Arithmetic of affine del Pezzo surfaces

Lyczak, J.T.

Author: Lyczak, J.T.
Title: Arithmetic of affine del Pezzo surfaces
Issue Date: 2019-10-01

Chapter 1

Brauer groups and the Brauer–Manin obstruction

In this chapter we will review important concepts in the theory of rational and integral points on schemes over number fields. We will start by describing several types of adelic rings associated to number fields. These rings are useful for studying all localizations k_v of a number field k simultaneously. In a similar manner one can study the k_v-points on a scheme X for all completions k_v of the number field k at the same time consider the so-called *adelic points* on a scheme X. This set will help us to study the set $X(k)$. In particular, we find a natural inclusion of the set of k-points on X in the adelic points.

In the cases where the set of adelic points is too big to yield information about the set of k-points one can turn to the *Brauer–Manin obstruction*. This technique uses the *Brauer group* of a scheme to identify a subset of the adelic points which contains $X(k)$. We will study the Brauer groups over schemes by first defining the Brauer group of a field in terms of central simple algebras. The definition of these algebras generalizes to sheaves of algebras on schemes. We will show that the Brauer group of a scheme thus constructed coincides with the second étale cohomology group of the scheme with values in $\mathbb{G}_{\mathrm{m}}$, assuming some conditions which will not be too restrictive for our purposes.

We proceed by defining the so-called *residue maps* and stating some of their relevant properties. These group homomorphism are an important tool in the computation and understanding of Brauer groups of local rings and also Brauer groups of schemes. These homomorphisms allow us to define the Brauer–Manin obstruction discussed above.

In the last section of this chapter we review some technical results on Brauer groups and residue maps which we will need throughout the thesis.

Very few results in this chapter are true for all schemes. The main goal will be to study schemes defined by polynomial equations with coefficients in a field. Such schemes will be called varieties. There are however a variety of definitions in use for this type of scheme. To avoid ambiguity we will fix the definition here for the remainder of the thesis.

DEFINITION 1.0.1. Let k be a field. A *variety over* k is a scheme over k which is separated and of finite type over k.

1.1 Adelic points, weak and strong approximation

Let k be a number field and denote the set of all non-trivial places of k by Ω_k. This set splits into the non-archimedean places Ω_k^{fin} and the archimedean places Ω_k^{∞}. For a finite subset $S \subseteq \Omega_k$ we write $\mathcal{O}_{k,S}$ or simply $\mathcal{O}_S$ for the elements of k which are v-integral for all finite $v \notin S$. In the case that S contains no finite places we recover the ring of integers $\mathcal{O}_k$ of k.

For a non-archimedean place v of k we write k_v for the completion of k at v and $\mathcal{O}_v$ for the v-integral elements in k_v. We will endow both k_v and $\mathcal{O}_v$ with the v-adic topology, which gives them the structure of topological rings.

We will study the rational points of a scheme X over k by considering the sets of points $X(k_v)$ for all $v \in \Omega_k$. We would like to study these sets of points simultaneously, but this approach loses too much of the global structure of $X(k)$. This is reminiscent of the fact that the product of all k_v does not reflect the important property that an element of k is v-integral for all but finitely many places v. This can be remedied by considering the elements in $\prod_v k_v$ which are v-integral for almost all v. The following definition constructs this ring using the restricted product [12, Section 13], which also incorporates the topologies on the respective completions of k and $\mathcal{O}_k$.

DEFINITION 1.1.1. Let k be a number field and let $T \subseteq \Omega_k$ be a finite set of places of k. The *ring of T-adeles of k* is the ring defined as the restricted product

$$\mathbb{A}_k^T := \prod_{v \in \Omega_k \setminus T}' (k_v, \mathcal{O}_v)$$

with respect to the integral elements $\mathcal{O}_v \subseteq k_v$ for all finite places v. We will endow $\mathbb{A}_k^T$ with the restricted product topology, which makes it into a topological ring.

If T is empty we suppress it in the notation and the terminology and we find the *ring of adeles* $\mathbb{A}_k$ of k.

We will often identify $\mathbb{A}_k$ with the subset of elements in $\prod_{v \in \Omega_k} k_v$ which are integral at all but finitely many places. Note that this product and its subset $\mathbb{A}_k$ are topological spaces, and that the inclusion is continuous, but the topology on the ring of adeles will be finer than the subspace topology.

For some applications we will be interested in adeles of k which are integral outside of a given set of places.

DEFINITION 1.1.2. Let S be a finite set of places of a number field k such that the archimedean places Ω_k^{∞} are contained in S. We define the *integral adelic points* *of k away from S* as

$$\mathbb{A}_{k,S} := \prod_{v \in S} k_v \times \prod_{v \in \Omega_k \setminus S} \mathcal{O}_v.$$

For a finite set $T \subseteq \Omega_k$ we can repeat the above construction while ignoring any primes in T. This produces the set $\mathbb{A}_{k,S}^T$ of *integral T-adelic points of k away from S*.

The set $\mathbb{A}_{k,S}$ is a subset of both $\mathbb{A}_k$ and $\prod_v k_v$. If we endow $\mathbb{A}_{k,S}$ with the product topology one can show that it is even an open subset of both $\mathbb{A}_k$ and the product $\prod_v k_v$. This shows that the topology on $\mathbb{A}_{k,S}$ coincides with subspace topology coming from either inclusion $\mathbb{A}_{k,S} \subseteq \mathbb{A}_k$ and $\mathbb{A}_{k,S} \subseteq \prod_v k_v$. Note the contrast with the ring of adeles; the topology on $\mathbb{A}_k$ differs from the subspace topology coming from the inclusion $\mathbb{A}_k \subseteq \prod_v k_v$.

The ring of adeles was defined to study rational points of a scheme X over k since the collection of point sets $X(k_v)$ does potentially not contain enough of the global structure of $X(k)$. This can be done by looking at the set $X(\mathbb{A}_k)$ of adelic points on X, which remembers more of the global structure of $X(k)$ than the collection of the $X(k_v)$. One possible setback is that *a priori* the topology on $X(k_v)$ cannot be recovered from $X(\mathbb{A}_k)$. Let us recall how the topology on $X(k_v)$ is defined. The idea is that for an affine scheme X of finite type over a topological ring R one can identify $X(R)$ with a subset of the R-points on an affine space $\mathbb{A}_R^n$, which is in bijection to R^n. Now we proceed by endowing R^n with the product topology and $X(R)$ with the subspace topology.

For a general finite type scheme over a topological ring R we cover X by affine opens X_i. In this case we will require that R is local so as to ensure that $X(R) = \bigcup X_i(R)$. Then we have topologies on the sets $X_i(R)$ and these topologies generate a topology on $X(R)$ and this will be the topology we will consider. One can show that this topology does not depend on the choice of the affine covering and that a morphism of schemes $X \to Y$ over R induces a continuous map $X(R) \to Y(R)$. For more information and the statement that these topologies are in some sense natural, one can consult [18].

To summarize, if R is a topological ring one can define a natural topology on the set $X(R)$ if X is affine over R, or R is a local ring and X is a separated scheme of finite type over R. Note that these constructions do not allow us to topologize the set of adelic points $X(\mathbb{A}_k^T)$, unless X happens to be affine over $\mathbb{A}_k^T$. The following proposition shows however that the set of adelic points on a variety admits a natural bijection to a set which comes with a topology.

PROPOSITION 1.1.3. *Let T be a finite set of places of a number field k and consider the generic fibre $X = \mathcal{X}_k$ of a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{k,T}$. If $v \notin T$ then $\mathcal{X}(\mathcal{O}_v) \to X(k_v)$ is injective and the natural map*

$$X(\mathbb{A}_k^T) \to \prod_{v \in \Omega_k \setminus T}' X(k_v)$$

is well-defined and bijective. The product in the codomain is the restricted product of the indicated factors with respect to $\mathcal{X}(\mathcal{O}_v)$ for $v \in \Omega_k^{\mathrm{fin}} \setminus T$.

