Arithmetic of affine del Pezzo surfaces
Lyczak, J.T.

Author: Lyczak, J.T.
Title: Arithmetic of affine del Pezzo surfaces
Issue Date: 2019-10-01

Chapter 2

Del Pezzo surfaces

In this chapter we will study the many different flavours of del Pezzo surfaces. First we will define and classify generalized del Pezzo surfaces. These surfaces are appropriately named as they generalize the important subclass of ordinary del Pezzo surfaces which were studied by del Pezzo in 1887 [22]. Next we proceed by studying the Picard group of these smooth surfaces and certain effective divisor classes with negative self-intersection. We will describe how to contract a collection of curves in such classes and we will give conditions for the constructed surface to be normal or even smooth.

Using these results we define singular del Pezzo surfaces. These normal projective surfaces are obtained from a generalized del Pezzo surface by contracting all curves with self-intersection equal to -2 and are the generalization of projectively embedded ordinary del Pezzo surfaces using the anticanonical bundle.

The next section describes a novel type of algebraic surface, namely the class of peculiar del Pezzo surfaces. These surfaces are again normal and obtained from contracting a subset of the curves with self-intersection -2 on a generalized del Pezzo surface. This shows that peculiar del Pezzo surfaces fit in between generalized and singular del Pezzo surfaces. This new type of surface was defined by the author to describe certain aspects of the geometry of both generalized and singular del Pezzo surfaces in a simpler manner.

We will show that the minimal desingularization of both a singular and a peculiar del Pezzo surface is a generalized del Pezzo surface. Using this result we can prove that there is a correspondence between generalized, peculiar and singular del Pezzo surfaces. For the classical case of ordinary del Pezzo surfaces the three notions coincide: an ordinary del Pezzo surface X is a generalized, peculiar and singular del Pezzo surface. Note the inescapable confusion: a singular del Pezzo need not be singular.

Now consider a projective family of ordinary del Pezzo surfaces over an open U of some base space B. If one can extend the family to B the fibres over $B \backslash U$ need not be ordinary del Pezzo surfaces as well. Depending on one's interpretation of ordinary del Pezzo surfaces these fibres can be generalized, peculiar or singular del Pezzo surfaces. The properties of ordinary del Pezzo surfaces with which we will work will be directly applicable to at least one of these three more general notions.

In the last section we give a short summary of the arithmetic of ordinary del Pezzo surfaces over a number field.

2.1 Preliminaries on geometry

Before we consider del Pezzo surfaces let us first define the following geometric objects and concepts. Recall that we defined a variety over a field k to be a scheme which is separated and of finite type over k.

DEFINITION 2.1.1. Let k be a field. A *curve over* k is a variety over k of pure dimension 1. A *surface over* k is a geometrically integral variety of dimension 2 over k. By a *curve* C *on a surface* S *over* k we mean a closed subscheme $C \subseteq S$ which is a curve over k.

Note that our definition of curve is less restrictive than our definition of surface. This stems from the fact that we will work with reasonably well-behaved surfaces. Practically all surfaces we will encounter are normal, and in this chapter all surfaces will be projective. A one-dimensional subscheme of such a surface can definitely be less elegant from a geometric point of view. For this reason we will want to allow curves to be reducible, non-reduced or singular.

Let us turn to our conventions on divisors. We will use the word divisor to refer to *Cartier divisors*. Since surfaces are integral by definition the *Picard group* $\operatorname{Pic} S$ of a surface S is isomorphic to both the group of isomorphism classes of line bundles and the group of linear equivalence classes of divisors. For a divisor $D \in \operatorname{Pic} X$ the associated line bundle is denoted by $\mathcal{L}(D)$. This line bundle comes with a specified rational section 1_D and the divisor D is effective precisely if 1_D lies in $H^0(S, \mathcal{L}(D))$.

Since most of our surfaces will be normal they are regular in codimension one and this makes it more convenient to also consider Weil divisors. Recall that a *prime divisor* on a surface S is just an integral curve on S, and a *Weil divisor* on S is an element of the free abelian group generated by all prime divisors on S. In this case Cartier divisors are precisely the locally principal Weil divisors. On a surface which is also smooth the two notions of divisors coincide completely.

Let us state the definition of the intersection product between two divisors.

DEFINITION 2.1.2. Let S be a surface which is proper over a field k. For an integral curve C on S and a divisor D on S their *intersection product* $C \cdot D$ is defined as the degree of the restriction of the line bundle $\mathcal{L}(D)$ to C.

We recall the definition of the degree of a line bundle $\mathcal{L}$ on a, possibly singular, integral curve C which is projective over a field k. First associate a Weil divisor to $\mathcal{L}$ following the procedure in Section 2.2 of [27]. This divisor is well-defined up to rational equivalence and the proper map $C \to \operatorname{Spec} k$ allows us to push this Weil divisor class forward to a Weil divisor class on $\operatorname{Spec} k$. The group of Weil divisors on $\operatorname{Spec} k$ modulo rational equivalence is canonically isomorphic to $\mathbb{Z}$. The integral number associated to $\mathcal{L}$ in this manner is what we define as the *degree* of $\mathcal{L}$ on C.

A complete treatment of intersection cycles and Weil divisor class groups can be found in [27]. We will use the following properties: the intersection product is linear in both arguments, it is preserved under arbitrary field extensions, and on a projective surface S the intersection pairing descends to a bilinear pairing on $\operatorname{Pic} S$.

Although it is not directly apparent the intersection product is important for both notions in the following definition.

DEFINITION 2.1.3. Let D be a divisor on a projective surface over a field k. We say that D is *nef* if for all integral curves C on X we have $C \cdot D \geq 0$. The divisor D is called *big* if the rational map $X \dashrightarrow \mathbb{P}^N_k$ associated to the complete linear system of a sufficiently large multiple of D defines a birational map from X to its scheme-theoretic image.

It is indeed clear that it can be checked numerically whether a divisor is nef. Corollary 2.2.8 in [36] shows that the same is true for big. We will however use the following proposition which gives a sufficient and necessary numerical condition for a divisor to be big and nef. It is similar to the Nakai–Moishezon criterion [33, Theorem V.1.10] for checking if a divisor on a surface is ample.

PROPOSITION 2.1.4. *Let D be a divisor on a projective surface X over a field k. The line bundle $\mathcal{L}(D)$ is big and nef precisely if $D^2 > 0$ and for all integral curves C on X we have $C \cdot D \geq 0$.*

Proof. Let us first prove the statement in the case that k is algebraically closed. We apply Theorem 2.2.16 of [36] to see that D is big and nef if and only if $D^2 > 0$ and $C \cdot D \geq 0$ for all integral curves C on X. Note that although the standing convention in [36] is that schemes are defined over $\mathbb{C}$ one can check that the statement is true over any algebraically closed field.

It is clear from the definition that being big is preserved under arbitrary field extensions. The same holds for being nef. We will prove this for the field extension $\bar{k}/k$.

Assume that the pullback $\bar{D}$ of D to $\bar{X} = X \times_k \bar{k}$ is nef. Let C be an integral curve on X and write $\bar{C}$ for its pullback to $\bar{X}$. Note that $\bar{C}$ defines an effective Weil divisor W on $\bar{X}$ and hence we find

$$C \cdot D = W \cdot \bar{D} \geq 0.$$

Now consider the case that D is nef on X and let C' be an integral curve on $\bar{X}$. The scheme-theoretic image C of C' under $\bar{X} \to X$ pulls back to a Weil divisor W on $\bar{X}$ which is supported on the finitely many conjugates C'_i of C' under the absolute Galois group G_k. Notes that this group acts transitively on the set of C'_i since C' is irreducible and trivially on W. Hence $\bar{D} \cdot C'_i$ and the multiplicity m_i

of C_i' in W is independent of i. Since the intersection pairing is preserved under base change we find that there exists a positive constant m such that

$$m(C_i' \cdot \bar{D}) = W \cdot \bar{D} = C \cdot D \geq 0$$

for all i. This proves that $\bar{D}$ is nef on X.

The result now follows from the fact that the intersection pairing is also preserved under field extensions. $\square$

This property shows that an ample divisor on a projective surface is both big and nef. The converse is not true; it need not even be the case that a big and nef line bundle is semiample. This means that no rational map $S \dashrightarrow \mathbb{P}_k^N$ associated to a multiple of a big and nef divisor extends to a morphism on the whole of S. For an example the reader is referred to [36, Section 2.3.A]. Practically all big and nef divisors we will encounter will however be semiample. Once we know that generalized del Pezzo surfaces are rational smooth projective surfaces with a big anticanonical divisor this can be explained by Lemma 2.6 in [54].

Now consider a semiample, big and nef divisor D on a surface X. This defines a birational morphism $X \to X' \subseteq \mathbb{P}_k^N$. One can show that the curves C on X which are contracted by this birational morphism are precisely those for which $C \cdot D = 0$.

We will need one more geometric concept. Recall that any smooth scheme X admits a canonical line bundle ω_X. We will write the associated divisor class as K_X. Now consider a normal scheme Y. The singular locus Σ is of codimension at least 2. This proves that there is an isomorphism between Weil divisors on Y and $Y \backslash \Sigma$. This allows us to define the canonical Weil divisor on the normal scheme Y. Take the closure of a canonical divisor $K_{Y\backslash\Sigma}$ in Y as a Weil divisor. We will denote this Weil divisor on Y by K_Y.

If the Weil divisor K_Y on a normal scheme is actually a Cartier divisor, then we denote the associated line bundle $\mathcal{L}(K_Y)$ by ω_Y.

2.2 Generalized and ordinary del Pezzo surfaces

We now have the terminology to define generalized and ordinary del Pezzo surfaces.

DEFINITION 2.2.1. A *generalized del Pezzo surface* is a smooth projective surface X over a field k for which the anticanonical divisor $-K_X$ is big and nef. The surface X is an *ordinary del Pezzo surface* if the anticanonical divisor is moreover ample.

The *degree* d of a generalized del Pezzo surface is defined as the canonical self-intersection number $d = K_X \cdot K_X$.

The following theorem classifies generalized del Pezzo surfaces over algebraically closed fields.

THEOREM 2.2.2. *Let X be a generalized del Pezzo surface over an algebraically closed field k. The surface X is rational and isomorphic to either $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$, the Hirzebruch surface $\mathbb{F}_2$, or there exists an integer $1 \leq r \leq 8$ such that X can be written as*

$$X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^2_k$$

where each map is the blowup in a reduced closed point which does not lie on a curve of self-intersection -2. The degree of these four types of generalized del Pezzo surfaces are respectively 9, 8, 8 and $9 - r$. In particular, the degree of a generalized del Pezzo surface is a positive integer $d \leq 9$.

Proof. The main ingredients for this proof come from [23], but in the classification the Hirzebruch surface $\mathbb{F}_2$ is erroneously left out. A corrected and completed classification can be found in [19, Proposition 0.4]. □

In this last case we say that X is the blowup of $\mathbb{P}^2_k$ in r points in *almost general position*. If we strengthen the condition that none of the centres of the blowups lies on a curve of self-intersection -2, to curves of self-intersection -1 we say that X is the blowup of the projective plane in r points in *general position*. Using this terminology we can identify the ordinary del Pezzo surfaces in the previous theorem.

THEOREM 2.2.3. *Let X be an ordinary del Pezzo surface over an algebraically closed field k. The surface X is isomorphic to either $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or a blowup of $\mathbb{P}^2_k$ in r points in general position for some $1 \leq r \leq 8$.*

Proof. See [41, Theorem 24.3]. □

The following proposition shows that a surface remains a del Pezzo surface after base extension. This means in particular that Theorem 2.2.2 and Theorem 2.2.3 can be used to classify del Pezzo surfaces over any field.

PROPOSITION 2.2.4. *Let X be a surface over a field k, and let K/k be a field extension. The surface X is a generalized del Pezzo surface precisely if $X_K = X \times_k K$ is a generalized del Pezzo surface. The same statement holds for ordinary del Pezzo surfaces.*

Proof. We have seen in the proof of Proposition 2.1.4 that both bigness and nefness are preserved under field extensions. A similar proof shows that the same holds for ampleness. □

We have seen that blowing up closed points is a principal operation one uses to produce generalized del Pezzo surfaces. The following proposition shows that we can always invert this process.

PROPOSITION 2.2.5. *Let X be a generalized del Pezzo surface X of degree d over a field k, and let L be a geometrically integral rational curve on X of self-intersection -1. There exists a generalized del Pezzo surface X' of degree $d+1$ together with a morphism $X \to X'$ such that the following properties are satisfied:*

» *L maps to a point $p \in X'(k)$; and*

» *the morphism* $X \to \mathrm{Bl}_p X'$ *obtained from the universal property of the blowup is an isomorphism.*

If X *is an ordinary del Pezzo surface, then so is* X'.

Proof. By Castelnuovo's Theorem [33, Theorem V.5.7] there is a smooth surface X' over k, a morphism $X \to X'$ and a point $p \in X'$ such that $X \to \mathrm{Bl}_p X'$ is an isomorphism and L is the exceptional curve of this blowup $\pi\colon X \to X'$. We only need to check that X' is indeed a generalized (respectively ordinary) del Pezzo surface. For generalized del Pezzo surfaces this can be done using Proposition 2.1.4.

