Into the darkness : forging a stable path through the gravitational landscape

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2 An Extended action for the effective field theory of dark energy: a stability analysis and a complete guide to the mapping at the basis of EFTCAMB

2.1 Introduction

In the present Chapter we propose an extension of the original EFT action for DE/MG [14, 15] by including extra operators with up to sixth order spatial derivatives acting on perturbations. This will allow us to cover a wider range of theories, e.g. Hořava gravity [35, 36], as shown in Refs. [37–39]. The latter model has recently gained attention in the cosmological context [39–58], as well as in the quantum gravity sector [35, 36, 59–61], since higher spatial derivatives have been shown to be relevant in building gravity models exhibiting powercounting and renormalizable behaviour in the ultra-violet regime (UV) [62–64].

We will work out a very general recipe that can be directly applied to any gravity theory with one extra scalar d.o.f. in order to efficiently map it into the EFT language, once the corresponding Lagrangian is written in the Arnowitt-Deser-Misner (ADM) formalism. We will pay particular attention to the different conventions by adapting all the calculations to the specific convention used in *EFTCAMB*, in order to provide a ready-to-use guide on the full mapping of models into this code. This method has already been used in Refs. [37, 65] and here we will further extend it by including the operators in our extended action. Additionally, we will revisit some of the already known mappings in order to accommodate the *EFTCAMB* conventions. Moreover, we will present for the first time the complete mapping of the covariant formulation of the GLPV theories [66, 67] into the EFT formalism. Subsequently, we will perform a detailed study of the stability conditions for the gravity sector of our extended

EFT action. For a restricted subset of EFT models such an analysis can already be found in the literature [14, 15, 65, 67, 68]. Doing this analysis will allow us to have a first glimpse at the viable parameter space of theories covered by the extended EFT framework and to obtain very general conditions to be implemented in *EFTCAMB*. In particular, we will compute the conditions necessary to avoid ghost instabilities and to avoid gradient instabilities, both for scalar and tensor modes. We will also present the condition to avoid tachyonic instabilities in the scalar sector. Finally, we will proceed to extend the ReParametrized Horndeski (RPH) basis, or α-basis, of Ref. [69] in order to include all the models of our generalized EFT action. This will require the introduction of new functions and we will proceed to comment on their impact on the kinetic terms and speeds of propagation of both scalar and tensor modes.

The work in this Chapter is based on [31]: *An Extended action for the effective field theory of dark energy: a stability analysis and a complete guide to the mapping at the basis of EFTCAMB* with N. Frusciante and A. Silvestri. In Section 2.2, we propose a generalization of the EFT action for DE/MG that includes all operators with up to six-th order spatial derivatives. In Section 2.3, we outline a general procedure to map any theory of gravity with one extra scalar d.o.f., and a well defined Jordan frame, into the EFT formalism. We achieve this through an interesting, intermediate step which consists of deriving an equivalent action in the ADM formalism, in Section 2.3.2, and work out the mapping between the EFT and ADM formalism, in Section 2.3.3. In order to illustrate the power of such method, in Section 2.4 we provide some mapping examples: minimally coupled quintessence, f(R)-theory, Horndeski/GG, GLPV and Hořava gravity. In Section 2.5, we work out the physical stability conditions for the extended EFT action, guaranteeing the avoidance of ghost and tachyonic instabilities and positive speeds of propagation for tensor and scalar modes. In Section 2.6, we extend the RPH basis to include the class of theories described by the generalized EFT action and we elaborate on the phenomenology associated to it. The last two sections are more or less independent, so the reader interested only in one of these can skip the other parts. Finally, in Section 2.7, we summarize and comment on our results.

2.2 An extended EFT action

The EFT framework for DE/MG models, introduced in Refs. [14, 15], provides a systematic and unified way to study the dynamics of linear perturbations in a wide range of DE/MG models characterized by an additional scalar d.o.f. and for which there exists a well defined Jordan frame [10, 11, 70–73]. The action is constructed in the unitary gauge as an expansion up to second order in perturbations around the FLRW background of all operators that are invariant under time-dependent spatial-diffeomorphisms. Each of the latter appear in the action accompanied by a time dependent coefficient. The choice of the unitary gauge implies that the scalar d.o.f. is "eaten" by the metric, thus it does not appear explicitly in the action. It can be made explicit by the Stükelberg technique which, by means of an infinitesimal time-coordinate transformation, allows one to restore the broken symmetry by introducing a new field describing the dynamic and evolution of the extra d.o.f.. For a detailed description of this formalism we refer the readers to Refs. [14, 15, 65, 74, 75]. In this Chapter we will always work in the unitary gauge.