Proof. See Section 2.3.1 and Exercise 3.4 in [46]. □

Recall that a variety X over k spreads out over an open dense subset U of Spec $\mathcal{O}_k$, see for example [46, Theorem 3.2.1], i.e. there is a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{k,T}$ for some finite set T such that $\mathcal{X}_k$ is isomorphic to X. This proves that any k-variety X is the generic fibre of an $\mathcal{O}_{k,T}$-scheme $\mathcal{X}$ and we can use Proposition 1.1.3 to describe the set of adelic points on X.

Note that the set $X(\mathbb{A}_k^T)$ only depends on the generic fibre X of $\mathcal{X}$, while the restricted product in Proposition 1.1.3 depends *a priori* on the choice of model $\mathcal{X}$ of X. However since for any two models one can identify the two sets of $\mathcal{O}_v$-points for all but finitely many v it follows from the definition of the restricted product that

$$\prod_{v\in\Omega_k\setminus T}' X(k_v)$$

is independent of the choice of model. Similarly one proves that the restricted product topology on this product is also independent of the choice of model $\mathcal{X}/\mathcal{O}_{k,T}$. Using the bijection in Proposition 1.1.3 we have produced a topology on the set of adelic points $X(\mathbb{A}_k)$.

DEFINITION 1.1.4. The *adelic topology* on the set $X(\mathbb{A}_k^T)$ of T-adelic points on a variety X over a number field k is the unique topology on this set such that the bijection in Proposition 1.1.3 is a homeomorphism.

There is an important corollary in the case that X is proper, because then the natural map $\mathcal{X}(\mathcal{O}_v) \to X(k_v)$ is an isomorphism for all v and the restricted product becomes the standard product.

COROLLARY 1.1.5. *Let X be a proper variety over a number field k. The natural map*

$$X(\mathbb{A}_k^T) \to \prod_{v\in\Omega_k\setminus T} X(k_v)$$

is a homeomorphism.

For two finite sets of places $T' \subseteq T$ the projection map $\mathbb{A}_k^{T'} \to \mathbb{A}_k^T$ is continuous. In particular we have a map

$$\mathbb{A}_k \to \mathbb{A}_k^T$$

for each such T. Composing with the diagonal map $k \to \mathbb{A}_k$ we get maps $X(k) \to X(\mathbb{A}_k^T)$. Usually, the set of adelic points $X(\mathbb{A}_k^T)$ is easier to work with than the set of rational points $X(k)$ in which we are primarily interested. This motivates our interest in the image of the map $X(k) \to X(\mathbb{A}_k^T)$.

DEFINITION 1.1.6. Let X be a variety over a number field k and let T be a finite set of places of k. If $X(k)$ is dense in $X(\mathbb{A}_k^T)$ we say that X satisfies *strong approximation away from T* and X satisfies *weak approximation away from T* if $X(k)$ is dense in $\prod_{v\in\Omega_k\setminus T} X(k_v)$.

The names of these two concepts seem counterintuitive; being dense in the subset $X(\mathbb{A}_k^T)$ seems weaker than being dense in the larger set $\prod_{v\in\Omega_k\setminus T} X(k_v)$. The terminology stems from the facts that the topology of $X(\mathbb{A}_k^T)$ is not the subspace topology as a subset of

$$\prod_{v\in\Omega_k\setminus T} X(k_v).$$

Using the definition of the restricted product topology on $X(\mathbb{A}_k^T)$ one can show that if X satisfies strong approximation, it will also satisfy weak approximation.

Now let X be a projective variety over a number field k. On such a scheme the notions of weak and strong approximation coincide by Corollary 1.1.5. We have seen that we study the set of adelic points on X, using a model of X over $\mathcal{O}_k$. If X comes with a projective embedding $X \subseteq \mathbb{P}_k^N$ we can define a model as follows; we have that X is a closed subset of $\mathbb{P}_k^N$ which is in turn the generic fibre of $\mathbb{P}_{\mathcal{O}_k}^N$. We can now construct a model $\mathcal{X}/\mathcal{O}_k$ by taking the closure of X in the projective space $\mathbb{P}_{\mathcal{O}_k}^N$. As for any model of a proper scheme one can again identify the $\mathcal{O}_k$-points of $\mathcal{X}$ with the k-points on X.

For a scheme which is not projective, the set of integral points will depend on the model. Hence, the set of integral points on a model can differ from the set of rational points. For this reason we will consider the following setup.

PROPOSITION 1.1.7. *Let S be a finite set of places of k which contains the infinite places Ω_k^∞ of k. Now let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}_{k,S}$. The natural map*

$$\mathcal{X}(\mathbb{A}_{k,S}) \to \prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$$

is a bijection. Both sets $\mathcal{X}(\mathbb{A}_{k,S})$ and $X(\mathbb{A}_k)$ are naturally subsets of $\prod_{v\in\Omega_k} X(k_v)$ and under this identification we have the inclusion $\mathcal{X}(\mathbb{A}_{k,S}) \subseteq X(\mathbb{A}_k)$. So we have the following chain of inclusions

$$\mathcal{X}(\mathcal{O}_{k,S}) \subseteq \mathcal{X}(\mathbb{A}_{k,S}) \subseteq X(\mathbb{A}_k),$$

where we embed the $\mathcal{O}_{k,S}$-points on $\mathcal{X}$ diagonally into $\mathcal{X}(\mathbb{A}_{k,S})$.

Proof. See Theorem 3.6 in [18]. □

Now let k be a number field and let S be a finite set of places containing the archimedean places Ω_k^∞ of k. If a scheme X does not have a point over any completion k_v of k we see that $\mathcal{X}(\mathbb{A}_{k,S})$ is empty for any model $\mathcal{X}/\mathcal{O}_k$. In this case we also conclude that there are no $\mathcal{O}_{k,S}$-points on $\mathcal{X}$.

By Proposition 1.1.7 we can check whether $\mathcal{X}(\mathbb{A}_{k,S})$ is non-empty by checking the existence of solutions over local fields and rings. However, if $\mathcal{X}(\mathcal{O}_{k,S})$ is empty it can still be that $\mathcal{X}(\mathbb{A}_{k,S})$ is non-empty. This motivates the following definition.

DEFINITION 1.1.8. A separated and finite type $\mathcal{O}_{k,S}$-scheme $\mathcal{X}$ satisfies the *S-integral Hasse principle* if the existence of $\mathbb{A}_{k,S}$-points on $\mathcal{X}$ implies the existence of $\mathcal{O}_{k,S}$-points.

Similarly, a k-variety X is said to satisfy the *Hasse principle* if the existence of adelic points on X implies the existence of k-points on X.

A scheme satisfies the Hasse principle if it either admits a global solution or if it is not locally soluble at at least one place. Usually we will consider the Hasse principle for family of schemes, which means that for each scheme in the family one of the two mutually exclusive statements holds, but not necessarily the same statement is true for all schemes in the family under consideration.

We will later see how the Brauer group of a scheme can in some cases be used to prove that a variety X does not satisfy weak or strong approximation, or that $X(k)$ or $\mathcal{X}(\mathcal{O}_k)$ is empty.

1.2 Central simple algebras over fields

In this section we will define the Brauer group of a field. The main notion will be that of a central simple algebra. The content of this section and much more can be found in [28].

DEFINITION 1.2.1. A *central simple algebra* over a field k is a finite-dimensional k-algebra A which is simple as a ring and central as a k-algebra, i.e. A has precisely two two-sided ideals, and the centre of A is the image of the natural inclusion $k \hookrightarrow A$.

Here are some examples of central simple algebras.

Example. The following algebras are central and simple over the indicated base field:

» any field over itself.

» the quaternions $\mathbb{H}$ over $\mathbb{R}$, i.e. the four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, ij$, such that multiplication is given by $i^2 = j^2 = -1$ and $ij = -ji$.

Now let A be a central simple algebra over a field k. The following algebras are also central and simple:

» the opposite algebra A^{opp} over k.