The line bundle $L - K_X$ is the pullback of the line bundle $-K_{X'}$ by Proposition V.3.3 in [33]. Let C be an integral curve on X'. The intersection number $-K_{X'} \cdot C$ equals $\pi^*(-K_{X'}) \cdot \pi^* C = (L - K_X) \cdot \pi^* C$. Since L is the exceptional curve of π we have $L \cdot \pi^* C = 0$, and since $-K_X$ is big and nef and $\pi^* C$ is an effective divisor we find $-K_X \cdot \pi^* C \geq 0$. We also see that

$$K_{X'}^2 = (\pi^* K_{X'})^2 = (L - K_X)^2 = L^2 - 2L \cdot K_X + K_X^2 = -1 + 2 + d = d + 1 > 0.$$

Proposition 2.1.4 now implies that $-K_{X'}$ is big and nef and hence X' is a generalized del Pezzo surface.

For ordinary del Pezzo surfaces we use the Nakai–Moishezon criterion [33, Theorem V.1.10]. The proof is similar to the proof above, but the inequality should be replaced by a strict inequality. □

These curves of negative self-intersection will turn out to be important. Before looking into them we will first consider their divisor classes in the next section.

2.3 Divisor classes of negative self-intersection

Since the Picard group of the blowup of a surface in a point is well understood, we can describe the Picard group of a generalized del Pezzo surface over an algebraically closed field.

PROPOSITION 2.3.1. *Let* X *be a generalized del Pezzo surface of degree* d *over an algebraically closed field* k. *If* X *is not isomorphic to* $\mathbb{P}^1_k \times \mathbb{P}^1_k$ *or* $\mathbb{F}_2$, *then there exist* $10 - d$ *divisor classes* $L_0, L_1, \ldots, L_r \in \mathrm{Pic}\, X$ *such that*

» $L_0^2 = 1$;

» $L_i^2 = -1$ *for* $i > 0$;

» $L_i \cdot L_j = 0$ *for* $i \neq j$; *and*

» $\mathrm{Pic}\, X$ *is freely generated by the* L_i.

The class of the anticanonical bundle $-K_X$ *equals* $3L_0 - \sum_{i=1}^{r} L_i$.

For any generalized del Pezzo surface X *of degree* d *we have* $\operatorname{Pic} X \cong \mathbb{Z}^{10-d}$.

Note that this proves that the lattice isomorphism class of the geometric Picard group $\operatorname{Pic} \bar{X}$ of a generalized del Pezzo surface X over any field only depends on the degree d.

Proof. See [23, II, Section 4]. □

The classes L_i for $i > 0$ have negative self-intersection, but these need not be all of them. Generalized del Pezzo surfaces do not have prime divisors of self-intersection smaller than -2 by nefness of $-K_X$ and the adjunction formula. With this in mind we will consider classes of self-intersection -1 and -2. To control the behaviour of curves representing these classes under the anticanonical embedding we add a condition on the intersection number of these classes with the canonical class.

DEFINITION 2.3.2. Suppose that X is a generalized del Pezzo surface over a field k and let s be either -1 or -2. An *s-class* is a divisor class D on $\bar{X}$ such that $D^2 = s$ and $D \cdot K_X = -2 - s$.

The following proposition shows that we could equally consider the base change to the separable closure.

PROPOSITION 2.3.3. *Let* X *be a smooth rational surface over a field* k*. The natural map* $\operatorname{Pic} X^{\mathrm{sep}} \to \operatorname{Pic} \bar{X}$ *is an isomorphism.*

Proof. Lemma 3.1 of [9] contains the same statement for K3 surfaces. The proof only uses that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. This is however also true for smooth rational surfaces. □

The following lemma shows that there are only finitely many s-classes on any generalized del Pezzo surface.

LEMMA 2.3.4. *Let* X *be a generalized del Pezzo surface over an algebraically closed field* k*. Then* X *has only finitely many* s*-classes.*

There are no s*-classes on* $\mathbb{P}^2_k$ *and* $\mathbb{P}^1_k \times \mathbb{P}^1_k$*. The only* s*-classes on the Hirzebruch surface* $\mathbb{F}_2$ *are the class of the base curve and its negative. The number of* s*-classes on the projective plane blown up in* r *points in almost general position can be found in Table A.*

Now suppose that X *is written as the blowup* $\pi\colon X \to \mathbb{P}^2_k$ *of the projective plane in* $r \leq 7$ *points in almost general position. The pushforward of an* s*-class on* X *to* $\mathbb{P}^2_k$ *is either trivial,* $\mathcal{O}(1)$ *or* $\mathcal{O}(2)$*.*

The last statement is not true for s-classes on generalized del Pezzo surfaces of degree 1 or 2; the pushforward of an s-class to the projective plane can also be the line bundle $\mathcal{O}(3)$.

r	8	7	6	5	4	3	2	1
$s=-1$	240	56	27	16	10	6	3	1
$s=-2$	240	126	72	40	20	8	2	0

Table A: The number of s-classes on the projective plane blown up in r points in almost general position.

Proof. The numbers of s-classes in the table follow from the description in Proposition 2.3.1 of the Picard group of the projective plane blown up in r points in almost general position. The details can be found in [23, II, Section 5]. Enumerating all s-classes on a generalized del Pezzo surface of degree $d \geq 3$ shows that for an s-class $R \in \operatorname{Pic} \bar{X}$ we have $0 \leq R \cdot L_0 \leq 2$, where L_0 is the first element of a basis of $\operatorname{Pic} \bar{X}$ as given in Proposition 2.3.1.

For the remaining cases the geometric Picard group is either $\mathbb{Z}$, $\mathbb{Z}^2$ with the Euclidean intersection pairing, or $\mathbb{Z} \cdot C + \mathbb{Z} \cdot F$ with the pairing given by $C^2 = -2$, $C \cdot F = 1$ and $F^2 = 0$, see for example Proposition 2.3 and Proposition 2.9 in [33]. In these cases the computation of the number of s-classes is straightforward. $\square$

Now let X be a generalized del Pezzo surface over a general field k. The absolute Galois group G_k of k acts naturally on $\bar{X}$ and this endows the geometric Picard group $\operatorname{Pic} \bar{X}$ with the structure of a G_k-module. An element $\sigma \in G_k$ acts on $\operatorname{Pic} \bar{X}$ in a specific way; it will preserve K_X and the intersection pairing. The following theorem shows that this cannot happen in many ways.

PROPOSITION 2.3.5. *Let X be a generalized del Pezzo surface of degree d over an algebraically closed field k and let A_X be the subgroup of $\operatorname{Aut}(\operatorname{Pic} X)$ consisting of the elements which preserve the canonical class K_X and the intersection pairing. The group A_X is finite.*

Now suppose that X is the blowup of $\mathbb{P}^2_k$ in $r = 9 - d$ points in almost general position. The group A_X permutes the -1-classes in $\operatorname{Pic} X$ and the induced map from A_X to the group consisting of the intersection pairing preserving permutations of the -1-classes is an isomorphism.

Proof. The finiteness of A_X is part a) of Théorème 2 in [23, II]. This result does not address the surfaces $\mathbb{F}_2$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$, but for those cases the statement is easily verified.

For the second statement recall that s-classes are defined in terms of their self-intersection and the intersection with the anticanonical class. It follows that A_X preserves s-classes by definition. Now consider the group homomorphism from A_X to the group consisting of the intersection pairing preserving permutations of the -1-classes. The surjectivity of this homomorphism follows from part b) of Proposition 5 in [23, II]. The injectivity is trivial if $d \geq 8$ and follows for $d \leq 7$ from the fact that the -1-classes generate $\operatorname{Pic} X$. $\square$

The second statement is also true for $\mathbb{F}_2$, but not for $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

Note that for generalized del Pezzo surfaces which are the blowup of the projective plane in r points in almost general position the group A_X only depends on the degree d, or equivalently r. For $r \leq 6$ this group has been identified in [41, Theorem 25.4] as the Weyl group W_r of a certain root system. For these del Pezzo surface we will write W_r instead of A_X, although one should be careful since W_r does not come with a fixed action on $\operatorname{Pic} X$. For more information about the groups W_r one can refer to §26 in [41].

Let X be a generalized del Pezzo surface over a general field k which is geometrically the blowup of the projective plane in $r \geq 3$ points in almost general position. After fixing the action of W_r on $\operatorname{Pic} \bar{X}$ the action of G_k on $\operatorname{Pic} \bar{X}$ induces a group homomorphism $G_k \to W_r$. The image of this homomorphism is the smallest subgroup W of W_r such that the action of G_k factors through the induced action of W on $\operatorname{Pic} \bar{X}$. This subgroup of W_r contains much information about the action of Galois on the geometric Picard group. An important corollary to Proposition 2.3.5 is that in this sense there are only finitely many possible actions of Galois on the geometric Picard group.

2.4 Curves of negative self-intersection

In the previous section we have studied the s-classes in the Picard group of a generalized del Pezzo surface. In this section we will study the integral curves in such divisor classes.

DEFINITION 2.4.1. Let X be a generalized del Pezzo surface over a field k. An integral curve C on $\bar{X}$ whose class in $\operatorname{Pic} \bar{X}$ is an s-class will be called a *geometric s-curve*. A curve C on X is called an *s-curve* if its base change $\bar{C} \subseteq \bar{X}$ to an algebraic closure $\bar{k}$ is a geometric s-curve.

Similar to Proposition 2.3.3 we could equivalently have used the separable closure of k instead of an algebraic closure. This fact becomes important once we start looking at the action of G_k on the geometric s-curves.

PROPOSITION 2.4.2. *Let X be a smooth rational surface over a field k. Any geometric s-curve is defined over k^{sep}.*

Proof. We again refer to [9]. The proof of Corollary 3.2 also proves this statement. □

By the adjunction formula we see that every s-curve is rational and the following results directly from Lemma 2.3.4.

LEMMA 2.4.3. *Let X be a generalized del Pezzo surface over a field k. Each s-class contains at most one geometric s-curve and in particular we see that there are finitely many geometric s-curves on X.*

Now suppose that X is a generalized del Pezzo surface which is written as the blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in $r \leq 7$ points in almost general position. Any

s-curve on X is contracted by π or it is the strict transform along π of a line or an integral conic on $\mathbb{P}^2_k$.

For generalized del Pezzo surfaces of degree 1 and 2 an s-curve can also be the strict transform of a cubic plane curve.

Proof. In general, a class of negative self-intersection has at most one irreducible representative. This proves the first statement together with Lemma 2.3.4. The second statement follows from the same lemma. □

Lemma 2.3.4 also shows that the study of geometric s-curves is trivial on a generalized del Pezzo surface which is geometrically isomorphic to $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or $\mathbb{F}_2$. So in this section X will be a surface given as the composition of blowups. In this case we can easily identify at least some of the geometric s-curves on X.

PROPOSITION 2.4.4. *Let X be a generalized del Pezzo surface over an algebraically closed field k, which is written as the blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in r points in almost general position. Let X_p be the fibre of π above a point $p \in X(k)$. Then X_p is either a single point or there exists a positive integer $m \leq r$ such that X_p is the union of -2-curves $E_1, E_2, \ldots, E_{m-1}$ and a -1-curve E_m which satisfy*

$$E_i \cdot E_j = \begin{cases} 1 & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, if X_p is a positive-dimensional fibre of $\pi\colon X \to \mathbb{P}^2_k$, then $\pi^{-1}p$ is a locally principal subscheme of X. The associated Cartier divisor of this subscheme equals

$$E_1 + E_2 + \ldots + E_{m-1} + E_m.$$

Proof. This follows by induction. It is obviously true for the generalized del Pezzo surface $\mathbb{P}^2_k$. Now suppose that the statement is true for a generalized del Pezzo surface X_{r-1} obtained from blowing up the projective plane in $r-1$ points in almost general postion. We know that X_r is the blowup of X_{r-1} in a closed point p_{r-1}. Let p_0 be the image of p_{r-1} in $\mathbb{P}^2_k$. The fibre of $X_r \to \mathbb{P}^2_k$ over any point $p' \neq p_0$ is isomorphic to the fibre over p' of $X_{r-1} \to \mathbb{P}^2_k$.

Now consider the fibre of X_r over p_0. The fibre F_{r-1} of $X_{r-1} \to \mathbb{P}^2_k$ over p_0 is a chain of a non-negative number of -2-curves and one -1-curve E. By Theorem 2.2.2 we see that p_{r-1} cannot lie on one of the -2-curves, so it lies on the -1-curve E. Blowing up this point, the strict transform of E becomes a -2-curve and the exceptional curve of the blowup $X_r \to X_{r-1}$ becomes the new -1-curve in the fibre $X_r \to \mathbb{P}^2_k$ over p_0.

The last statement follows again by induction. □

In the notation of this proposition, we will be more interested in the divisor

$$(\pi^{-1}p)^{\text{pec}} = E_1 + 2E_2 + \ldots + (m-1)E_{m-1} + mE_m$$

above the point p than in the divisor $\pi^{-1}p$. The reason for these unlikely multiplicities will become apparent in the following sections.

DEFINITION 2.4.5. Let X be a generalized del Pezzo surface over a field k, given as the blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in r points in almost general position. The sum of the positive-dimensional fibres of π is denoted by E_π and is called the *exceptional divisor of* π. The sum of $(\pi^{-1}p)^{\mathrm{pec}}$ over all points p with a positive-dimensional fibre is called the *peculiar divisor of* π and denoted by E_π^{pec}.