The original EFT action introduced in Refs. [14, 15], and its follow ups in Refs. [65, 75–77], cover most of the theories of cosmological interest, such as Horndeski/GG [78, 79], GLPV [66] and low-energy Hořava [35, 36]. However, operators with higher order spatial derivatives are not included. On the other hand, theories which exhibit higher than second order spatial derivatives in the field equations have been gaining attention in the cosmological context [37, 38, 53, 64, 76], moreover, they appear to be interesting models for quantum gravity as well [35, 36, 59–62]. As long as one deals with scales that are sufficiently larger than the non-linear cutoff, the EFT formalism can be safely used to study these theories. In the following, we propose an extended EFT action that includes operators up to sixth order in spatial derivatives:

$$
\begin{aligned}
\mathcal{S}_{EFT} = \int d^4x\sqrt{-g}\,\Big[&\frac{m_0^2}{2}(1+\Omega(t))R + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 \\
&-\frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta\mathcal{R} \\
&+m_2^2(t)h^{\mu\nu}\partial_\mu g^{00}\partial_\nu g^{00} + \frac{\bar{m}_5(t)}{2}\delta\mathcal{R}\delta K + \lambda_1(t)(\delta\mathcal{R})^2 + \lambda_2(t)\delta\mathcal{R}^\mu_\nu\delta\mathcal{R}^\nu_\mu \\
&+\lambda_3(t)\delta\mathcal{R}h^{\mu\nu}\nabla_\mu\partial_\nu g^{00} + \lambda_4(t)h^{\mu\nu}\partial_\mu g^{00}\nabla^2\partial_\nu g^{00} + \lambda_5(t)h^{\mu\nu}\nabla_\mu\mathcal{R}\nabla_\nu\mathcal{R} \\
&+\lambda_6(t)h^{\mu\nu}\nabla_\mu\mathcal{R}_{ij}\nabla_\nu\mathcal{R}^{ij} + \lambda_7(t)h^{\mu\nu}\partial_\mu g^{00}\nabla^4\partial_\nu g^{00} \\
&+\lambda_8(t)h^{\mu\nu}\nabla^2\mathcal{R}\nabla_\mu\partial_\nu g^{00}\Big]\,, \qquad (2.1)
\end{aligned}
$$

where m_0^2 is the Planck mass, g is the determinant of the four dimensional metric $g_{\mu\nu}$, $h^{\mu\nu} = (g^{\mu\nu} + n^\mu n^\nu)$ is the spatial metric on constant-time hypersurfaces, n_μ is the normal vector to the constant-time hypersurfaces, δg^{00} is the perturbation of the upper time-time component of the metric, R is the trace of the four dimensional Ricci scalar, $\mathcal{R}_{\mu\nu}$ is the three dimensional Ricci tensor and $\mathcal{R}$ is its trace, $K_{\mu\nu}$ is the extrinsic curvature and K is its trace and $\nabla^2 = \nabla_\mu\nabla^\mu$ with ∇_μ being the covariant derivative constructed with $g_{\mu\nu}$. The coefficients $\{\Omega, \Lambda, c, M_2^4, \bar{M}_1^3, \bar{M}_2^2, \bar{M}_3^2, \hat{M}^2, m_2^2, \bar{m}_5, \lambda_i\}$ (with $i = 1$ to 8) are free functions of time and hereafter we will refer to them as *EFT functions.* $\{\Omega, \Lambda, c\}$ are usually called background EFT functions as they are the only ones contributing to both the background and linear perturbation equations, while the others enter only at the level of perturbations. Let us notice that the operators corresponding to $\bar{m}_5, \lambda_{1,2}$ have already been considered in Ref. [65], while the remaining operators have been introduced by some of the authors of this paper in Ref. [39], where it is shown that they are necessary to map the high-energy Hořava gravity action [64] in the EFT formalism.

The EFT formalism offers a unifying approach to study large scale structure (LSS) in DE/MG models. Once implemented into an Einstein-Boltzmann solver like CAMB [20], it clearly provides a very powerful software with which to test gravity on cosmological scales. This has been achieved with the patches *EFTCAMB/EFTCosmoMC*, introduced in Refs. [22–24]. This software can be used in two main realizations: the *pure EFT* and the *mapping EFT*. The former corresponds to an agnostic exploration of dark energy, where the user can turn on and off different EFT functions and explore their effects on the LSS. In the latter case instead, one specializes to a model (or a class of models, e.g. $f(R)$ gravity), maps it into the EFT functions and proceed to study the corresponding dynamics of perturbations. We refer the reader to Ref. [80] for technical details of the code.