» the matrix ring $\text{Mat}_n(A)$ over k for each positive n, in particular $\text{Mat}_n(k)$.

» the base change $A \otimes_k K$ over K for any finite field extension K/k.

The statement follows directly from the definition and a direct computation for the first three cases. For the fourth case one can use the fact that there is a correspondence between the ideals of A and $\mathrm{Mat}_n(A)$, and that the centre $Z(\mathrm{Mat}_n(A))$ consists of the diagonal matrices aI_n where $a \in Z(A)$ and I_n is the identity matrix in $\mathrm{Mat}_n(A)$. The proof that $A \otimes_k K$ is central and simple over K involves some manipulations of the tensor product. See for example [28, Lemma 2.2.2].

The following theorem is an important result if one wants to classify central simple algebras.

THEOREM 1.2.2 (Wedderburn's theorem). *Let A be a finite-dimensional simple k-algebra. There exists a unique integer $r > 0$ and a division k-algebra D such that*

$$\mathrm{Mat}_r(D) \cong A.$$

Proof. See [28, Theorem 2.1.3]. □

This theorem allows us to classify all central simple algebras over two important classes of fields.

COROLLARY 1.2.3. *A central simple algebra A over a finite field k is isomorphic to a matrix algebra $\mathrm{Mat}_r(k)$ for some positive integer r.*

Proof. Let D be the division ring such that $\mathrm{Mat}_r(D) \cong A$, hence $Z(D) = k$. Since k is a finite field and D is a finite-dimensional k-algebra we see that D has finitely many elements. A famous theorem by Wedderburn says that every finite division ring is a field showing in particular that D is commutative and we find $D = Z(D) = k$. □

COROLLARY 1.2.4. *Let k be an algebraically closed field. Any central simple algebra over k is a matrix algebra $\mathrm{Mat}_r(k)$.*

Proof. Let A be a central simple algebra over the field k and let D be the division ring such that $A \cong \mathrm{Mat}_r(D)$. We will prove that the natural morphism $k \hookrightarrow D$ is surjective. Pick an element $d \in D$. Since A is of finite dimension over k we see that there is relation over k between finitely many of the $1, d, d^2, \ldots$. This proves that $k(d)$ is a field extension of k of finite degree. Since k is algebraically closed we conclude $d \in k$. □

Corollary 1.2.4 implies the following result about the dimension of a central simple algebra over a general field.

COROLLARY 1.2.5. *The dimension of a central simple algebra over a field is always a square.*

Proof. Let A be a central and simple algebra over a field k. Since $\dim_k A = \dim_{\bar{k}} \left(A \otimes_k \bar{k}\right)$ we can assume that k is algebraically closed. By Corollary 1.2.4 we see that there is a positive integer r and an isomorphism $\mathrm{Mat}_r(k) \cong A$. The statement now follows from the fact $\dim_k \mathrm{Mat}_r(k) = r^2$. □

We have already seen that the finite-dimensional matrix algebras over a field are always central and simple. Hence central simple algebras over finite fields and algebraically closed fields are as simple as they possibly can be. Although the matrix algebras are in some sense the trivial central simple algebras, they play a pivotal role in the study of all central simple algebras. We have for example seen that after base changing to an appropriate field extension every central simple algebra becomes isomorphic to a matrix algebra. One can study a central algebra by considering these fields. This motivates the following definition.

DEFINITION 1.2.6. Let A be a central simple algebra over a field k. A *splitting field* for A is a field extension K/k, such that there exist a positive integer n and an isomorphism

$$A \otimes_k K \cong \mathrm{Mat}_n(K).$$

We have seen that any algebraic closure of k is a splitting field for any central simple algebra over k. The following theorem shows that each central simple algebra has a separable splitting field of finite degree and that any finite-dimensional algebra with this property is both central and simple.

THEOREM 1.2.7. *Let A be a finite-dimensional algebra over a field k. Then A is central and simple precisely if there exists a separable field extension K/k of degree n such that*

$$A \otimes_k K \cong \mathrm{Mat}_n(K).$$

Proof. This is a combination of Theorem 2.2.1 and Theorem 2.2.7 in [28]. □

The following result follows immediately from this theorem.

COROLLARY 1.2.8. *Let A and B be two central simple algebras over a field k. The tensor product $A \otimes_k B$ is a central simple algebra over k.*

Proof. By Theorem 1.2.7 we see that both A and B have a separable splitting field of finite degree over k. This implies that the compositum K of these two fields is also separable and splits both central simple algebras. We then see that

$$(A \otimes_k B) \otimes_k K \cong (A \otimes_k K) \otimes_K (B \otimes_k K) \cong \mathrm{Mat}_m(K) \otimes \mathrm{Mat}_n(K) \cong \mathrm{Mat}_{mn}(K)$$

and we conclude from Theorem 1.2.7 that $A \otimes_k B$ is a central simple algebra over the field k. □

A splitting field of a central simple algebra can be taken to be a splitting field of the associated division ring D from Theorem 1.2.2. The following result tells us where to look for a finite separable splitting field for division rings.

PROPOSITION 1.2.9. *Let A be central simple algebra over k. The algebra A is split by any subfield K of A, i.e. a subalgebra of A which is also a field, satisfying*

$$[K\colon k]^2 = \dim_k A.$$

A central division k-algebra D contains a separable element x of degree $\sqrt{\dim_k D}$, and hence $k(x)$ is a splitting field for all central simple algebras $\mathrm{Mat}_r(D)$ over k.

Proof. The two statements are precisely Proposition 2.2.9 and Proposition 2.2.10 in [28]. □

Using Corollary 1.2.8 we can prove the following result, which is very useful for generalizing central simple algebras to rings and even schemes.

THEOREM 1.2.10. *Let A be a finite-dimensional algebra over a field k. The morphism*

$$\varphi\colon A \otimes_k A^{\mathrm{opp}} \to \mathrm{End}_k(A), \quad \sum_i a_i \otimes b_i \mapsto \left(x \mapsto \sum_i a_i x b_i \right)$$

of vector spaces over k is an isomorphism precisely when A is a central simple algebra over k.

Proof. Suppose that A is a central simple algebra. Then by Corollary 1.2.8 we see that so is $A \otimes_k A^{\mathrm{opp}}$. Because φ is not identically zero, we conclude that the kernel of φ is trivial. By comparing the dimensions we see that φ must be bijective and hence an isomorphism of k-algebras.

Now suppose that φ is an isomorphism of algebras over k. If A is not central, then pick an $a \in Z(A) \backslash k$ and note that $a \otimes 1 - 1 \otimes a$ is non-zero and lies in the kernel of φ. This contradicts the existence of such an a and we conclude that A is central. Now let I be a two-sided ideal of A. Considering I as a subspace of A we find a subspace J of A such that $I \oplus J \cong A$. The morphism

$$\varphi\colon (I \oplus J) \otimes (I \oplus J) \to \mathrm{End}_k(A)$$

restricts to an injective linear map

$$(I \otimes I) \oplus (I \otimes J) \oplus (J \otimes I) \to \mathrm{Hom}_k(A, I).$$

Now let n, i and j be the respective dimensions of A, I and J over k, then we find that $i^2 + ij + ji \leq ni$. So either $i = 0$ or $i + j + j \leq n$ and we see that either $i = 0$ or $j = 0$. These cases correspond to I being either the zero ideal or the whole of A. □

Now we can define an important invariant for fields.

DEFINITION 1.2.11. Let k be a field and A and B two central simple algebras over k. We say that A and B are *similar* if there are integers m and n, such the central simple algebras $\mathrm{Mat}_m(A)$ and $\mathrm{Mat}_n(B)$ over k are isomorphic.

The *Brauer group* $\mathrm{Br}\, k$ *of* k is the set of similarity classes of central simple algebras over k. Let $[A]$ and $[B]$ be two classes of the Brauer group represented by central simple algebras. Multiplication on $\mathrm{Br}\, k$ is defined by

$$[A] \cdot [B] = [A \otimes_k B]$$

and inverses are given by $[A]^{-1} = [A^{\mathrm{opp}}]$.