Note that the exceptional and the peculiar divisor of π have the same support. The number of prime divisors in this support equals $9 - d$, where d is the degree of X. By comparing with the numbers in Table A on page 32 we see that there can be more s-curves than those in the support of E_π. However, any s-curve which is not in E_π will be the strict transform of a plane curve C. This curve C is either a line or a smooth conic by Lemma 2.4.3. Assume for a moment that k is an algebraically closed field. We can decompose the morphism into blowups $X = X_r \to X_{r-1} \to \ldots \to X_0 = \mathbb{P}^2_k$ with centres $x_i \in X_i(k)$ and consider the strict transform $C_i \subseteq X_i$ of $C \subseteq \mathbb{P}^2_k$ at every level. By induction we see that the self-intersection of $\tilde{C} = C_r$ can be computed as follows

$$\tilde{C}^2 = C^2 - \#\{i \mid x_i \in C_i(k)\}$$

since $C_{i+1}^2 = C_i^2 - 1$ if $x_i \in C_i(k)$ and $C_{i+1}^2 = C_i^2$ otherwise. Here we have used that C and hence each C_i is a smooth curve.

So if X is given as a composition of blowups of the projective plane, then we can usually identify the remaining s-curves. For ordinary del Pezzo surfaces, this is even more straightforward as the following proposition shows.

PROPOSITION 2.4.6. *On ordinary del Pezzo surfaces every -1-class contains a geometric -1-curve and there are no geometric -2-curves.*

Proof. It is enough to assume that k is algebraically closed. For an ordinary del Pezzo surface $\pi\colon X \to \mathbb{P}^2_k$, written as the blowup of r points in general position, each -1-class is either one of the r irreducible components of E_π or the strict transform of a plane curve of prescribed degree and multiplicities at the blowup centres. These curves are all different and one can count that the number of these curves equals the number of -1-classes. For details see [41, Theorem 26.2]. □

With a little more work one can adapt the proof to show that all -1-classes on a generalized del Pezzo surface are effective over an algebraic closure. However, not every -1-class needs to be represented by a prime divisor, i.e. an integral curve. For example, the class of a connected component of the divisor E_π in Definition 2.4.5 represents a -1-class. However, such a component need not be a prime divisor.

Now let us consider the same problem for -2-classes. We will restrict to generalized del Pezzo surfaces of degree $d \leq 7$ to ensure that there are -2-classes. We already saw that there are no -2-curves on ordinary del Pezzo surfaces. For generalized del Pezzo surfaces we see in general that not all -2-classes will be effective; the negative $-R$ of a -2-class R is also a -2-class, and R and $-R$

cannot both be effective. The following lemma gives an upper bound for the number of -2-curves.

LEMMA 2.4.7. *Let X be a generalized del Pezzo surface of degree d over a field k. The number of -2-curves on X is at most $9-d$.*

Proof. We will prove the result in the case that k is algebraically closed. The general results follow directly from there.

The result is trivial if $d = 9$ and for $d = 8$ there is at most one -2-curve in all possible cases as one can easily check. If $d \leq 7$ then X is the blowup of the projective plane in $r = 9-d$ points in almost general position and the geometric Picard group $\operatorname{Pic} X$ only depends on the degree d. The intersection product on $\operatorname{Pic} X$ defines a negative definite pairing on the orthogonal complement $K_X^{\perp}$ in $\mathbb{R} \otimes \operatorname{Pic} X$ by Proposition 25.2 in [41]. Now let $R_1, \dots R_t$ be t distinct integral curves with self-intersection -2 on X. We will prove that they are linearly independent in $K_X^{\perp} \subseteq \mathbb{R} \otimes \operatorname{Pic} X$.

Suppose that $\sum_{i \in I} \alpha_i R_i = \sum_{j \in J} \alpha_j R_j$ in $K_X^{\perp}$ where all α_i and α_j are positive and the index sets $I, J \subseteq \{1, 2, \dots, t\}$ are disjoint. This implies that $R_i \cdot R_j \geq 0$ for all $i \in I$ and $j \in J$ and we find

$$\left(\sum_{i \in I} \alpha_i R_i\right)^2 = \left(\sum_{i \in I} \alpha_i R_i\right)\left(\sum_{j \in J} \alpha_j R_j\right) \geq 0.$$

We see that $\sum_{i \in I} \alpha_i R_i = 0$ and hence $\alpha_i = 0$ for all i. Similarly we find that $\alpha_j = 0$ for all $j \in J$.

We conclude that $R_1, \dots R_t$ are linearly independent in $K_X^{\perp}$ which is of dimension $9-d$. □

Now that we have discussed the number of s-classes on generalized del Pezzo surfaces we will look at the possible geometric configurations. Let X be a generalized del Pezzo surface of degree $d \leq 7$ over an algebraically closed field k. Consider the graph whose vertices are the -1-curves on X. Between two distinct vertices corresponding to the -1-curves L and L' we have precisely $L \cdot L'$ edges. Since L and L' are integral and different we see that this is a non-negative integer. This graph is called the *intersection graph of -1-curves on X*.

We could also have looked at the *intersection graph of the -1-classes on X*. It follows from Lemma 2.3.1 that the Picard groups of two generalized del Pezzo surfaces of the same degree are isomorphic as lattices. This proves that the intersection graph of the -1-classes on a generalized del Pezzo surface depends only on the degree of X. We conclude that the intersection graph of all -1-curves is a subgraph of the intersection graph of all -1-classes. Proposition 2.4.6 implies that these graphs even coincide on an ordinary del Pezzo surface.

We could have also defined these objects starting from all s-curves or even all s-classes on X. Note that for the s-classes this does not produce a graph; the intersection number between two different -2-classes can be negative. Because of the obvious analogy with the situation of -1-classes we will stick with the terminology of graphs.

It follows as above that this intersection graph of the s-classes again only depends on the degree d of X if $d \leq 7$. This need no longer be true if we consider the intersection graph of all s-curves on X; note that this is an actual graph. We do have the following result if we restrict to the -2-curves.

LEMMA 2.4.8. *Let $\mathcal{R}$ be a set of -2-curves on a generalized del Pezzo surface over a field k. Any connected component of the intersection graph on the elements of $\mathcal{R}$ is a subgraph of the graph $\mathcal{G}$ shown in Figure I.*

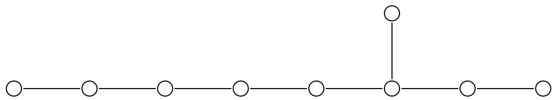

Figure I: The graph $\mathcal{G}$.

Note that the complete graph $\mathcal{G}$ is not possible by Lemma 2.4.7.

Proof. Again we can assume that k is algebraically closed. Since k is algebraically closed we can find a point on X which does not lie on any s-curve. We blow up X in this point and by induction we see that there is a generalized del Pezzo surface $W \to X$ of degree 1 with the same intersection graph of -2-curves as X. This shows that we can assume that X is a generalized del Pezzo surface of degree 1.

The anticanonical map of X has a single base point $p \in X(k)$ [23, Proposition III.2] and hence all effective anticanonical divisors on X pass through p. The anticanonical map on X defines a rational map $X \dashrightarrow \mathbb{P}^1_k$ which is defined away from p. If we blow up p on X we find a smooth surface $X' = \mathrm{Bl}_p\, X \dashrightarrow \mathbb{P}^1_k$. Since blowing up separates the effective anticanonical divisors at p, the rational map on X extends to a morphism $X' \to \mathbb{P}^1_k$. This makes X' into an elliptic surface with a section given by the exceptional divisor of the blowup $X' \to X$.

Now let R be a -2-curve on X and let R' be the pullback of R back to X'. Since $R \cdot K_X = 0$ we see that p does not lie on R. This also proves that R' lies in a fibre of $X' \to \mathbb{P}^1_k$. This fibre is the strict transform of an effective anticanonical divisor D on X. By Corollaire IV.2 in [23] this effective anticanonical divisor contains the connected component of R in the intersection graph of -2-curves on X. Let us write S for the sum of all -2-curves in this connected component of R. The same corollaire also shows that there is a unique -1-curve L on X such that $D = S + L$. Since p does not lie on any -2-curve it lies on L and this shows that the fibre of $X' \to \mathbb{P}^1_k$ containing R' is a sum of -2-curves, namely the sum of the strict transform of S and the strict transform of L.

The possible singular fibres of elliptic surfaces are classified and can for example be found in [48, Table 15.1]. To recover the connected components of -2-curves one still has to remove one component: the strict transform of L. We find the possible graphs A_n and D_n for any positive integer n, and E_6, E_7 and E_8.

One now uses Lemma 2.4.7 to conclude the proof. □

Now consider a del Pezzo surface over a general field k. We have looked at the action of the absolute Galois group G_k on the s-classes. This action induces an action on the geometric s-curves on X. Understanding this action is important for the following two results.

PROPOSITION 2.4.9. *Let X be a generalized del Pezzo surface of degree d over a field k. Suppose that $\mathcal{L}$ is a Galois-invariant set of -1-curves on $\bar{X}$ such that the curves in $\mathcal{L}$ are pairwise skew, i.e. $L_1 \cdot L_2 = 0$ for all $L_1 \neq L_2$ in $\mathcal{L}$. There exists a unique generalized del Pezzo surface X' defined over k such that $\bar{X}'$ is obtained from $\bar{X}$ by contracting the curves in $\mathcal{L}$. The degree of X' is $d + \#\mathcal{L}$.*

If X is an ordinary del Pezzo surface, then so is X'.

By contracting curves which are not defined over the base field k, we mean that $\bar{X}'$ is obtained from $\bar{X}$ by contracting the elements of $\mathcal{L}$. The proof is by contracting the -1-curves on $\bar{X}$ and then descending this surface back along $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$. We actually need not pass to an algebraic closure $\bar{k}$ of k; any field K over which all the lines in $\mathcal{L}$ are defined will do. This ensures that we can assume that K/k is a finite Galois extension and this makes the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$ into an fpqc cover.

We will use the following equivalent formulation of an fpqc descent datum along a Galois extension of fields.

PROPOSITION 2.4.10. *Let K/k be a finite Galois extension and let G be the Galois group $\mathrm{Gal}(K/k)$. Let $\sigma^* \colon \operatorname{Spec} K \to \operatorname{Spec} K$ be the isomorphism induced by the element $\sigma \in G$.*

Let $\tilde{X}$ be a scheme over K. An fpqc descent datum on $\tilde{X}$ relative to K/k is equivalent to a set of isomorphisms of schemes

$$\tilde{\sigma} \colon \tilde{X} \to \tilde{X},$$

indexed by $\sigma \in G$ such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\sigma}} & \tilde{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\sigma^*} & \operatorname{Spec} K \end{array}$$

commutes and $\tilde{\sigma}\tilde{\tau} = \widetilde{\tau\sigma}$ for all σ and τ in G.

Proof. See Proposition 4.4.2 in [46]. □

Let X be a scheme over the field k. In this interpretation the morphisms $\tilde{\sigma}$ corresponding to the effective descent datum on the base change X_K are given by the base change of $\sigma^* \colon \operatorname{Spec} K \to \operatorname{Spec} K$ along $X_K \to \operatorname{Spec} K$.

Proof of Proposition 2.4.9. Let K/k be a finite Galois extension over which all the lines in $\mathcal{L}$ are defined. On the surface X_K we can blow down each curve in $\mathcal{L}$ to obtain a smooth surface $\tilde{X}$ over K [33, Theorem V.5.7] with $x_i \in \tilde{X}(K)$ the image of the contracted curves. Let $\gamma\colon X_K \to \tilde{X}$ be the morphism which contracts all curves in $\mathcal{L}$.

We will make the effective descent datum of X_K into a descent datum on $\tilde{X}$. The morphism $\tilde{\sigma}\colon X_K \to X_K$ associated to an element $\sigma \in \mathrm{Gal}(K/k)$ fits into the commutative diagram in (2.1).

$$
\begin{tikzcd}
X_K \arrow[rr, "\tilde{\sigma}"] \arrow[dd] \arrow[dr, "\gamma"] & & X_K \arrow[dd] \arrow[dr, "\gamma"] & \\
& \tilde{X} \arrow[dl] & & \tilde{X} \arrow[dl] \\
\operatorname{Spec} K \arrow[rr, "\sigma^*"] & & \operatorname{Spec} K &
\end{tikzcd}
\tag{2.1}
$$

Since γ is a birational morphism and $\tilde{\sigma}$ an isomorphism we find a birational map $\tilde{\sigma}\colon \tilde{X} \dashrightarrow \tilde{X}$ which makes the diagram in (2.1) commute. This map $\tilde{\sigma}$ is clearly defined on the complement of the points x_i and the composition $\gamma \circ \tilde{\sigma}$ contracts each curve in $\mathcal{L}$ to a closed point. We conclude from [50, Tag 0C5J] that $\tilde{\sigma}$ descends to an actual morphism $\tilde{\sigma}\colon \tilde{X} \to \tilde{X}$. This morphism makes the diagram in (2.2) commute.

$$
\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{\sigma}"] \arrow[d] & \tilde{X} \arrow[d] \\
\operatorname{Spec} K \arrow[r, "\sigma^*"] & \operatorname{Spec} K
\end{tikzcd}
\tag{2.2}
$$

The morphisms $\tilde{\sigma}\tilde{\tau}$ and $\widetilde{\tau\sigma}$ restrict to the same automorphisms on $\tilde{X}\backslash\{x_i\}$. This proves that the morphisms themselves are the same, since they agree on an open dense subset.