There are some key virtues of *EFTCAMB* which make it a very interesting tool to constrain gravity on cosmological scales. One is the possibility of imposing powerful yet general conditions of stability at the level of the EFT action, which makes the exploration of the parameter space very efficient [23]. We will elaborate on this in Section 2.5. Another, is the fact that a vast range of specific models of DE/MG can be implemented *exactly* and the corresponding dynamics of perturbations be evolved, in the same code, guaranteeing unprecedented accuracy and consistency.

In order to use *EFTCAMB* in the mapping mode it is necessary to determine the expressions of the EFT functions corresponding to the given model. Several models are already built-in in the currently public version of *EFTCAMB*. This Chapter offers a complete guide on how to map specific models and classes of models of DE/MG all the way into the EFT language at the basis of *EFTCAMB*, whether they are initially formulated in the ADM or covariant formalism; all this, without the need of going through the cumbersome expansion of the models to quadratic order in perturbations around the FLRW background.

2.3 From a General Lagrangian in ADM formalism to the EFToDE/MG

In this Section we use a general Lagrangian in the ADM formalism which covers the same class of theories described by the EFT action (2.1). This will allow us to make a parallel between the ADM and EFT formalisms, and to use the former as a convenient platform for a general mapping description of DE/MG theories into the EFT language. In particular, in Section 2.3.1 we will expand a general ADM action up to second order in perturbations, in Section 2.3.2 we will write the EFT action in ADM form and, finally, in Section 2.3.3 we will provide the mapping between the two.

2.3.1 A General Lagrangian in ADM formalism

Let us introduce the 3+1 decomposition of spacetime typical of the ADM formalism, for which the line element reads:

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{2.2}$$

where $N(t,x^i)$ is the lapse function, $N^i(t,x^i)$ the shift and $h_{ij}(t,x^i)$ is the three dimensional spatial metric. We also adopt the following definition of the normal vector to the hypersurfaces of constant time and the corresponding extrinsic curvature:

$$n_\mu = N\delta_{\mu 0}, \qquad K_{\mu\nu} = h^\lambda_\mu \nabla_\lambda n_\nu. \tag{2.3}$$

The general Lagrangian we use in this Section has been proposed in Ref. [37] and can be written as follows:

$$L = L(N, \mathcal{R}, \mathcal{S}, K, \mathcal{Z}, \mathcal{U}, \mathcal{Z}_1, \mathcal{Z}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; t)\,, \tag{2.4}$$

where the above geometrical quantities are defined as follows:

$$\begin{aligned} &\mathcal{S} = K_{\mu\nu}K^{\mu\nu}\,,\ \mathcal{Z} = \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\,,\ \mathcal{U} = \mathcal{R}_{\mu\nu}K^{\mu\nu}\,,\ \mathcal{Z}_1 = \nabla_i\mathcal{R}\nabla^i\mathcal{R}\,,\\ &\mathcal{Z}_2 = \nabla_i\mathcal{R}_{jk}\nabla^i\mathcal{R}^{jk}\,, \alpha_1 = a^i a_i\,,\ \alpha_2 = a^i\Delta a_i\,,\ \alpha_3 = \mathcal{R}\nabla_i a^i\,,\\ &\alpha_4 = a_i\Delta^2 a^i\,,\ \alpha_5 = \Delta\mathcal{R}\nabla_i a^i, \end{aligned} \tag{2.5}$$

with $\Delta = \nabla_k\nabla^k$ and a^i is the acceleration of the normal vector, $n^\mu\nabla_\mu n_\nu$. ∇_μ and ∇_k are the covariant derivatives constructed respectively with the four dimensional metric, $g_{\mu\nu}$ and the three metric, h_{ij}.

The operators considered in the Lagrangian (2.4) allow to describe gravity theories with up to sixth order spatial derivatives, therefore the range of theories covered by such a Lagrangian is the same as the EFT action proposed in Section 2.2. The resulting general action, constructed with purely geometrical quantities, is sufficient to cover most of the candidate models of modified gravity [10, 11, 70–73].