Let K/k be a finite Galois extension. The *Brauer group of k relative to K* is the subset of $\mathrm{Br}\, k$ consisting of the classes which are split by K and is denoted by $\mathrm{Br}(K/k)$.

Multiplication in $\operatorname{Br} k$ is clearly associative and commutative. It follows from $[A] \cdot [k] = [A \otimes_k k] = [A]$ that the class of k is the unit element. Furthermore, Theorem 1.2.10 shows that $[A] \cdot [A]^{-1} = [A] \cdot [A^{\mathrm{opp}}] = [A \otimes A^{\mathrm{opp}}] = [\operatorname{End}_k(A)]$. Now let n^2 be the dimension of A over k then the algebra of k-linear endomorphism of A is isomorphic to $\operatorname{Mat}_{n^2}(k)$ after choosing a k-linear basis for A. So we find $[A] \cdot [A]^{-1} = [\operatorname{Mat}_{n^2}(k)] = [k]$. This shows that $[A]^{-1}$ is the multiplicative inverse of $[A]$.

Note that similar central simple algebras are split by the same fields. Now let K/k be a finite Galois extension. The Brauer group $\operatorname{Br}(K/k)$ of k relative to K is a subgroup of $\operatorname{Br} k$, and it follows from Theorem 1.2.7 that the Brauer group of k is the union over all finite Galois extensions of these subgroups.

Sometimes the following cohomological interpretation of the Brauer group is in some cases more useful than the definition using central simple algebras.

THEOREM 1.2.12. *Let k be a field and let K be a finite Galois extension. There are natural maps*

$$\operatorname{Br}(K/k) \to \mathrm{H}^2(\operatorname{Gal}(K/k), K^\times) \quad \textit{and} \quad \operatorname{Br} k \to \mathrm{H}^2(k, k^{\mathrm{sep},\times})$$

which are isomorphisms of groups.

Proof. We will sketch the construction of these isomorphisms. The details can be found in Sections 2.4 and 4.4 in [28]. One can find this precise statement there as Theorem 4.4.7.

We start by recalling that central simple algebras over k can be classified as the k-algebras which are forms of $\operatorname{Mat}_n(k)$ by Theorem 1.2.7. Using the fact that the automorphism group of $\operatorname{Mat}_n(K)$ is $\operatorname{PGL}_n(K)$ one can show that the set of isomorphism classes of central simple algebras over k split by K is isomorphic as a pointed set to $\mathrm{H}^1(\operatorname{Gal}(K/k), \operatorname{PGL}_n(K))$. Using the short exact sequence

$$0 \to K^\times \to \operatorname{GL}_n(K) \to \operatorname{PGL}_n(K) \to 0$$

we obtain a map $\mathrm{H}^1(\operatorname{Gal}(K/k), \operatorname{PGL}_n(K)) \to \mathrm{H}^2(\operatorname{Gal}(K/k), K^\times)$. One proceeds by showing that these maps are compatible and combine to give an isomorphism of groups

$$\operatorname{Br}(K/k) \to \mathrm{H}^2(\operatorname{Gal}(K/k), K^\times).$$

For the second statement one uses the fact that every central simple algebra over k is split by a finite Galois extension K. This proves that the isomorphism

$$\operatorname{Br} k \to \mathrm{H}^2(k, k^{\mathrm{sep},\times})$$

is the limit of the first isomorphism over all finite Galois extensions K of k. □

A consequence of this theorem is that we can bound the order of an element of the Brauer group in terms of its splitting field.

COROLLARY 1.2.13. *Let K/k be a finite Galois extension of degree n and suppose that A is a central simple algebra over k which is split by K. In the Brauer group* $\operatorname{Br} k$ *we have $n[A] = 0$.*

Proof. Since $\operatorname{Gal}(K/k)$ is of order n, we see that $n\mathrm{H}^2(\operatorname{Gal}(K/k), K^\times) = 0$ by a classical result in group cohomology, see [43, Proposition II.1.31] for example. So $[A]$ has order dividing n in $\operatorname{Br}(K/k) \subseteq \operatorname{Br} k$. □

We can rephrase Corollaries 1.2.3 and 1.2.4 directly in terms of Brauer groups.

PROPOSITION 1.2.14. *We have* $\operatorname{Br} k = 0$ *when k is algebraically closed or a finite field.*

Let us now give a first example of a field whose Brauer group is non-trivial.

PROPOSITION 1.2.15. *It holds that* $\operatorname{Br} \mathbb{R} = \mathbb{Z}/2\mathbb{Z}$ *and it is generated by the class of the only non-trivial central division algebra over $\mathbb{R}$, namely the quaternions* $\mathbb{H}$.

Proof. Let us write $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$. Since every central simple algebra over $\mathbb{R}$ is split by $\mathbb{C}$ we see that $\operatorname{Br} \mathbb{R} = \operatorname{Br}(\mathbb{C}/\mathbb{R}) = \mathrm{H}^2(G, \mathbb{C}^\times)$ by Theorem 1.2.12. Group cohomology of cyclic groups is well-understood, see for example [58, Theorem 6.2.2], and we find

$$\mathrm{H}^2(G, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^G / |\mathbb{C}^\times| = \mathbb{R}^\times / \mathbb{R}_{>0}$$

which is naturally isomorphic to $\mathbb{Z}/2\mathbb{Z}$. One can check that $\mathbb{R}$ and $\mathbb{H}$ are division rings whose centres equal $\mathbb{R}$, so their classes are different elements of $\operatorname{Br} \mathbb{R}$, which proves the statement. □

1.3 Cyclic algebras

We will now consider an important class of algebras.

DEFINITION 1.3.1. Let K/k be a cyclic extension of degree n. Fix a generator $\sigma \in \operatorname{Gal}(K/k)$ and an element $a \in k^\times$. We define a k-algebra $(a, K/k, \sigma)$ as follows: consider a K-vector space with basis $1, x, \ldots, x^{n-1}$ endowed with a multiplication determined by the properties $x^n = a$ and $cx^i \cdot dx^j = c\sigma^i(d)x^{i+j}$ for $c, d \in K$. We call this algebra a *cyclic algebra* over k.

When no confusion is possible we might suppress either the field extension or the generator in the notation.

PROPOSITION 1.3.2. *A cyclic algebra $A = (a, K/k, \sigma)$ over a field is a central simple algebra, which is split by the field K.*

The map $k^\times \to$ $\operatorname{Br} k$ *which maps an element $a \in k^\times$ to the class of $(a, K/k, \sigma)$ is a homomorphism of groups. The kernel of this morphism equals* $\operatorname{Nm}_{K/k}(K^\times)$. *In particular, we see that $(a, K/k, \sigma)$ splits over k precisely when a is a norm from K, i.e. $a \in \operatorname{Nm}_{K/k}(K^\times)$.*

Proof. One can check that A is central and simple using Theorem 1.2.10. We also see that K is a subfield of A and by Proposition 1.2.9 we find that K splits A.

The last statement follows from Equation 1.5.21 and Proposition 1.5.23 in [46]. □

The only non-split central simple algebra we have seen so far is an example of a cyclic algebra.

Example. Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denote complex conjugation. Then the map

$$(-1, \mathbb{C}/\mathbb{R}, \sigma) \mapsto \mathbb{H}$$

defined by mapping i to i, and x to j is an isomorphism of algebras over $\mathbb{R}$.

This example illustrates the following general fact, which gives us a condition for when a central simple algebra can be written using only cyclic algebras.

THEOREM 1.3.3 (Merkurjev–Suslin). *Let k be a field which contains a primitive n-th root of unity. An element of* $\mathrm{Br}\, k$ *whose order divides n is the product of classes of cyclic algebras over k.*

Proof. See [28, Theorem 2.5.7]. □

We will be more interested in the following fields, which need not satisfy the condition of the previous statement, but do satisfy an even stronger property.