This proves both conditions of Proposition 2.4.10 for the set of isomorphisms $\tilde{\sigma}\colon \tilde{X} \to \tilde{X}$. It follows that there is a surface X' over k such that X'_K is isomorphic to $\tilde{X}$ over K. The surface X' is a generalized del Pezzo surface by Proposition 2.2.4, because X'_K is a generalized del Pezzo surface. The same proposition also proves that X' is an ordinary del Pezzo surface if X is an ordinary del Pezzo surface.

The morphism $\gamma\colon X_K \to \tilde{X}$ commutes with the Galois descent morphisms $\sigma^*\colon \tilde{X} \to \tilde{X}$ and hence descends to a morphism $X' \to X$ [46, Theorem 4.3.5(i)]. It follows directly that on the complement of the -1-curves in $\mathcal{L}$ this morphism is an isomorphism onto its image, because this is the case for its base change morphism $X_K \to \tilde{X}$. $\square$

We also have the following result about contracting -2-curves, but unlike the case of -1-curves the constructed surface can be singular. Because the possible configurations of contracted curves is limited by Proposition 2.4.9 the surface will however still be normal.

PROPOSITION 2.4.11. *Let $\mathcal{R}$ be a Galois-invariant set of -2-curves on a generalized del Pezzo surface X over a field k. There is a normal surface Y together with a birational proper morphism $\gamma\colon X \to Y$ such that*

» *the integral curves on $\bar{X}$ which are contracted by γ are precisely the elements $R \in \mathcal{R}$;*

» *the Weil divisor K_Y on Y is a Cartier divisor; and*

» *the pullback of the associated line bundle ω_Y along γ is the canonical line bundle ω_X on X.*

Proof. The proof is similar to the proof of Proposition 2.4.9. Let K be a finite Galois extension over which all the -2-curves in $\mathcal{R}$ are defined. We will first contract the -2-curves on X_K.

Let us consider the intersection matrix $(R_i \cdot R_j)$ of the -2-curves of $\mathcal{R}$. We will prove that this matrix is negative definite. This statement is true for -2-curves with intersection graph A_8, D_8 and E_8. It is easily checked that any connected subgraph of the graph $\mathcal{G}$ in Figure I is a subgraph of one of these three graphs. Using Lemma 2.4.8 and the fact that any principal minor of a negative definite matrix is again negative definite we see that this property is satisfied for any collection of geometric -2-curves on a generalized del Pezzo surface.

Since the matrix $(R_i \cdot R_j)$ is negative definite we can apply Theorem 2.7 in [1]. This produces a normal surface $\tilde{X}$ together with a proper birational morphism $\tilde{\gamma}\colon X_K \to \tilde{X}$ which contracts precisely the -2-curves in $\mathcal{R}$. As before we can descend this to a morphism $X \to Y$ over k, which is an isomorphism away from the contracted -2-curves. Using Corollaire 9.10 in [31] we see that Y is normal, because $\tilde{X}$ is.

The statements about the canonical divisor and canonical line bundle on Y also follow from Theorem 2.7 in [1]. □

2.5 Effective anticanonical divisors

In this section we will give several characterizations of the effective anticanonical divisors on a generalized del Pezzo surface X. We will need the following result which we will state as a lemma.

LEMMA 2.5.1. *Let $\pi\colon S' \to S$ be a proper birational morphism of projective surfaces and assume that S is normal. The natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_{S'}$ is an isomorphism.*

More generally, the natural morphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$ coming from the adjunction between π_* and π^* is an isomorphism for any quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. The isomorphism $\mathcal{O}_S \to \pi_*\mathcal{O}_{S'}$ follows from [50, Tag 0AY8]. This proves the first claim. We now have the following chain of isomorphisms

$$\begin{aligned} \pi_*\pi^*\mathcal{F} &\cong \pi_*\left(\pi^*\mathcal{F}\otimes\mathcal{O}_{S'}\right) \\ &\cong \mathcal{F}\otimes\pi_*\mathcal{O}_{S'} \\ &\cong \mathcal{F}\otimes\mathcal{O}_S \\ &\cong \mathcal{F}, \end{aligned}$$

where we have used the projection formula [33, Exercise III.8.3] in the second isomorphism. One can check locally that this isomorphism is given by the natural morphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$ coming from the adjunction between π_* and π^*. □

We can now prove the following proposition and state the main definition of this section. This also explains the coefficients in the definition of E_π^{pec} in Definition 2.4.5.

PROPOSITION 2.5.2. *Let X be a generalized del Pezzo surface of degree d over a field k together with a birational morphism $\pi\colon X \to \mathbb{P}^2_k$.*

There is an isomorphism of line bundles $\pi^*\omega_{\mathbb{P}^2_k} = \omega_X \otimes \mathcal{L}(-E_\pi^{\mathrm{pec}})$ over X and an isomorphism of k-vector spaces

$$\mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee) \to \mathrm{H}^0(X, \omega_X^\vee \otimes \mathcal{L}(E_\pi^{\mathrm{pec}})).$$

This last map is given on divisors by mapping a divisor C to its total transform π^*C. Here we have identified the complete linear system $|D|_X$ of a divisor D on X with the global sections $\mathrm{H}^0(X, \mathcal{L}(D))$ up to scaling by elements in $k^\times$.

The morphism $\mathrm{H}^0(X, \omega_X^\vee) \to \mathrm{H}^0(X, \omega_X^\vee \otimes \mathcal{L}(E_\pi^{\mathrm{pec}}))$ defined by taking the tensor product with the designated global section $1_{E_\pi^{\mathrm{pec}}} \in \mathrm{H}^0(X, \mathcal{L}(E_\pi^{\mathrm{pec}}))$ is injective. This injection is given on divisors by adding the effective divisor E_π^{pec} on X.

Proof. The statement is purely geometric so we can assume that π decomposes as the blowup $X = X_r \to X_{r-1} \to \ldots \to X_1 \to \mathbb{P}^2_k$ of the projective plane in $r = 9 - d$ points in almost general position, where the centre of each blowup is a closed k-point.

Define $\pi_i\colon X_i \to \mathbb{P}^2_k$ to be the composition of the first i blowups. So we get that π_0 is the identity on the projective plane and that $\pi_r = \pi$. One can now prove by induction that $\pi_i^*\omega_{\mathbb{P}^2_k} = \omega_{X_i} \otimes \mathcal{L}(-E_{\pi_i}^{\mathrm{pec}})$. The base case is trivial and for the induction step one uses Proposition V.3.3 in [33].

We apply Lemma 2.5.1 to the quasi-coherent sheaf $\omega_{\mathbb{P}^2_k}^\vee$. We find an isomorphism $\omega_{\mathbb{P}^2_k}^\vee \to \pi_*\pi^*\omega_{\mathbb{P}^2_k}^\vee$. Since the global sections of the pushforward $\pi_*\mathcal{F}$ are the same as the global sections of the original sheaf $\mathcal{F}$ we conclude that there is an isomorphism on global sections

$$\mathrm{H}^0(\mathbb{P}^2_k, \omega_{\mathbb{P}^2_k}^\vee) \to \mathrm{H}^0(X, \pi^*\omega_{\mathbb{P}^2_k}^\vee)$$

induced by π. If we consider the associated divisors of a global section in the domain and codomain of this isomorphism we see that a Cartier divisor associated to a global section in $\mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee)$ gets sent to the Cartier divisor on X locally defined by the same functions after identifying $\kappa(X)$ and $\kappa(\mathbb{P}^2_k)$ along π. This maps a divisor C on $\mathbb{P}^2_k$ to the divisor on X which by definition is the total transform of C along π.

Now consider the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{L}(E^{\mathrm{pec}}_\pi)$ which maps the unit of $\mathcal{O}_X$ to $1_{E^{\mathrm{pec}}_\pi}$. A locally free sheaf is flat so we find the inclusion of line bundles $\omega^\vee_X \hookrightarrow \omega^\vee_X \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi)$. This induces an inclusion on global sections. The last statement about divisors follows from considering the inclusion $\mathrm{H}^0(X, \omega^\vee_X) \to \mathrm{H}^0(X, \omega^\vee_X \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi))$ locally. $\square$

DEFINITION 2.5.3. Let $V_X \subseteq \mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee)$ be the image of the composition of the inclusion $\mathrm{H}^0(X, \omega^\vee) \hookrightarrow \mathrm{H}^0(X, \omega^\vee \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi))$ with the isomorphism

$$\mathrm{H}^0(X, \omega^\vee \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi)) \xrightarrow{\cong} \mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee).$$

Since $\omega_{\mathbb{P}^2_k}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(-3)$ we can identify global sections of the anticanonical bundle with homogeneous cubic polynomials in three variables. We will consider the linear subsystem of cubic plane curves associated to these polynomials. The next proposition describes the relation between this linear subsystem $|V_X|_{\mathbb{P}^2_k}$ of cubics on $\mathbb{P}^2_k$ and the effective anticanonical divisors on X. Note that by Definition 2.5.3 both linear systems are of dimension $9 - d$.

PROPOSITION 2.5.4. *Let $\pi \colon X \to \mathbb{P}^2_k$ be a birational morphism from a generalized del Pezzo surface to the projective plane. Let $C \subseteq \mathbb{P}^2_k$ be a cubic plane curve and let $\tilde{C}$ be the strict transform of C along π. The following three statements are equivalent.*

(i) The divisor C lies in the linear subsystem $|V_X|_{\mathbb{P}^2_k}$.

(ii) The anticanonical divisor $\pi^*C - E^{\mathrm{pec}}_\pi$ on X is effective.

(iii) There exists an effective divisor D on X supported on the peculiar divisor E^{pec}_π of π such that the class of $\tilde{C} + D$ in the Picard group of X is the anticanonical class $-K_X$.

Note that $\pi^*C - E^{\mathrm{pec}}_\pi$ is an anticanonical divisor for any cubic plane curve C. Statement (ii) is purely about the effectiveness of this divisor. We will also use that $\pi^*C - \tilde{C}$ is an effective divisor on X whose prime divisors lie in the support of E^{pec}_π.

Proof. The equivalence between (i) and (ii) follows directly from Definition 2.5.3.

Assume statement (ii) holds for a cubic plane curve C. The divisor $D = \pi^*C - E^{\mathrm{pec}}_\pi - \tilde{C}$ then satisfies the conditions of (iii). Now assume (iii). We have two anticanonical divisors $\pi^*C - E^{\mathrm{pec}}_\pi$ and $\tilde{C} + D$. That implies that the difference $(\pi^*C - \tilde{C}) - E_\pi^{\text{pec}} - D$ is a principal divisor. Since $\pi^*C - \tilde{C}$ and D are supported on E_π^{pec} we see that the same holds for $(\pi^*C - \tilde{C}) - E_\pi^{\text{pec}} - D$. Now consider a function $f \in \kappa(X)$ such that $\operatorname{div}_X f = (\pi^*C - \tilde{C}) - E_\pi^{\text{pec}} - D$. We see that f has a trivial divisor on $X \backslash E_\pi^{\text{pec}}$. Using the birational morphism $\pi\colon X \to \mathbb{P}_k^2$ we find a function $f \in \kappa(\mathbb{P}_k^2)$ which has a trivial divisor on the complement of a finite set of points. Since $\mathbb{P}_k^2$ is normal, we see that $\operatorname{div}_{\mathbb{P}_k^2} f = 0$ and hence f must be constant both in $\kappa(\mathbb{P}_k^2)$ and in $\kappa(X)$. This shows that $(\pi^*C - \tilde{C}) - E_\pi^{\text{pec}} - D = 0$ and hence $\pi^*C - E_\pi^{\text{pec}} = \tilde{C} + D$ is an effective anticanonical divisor. □

Note that the proof also shows that the complementary divisor D in (iii) is unique.

We will see in Proposition 2.7.16 another way to identify the cubic plane curves in the linear system of V_X.

2.6 Singular del Pezzo surfaces

A generalized del Pezzo surface comes by definition with a morphism to a projective space, namely the morphism associated to the complete linear system of a sufficiently large multiple of the anticanonical divisor. For del Pezzo surfaces of high degree it is even enough to consider $-K_X$ itself.

THEOREM 2.6.1. *Let X be a generalized del Pezzo surface of degree d over a field k. If $d \geq 3$ then the complete linear system $|-K_X|$ does not have base points and $-K_X$ is very ample outside of the -2-curves. The associated morphism contracts all -2-curves on X and embeds the obtained surface as a degree d surface in $\mathbb{P}_k^d$.*

Proof. See Proposition V.1 of [23]. □

COROLLARY 2.6.2. *The anticanonical map embeds an ordinary del Pezzo surface of degree $d \geq 3$ as a smooth surface in $\mathbb{P}_k^d$ of degree d.*

For ordinary del Pezzo surfaces of degrees 1 and 2 it is known that although the class $-K_X$ is ample it will not be very ample. The smallest multiples of the anticanonical line bundle which are very ample are $-3K_X$ and $-2K_X$ respectively. Theorem 2.6.1 is also true for generalized del Pezzo surfaces of degree 1 and 2 if one considers these multiples of the anticanonical line bundles.