We shall now proceed to work out the mapping of Lagrangian (2.4) into the EFT formalism. The procedure that we will implement in the following retraces that of Refs. [37, 65]. However, there are some tricky differences between the EFT language of Ref. [65] and the one at the basis of *EFTCAMB* [22, 23]. Most notably the different sign convention for the normal vector, n_μ, and the extrinsic curvature, $K_{\mu\nu}$ (see Eq. (2.3)), a different notation for the conformal coupling and the use of δg^{00} in the action instead of g^{00}, which changes the definition of some EFT functions. It is therefore important that we present all details of the calculation as well as derive a final result which is compatible with *EFTCAMB*. In particular, the results of this Section account for the different convention for the normal vector.

We shall now expand the quantities in the Lagrangian (2.4) in terms of perturbations by considering for the background a flat FLRW metric of the form:

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{2.6}$$

where $a(t)$ is the scale factor. Therefore, we can define:

$$\begin{aligned}
&\delta K_{\mu\nu} = Hh_{\mu\nu} + K_{\mu\nu}\,, \quad \delta\mathcal{S} = \mathcal{S} - 3H^2 = -2H\delta K + \delta K^\mu_\nu \delta K^\nu_\mu\,, \\
&\delta K = 3H + K\,, \quad \delta\mathcal{U} = -H\delta\mathcal{R} + \delta K^\mu_\nu\delta K^\nu_\mu\,, \quad \delta\alpha_1 = \partial_i\delta N\partial^i\delta N\,, \\
&\delta\alpha_2 = \partial_i\delta N\nabla_k\nabla^k\partial^i\delta N\,, \qquad \delta\alpha_3 = \mathcal{R}\nabla_i\partial^i\delta N\,, \quad \delta\alpha_4 = \partial_i\delta N\Delta^2\partial^i\delta N\,, \\
&\delta\alpha_5 = \Delta^2\mathcal{R}\nabla_i\partial^i\delta N\,, \quad \delta\mathcal{Z}_1 = \nabla_i\delta\mathcal{R}\nabla^i\delta\mathcal{R}\,, \quad \delta\mathcal{Z}_2 = \nabla_i\delta\mathcal{R}_{jk}\nabla^i\delta\mathcal{R}^{jk}
\end{aligned} \tag{2.7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and ∂_μ is the partial derivative w.r.t. the coordinate x^μ. The operators $\mathcal{R}, \mathcal{Z}$ and $\mathcal{U}$ vanish on a flat FLRW background, thus they contribute only to perturbations, and for convenience we can write $\mathcal{R} = \delta\mathcal{R} = \delta_1\mathcal{R} + \delta_2\mathcal{R}$, $\mathcal{Z} = \delta\mathcal{Z}$, $\mathcal{U} = \delta\mathcal{U}$, where $\delta_1\mathcal{R}$ and $\delta_2\mathcal{R}$ are the perturbations of the Ricci scalar respectively at first and second order. We now proceed with a simple expansion of the Lagrangian (2.4) up to second order:

$$\begin{aligned}
\delta L &= \bar{L} + L_N\delta N + L_K\delta K + L_\mathcal{S}\delta\mathcal{S} + L_\mathcal{R}\delta\mathcal{R} + L_\mathcal{U}\delta\mathcal{U} + L_\mathcal{Z}\delta\mathcal{Z} + \sum_{i=1}^{5} L_{\alpha_i}\delta\alpha_i \\
&+ \sum_{i=1}^{2} L_{\mathcal{Z}_i}\delta\mathcal{Z}_i + \frac{1}{2}\left(\delta N\frac{\partial}{\partial N} + \delta K\frac{\partial}{\partial K} + \delta\mathcal{S}\frac{\partial}{\partial\mathcal{S}} + \delta\mathcal{R}\frac{\partial}{\partial\mathcal{R}} + \delta\mathcal{U}\frac{\partial}{\partial\mathcal{U}}\right)^2 L \\
&+ \mathcal{O}(3),
\end{aligned} \tag{2.8}$$

where $\bar{L}$ is the Lagrangian evaluated on the background and $L_X = \partial L/\partial X$ is the derivative of the Lagrangian w.r.t the quantity X. It can be shown that by considering the perturbed quantities in (2.7) and, after some manipulations, it is possible to obtain the following expression for the action up to second order in perturbations:

