There are several ways to move from equations describing curves to equations describing surfaces. The first way is to add a third variable, for which we will use z. For example, the equation $x^3 + y^3 + z^3 = 4$ describes a surface which has many solutions such as

$$\left(\frac{1}{21}, \frac{5}{3}, -\frac{6}{7}\right).$$

Another way to write down equations for surfaces is by not only increasing the number of variables, but also the number of equations. Consider the system of equations with the four variables w, x, y en z:

$$\begin{cases} w^2 + x^2 + y^2 = 6, \\ w^2 + xy + yz = 2. \end{cases}$$

We can then also study the quadruple (w, x, y, z) for which *both* equations are satisfied simultaneously. One can check that $(1, 2, 1, -1)$ and $(\frac{1}{5}, -2, -\frac{7}{5}, -\frac{21}{5})$ are solutions to this set of equations.

These two surfaces are examples of so-called *ample log K3 surfaces*. Occasionally, ample log K3 surfaces are referred to, albeit slightly imprecisely, as *affine del Pezzo surfaces* which is the term used in the title of this thesis.

Another example of an ample log K3 surface is given by the system of five equations

$$\begin{cases} 1 + 1952wy + 412071wz = \\ \qquad 121v + 492x + 52838z + vx + 22vy + 971038xz + 1771110z^2, \\ w + 88wy + 20504wz + 16xy = \\ \qquad\qquad 11v + 22x + 4y + 3267z + vy + 39017xz + 78089z^2, \\ v + x + 169z + 220z^2 = w^2 + 4wy + 451wz + 500xz, \\ y + 121wz = 11z + wx + 4xy + 363xz + 594z^2, \\ z + x^2 + 40xz + 55z^2 = wy + 11wz \end{cases} \tag{$\star$}$$

in the five variables v, w, x, y and z. The surface it describes has many rational points, such as

$$\left(-20, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 0\right) \quad \text{and} \quad \left(\frac{23}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, 0\right).$$

Although the equations are quite hideous this surface is geometrically very pleasing. One of its geometrically interesting features of this surface is even shown on the cover: although the surface is very twisted it is possible to walk in a straight line along this surface. Note that there are precisely ten such straight lines.

The number of lines is an aspect that distinguishes this surface from the previous two examples, since one can show that those surfaces contain respectively 27 and 16 lines. The reader is advised at this point to look up a picture of the 27 lines on the 'Clebsch surface' online.

We will consider ample log K3 surfaces with 10 lines. There are technical reasons we will not go into to call these surfaces *ample log K3 surfaces of degree* 5. In contrast the other examples are log K3 surfaces of degree 3 and 4, respectively. So now we used the geometric property of the number of lines to classify equations.

Integral points

Let us now focus on the ample log K3 surface of degree 5 defined by the system $(\star)$. We saw that it has rational points. A next question is whether some of these rational solutions are in fact solutions in integers. In this case will speak of *integral points* and *integral solutions*.

The two rational solutions mentioned above are clearly not integral, but this is in no way a guarantee that all rational solutions are not integral. But one would do well to also consider the possibility that there are no integral solutions at all!

Any technique which proves that there can not be any points is called an *obstruction*. One possible obstruction to the existence of solutions was developed by Manin. He used a technical object called the *Brauer group* of the system of equations. If such an obstruction excludes the existence of integral solutions one says that there is a *Brauer–Manin obstruction to the existence of integral points*. In that case we can conclude that there are no integral solutions. This is what I did in this thesis for the system ($\star$).

THEOREM 1. *There is a Brauer–Manin obstruction to the existence of integral solutions to the system of equations* ($\star$). *This proves that the system does not have an integral solution.*

Chapter 4 gives a general method for deciding whether an ample log K3 surface of degree 5 admits a Brauer–Manin obstruction. Using this method, I found infinitely many systems of equations which do not admit an integral solution.

Bounding the Brauer group

One can now also wonder if it would be possible to have a computer determine whether or not a system of equations has a Brauer–Manin obstruction. Usually this depends on the complexity of the Brauer group. Unfortunately it is known that the Brauer group of surfaces can be arbitrarily complex, in the sense that for every surface there is another surface for which the Brauer group is more complex. The main theorem of Chapter 3 states that this is not the case for ample log K3 surface.

THEOREM 2. *There is a bound on the complexity of the Brauer group of ample log K3 surfaces.*

This bound shows that the Brauer group of an ample log K3 surface can not be arbitrarily complex. This is important information for developing an algorithm that determines whether there is Brauer–Manin obstruction to the existence of integral solutions to complicated systems of equations such as the system ($\star$).