We will now consider the image of a generalized del Pezzo surface under this morphism to a projective space over k. Such a surface will be normal by Proposition 2.4.11.

DEFINITION 2.6.3. The projective normal surface obtained from contracting all the -2-curves on a generalized del Pezzo surface X is called a *singular del Pezzo surface*.

Note the unfortunate terminology: an ordinary del Pezzo surface X is also a singular del Pezzo surface, although X is actually non-singular. In general the morphism $X \to Y$ is not an isomorphism, but it will be a birational morphism. We have seen in Proposition 2.4.11 how to construct Y from X. One can also recover the generalized del Pezzo surface X from the singular del Pezzo surface Y, but we will need the following notion.

DEFINITION 2.6.4. Let Y be a scheme. A scheme X together with a proper birational map $\gamma\colon X \to Y$ is called a *desingularization* of Y if X is regular.

A desingularization $\gamma\colon X \to Y$ of Y is said to be a *minimal desingularization* if for any desingularization X' the morphism $X' \to Y$ factors through $X \to Y$.

Finding desingularizations can be quite hard in general and even minimal desingularizations need not exist [3, Section 3]. For surfaces they are well enough understood and we have the following proposition.

PROPOSITION 2.6.5. *Let Y be a surface over a field k. Suppose that Y has a desingularization $\gamma\colon X \to Y$. Then Y has a unique minimal desingularization.*

A desingularization $X \to Y$ is minimal if and only if all integral curves E on X which map to a point on Y satisfy $E^2 \leq -2\chi(E)$.

Because of this result we will usually talk about the minimal desingularization of a surface instead of a minimal desingularization.

Proof. See Corollary 27.3 in [37]. □

This gives us the result we were looking for.

COROLLARY 2.6.6. *Let X be a generalized del Pezzo surface over a field k and let Y be the associated singular del Pezzo surface over k. The morphism $\gamma\colon X \to Y$ which contracts all -2-curves on X is the minimal desingularization of Y.*

Proof. The surface X is smooth over k and we see by [33, Corollary II.4.8] that the morphism γ is proper. This shows that X is a desingularization of Y by definition. We will show that X is the minimal desingularization of Y.

The integral curves on X mapping to a point on Y are precisely the -2-curves on X. A -2-curve $E \subseteq X$ satisfies $\chi(E) = 1$ because E is a smooth curve of genus 0. This means that we have $E^2 \leq -2\chi(E)$ and we conclude from Proposition 2.6.5 that X is the minimal desingularization of Y. □

A singular del Pezzo surface will only have isolated singularities since it is normal. Such a singularity p on a singular del Pezzo surface Y can be studied by looking at the fibre of p of the minimal desingularization $\gamma\colon X \to Y$; the type of singularity is encoded by the graph of intersection of the components of the fibre $\gamma^{-1}p$.

COROLLARY 2.6.7. *A singularity on a singular del Pezzo surface over an algebraic closed field is an isolated singularity of type A_n or D_n for $n \leq 8$, or E_6, E_7 or E_8.*

Proof. This corollary is proved by listing all connected subgraphs of the graph in Lemma 2.4.8. □

2.7 Peculiar del Pezzo surfaces

The first type of del Pezzo surfaces to be studied were the surfaces which with our definitions would be called ordinary del Pezzo surfaces of degree $d \geq 3$. Del Pezzo in [22] used Corollary 2.6.2 as his definition; he studied smooth surfaces in $\mathbb{P}_k^d$ of degree d. In this terminology singular del Pezzo surfaces appear to be a natural generalization of ordinary del Pezzo surfaces. On the other hand generalized del Pezzo surface seem to be an obvious extension if one considers ordinary del Pezzo surfaces as the projective plane blown up in r points in general position.

In this section we will study a new type of del Pezzo surfaces: peculiar del Pezzo surfaces. We will see that they fit in between the classes of singular and generalized del Pezzo surfaces. They generalize the notion of ordinary del Pezzo surfaces in the following way.

Let X be an ordinary del Pezzo surface over a field k and suppose that it is explicitly written as the blowup $X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}_k^2$ of the projective plane in r points in general position. As the centre $p_i \in X_i(k)$ of the blowup $X_{i+1} \to X_i$ does not lie on a curve with negative self-intersection on X_i, we find that for $0 \leq j < i$ the point p_i does not map to p_j under $X_i \to X_j$. Let Z be the union of the images of all p_i in $X_0 = \mathbb{P}_k^2$. By the commutativity of blowing up two closed subschemes [24, Lemma IV-41] we find that X and $\mathrm{Bl}_Z \mathbb{P}_k^2$ are isomorphic.

This approach has the following consequence: we do not need the intermediate surfaces X_i with $0 < i < r$ to study X; the geometry of X is completely determined by information on $\mathbb{P}_k^2$. For example, the -1-curves on an ordinary del Pezzo surface X of degree $d \geq 3$ are either a component of the exceptional divisor of $\beta\colon X = \mathrm{Bl}_Z \mathbb{P}_k^2 \to \mathbb{P}_k^2$ or the strict transform along β of a line on $\mathbb{P}_k^2$ which meets Z in two points or a conic which meets Z in five points. Recall that for ordinary del Pezzo surfaces of degree 1 or 2 we would also have to consider cubic plane curves. In any case, the intersection graph of all s-curves on an ordinary del Pezzo surface is determined by the configuration of Z on the projective plane.

Let us mimic the construction of Z for a generalized del Pezzo surface X written as the blowup $X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}_k^2$ of the projective plane in r points in almost general position: let q_α be the images of the blowup centres $p_i \in X_i$ under the composition $X_i \to \mathbb{P}_k^2$. We will let n_α be the number of i such that p_i maps to q_α. We would want Z to be a zero-dimensional scheme supported in the q_α such that the geometrically irreducible component at each q_α is of length n_α. This presents two problems.

The first problem is that $\mathrm{Bl}_Z \mathbb{P}_k^2$ will not be isomorphic to X in general. Consider the first example in Section VI.2.3 from [24]. They consider the blowup of the projective plane in the non-reduced point defined by (x^2, y). This ideal contain the information of the origin p_0 and the direction v defined at p_0 by the line $y = 0$. If we blow up the point p_0 we get a surface $X_1 \to \mathbb{P}^2_k$ with an exceptional divisor E_1. Now consider v as a k-point on E_1. If we blowup X_1 in v, then we get a scheme $X_2 \to X_1$ with the exceptional divisor E_2. Let us denote the strict transform of E_1 along $X_2 \to X_1$ also by E_1. If we compose the two blowup morphisms we get a birational morphism $\pi\colon X = X_2 \to X_1 \to X_0 = \mathbb{P}^2_k$ with peculiar divisor $E_\pi^{\mathrm{pec}} = E_1 + 2E_2$. The morphism π restricts to an isomorphism

$$X \backslash E_\pi^{\mathrm{pec}} \to \mathbb{P}^2_k \backslash p_0.$$

Proposition IV.40 in [24] states that $\mathrm{Bl}_Z \mathbb{P}^2_k$ is obtained from X by contracting the -2-curve E_1. So in this case $\mathrm{Bl}_Z \mathbb{P}^2_k$ is not the generalized del Pezzo surface we started with, but rather the associated singular del Pezzo surface. For some zero-dimensional schemes Z the blowup $\mathrm{Bl}_Z \mathbb{P}^2_Z$ will be neither a generalized or a singular del Pezzo surface. This is where the peculiar del Pezzo surfaces come in.

The second problem is that the support and local lengths do not determine the zero-dimensional subscheme uniquely. It would if we could embed Z into a smooth curve. So let us recall that the associated zero-dimensional scheme Z associated to an ordinary del Pezzo surface naturally lies on a certain important cubic plane curve.

It follows from Proposition 2.3.1 that the pushforward $\beta_* D$ of an effective anticanonical divisor D along $\beta\colon \mathrm{Bl}_Z \mathbb{P}^2_k \to \mathbb{P}^2_k$ is a cubic curve. One can prove that this cubic curve passes through Z. This even defines a bijection between the effective anticanonical divisors on the ordinary del Pezzo surface $\mathrm{Bl}_Z \mathbb{P}^2_k$ and the cubic curves passing through Z. This proves that Z could equivalently be defined as the intersection of all these cubic curves.

Now consider a generalized del Pezzo surface X and construct the points q_α with the multiplicities n_α. With this data we could determine Z if the pushforward of an anticanonical divisor on X were a cubic curve which passes through each q_α and is furthermore smooth at these points. This is precisely the following result.

LEMMA 2.7.1. *Let $X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^2_k$ be a generalized del Pezzo surface over a field k written as the blowup of the projective plane in r points in almost general position. Let $p_i \in X_i(k)$ be the centre of the blowup $X_i \to X_{i-1}$. There exists an irreducible cubic curve $C \subseteq \mathbb{P}^2_k$ such that the strict transform of C_i along $X_i \to X_0 = \mathbb{P}^2_k$ passes through p_i and is also smooth at p_i.*

For generalized del Pezzo surfaces over fields of characteristic zero one can even find an irreducible curve C which is everywhere smooth using Bertini's theorem. We include the more general result so that our results are true over any field.

Proof. This is part (b') of Théorème III.1 in [23] in the version stated in the introduction of part IV. $\square$

So let us first look into zero-dimensional schemes on curves and define what it means for such a subscheme to be in almost general position.

DEFINITION 2.7.2. A zero-dimensional subscheme Z on a surface S is called *curvilinear* if it can be embedded in a curve $C \subseteq S$ which is smooth in the support of Z.

Let k be a field and consider a curvilinear subscheme $Z \subseteq \mathbb{P}_k^2$ of degree $r \leq 6$. We say that Z lies *in almost general position* if the scheme-theoretic intersection of Z with any integral curve L of degree 1 satisfies $\deg(Z \cap L) \leq 3$.

If Z is reduced and we have

$$\deg(Z \cap D) \leq \begin{cases} 2 & \text{if } \deg D = 1; \\ 5 & \text{if } \deg D = 2 \end{cases}$$

for all effective divisors D of degree at most 2, we say that Z lies *in general position*.

In the definition of almost general position one would want to add the condition that for all curves C of degree 2 we have $\deg(Z \cap C) \leq 6$. Since we have restricted to $r \leq 6$ this condition is trivially satisfied. If one considers zero-dimensional subschemes of degrees 7 and 8 in almost general position, one would require that $\deg(Z \cap C) \leq 6$. A more complication condition would also be needed on cubic plane curves.

A zero-dimensional scheme supported on a plane curve of low degree will always lie in almost general position.

LEMMA 2.7.3. *Let k be a field and Z a zero-dimensional supported in the smooth locus of a cubic plane curve $C \subseteq \mathbb{P}_k^2$. Then Z lies in almost general position.*

To construct curvilinear subschemes more generally one can use the fact that a geometrically irreducible zero-dimensional subscheme on a given smooth curve C is uniquely defined by its support and its degree. This warrants the following definition.

DEFINITION 2.7.4. Let m be a positive integer and C a curve which lies on a surface S over a field k. Fix a k-point x on S which is smooth as a point of both C and S over k. We will write $\mathcal{I}_{C,x,m}$ for the ideal sheaf defining the unique zero-dimensional subscheme of C of degree m which is supported at x.

When $X \to \mathbb{P}_k^2$ is the blowup of the projective plane in r points in general position we will consider the degree r zero-dimensional subscheme Z of $\mathbb{P}_k^2$ such that $\mathrm{Bl}_Z \mathbb{P}_k^2$ is isomorphic to X over $\mathbb{P}_k^2$. This proves that for ordinary del Pezzo surfaces we can freely shift between the set of points and the subscheme Z.

The main objective of this section is to relate the blowup $X' = \mathrm{Bl}_Z \mathbb{P}_k^2$ for a zero-dimensional scheme Z in almost general position to generalized and singular del Pezzo surfaces. Let us start by giving these surfaces X' a name.

DEFINITION 2.7.5. Let k be a field. A *peculiar del Pezzo surface* over k is the blowup of $\mathbb{P}_k^2$ in a subscheme $Z \subseteq \mathbb{P}_k^2$ in almost general position. The degree of a peculiar del Pezzo surface is $9 - \deg Z$.

We have seen that ordinary del Pezzo surfaces are also peculiar del Pezzo surfaces. Note however that peculiar del Pezzo surface can be singular surfaces, but on the other hand they are not necessarily singular del Pezzo surfaces. We will now describe the relation between generalized and peculiar del Pezzo surfaces.

PROPOSITION 2.7.6. *Let X' be a peculiar del Pezzo surface of degree $9 - r$ over a field k. It has the minimal desingularization X and the surface X is a generalized del Pezzo surface given as the blowup of $\mathbb{P}^2_k$ in r points in almost general position. Furthermore, the del Pezzo surfaces X' and X have the same degree.*

For the proof we will need several lemmas. First we will describe the blowup of a surface in a curvilinear subscheme. Then we will consider how the geometry of a surface changes under a blowup in a possibly non-reduced curvilinear scheme.