PROPOSITION 1.3.4. *Let k be a local or a global field. Every central simple algebra over k is a cyclic algebra.*

Proof. This is the content of Theorem 1.5.34 (iii) and Theorem 1.5.36 (iii) in [46]. □

We will see a complete description of Brauer groups of local and global fields in Proposition 1.5.3 and Proposition 1.5.4.

1.4 Azumaya algebras over schemes

We will now generalize the notion of a central simple algebra over a field to the notion of an Azumaya algebra over a scheme. One can think of such an algebra as a family of central simple algebras. The following proposition shows that under this interpretation Theorem 1.2.7 and Theorem 1.2.10 are satisfied locally.

PROPOSITION 1.4.1. *Let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra on a scheme X, such that all stalks $\mathcal{A}_x$ are non-trivial. The following statements are equivalent.*

(i) The sheaf $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and for each point $x \in X$ the fibre $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra over $\kappa(x)$.

(ii) There is an étale covering $U_i \to X$ such that for each i there is an $r_i > 0$ and an isomorphism

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i}).$$

(iii) The sheaf $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and the morphism

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{opp}} \to \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{A})$$

defined on sections by sending $a \otimes b$ to the endomorphism $x \mapsto axb$ is an isomorphism.

Proof. See Definition 6.6.12 in [46]. □

A sheaf satisfying one, and hence all, of the statements in Proposition 1.4.1 is the generalization of central simple algebra with which we will work.

DEFINITION 1.4.2. An *Azumaya algebra* over a scheme X is a coherent $\mathcal{O}_X$-algebra with non-zero stalks satisfying the equivalent statements in Proposition 1.4.1.

Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ on X are called *similar* if there are locally free coherent $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ with non-zero stalks such that there is an isomorphism

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathscr{E}nd(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathscr{E}nd(\mathcal{E}')$$

of $\mathcal{O}_X$-algebras.

The *Azumaya Brauer group* $\mathrm{Br}_{\mathrm{Az}}\, X$ of a scheme X is the set of similarity classes of Azumaya algebras on X. The product of two classes $[\mathcal{A}]$ and $[\mathcal{B}]$ is defined as $[\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}]$. The class $[\mathcal{O}_X]$ is a two-sided identity element for this multiplication and an inverse of a class $[\mathcal{A}]$ is given by $[\mathcal{A}]^{-1} := [\mathcal{A}^{\mathrm{opp}}]$. This makes $\mathrm{Br}_{\mathrm{Az}}$ into a functor $\mathbf{Sch} \to \mathbf{Ab}^{\mathrm{opp}}$.

Because we require $\mathcal{A}$ to be locally free, the rank of $\mathcal{A}_x$ as an $\mathcal{O}_{X,x}$-module is locally constant. So on connected components of X we see that the rank is the square of a positive integer, just as the dimension of central simple algebras over fields. There are more similarities between Azumaya algebras on schemes and central simple algebras for fields. It might for example be convenient to have a cohomological interpretation of the Azumaya Brauer group of a scheme as we did for fields in Theorem 1.2.12.

DEFINITION 1.4.3. Let X be a scheme. The *cohomological Brauer group* $\mathrm{Br}\, X$ is defined as $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$, which defines a functor $\mathbf{Sch} \to \mathbf{Ab}^{\mathrm{opp}}$.

Unlike for fields, the cohomological Brauer group is not always isomorphic to the Azumaya Brauer group. There is however a natural transformation

$$\mathrm{Br}_{\mathrm{Az}} \to \mathrm{Br}$$

constructed similarly as in the case for fields: the isomorphism classes of Azumaya algebras over a scheme X of rank r^2 are classified by the cohomology group $\mathrm{H}^1(X, \mathrm{PGL}_r)$. The short exact sequence

$$0 \to \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}_r \to \mathrm{PGL}_r \to 0$$

of étale sheaves on X gives rise to a homomorphism

$$\mathrm{H}^1(X, \mathrm{PGL}_r) \to \mathrm{H}^2(X, \mathbb{G}_\mathrm{m}) = \operatorname{Br} X.$$

One can check that the image in $\operatorname{Br} X$ of an Azumaya algebra $\mathcal{A}$ in $\mathrm{H}^1(X, \mathrm{PGL}_r)$ only depends on its class $[\mathcal{A}]$ in $\operatorname{Br}_{\mathrm{Az}} X$. This gives us a map $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ for any scheme X.

PROPOSITION 1.4.4. *Consider the family of homomorphisms* $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ *constructed above.*

(a) *This defines a natural transformation* $\operatorname{Br}_{\mathrm{Az}} \to \operatorname{Br}$ *of functors* $\mathbf{Sch} \to \mathbf{Ab}^{\mathrm{opp}}$.

(b) *The homomorphism* $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ *is injective for every scheme X.*

(c) *Let k be a field. The three notions of the Brauer group of k, namely* $\operatorname{Br} k$, $\operatorname{Br}(\operatorname{Spec} k)$ *and* $\operatorname{Br}_{\mathrm{Az}}(\operatorname{Spec} k)$, *are all naturally isomorphic.*

(d) *Assume that X has an ample line bundle. The natural map* $\operatorname{Br}_{\mathrm{Az}} X \to (\operatorname{Br} X)_{\mathrm{tors}}$ *is an isomorphism.*

We see that $\operatorname{Br} k$ and $\operatorname{Br}(\operatorname{Spec} k)$ are naturally isomorphic and we will keep with the convention that an affine scheme will be denoted by its coordinate ring if no confusion is likely, in particular we will write $\operatorname{Br} R$ for the Brauer group of the affine scheme $\operatorname{Spec} R$ for any commutative ring R.

Proof. The fact that each map $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ is an injective group homomorphism can be found in [46, Theorem 6.6.17(i)]. A first isomorphism

$$\operatorname{Br} k \to \operatorname{Br}(\operatorname{Spec} k)$$

for the third statement follows from Theorem 1.2.12. It is quite straightforward to exhibit a natural correspondence between Azumaya algebras on $\operatorname{Spec} k$ and central simple algebras over k which preserves similarity of algebras. This proves the existence of the isomorphism $\operatorname{Br} k \to \operatorname{Br}_{\mathrm{Az}}(\operatorname{Spec} k)$. The last statement is Theorem 6.6.17(iii) in [46]. □

Here are some more properties of the two notions of Brauer groups.

PROPOSITION 1.4.5. *Let X be a scheme.*

(a) *Suppose that a class* $[\mathcal{A}] \in \operatorname{Br}_{\mathrm{Az}} X$ *is represented by an Azumaya algebra of rank r^2. Then $r[\mathcal{A}]$ is zero in* $\operatorname{Br}_{\mathrm{Az}} X$.

(b) *The Azumaya Brauer group of a scheme with finitely many connected components is torsion.*

For the next two statement we require that X is a regular integral noetherian scheme.

(c) *The natural map* $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ *is injective.*

(d) *The cohomological Brauer group* $\operatorname{Br} X$ *is a torsion group.*

Proof. The first two statements are precisely Theorem 6.6.17(ii) in [46]. For the last two statement we refer to Theorem 6.6.7 in [46]. □

So, in many cases we can identify all the types of Brauer groups we have seen so far and in those cases we will simply write $\operatorname{Br} X$ for this group. In particular we have the following important corollary to the previous proposition.

COROLLARY 1.4.6. *Let X be a regular quasi-projective variety over a field then we have*

$$\operatorname{Br} X \cong (\operatorname{Br} X)_{\mathrm{tors}} \cong \operatorname{Br}_{\mathrm{Az}} X.$$

Now that we have defined Brauer groups for general schemes, we have in particular done so for rings. Let us consider the following example.

PROPOSITION 1.4.7. *Let R be a complete local ring with maximal ideal m. The natural morphism*

$$\operatorname{Br} R \to \operatorname{Br}(R/m)$$

is an isomorphism.

Proof. See Proposition 6.9.1 in [46]. □

Using the fact that the Brauer group of a finite field is trivial, see Proposition 1.2.14, we find the following result.