LEMMA 2.7.7. *Let m be a positive integer and C a curve which lies on a surface S over a field k. Fix a k-point x on S which is smooth as a point of both C and S over k. Let Z be the zero-dimensional subscheme of S defined by the ideal sheaf $\mathcal{I}_{C,x,m}$.*

The blowup $B = \mathrm{Bl}_Z S$ of S in the subscheme Z can be computed as follows: define $S_0 = S$, $x_0 = x$, $C_0 = C$ and recursively the blowup $\pi_{i+1}\colon S_{i+1} \to S_i$ in x_i with exceptional curve E_{i+1}, the strict transform C_{i+1} of C_i along π_{i+1}, and x_{i+1} the unique intersection between E_{i+1} and C_{i+1}. Then B is obtained from S_m by contracting the strict transforms of E_i for all $1 \leq i \leq m-1$.

This construction also shows that the positive-dimensional fibres of $S_m \to S$ are of the same form as described in Proposition 2.4.4. It now follows from Proposition 2.6.5 that S_m is the minimal desingularization of B.

Proof. Consider the completed local ring $\widehat{\mathcal{O}}_{S,x}$ at x and let $\widehat{x}$, $\widehat{C}$ and $\widehat{Z}$ be the pullbacks of x, C and Z along the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{S,x} \to S$. As $\widehat{Z}$ is the subscheme of $\widehat{C}$ of length m which is supported at $\widehat{x}$ and we recover a situation similar to the one in the lemma; we will compute the blowup of the two-dimensional scheme $\operatorname{Spec} \widehat{\mathcal{O}}_{S,x}$ in the zero-dimensional scheme $\widehat{Z}$, which is contained in $\widehat{C}$ and supported in the point $\widehat{x}$. We will first prove the result in this case. To that end let $\widehat{\pi}_i$, $\widehat{S}_i$, $\widehat{x}_i$ and $\widehat{C}_i$ be the objects mentioned in the lemma when applied to computing the blowup $\widehat{B} = \mathrm{Bl}_{\widehat{Z}} \widehat{S}$.

By smoothness we can choose an isomorphism $\widehat{\mathcal{O}}_{S,x} \to k[[u,v]]$ such that $\widehat{C}$ is given by the vanishing of v. Then Z is given by the ideal (u^m, v). Using these equations one can prove by explicit calculations that $\widehat{S}_m \to \widehat{B}$ is the contraction of all but the last of the exceptional curves $\widehat{E}_i$. A particularly nice way to prove this is using toric geometry: the blowup in the ideal (u^m, v) gives us the completion of a singular toric variety covered by two affine charts. To find a desingularization of $\widehat{B}$ one can use the procedure in [21, Section 10.1], which coincides with the process described in the lemma.

Blowing up commutes with flat base change, so we have the following tower of cartesian squares combining the situation above with the general case.

$$
\begin{tikzcd}
\widehat{S}_m \arrow[rr] \arrow[d, "\widehat{\pi}_m"'] \arrow[dr] & & S_m \arrow[d, "\pi_m"'] & \\
\vdots \arrow[d, "\widehat{\pi}_2"'] & \widehat{B} \arrow[rr] \arrow[ddl, "\widehat{\beta}"'] & \vdots \arrow[d, "\pi_2"'] & B \arrow[ddl, "\beta"'] \\
\widehat{S}_1 \arrow[rr] \arrow[d, "\widehat{\pi}_1"'] & & S_1 \arrow[d, "\pi_1"'] & \\
\widehat{S}_0 = \operatorname{Spec} \widehat{\mathcal{O}}_{S,x} \arrow[rr] & & S_0 = S &
\end{tikzcd}
$$

Let U be the complement of x in S and identify it with the corresponding open $U \times_S S_m \subseteq S_m$. Now consider the map $U \coprod \widehat{S}_0 \to S_0$, which is an fpqc cover, so the same holds for the base change $U \coprod \widehat{S}_m \to S_m$. The fibred product of $U \coprod \widehat{S}_m$ over S_m with itself is the disjoint union of the S_m-schemes $U \times_{S_m} U$, $\widehat{S}_m \times_{S_m} \widehat{S}_m$ and $U \times_{S_m} \widehat{S}_m$. The projection maps of the first product are isomorphisms as U is an open of S_m. Because taking the fibred product of $T \to S$ with the completion of S in x we get the completion of T in the pullback of x we see that the projections $\widehat{S}_0 \times_{S_0} \widehat{S}_0 \to \widehat{S}_0$ are isomorphisms too. Pulling back this isomorphism to S_m we find that $\widehat{S}_m \times_{S_m} \widehat{S}_m$ and $\widehat{S}_m$ are also naturally isomorphic. Lastly, $U \times_{S_m} \widehat{S}_m$ is the complement of the closed point in $\widehat{S}_m$. So we have maps $U, \widehat{S}_m \to B$ over S_m which agree on the product over S_m described above. As representable functors on the category of S-schemes are sheaves in the fpqc topology [25, Theorem 2.55] the morphism $U \coprod \widehat{S}_m \to B$ descends to the morphism $S_m \to B$ of S-schemes we were looking for. □

We have the composition $S_m \to B \to S_0$ and similarly to the proof of Proposition 2.4.4 we see that the positive-dimensional fibre of $S_m \to S$ is a union of -2-curves and one -1-curve. We conclude that the desingularization $S_m \to B$ contracts a chain of -2-curves and Proposition 2.6.5 proves that S_m is the minimal desingularization of B. Similarly to the proof of Proposition 2.4.11 one can prove that if S_m is a projective normal surface, then so is B.

We will also need to know how intersection numbers of curves on S behave under pullback to S_m. The following lemma describes local intersection numbers for the blowup of S in a closed point. For a reference on the notions of intersection numbers and multiplicities of curves one can consult Sections 3.2 and 3.3 in [26].

LEMMA 2.7.8. *Consider a surface S over a field k with a smooth k-point x. Let m be a positive integer, $C \subseteq S$ a curve on S which is smooth at x, and $Z \subseteq C$ the zero-*

dimensional subscheme defined by $\mathcal{I}_{C,x,m}$. Consider an integral curve D on S such that any common component of C and D does not pass through x. It holds that

$$\deg(D \cap Z) = \min(i_x(C,D), \deg Z).$$

Let E be the exceptional divisor of the blowup $\pi\colon S' \to S$ of S at x. Define $\tilde{C}$ and $\tilde{D}$ as the strict transforms of C and D along π, and let x' be the unique intersection point of $\tilde{C}$ with E. If s is the multiplicity of D at x then we have

$$i_x(C,D) = i_{x'}(\tilde{C},\tilde{D}) + s.$$

Proof. We will work with the completed local rings $\widehat{\mathcal{O}}_{S,x}$ and $\widehat{\mathcal{O}}_{S',x'}$. We can pick coordinates u and v for $\widehat{\mathcal{O}}_x$, such that C is given by $v = 0$, and hence Z is given by $u^m = 0 = v$. Then we can use coordinates u and V on $\widehat{\mathcal{O}}_{x'}$, such that the map $\widehat{\mathcal{O}}_x \to \widehat{\mathcal{O}}_{x'}$ induced by π maps u to u and v to Vu. Now let D be defined at x by a polynomial $g(u,v)$.

By definition we have

$$\deg(D \cap Z) = \dim_k \widehat{\mathcal{O}}_x/(u^m, v, g) = \dim_k \widehat{\mathcal{O}}_x/(u^m, v, g(u,0)),$$

$$\deg Z = \dim_k \widehat{\mathcal{O}}_x/(u^m, v) = m$$

and

$$i_x(C,D) = \dim_k \widehat{\mathcal{O}}_x/(v,g) = \dim_k \widehat{\mathcal{O}}_x/(v, g(u,0))$$

which proves the first statement.

For the second statement we interpret

$$i_x(C,D) = \dim_k \widehat{\mathcal{O}}_x/(v, g(u,0))$$

as the smallest t such that u^t is a monomial of g.

Since C is smooth at x and E is defined by the vanishing of u we see that $\tilde{C}$ is defined by $V = 0$. Similarly, $\tilde{D}$ is given by the vanishing of the polynomial $\tilde{g} = \frac{g(u,Vu)}{u^s}$. So we have

$$i_{x'}(\tilde{C},\tilde{D}) = \dim_k \widehat{\mathcal{O}}_{x'}/(V, \tilde{g}).$$

This is the smallest integer t' such that $u^{t'}$ is a monomial of $\tilde{g}$. We have a correspondence between monomials of g and $\tilde{g}$ by associating $u^\alpha v^\beta$ to $u^{\alpha+\beta-s}V^\beta$. This implies that $t = t' + s$. □

We can now compute the self-intersection of the strict transform $\tilde{D} \subseteq S_m$ of an integral curve $D \subseteq S$, which we will need to prove Proposition 2.7.6.

LEMMA 2.7.9. *Let $Z \subseteq S$ be a curvilinear subscheme of degree r on a smooth surface S over a field k. Let $X' \to S$ be the blowup of S in Z and let $X \to X'$ be its minimal desingularization. For an integral curve D on S we define $\tilde{D}$ to be the strict transform of D along $X \to X' \to S$. If D is smooth in the support of Z we have*

$$\tilde{D}^2 = D^2 - \deg(D \cap Z).$$

Proof. Let $C \subseteq S$ be a curve on S which contains Z and is smooth in the support of Z. It is enough to prove the result in the case that Z is supported in a single point $x \in C$. We use the notation of Lemma 2.7.7: the morphism $X \to X' \to S$ equals the composition $X = S_m \to S_{m-1} \to \ldots \to S_1 \to S_0 = S$ of m blowups in k-points $x_i \in S_i(k)$. Let $E_i \subseteq S_i$ be the exceptional curve of $S_i \to S_{i-1}$.

We also define $C_i, D_i \subseteq S_i$ as the strict transforms of C and D along $S_i \to S_0$. We see that $x_i \in C_i$. We will determine the i for which x_i lies on D_i.

We will prove by induction that

$$i_{x_i}(C_i, D_i) = \max(i_{x_0}(C_0, D_0) - i, 0).$$

Indeed, D_i is either smooth in x_i or it does not pass through x_i which correspond to the conditions $i_{x_i}(C_i, D_i) > 0$ and $i_{x_i}(C_i, D_i) = 0$. The result now follows from Lemma 2.7.8; $i_{x_i}(C_i, D_i)$ decreases by 1 as i increases by 1 until it is zero.

Define $m' = \deg(Z \cap D)$. Note that we have proved that D_i passes through x_i for $i = 0, 1, \ldots, m' - 1$, but not for $i > m'$. Since D is smooth at each of the x_i for $i < m'$, we find $D_i^2 = D_0^2 - i$ for $i < m'$. From the fact that D_i does not pass through x_i for $i \geq m'$ it follows that $D_i^2 = D_0^2 - m'$ for those i. Since $Z \cap D \subseteq Z$ we see that $m' = \deg(Z \cap D) \leq \deg Z = m$ and from the first result in Lemma 2.7.8 we conclude

$$\tilde{D}^2 = D_n^2 = D_0^2 - m' = D^2 - \deg(D \cap Z). \qquad \square$$

While proving this last lemma we have also proved the following statement.

COROLLARY 2.7.10. *Let k be a field, m a positive integer, and C a curve on a surface S over k. Let $x \in C(k)$ be a point which is smooth as a point of C and of S. Define Z to be the curvilinear scheme of S corresponding to the ideal $\mathcal{I}_{C,x,m}$. Consider $X' = \mathrm{Bl}_Z S$ and let $X \to X'$ be its minimal desingularization. Now write $E_1 + E_2 + \ldots + E_m$ for the unique positive-dimensional fibre over $\pi\colon X \to S$ as in Proposition 2.4.4. Let $D \subseteq S$ be a curve which passes through x and is smooth at x. The strict transform $\tilde{D} \subseteq X$ of D along π passes through exactly one E_i namely the one with $i = \deg(D \cap Z)$.*

We will now prove that the minimal desingularization of a peculiar del Pezzo surface is a generalized del Pezzo surface.

Proof of Proposition 2.7.6. Fix a zero-dimensional subscheme $Z \subseteq \mathbb{P}^2_k$ in almost general position such that X' is isomorphic to $\mathrm{Bl}_Z \mathbb{P}^2_k$ and let $\pi'\colon X' \to \mathbb{P}^2_k$ be the composition of this isomorphism with the blowup morphism. Now let K be a finite Galois extension of k such that the geometric components of Z are defined over K. We can apply Lemma 2.7.7 to each component of Z and find a smooth surface over K, which descends to a smooth surface X over k with a map $\gamma'\colon X \to X'$. Proposition 2.6.5 shows that γ' is the minimal desingularization of the peculiar del Pezzo surface X' since it is so locally.

Now fix an algebraic closure $\bar{k}$ of k and let $\bar{X}$ be the base change of X to $\bar{k}$. We will prove that there are no integral curves on $\bar{X}$ with self-intersection smaller than -2. Suppose that there is an integral curve $\tilde{D}$ on $\bar{X}$ with self-intersection less than -2. Fix a decomposition $\bar{X} = X_r \to X_{r-1} \to \ldots X_1 \to X_0 = \mathbb{P}^2_{\bar{k}}$ into blowups in $\bar{k}$-points $x_i \in X_i(\bar{k})$. Let n be the smallest positive integer such that there is an integral curve $D' \subseteq X_n$ passing through x_n, such that $D'^2 = -2$. Let $D = \pi_* D'$ be the pushforward of D' to the projective plane. Now $\tilde{D}$ is the strict transform of D along $X \to \mathbb{P}^2_{\bar{k}}$. This implies that $\tilde{D}^2 < D'^2 = -2$. By minimality of n we have that X_n is a generalized del Pezzo surface over $\bar{k}$ of degree $d \geq 3$ and hence D is either a line or a conic on $\mathbb{P}^2_{\bar{k}}$ by Lemma 2.4.3. Because D is integral it is smooth and we can apply Lemma 2.7.9.