COROLLARY 1.4.8. *Let $\mathcal{O}$ be the ring of integers in a non-archimedean local field. Then* $\operatorname{Br} \mathcal{O} = 0$ *and in particular* $\operatorname{Br} \mathbb{Z}_p = 0$.

Let us now look at some examples of Brauer groups of, not necessarily affine, schemes.

PROPOSITION 1.4.9. *Suppose that C is a smooth integral curve over an algebraically closed field then* $\operatorname{Br} C = 0$.

Proof. By Tsen's theorem [46, Theorem 1.5.33] we see that the Brauer group of a field of transcendence degree 1 over an algebraically closed field is trivial. By the inclusion $\operatorname{Br} C \to \operatorname{Br} \kappa(C)$ from Proposition 1.4.5 we see that $\operatorname{Br} C$ must be trivial too. □

Most Brauer groups we have seen thus far are trivial. Starting from a field k with a non-trivial Brauer group one can construct examples of schemes with a non-trivial Brauer group.

PROPOSITION 1.4.10. *Suppose that the k-scheme $\pi \colon X \to \operatorname{Spec} k$ admits a k-point. The natural morphism $\pi^* \colon \operatorname{Br} k \to \operatorname{Br} X$ is injective.*

Proof. A k-point p on X is a section of the structure morphism π. This shows that the composition

$$\operatorname{Br} k \xrightarrow{\pi^*} \operatorname{Br} X \xrightarrow{p^*} \operatorname{Br} k$$

is the identity on $\operatorname{Br} k$ and π^* must be injective. □

The following lemma gives us another situation when the induced map between Brauer groups coming from a morphism between two schemes is injective.

LEMMA 1.4.11. *Let $X \to Y$ be a birational morphism between two regular integral noetherian schemes. The induced morphism* $\operatorname{Br} Y \to \operatorname{Br} X$ *is injective.*

Proof. By Proposition 1.4.5 the morphisms $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ and $\operatorname{Br} Y \to \operatorname{Br} \kappa(Y)$ are injective and we have an isomorphism $\operatorname{Br} \kappa(Y) \to \operatorname{Br} \kappa(X)$ induced by the birational morphism $X \to Y$. By functoriality we have the following commutative square.

$$\begin{array}{ccc} \operatorname{Br} Y & \longrightarrow & \operatorname{Br} \kappa(Y) \\ \big\downarrow & & \big\downarrow \cong \\ \operatorname{Br} X & \longrightarrow & \operatorname{Br} \kappa(X) \end{array}$$

The injectivity of $\operatorname{Br} Y \to \operatorname{Br} X$ is immediate. $\square$

This lemma is one of the main ingredients for the following proposition.

PROPOSITION 1.4.12. *Let k be a field of characteristic* 0 *and n a positive integer. The natural morphisms* $\operatorname{Br} k \to \operatorname{Br} \mathbb{A}^n_k$ *and* $\operatorname{Br} k \to \operatorname{Br} \mathbb{P}^n_k$ *are isomorphisms.*

Proof. It follows from Proposition 1.4.9 and [4, Theorem 7.5] that the morphism $\operatorname{Br} L \to \operatorname{Br} \mathbb{A}^1_L$ is an isomorphism for a perfect field L. The first statement follows from induction by considering $\mathbb{A}^n_k$ as the affine line over $\mathbb{A}^{n-1}_k$. It is this step where the condition on the characteristic comes in. Details can be found in [13, Proposition 1.3].

For the last statement, one notes that the embedding $\mathbb{A}^n_k \to \mathbb{P}^n_k$ satisfies the conditions of Lemma 1.4.11. $\square$

Even for a birational map we have the following result for proper schemes.

PROPOSITION 1.4.13. *Let X and X' be birational regular integral noetherian schemes which are projective over a field of characteristic* 0*. The Brauer groups* $\operatorname{Br} X$ *and* $\operatorname{Br} X'$ *are isomorphic.*

Proof. This follows from Corollaire 7.5 in [30]. $\square$

1.5 Residue and invariant maps

Now we will study residue maps. These are an important tool for working with Brauer groups of local and global fields. They will also turn out to be relevant when studying the Brauer group of a scheme $X \backslash C$ which is the complement of divisor $C \subseteq X$. The more general notion is the following.

PROPOSITION 1.5.1. *Let R be a discrete valuation ring with fraction field K and perfect residue field $\mathbb{F}$. There exists an exact sequence*

$$0 \to \operatorname{Br} R \to \operatorname{Br} K \xrightarrow{\partial_R} \mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

The map ∂_R is called the residue map of R.

Proof. This proposition is exactly Proposition 6.8.1 of [46]. The construction of the residue map ∂_R can be found there too. □

If $\mathbb{F}$ is not perfect we lose the surjectivity of ∂_R, but the rest remains true after excluding the p-primary parts of all groups under consideration. See for a more complete treatment Section 6.8.1 in [46].

DEFINITION 1.5.2. Let k be a non-archimedean local field with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. The absolute Galois group of $\mathbb{F}$ is $\widehat{\mathbb{Z}}$ with $\mathrm{Frob}_{\mathbb{F}}$ as a topological generator. We can identify the cohomology group $\mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z})$ with the group $\mathrm{Hom}_{\mathrm{cnt}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ of 1-cocycles, since the action of $\widehat{\mathbb{Z}}$ on $\mathbb{Q}/\mathbb{Z}$ is trivial. The *invariant map* inv_k of k is the composition

$$\operatorname{Br} k \xrightarrow{\partial_{\mathcal{O}}} \mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathrm{cnt}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z},$$

where the isomorphism $\mathrm{Hom}_{\mathrm{cnt}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is defined by evaluating at $\mathrm{Frob}_{\mathbb{F}}$. When k is either $\mathbb{R}$ or $\mathbb{C}$ we define inv_k to be the unique injective map $\operatorname{Br} k \to \mathbb{Q}/\mathbb{Z}$.

Using these invariant maps we can describe the Brauer groups of both local and global fields.

PROPOSITION 1.5.3. *Let k be a local field. The invariant maps $\mathrm{inv}_k \colon \operatorname{Br} k \hookrightarrow \mathbb{Q}/\mathbb{Z}$ satisfy*

$$\operatorname{Im} \mathrm{inv}_k = \begin{cases} 0 & \text{if } k = \mathbb{C}; \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } k = \mathbb{R}, \text{ and} \\ \mathbb{Q}/\mathbb{Z} & \text{if } k \text{ is non-archimedean.} \end{cases}$$

If k'/k is a finite extension of local fields, then the diagram in (1.1) *commutes.*

$$\begin{tikzcd} \operatorname{Br} k \arrow[r, "\mathrm{inv}_k"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cdot[k' : k]"] \\ \operatorname{Br} k' \arrow[r, "\mathrm{inv}_{k'}"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd} \tag{1.1}$$

Proof. See [46, Theorem 1.5.34]. □

PROPOSITION 1.5.4. *Let k be a global field. An element $A \in \operatorname{Br} k$ is split by k_v for all but finitely many places v.*

Let inv_v *be the invariant map associated to the completion* k_v *of* k *using the place* v. *We have the following exact sequence*

$$0 \to \operatorname{Br} k \to \bigoplus_v \operatorname{Br} k_v \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

Proof. See [46, Theorem 1.5.36]. □

For more information one can consult Section 1.5.9 in [46]. Here one can find a construction of a cyclic algebra over a fixed local field with a given invariant. Also a construction is given for a cyclic algebra over a global field with a given set of local invariants. This last construction works precisely if the local invariants sum to 0 in $\mathbb{Q}/\mathbb{Z}$, which is the best possible result given the short exact sequence in Proposition 1.5.4.

1.6 The Brauer–Manin obstruction

In this section we fix a number field k and k-variety X.

PROPOSITION 1.6.1. *Let* v *be a place of* k *and* $\mathcal{A}$ *an Azumaya algebra on* X.

(a) The evaluation map $\mathrm{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v$ *is locally constant in the* v*-adic topology on* $X(k_v)$.