If D is a line we see that

$$\deg(D \cap Z) = D^2 - \tilde{D}^2 > 1 - (-2) = 3$$

which contradicts Z lying in almost general position. If D is an integral conic we have

$$\deg(D \cap Z) = D^2 - \tilde{D}^2 > 4 - (-2) = 6,$$

which contradicts $\deg Z \leq 6$.

This implies that $\bar{X}$ and hence X is a generalized del Pezzo surface and its degree is also $9 - r = d$. □

Note that for every peculiar del Pezzo surface X' there is a birational morphism from X' to the projective plane, and by composition we find a birational morphism $X \to \mathbb{P}^2_k$ for its associated generalized del Pezzo surface. The next proposition shows that associating a generalized del Pezzo surface to a peculiar del Pezzo surface in this manner defines a bijection.

PROPOSITION 2.7.11. *Let $\pi\colon X \to \mathbb{P}^2_k$ be a birational morphism from a generalized del Pezzo surface X of degree $d \geq 3$ to the projective plane. The natural map*

$$\pi^* \colon \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}(1)) \to \mathrm{H}^0(X, \pi^*\mathcal{O}(1))$$

is an isomorphism of k-vector spaces, which identifies the complete linear systems of $\mathcal{O}(1)$ and $\mathcal{L} = \pi^*\mathcal{O}(1)$. The morphism π is associated to the complete linear system of $\mathcal{L}$. The line bundle $\mathcal{L} \otimes \omega_X^\vee$ is big and nef and the image X' of the complete linear system of a sufficiently large multiple of $\mathcal{L} \otimes \omega_X^\vee$ is a peculiar del Pezzo surface.

One can show that the peculiar del Pezzo surface is the image of the complete linear system associated to the line bundle $\mathcal{L} \otimes \omega_X^\vee$ itself. For del Pezzo surfaces of degree 1 and 2 with a birational morphism $\pi\colon X \to \mathbb{P}^2_k$ the contraction of all -2-curves in the support of the exceptional divisor E_π of π is given by the powers $(\mathcal{L} \otimes \omega_X^\vee)^3$ and $(\mathcal{L} \otimes \omega_X^\vee)^2$ respectively.

Proof of Proposition 2.7.11. The isomorphism

$$\pi^* \colon \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}(1)) \to \mathrm{H}^0(X, \pi^*\mathcal{O}(1))$$

follows from the isomorphism $\mathcal{O}(1) \to \pi_*\pi^*\mathcal{O}(1)$ in Lemma 2.5.1. This also proves that π is associated to the complete linear system of $\mathcal{L}$.

In particular we see that the complete linear system of $\mathcal{L}$ is non-empty. Let Λ be an effective divisor in the complete linear system of $\mathcal{L}$ and let $-K_X$ be an effective anticanonical divisor on X. It follows from Definition 2.1.3 that $\Lambda - K_X$ is nef, since Λ and $-K_X$ are both nef. By Proposition 2.1.4 we only need to compute

$$(\Lambda - K_X)^2 = \Lambda^2 - 2K_X \cdot \Lambda + K_X^2 = 1 + 2 \cdot 3 + d = d + 7 > 0$$

to conclude that $\Lambda - K_X$ is also big. Here we have used that Λ is the pullback of some line $L \subseteq \mathbb{P}^2_k$ and by the projection formula we find

$$K_X \cdot \Lambda = K_X \cdot \pi^* L = \pi_* K_X \cdot L = K_{\mathbb{P}^2_k} \cdot L = -3.$$

Let $\gamma' \colon X \to X'$ be the birational proper morphism associated to a sufficiently large multiple of $\Lambda - K_X$. The integral curves $C \subseteq X$ which are contracted are those for which $C(\Lambda - K_X) = 0$. We see by Proposition 2.1.4 that $C \cdot \Lambda \geq 0$ and $C \cdot -K_X \geq 0$ and it follows that the curves contracted by γ' are the curves which are contracted by both π and the anticanonical map. These are precisely the -2-curves in the support of E_π.

Now let X' be the image of the anticanonical map on X. We will need to prove that X' is a peculiar del Pezzo surface. We will first show that X' is the blowup of the projective plane in a curvilinear subscheme Z. Note that the statement is purely geometric so we may assume that k is algebraically closed.

By Lemma 2.7.1 there exists a cubic curve $C \subseteq \mathbb{P}^2_k$ such that its strict transform $C_i \subseteq X_i$ is smooth at p_i for all i. Note that in particular C must be reduced as it is irreducible, locally principal and smooth in at least one point.

Let us write the birational morphism $X \to \mathbb{P}^2_k$ as the composition $X = X_r \to X_{r-1} \to \ldots \to X_0 = \mathbb{P}^2_k$ of blowups in closed points $p_i \in X_i(k)$. Consider the multiset I consisting of the images of the points p_i in $\mathbb{P}^2_k$. This means that I is a set of smooth points q_α on C with some multiplicities n_α. Now let Z_α be the zero-dimensional subscheme of C defined by $\mathcal{I}_{C,q_\alpha,n_\alpha}$ and set $Z = \cup Z_\alpha$. It follows from Lemma 2.7.7 that X' is the blowup of $\mathbb{P}^2_k$ in Z.

We will prove that Z lies in almost general position. For a line L on the projective plane we define $\tilde{L}$ to be the strict transform. Since L is smooth we may apply Lemma 2.7.9 and we find

$$\deg(Z \cap L) = L^2 - \tilde{L}^2 \leq 1 - (-2) = 3$$

so Z lies in almost general position. □

The zero-dimensional scheme Z constructed in the proof will be called *the zero-dimensional scheme $Z \subseteq \mathbb{P}^2_k$ associated to the morphism $X \to \mathbb{P}^2_k$*. The scheme Z does not only depend on X, but also on the morphism to the projective plane. When it is clear from the context how X is given as the blowup of the projective plane in r points in almost general position we will say Z is the associated subscheme of X.

In many cases we will also need the curve C which is smooth in the support of Z. Usually the existence is enough, but occasionally we will assume that C is a cubic curve, which is allowed as one can see in the proof above.

Let us put together the main results on peculiar del Pezzo surfaces we have proved thus far.

COROLLARY 2.7.12. *Fix an integer $1 \leq d \leq 9$ and let k be a field. There is a bijection from the set of isomorphism classes of peculiar del Pezzo surfaces to the set of isomorphism classes of generalized del Pezzo surfaces of degree d with a birational morphism to $\mathbb{P}^2_k$ defined by mapping a peculiar del Pezzo surface to its minimal desingularization. The inverse is given by sending a generalized del Pezzo surface with a birational morphism $\pi\colon X \to \mathbb{P}^2_k$ to the peculiar del Pezzo surface obtained by contracting all -2-curves on X which are contracted by π.*

Proof. This follows from Proposition 2.7.6 and Proposition 2.7.11. □

We also have the following result.

COROLLARY 2.7.13. *A peculiar del Pezzo surface $\mathrm{Bl}_Z \mathbb{P}^2_k$ over a field k is a nonsingular surface precisely if Z is reduced.*

Proof. Let Z be a curvilinear subscheme of the projective plane. We can compute the blowup $\mathrm{Bl}_Z \mathbb{P}^2_k$ using Lemma 2.7.7 for each geometrically irreducible component of Z. It follows that it is smooth precisely if all geometrically irreducible components of Z are of degree 1. This is equivalent to Z being reduced. □

As for most statements about generalized del Pezzo surfaces, we are interested in the application to ordinary del Pezzo surfaces.

COROLLARY 2.7.14. *A peculiar del Pezzo surface $\mathrm{Bl}_Z \mathbb{P}^2_k$ over a field k is an ordinary del Pezzo surface precisely if Z lies in general position.*

Proof. It is clear that Z lies in (almost) general position precisely if $\bar{Z}$ lies in (almost) general position. This together with Proposition 2.2.4 implies that we can assume that k is algebraically closed.

Write $X' = \mathrm{Bl}_Z \mathbb{P}^2_k$ and let $X \to X'$ be the minimal desingularization of X'.

Suppose that X' is an ordinary del Pezzo surface. This implies that there are no -2-curves on X. This shows that Z is reduced, otherwise there are -2-curves in the fibres contracted by $X \to X'$ by Lemma 2.7.7. We saw in Lemma 2.3.4 that a -2-curve on X is the strict transform of a line or a conic along $X \to \mathbb{P}^2_k$. Lemma 2.7.9 now shows that there are no curves of self-intersection less than -1 precisely if Z lies in general position.

If follows similarly as discussed in the introduction that if Z lies in general position, then $X' = X$ is an ordinary del Pezzo surface. □

Let $X \to \mathbb{P}^2_k$ be a birational morphism from a generalized del Pezzo surface to the projective plane. Let X' and Y be the corresponding peculiar and singular del Pezzo surfaces. Then there exist birational morphisms $X \xrightarrow{\gamma'} X' \xrightarrow{\gamma''} Y$ such that the composition γ contracts all -2-curves on X. We have also seen that γ' contracts the -2-curves in E_π^{pec} and hence γ'' contracts precisely the remaining -2-curves.

We have also seen in Corollary 2.7.14 that if Z lies in general position then $\mathrm{Bl}_Z \mathbb{P}^2_k$ is an ordinary del Pezzo surface. This implies that γ, γ' and γ'' are isomorphisms. Corollary 2.7.13 shows that γ' is an isomorphism precisely if Z is reduced. Note that Z being a reduced is one of the conditions in the definition for Z to lie in general position. One could wonder what the relation is between the second condition and γ''. It is what one would expect: γ'' is an isomorphism precisely if $\deg(Z \cap L) \leq 2$ for lines on $\mathbb{P}^2_k$ and $\deg(Z \cap C) \leq 5$ for conics. We will not need this precise result, but the following corollary will be useful.

PROPOSITION 2.7.15. *Let $Z \subseteq \mathbb{P}^2_k$ be a zero-dimensional subscheme of degree* $\deg Z = r \leq 5$ *in almost general position. If Z lies on a conic, then the peculiar del Pezzo surface $X' = \mathrm{Bl}_Z \mathbb{P}^2_k$ is a singular del Pezzo surface.*

Proof. Let $X \to X'$ be the minimal desingularization of X'. This implies that X is a generalized del Pezzo surface. The morphism $X \to X'$ contracts precisely the -2-curves in the fibres of $\pi\colon X \to \mathbb{P}^2_k$. We will show that there are no other -2-curves. Suppose that $\tilde{D} \subseteq X$ is a -2-curve. The pushforward $D = \pi_*\tilde{D}$ is a line or a smooth conic on the projective plane. In either case we have $\deg(D \cap Z) \leq \deg(D \cap C) = 2 \deg D$. We now use Lemma 2.7.9 to find

$$-2 = \tilde{D}^2 = D^2 - \deg(D \cap Z) \geq \deg D(\deg D - 2) \geq (\deg D - 1)^2 - 1$$

which is a contradiction for any degree of D. □

An advantage of introducing peculiar del Pezzo surfaces is the following generalization of a well-known fact of ordinary del Pezzo surfaces to generalized ones. It allows us to identify the linear system V_X of cubics in Proposition 2.5.4 in terms of Z.

PROPOSITION 2.7.16. *Let X be a generalized del Pezzo surface over a field k given as a blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in r points in almost general position. Let $Z \subseteq \mathbb{P}^2_k$ be the associated zero-dimensional subscheme such that $X' = \mathrm{Bl}_Z \mathbb{P}^2_k$ is the associated peculiar del Pezzo surface of $X \to \mathbb{P}^2_k$.*

The linear system of the cubic curves passing through Z is the linear system $|V_X|_{\mathbb{P}^2_k}$ described in Proposition 2.5.4.

We will use the following result.

LEMMA 2.7.17. *Let Z be a curvilinear scheme of degree r supported in a smooth k-point x on a surface S. Let $\pi\colon X \to X' \to S$ be the composition of the minimal desingularization X of $X' = \mathrm{Bl}_Z S$ and the blowup morphism $X' \to S$. Let E_i for $1 \leq i \leq r$ be the components of the exceptional divisor E_π of π as described in Proposition 2.4.4. In particular, $E_i^2 = -2$ for $1 \leq i < r$ and $E_r^2 = -1$.*

Let C be an effective Cartier divisor on S and let π^*C be its pullback to X. The following statements are equivalent.