(b) For an $(x_v) \in X(\mathbb{A}_k)$ *we have that* $\mathrm{ev}_{\mathcal{A}}(x_v)$ *is trivial for almost all* v.

(c) For an adele (x_v) *in the image of* $X(k) \to X(\mathbb{A}_k)$ *we have that*

$$\sum_v \mathrm{inv}_v \, \mathcal{A}(x_v) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Proof. These are Proposition 8.2.9, Proposition 8.2.1 and Proposition 8.2.2 in [46]. □

The last statement in Proposition 1.6.1 gives us a property of the elements in the image of $X(k)$ in $X(\mathbb{A}_k)$. Although this property can be shared with other adelic points than those coming from $X(k)$, it motivates the following definition.

DEFINITION 1.6.2. Consider a scheme X over a number field k. For an element $\mathcal{A} \in \operatorname{Br} X$ and $x = (x_v) \in X(\mathbb{A}_k)$ define

$$(\mathcal{A}, x) = \sum_{v \in \Omega_k} \mathrm{inv}_v \, \mathcal{A}(x_v).$$

This map $\operatorname{Br} X \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is called the *Brauer–Manin pairing of* X.

For a subset $B \subseteq \operatorname{Br} X$ the *Brauer–Manin set associated to* B is defined as

$$X(\mathbb{A}_k)^B := \{x \in X(\mathbb{A}_k) \mid (\mathcal{A}, x) = 0 \text{ for all } \mathcal{A} \in B\}.$$

The Brauer–Manin sets give an inclusion-reversing map from subsets of $\operatorname{Br} X$ to subsets of $X(\mathbb{A}_k)$ and satisfy the following properties.

COROLLARY 1.6.3. *Let $\mathcal{A}$ be an element of* $\operatorname{Br} X$ *and let* $B \subseteq \operatorname{Br} X$ *be a subset.*

(a) The map $\sum \operatorname{inv}_v \mathcal{A} \colon X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is locally constant and the Brauer–Manin set $X(\mathbb{A}_k)^{\mathcal{A}}$ associated to $\mathcal{A}$ is open and closed in $X(\mathbb{A}_k)$.

(b) The subset $X(\mathbb{A}_k)^B$ of $X(\mathbb{A}_k)$ is closed.

(c) We have the following inclusions

$$X(k) \subseteq \overline{X(k)} \subseteq X(\mathbb{A}_k)^B \subseteq X(\mathbb{A}_k).$$

Where we have written $\overline{X(k)}$ for the topological closure of $X(k)$ in $X(\mathbb{A}_k)$.

Some subsets of the Brauer group of a scheme are of particular interest.

DEFINITION 1.6.4. The classes of $\operatorname{Br} X$ in the image of $\operatorname{Br} k \to \operatorname{Br} X$ are called *constant classes*, which form the subgroup $\operatorname{Br}_0 X \subseteq \operatorname{Br} X$. The *algebraic Brauer group* $\operatorname{Br}_1 X$ *of* X is the subgroup consisting of the classes in the kernel of the natural homomorphism $\operatorname{Br} X \to \operatorname{Br} \bar{X}$. The elements of $\operatorname{Br}_1 X$ are called *algebraic classes* and an element of $\operatorname{Br} X \setminus \operatorname{Br}_1 X$ is called a *transcendental class*.

For the Brauer–Manin set associated to either $\operatorname{Br}_1 X$ or $\operatorname{Br} X$ we omit the X from the notation and write $X(\mathbb{A}_k)^{\operatorname{Br}_1}$ and $X(\mathbb{A}_k)^{\operatorname{Br}}$ respectively. The following proposition shows why we do not introduce this notation for $\operatorname{Br}_0 X$. It also shows that in some cases the algebraic part of the Brauer group can be explicitly calculated.

PROPOSITION 1.6.5. *(a) We have* $\operatorname{Br}_0 X \subseteq \operatorname{Br}_1 X \subseteq \operatorname{Br} X$.

(b) The constant Brauer classes of X lie in the right kernel of the Brauer–Manin pairing, i.e. $X(\mathbb{A}_k)^{\operatorname{Br}_0 X} = X(\mathbb{A}_k)$.

(c) Let X be a variety over a number field k such that all global invertible functions over an algebraic closure are constant, i.e. $\mathbb{G}_{\mathrm{m}}(X^{\mathrm{sep}}) = k^{\mathrm{sep},\times}$. There is an isomorphism

$$\operatorname{Br}_1 X / \operatorname{Br}_0 X \cong \mathrm{H}^1(G_k, \operatorname{Pic} X^{\mathrm{sep}}).$$

Proof. For the first statement we only need to prove that constant classes are algebraic. This follows from the fact that the Brauer group of an algebraically closed field is trivial, see 1.2.14. The second statement follows from the fact that the sum of local invariants of $A \in \operatorname{Br} k$ is always zero, Proposition 1.5.4. The third statement follows from the Hochschild–Serre spectral sequence for details one is referred to [46, Corollary 6.7.8]. □

These Brauer–Manin sets allow us in some cases to prove that a scheme X does not satisfy the Hasse principle or weak approximation.

PROPOSITION 1.6.6. *Let X be a scheme over a number field k and let B be a subset of* $\operatorname{Br} X$.

- *If $X(\mathbb{A}_k)^B = \emptyset$ then $X(k) = \emptyset$. If $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^B = \emptyset$ we say that there is a Brauer–Manin obstruction to the Hasse principle on X.*
- *If $X(\mathbb{A}_k)^B \subsetneq X(\mathbb{A}_k)$ then X does not satisfy strong approximation. We say that there is a Brauer–Manin obstruction to strong approximation on X.*

We call either obstruction an algebraic obstruction if it occurs for a $B \subseteq \operatorname{Br}_1 X$.

The Brauer group can also be used to explain the absence of integral points.

DEFINITION 1.6.7. Consider a finite set S of places of a number field k and suppose that S contains the infinite places Ω_k^∞. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}_{k,S}$ and let $B \subseteq \operatorname{Br} X$ be a subgroup of the Brauer group of the generic fibre $X = \mathcal{X}_k$. The *integral Brauer–Manin set associated to B* is denoted by $\mathcal{X}(\mathbb{A}_{k,S})^B$ and defined as

$$X(\mathbb{A}_k)^B \cap \mathcal{X}(\mathbb{A}_{k,S}),$$

where the intersection is taken in $X(\mathbb{A}_k)$.

PROPOSITION 1.6.8.

- *We have the following chain of inclusions*

$$\mathcal{X}(\mathcal{O}_{k,S}) \subseteq \mathcal{X}(\mathbb{A}_{k,S})^B \subseteq \mathcal{X}(\mathbb{A}_{k,S}) \subseteq X(\mathbb{A}_k).$$

- *If $\mathcal{X}(\mathbb{A}_{k,S})^B = \emptyset$ then $\mathcal{X}(\mathcal{O}_{k,S}) = \emptyset$. If also $\mathcal{X}(\mathbb{A}_{k,S}) \neq \emptyset$ we say that there is a Brauer–Manin obstruction to the S-integral Hasse principle on $\mathcal{X}$.*

Note that even if there is no Brauer–Manin obstruction to the integral Hasse principle, the absence of integral points can in some cases be explained by proving that $X(k)$ is the empty set.

1.7 The purity theorem and the ramification locus

We have seen that the residue map can be used to compute the Brauer groups of certain rings. We will now apply residue maps in the setting of schemes; in Proposition 1.4.5 we have seen that for a regular integral noetherian scheme X the natural map $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ is injective. Not every element of $\operatorname{Br} \kappa(X)$ will necessarily define an Azumaya algebra on X, but it will always do so on a dense open subset.

PROPOSITION 1.7.1. *Let $\mathcal{A} \in \operatorname{Br} \kappa(X)$ be a central simple algebra over the function field of a regular integral noetherian scheme X over a field of characteristic* 0. *There exists an open and dense subset U, such that $\mathcal{A}$ lies in the image* $\operatorname{Br} U \to \operatorname{Br} \kappa(X)$.