(i) *The zero-dimensional scheme Z lies on C.*

(ii) *The multiplicity of π^*C on E_r is at least r.*

(iii) *The multiplicity of π^*C on E_i is at least i for all $1 \leq i \leq r$.*

Proof. We work with the completed local ring $\widehat{\mathcal{O}}_x \cong k[[u,v]]$ where the isomorphism is chosen such that Z is defined by the ideal (u^r, v). Define $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_x$ and $\widehat{C} = C \times_S \widehat{S}$. Since localization is flat we can compute the multiplicities of π^*C along E_i by pulling the everything back to $\widehat{X} = X \times_S \widehat{S}$. The scheme $\widehat{X}$ contains affine opens $U_i = \operatorname{Spec} k[[u, V_i]]$ such that the map $U_i \to \widehat{X} \to \widehat{S}$ induced by π is defined by $u \mapsto u$ and $V_i \mapsto u^i v$. The open U_i contains the generic point of E_i and E_i is defined by the vanishing of u on this open.

Now let $\widehat{C}$ be defined by the vanishing of $f \in k[[u,v]]$. The lemma is equivalent to the following immediate statement. Let $0 \leq i \leq r$ be an integer. We have that f lies in the ideal (u^i, v) precisely if the multiplicity of u in $f(u, u^i v)$ is at least i. □

Proof of Proposition 2.7.16. We will prove that the linear subsystems of cubics passing through Z and the linear system $|V_X|_{\mathbb{P}^2_k}$ described in Proposition 2.5.4 are the same. We will do so by showing that a cubic curve $C \subseteq \mathbb{P}^2_k$ passes through Z precisely if $E_\pi^{\mathrm{pec}} \leq \pi^*C$. This follows from applying Lemma 2.7.17 to each point in the support of Z. □

We conclude the following generalization of Corollary V.4.4 in [33].

COROLLARY 2.7.18. *Let $Z \subseteq \mathbb{P}^2_k$ be a zero-dimensional subscheme in almost general position of degree* $\deg Z \leq 6$. *The linear subsystem of cubics through Z is of dimension* $9 - \deg Z$.

Consider the situation where the curvilinear subscheme $Z \subseteq S$ of degree r is supported in a single point x. Let X' be the blowup $\operatorname{Bl}_Z S$, $X \to X'$ the minimal desingularization and $\pi\colon X \to X' \to S$ the composition of these two morphisms. Recall from Proposition 2.4.4 that the fibre of π above x is the union of -2-curves and a single -1-curve E_r. In the above proof we have identified the effective Cartier divisors $C \subseteq S$ which satisfy $\pi^*C \geq E_\pi^{\mathrm{pec}}$ as the locally principal curves passing through Z. This means that $\pi^*C - E_\pi^{\mathrm{pec}}$ is an effective divisor. We would like to be able to describe when this effective divisor is supported on the -1-curve E_r on E_π^{pec}.

LEMMA 2.7.19. *An effective Cartier divisor $C \subseteq S$ satisfies $\pi^*C \geq E_\pi^{\mathrm{pec}} + E_r$ if and only if $\mathcal{I}_C \subseteq \mathcal{I}_x \mathcal{I}_{C,x,r}$.*

Proof. We will use the notation in the proof of Lemma 2.7.17. We will need to show that the multiplicity of π^*C along E_r is at least $r+1$ precisely if $\mathcal{I}_C$ is a ideal subsheaf of $\mathcal{I}_x \mathcal{I}_{C,x,r}$. The multiplicity of π^*C along E_r is the multiplicity of u in $f(u, u^r V_r)$. This is at least $r+1$ if and only if $f \in (u^{r+1}, uv, v) = (u,v)(u^r, v)$. The ideals (u,v) and (u^r, v) define the respective ideals $\mathcal{I}_x$ and $\mathcal{I}_{C,x,r}$ on S. □

In this section we have defined peculiar del Pezzo surfaces associated to zero-dimensional subschemes in almost general position. We have also seen how to construct the associated generalized del Pezzo surface from just this subscheme. The last lemma of this section tells us how to determine the associated singular del Pezzo surface.

LEMMA 2.7.20. *Let $Z \subseteq \mathbb{P}^2_k$ be a zero-dimensional subscheme in almost general position of degree $r \leq 6$.*

The closure of the image Y of the birational map $\mathbb{P}^2_k \dashrightarrow Y \subseteq \mathbb{P}^d_k$ defined by the linear subsystem of cubics passing through Z is the singular del Pezzo surface Y associated to Z.

Proof. Let $\pi\colon X \to \mathbb{P}^2_k$ be the generalized del Pezzo surface associated to Z. Since the degree of X is $9 - r \geq 3$ the image of the anticanonical morphism $\gamma\colon X \to \mathbb{P}^d_k$ is the associated singular del Pezzo surface Y.

Note that π restricts to an isomorphism $\pi^{-1}\colon \mathbb{P}^2_k \backslash Z \xrightarrow{\cong} X \backslash E_\pi$. The pullback of π^{-1} identifies the divisors in V_X restricted to $\mathbb{P}^2_k \backslash Z$ with the divisors in $\mathrm{H}^0(X, \omega^\vee)$ restricted to $X \backslash E_\pi$. This proves that the birational map $\gamma \circ \pi^{-1}$ is defined by the cubics in V_X, i.e. the cubics which pass through Z. To conclude the proof we note that the closure of the scheme-theoretic image of this map is the same as the scheme-theoretic image of the map γ. $\square$

2.8 Arithmetic of del Pezzo surfaces

In Chapters 3 and 4 we will study arithmetic properties of surfaces which are closely related to del Pezzo surfaces. We will use some results on the arithmetic of ordinary del Pezzo surfaces over number fields which we will review in this section. The aim is not to give a complete overview of the topic. The focus lies on the results we will need later on. A more complete overview of the arithmetic of del Pezzo surfaces can be found in [55] which is the main source for this section. For similar results on singular del Pezzo surfaces one is referred to [7] and [19].

THEOREM 2.8.1. *Let X be an ordinary del Pezzo surface of degree $d \geq 5$ over a field k. If $X(k)$ is non-empty then X is birational over k to $\mathbb{P}^2_k$.*

Now let k be a number field. Then X satisfies both weak approximation and the Hasse principle.

Proof. The proof is different for each degree. A very clear and detailed exposition can be found in [55, Lecture 2]. $\square$

The proofs for the cases $d = 5$ and $d = 7$ also yield the following important result.

PROPOSITION 2.8.2. *Let X be an ordinary del Pezzo surface of degree 5 or 7 over a field k. Then X has a k-rational point and hence X is birational over k to $\mathbb{P}^2_k$.*

Proof. See Lecture 2 in [55] for the proof of the existence of the rational points. The last statement now follows from Theorem 2.8.1. □

Note that this proposition also shows that ordinary del Pezzo surfaces of degree 5 and 7 satisfy the Hasse principle trivially. This is not the case for ordinary del Pezzo surfaces of degrees 6, 8 and 9.

Example. Let k be a number field and let $C \subseteq \mathbb{P}^2_k$ be a conic such that $C(k) = \emptyset$. Since $\bar{C}$ is isomorphic to $\mathbb{P}^1_{\bar{k}}$ we see that $X = C \times C$ is an ordinary del Pezzo surface of degree 8 over k, such that $X(k) = \emptyset$.

Also, any ordinary del Pezzo surface X degree 9 is a Brauer–Severi variety, i.e. $\bar{X} \cong \mathbb{P}^2_{\bar{k}}$. By a result of Châtelet we see that X is isomorphic to $\mathbb{P}^2_k$ precisely when $X(k) \neq \emptyset$, see for example [46, Proposition 4.5.10]. The existence of a surface X over k with these properties follows from the correspondence between isomorphism classes of Severi–Brauer varieties of dimension 2 and central simple algebras over k of dimension 3^2, see [46, Section 4.5.1].

For ordinary del Pezzo surfaces of low degree it is known that weak approximation or the Hasse principle need not hold, except for the following case.

PROPOSITION 2.8.3. *Let X be a generalized del Pezzo surface of degree* 1 *over a number field k. Then $X(k) \neq \emptyset$ and X satisfies the Hasse principle trivially.*

Proof. We see from [23, Proposition III.2] that the anticanonical map on a generalized del Pezzo surface of degree 1 has a unique base point. This base point is hence defined over k which proves that $X(k) \neq \emptyset$. □

For the degrees $2 \leq d \leq 4$ surfaces are known for which the Hasse principle does not hold, and we also have counterexamples for weak approximation on ordinary del Pezzo surfaces of degree $1 \leq d \leq 4$. These counterexamples are clearly presented in Table 3 of [55].

One might wonder if the failure of the Hasse principle or weak approximation can be explained by a Brauer–Manin obstruction. Let us to that end compute the Brauer groups of del Pezzo surfaces over a number field. We first have the following general result, which shows that the transcendental part of the Brauer group of a generalized del Pezzo surface over a field of characteristic 0 is trivial.

PROPOSITION 2.8.4. *Let X be a rational projective smooth variety over an algebraically closed field k of characteristic* 0. *The Brauer group* $\operatorname{Br} X$ *is trivial.*

Proof. Since X is rational it is by definition birational to $\mathbb{P}^n_k$ where n is the dimension of X. This shows that X is irreducible and we can apply Proposition 1.4.13. This shows that $\operatorname{Br} \mathbb{P}^2_k$ and $\operatorname{Br} X$ are isomorphic. By Proposition 1.4.12 and Proposition 1.2.14 we see that $\operatorname{Br} X$ must be trivial. □

COROLLARY 2.8.5. *Let X be a generalized del Pezzo surface over a field k of characteristic zero. We have* $\operatorname{Br} X = \operatorname{Br}_1 X$.

Degree	Possibilities for $\operatorname{Br} X/\operatorname{Br} k$
any $\mathbf{d}$	1
$\mathbf{d} \leq \mathbf{4}$	$2 \quad 2^2$
$\mathbf{d} \leq \mathbf{3}$	$3 \quad 3^2$
$\mathbf{d} \leq \mathbf{2}$	$2^3 \quad 2^4 \quad 2^5 \quad 2^6 \quad 2 \cdot 4 \quad 2^2 \cdot 4 \quad 4 \quad 4^2$
$\mathbf{d} = \mathbf{1}$	$2^7 \quad 2^8 \quad 3^3 \quad 3^4 \quad 2 \cdot 4^2 \quad 2^2 \cdot 4^2 \quad 2^3 \cdot 4$
	$2^4 \cdot 4 \quad 5 \quad 5^2 \quad 2 \cdot 6 \quad 3 \quad 6 \quad 6^2$

Table B: Possible group structures of $\operatorname{Br} X/\operatorname{Br} k$. For example, $2^2 \cdot 4$ means $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$.

Proof. By Proposition 2.8.4 we see that $\operatorname{Br} \bar{X}$ is trivial. This implies that

$$\operatorname{Br}_1 X = \ker(\operatorname{Br} X \to \operatorname{Br} \bar{X}) = \operatorname{Br} X. \qquad \square$$

We now use Proposition 1.6.5 to compute the algebraic Brauer group modulo constants of a del Pezzo surface.

PROPOSITION 2.8.6. *Let X be a generalized del Pezzo surface over a number field k. The Brauer group modulo constants* $\operatorname{Br} X/\operatorname{Br} k$ *is one of the groups in Table B.*

Proof. This result was proved by Manin for $d \geq 5$ [41], Swinnerton-Dyer for $d = 4$ and 3 [52], and $d = 2$ and 1 by Corn in [20, Theorem 1.4.1] for ordinary del Pezzo surfaces. The result also holds since the geometric Picard group of a generalized del Pezzo surface of degree $d \geq 7$ depends only on d.

During a calculation done for Proposition 3.2.1 the possible Brauer groups modulo constants were recomputed by a MAGMA program available at [39]. For completeness we will explain the code.

Since G_k acts on $\operatorname{Pic} \bar{X}$ it factors through the Weyl group W_{9-d}. Let $W \subseteq W_{9-d}$ be the minimal subgroup of W_{9-d} through which this action factors. By the inflation–restriction sequence we find that the inflation morphism

$$\mathrm{H}^1(W, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(G_k, \operatorname{Pic} \bar{X})$$

is an isomorphism, since the kernel of $G_k \to W$ acts trivially on the geometric Picard group. One now enumerates the subgroups W of W_{9-d} and compute the first cohomology group of the action of W on $\operatorname{Pic} \bar{X}$.

The computation is simplified by a general result in group cohomology: the cohomology group $\mathrm{H}^1(W, \operatorname{Pic} \bar{X})$ only depends on the conjugacy class of W in W_{9-d} by [58, Example 6.7.7].

The possible groups we find are precisely the groups in Table B. $\square$

One sees that it is only possible to have a Brauer–Manin obstruction to the Hasse principle or weak approximation on a del Pezzo surface if $d \leq 4$. Of course, for $d \geq 5$ we would not expect this since we know from Theorem 2.8.1 that ordinary del Pezzo surfaces of large degree satisfy both the Hasse principle and weak approximation.

All the known examples of failure of either of these local–global principles are explained by a Brauer–Manin obstruction and this led Colliot-Thélène and Sansuc [14, page 174] to ask the following question.

QUESTION 2.8.7. Let X be an ordinary del Pezzo surface of degree $d \leq 4$ over a number field k. If $X(k) = \emptyset$ does this imply $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$? Also, if the subset $X(k) \subseteq X(\mathbb{A}_k)$ is not dense, does this imply $X(\mathbb{A}_k)^{\mathrm{Br}} \subsetneq X(\mathbb{A}_k)$?

In other words, is the Brauer–Manin obstruction the only obstruction to weak approximation and the Hasse principle on ordinary del Pezzo surfaces?