Proof. Consider the inverse filtered system of schemes given by open dense subsets $U_i \subseteq X$. We know that $\lim_{\leftarrow} U_i \cong \operatorname{Spec} \kappa(X)$. Corollaire VII.5.9 in [2] tells us that for such a system the natural morphism $\lim_{\rightarrow} \operatorname{Br} U_i \to \operatorname{Br} \kappa(x)$ is an isomorphism. This shows that an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$ comes from an element of $\operatorname{Br} U$ for some open dense $U \subseteq X$. □

A next question might be whether one can enlarge such an open subset U. The following two results answer this question in general.

THEOREM 1.7.2 (Purity theorem). *Let X be a noetherian regular integral scheme over a field of characteristic* 0 *and consider an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$. There exists an open subscheme $U \subseteq X$ whose complement is either empty or of pure codimension* 1*, such that $\mathcal{A}$ lies in the image of* $\operatorname{Br} U \to \operatorname{Br} \kappa(X)$.

Proof. Let $U' \subseteq X$ be an open and dense subscheme such that $\mathcal{A}$ lies in the image of $\operatorname{Br} U' \to \operatorname{Br} \kappa(X)$. Consider the complement $Z' = X \backslash U'$ and let Z be the union of the irreducible components of Z' of codimension 1. Then by Corollaire 6.2 in [30] we see that the map $\operatorname{Br}(X \backslash Z) \to \operatorname{Br}(X \backslash Z') = \operatorname{Br} U'$ is an isomorphism. This proves that one can take $U = X \backslash Z$. □

For a cyclic algebra $\mathcal{A} = (g, \kappa(X_K)/\kappa(X), \sigma)$ over the field $\kappa(X)$ Theorem 1.7.2 can be made explicit; the Azumaya class of $\mathcal{A}$ lies in the Brauer group of the dense open subset where g is defined and invertible. The following lemma allows us to check when this class actually lives in the subgroup $\operatorname{Br} X$.

LEMMA 1.7.3. *Consider a smooth and geometrically integral variety X over a field k satisfying $\mathbb{G}_{\mathrm{m}}(\bar{X}) = \bar{k}^{\times}$. Fix a finite cyclic extension K/k, a generator $\sigma \in \operatorname{Gal}(K/k)$, and an element $g \in \kappa(X)^{\times}$.*

The cyclic algebra $\mathcal{A} = (g, \kappa(X_K)/\kappa(X), \sigma)$ lies in the image of $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ *precisely if* $\operatorname{div} g = N_{K/k}(D)$ *for some divisor D on X_K. If k, and hence K, is a number field, and X is everywhere locally soluble then $\mathcal{A}$ is constant exactly when D can be taken to be principal.*

Proof. This lemma is similar to Proposition 4.17 from [5]. The difference is that the projectivity assumption is replaced by the weaker condition $\mathbb{G}_{\mathrm{m}}(\bar{X}) = \bar{k}^{\times}$. One can check that under this assumption the proof presented in [5] is still valid. □

This result exhibits a general principle. To explicitly write down classes of $\operatorname{Br} X$ one usually uses the inclusion $\operatorname{Br} X \hookrightarrow \operatorname{Br} \kappa(X)$ and starts with a central simple algebra over the field $\kappa(X)$. Finding an open subscheme U of X for which $\mathcal{A}$ comes from $\operatorname{Br} U$ is usually straightforward. We would want to identify the irreducible components of the complement Z of U in X on which we can extend the Azumaya algebra. We have seen in the proof of Theorem 1.7.2 that we can extend $\mathcal{A}$ on any irreducible components of Z of codimension at least 2. For the irreducible components of Z of codimension 1 we can use residue maps.

THEOREM 1.7.4. *Suppose that X is a regular integral noetherian scheme over a field of characteristic* 0*. For each point x of X of codimension* 1 *we have a residue map*

$$\partial_x \colon \operatorname{Br} \kappa(X) \to \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Fix an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$. The residue $\partial_x(\mathcal{A})$ is trivial for all but finitely many codimension 1 *points $x \in X^{(1)}$ and the following sequence is exact*

$$0 \to \operatorname{Br} X \to \operatorname{Br} \kappa(X) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Proof. See Theorem 6.8.3 in [46]. □

Again, one can show a more general statement for schemes not necessarily defined over a field, although one might need to exclude the p-primary part of all groups if there exist codimension 1 points x for which $\kappa(x)$ is imperfect.

In the proof of Proposition 1.7.1 we have seen the important result that the Brauer group does not change when we cut out a subscheme of codimension at least 2. When we cut out a subscheme of codimension 1, the Brauer group can change and the residue maps allow us to quantify this change.

PROPOSITION 1.7.5. *Let D be a geometrically irreducible regular divisor on a regular integral noetherian scheme X over a field of characteristic* 0*. Write U for the complement of D in X. The image of the residue map $\partial_D \colon \operatorname{Br} U \to \mathrm{H}^1(\kappa(D), \mathbb{Q}/\mathbb{Z})$ lies in the subgroup $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$ and the sequence in* (1.2) *is exact.*

$$0 \to \operatorname{Br} X \to \operatorname{Br} U \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \tag{1.2}$$

Proof. See Corollaire 6.2 in [30]. □

This sequence is only functorial in some cases.

PROPOSITION 1.7.6. *Let $f \colon X' \to X$ be a morphism between two regular integral noetherian schemes over a field of characteristic* 0*. Let D be a regular integral divisor on X, such that $D' = f^{-1}(D)$ is also regular and integral. Define U and U' as the complement of D in X and D' in X' respectively. The diagram in* (1.3) *is commutative.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Br} X & \longrightarrow & \operatorname{Br} U & \longrightarrow & \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Br} X' & \longrightarrow & \operatorname{Br} U' & \longrightarrow & \mathrm{H}^1_{\text{ét}}(D', \mathbb{Q}/\mathbb{Z}) \end{array} \tag{1.3}$$

Proof. The rows are exact by Proposition 1.7.5.

Now consider for a regular closed subscheme D of codimension 1 on a regular scheme X the first terms of the Gysin sequence, see for example Corollary 5.2 in [8],

$$0 \to \mathrm{H}^2_{\text{ét}}(X, \mu_n) \to \mathrm{H}^2_{\text{ét}}(X \backslash D, \mu_n) \to \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n\mathbb{Z}).$$

In the discussion following Lemma 5.4 in [8] we see that the Gysin sequence is functorial for a morphism $f \colon X' \to X$ for which $D' = f^{-1}(D)$ is also regular and of codimension 1.

One can derive the exact sequence in Proposition 1.7.5 from the n-torsion from the Gysin sequence, i.e.

$$0 \to \operatorname{Br} X[n] \to \operatorname{Br} U[n] \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})[n]$$

is exact. Note however that the map $\mathrm{H}^2_{\text{ét}}(X \backslash D, \mu_n) \to \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n\mathbb{Z})$ can differ from the residue map $\operatorname{Br} U[n] \to \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})[n]$ by a sign. Furthermore, the functoriality of the Gysin sequence carries over to these sequences and we have the commutative diagram as shown in (1.4).

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Br} X[n] \arrow[r] \arrow[d] & \operatorname{Br} U[n] \arrow[r] \arrow[d] & \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})[n] \arrow[d] \\
0 \arrow[r] & \operatorname{Br} X'[n] \arrow[r] & \operatorname{Br} U'[n] \arrow[r] & \mathrm{H}^1_{\text{ét}}(D', \mathbb{Q}/\mathbb{Z})[n]
\end{tikzcd} \tag{1.4}$$

To prove the proposition we recall that the Brauer group of a regular integral noetherian scheme is torsion, see Proposition 1.4.5. □

Consider an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$ and let Z be the union of all irreducible curves D on X for which $\partial_D(\mathcal{A})$ is non-zero. The results in this section show that $\mathcal{A}$ lies in the image of $\operatorname{Br} U \to \operatorname{Br} \kappa(X)$ for $U = X \backslash Z$. The closed subscheme Z is called the *ramification locus* of $\mathcal{A}$.