$$\begin{aligned} \mathcal{S}_{ADM} &= \int d^4x\sqrt{-g}\Big[\bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} + (L_N - \dot{\mathcal{F}})\delta N + \left(\dot{\mathcal{F}} + \frac{1}{2}L_{NN}\right)(\delta N)^2 \\ &+ L_{\mathcal{S}}\delta K^\nu_\mu \delta K^\mu_\nu + \frac{1}{2}\mathcal{A}(\delta K)^2 + \mathcal{B}\delta N\delta K \mathcal{C}\delta K\delta\mathcal{R} + \mathcal{D}\delta N\delta\mathcal{R} + \mathcal{E}\delta\mathcal{R} \\ &+ \frac{1}{2}\mathcal{G}(\delta\mathcal{R})^2 + L_{\mathcal{Z}}\delta\mathcal{R}^\mu_\nu\mathcal{R}^\nu_\mu + L_{\alpha_1}\partial_i\delta N\partial^i\delta N + L_{\alpha_2}\partial_i\delta N\nabla_k\nabla^k\partial^i\delta N \\ &+ L_{\alpha_3}\mathcal{R}\nabla_i\partial^i\delta N + L_{\alpha_4}\partial_i\delta N\Delta^2\partial^i\delta N + L_{\alpha_5}\Delta\mathcal{R}\nabla_i\partial^i\delta N \\ &+ L_{\mathcal{Z}_1}\nabla_i\delta\mathcal{R}\nabla^i\delta\mathcal{R} + L_{\mathcal{Z}_2}\nabla_i\delta\mathcal{R}_{jk}\nabla^i\delta\mathcal{R}^{jk}\Big] , \end{aligned} \tag{2.9}$$

where:

$$\begin{aligned} &\mathcal{A} = L_{KK} + 4H^2 L_{\mathcal{SS}} - 4HL_{SK}, \\ &\mathcal{B} = L_{KN} - 2HL_{\mathcal{S}N}, \\ &\mathcal{C} = L_{KR} - 2HL_{\mathcal{S}R} + \frac{1}{2}L_{\mathcal{U}} - HL_{K\mathcal{U}} + 2H^2 L_{\mathcal{SU}}, \\ &\mathcal{D} = L_{N\mathcal{R}} + \frac{1}{2}\dot{L}_{\mathcal{U}} - HL_{N\mathcal{U}}, \\ &\mathcal{E} = L_{\mathcal{R}} - \frac{3}{2}HL_{\mathcal{U}} - \frac{1}{2}\dot{L}_{\mathcal{U}}, \\ &\mathcal{F} = L_K - 2HL_{\mathcal{S}}, \\ &\mathcal{G} = L_{\mathcal{RR}} + H^2 L_{\mathcal{UU}} - 2HL_{\mathcal{RU}}. \end{aligned} \tag{2.10}$$

Here and throughout the Chapter, unless stated otherwise, dots indicate derivatives w.r.t. cosmic time, t. The above quantities are general functions of time evaluated on the background. In order to obtain action (2.9), we have followed the same steps as in Refs. [37, 65], however, there are some differences in the results due to the different convention that we use for the normal vector (Eq. (2.3)). As a result the differences stem from the terms which contain K and $K_{\mu\nu}$. More details are in Appendix 2.8, where we derive the contribution of δK and $\delta\mathcal{S}$, and in Appendix 2.9, where we explicitly comment and derive the perturbations generated by $\mathcal{U}$.

Finally, we derive the modified Friedmann equations considering the first order action, which can be written as follows:

$$\mathcal{S}^{(1)}_{ADM} = \int d^4x\left[\delta\sqrt{h}(\bar{L} + 3H\mathcal{F} + \dot{\mathcal{F}}) + a^3(L_N + 3H\mathcal{F} + \bar{L})\delta N + a^3\mathcal{E}\delta_1\mathcal{R}\right], \tag{2.11}$$

where $\delta_1\mathcal{R}$ is the contribution of the Ricci scalar at first order. Notice that we used $\sqrt{-g} = N\sqrt{h}$, where h is the determinant of the three dimensional metric. It is straightforward to show that by varying the above action w.r.t. δN and $\delta\sqrt{h}$, one finds the Friedmann equations:

$$\begin{aligned} L_N + 3H\mathcal{F} + \bar{L} = 0\,, \\ \bar{L} + 3H\mathcal{F} + \dot{\mathcal{F}} = 0. \end{aligned} \tag{2.12}$$

Hence, the homogeneous part of action (2.9) vanishes after applying the Friedmann equations.

2.3.2 The EFT action in ADM notation

We shall now go back to the EFT action (2.1) and rewrite it in the ADM notation. This will allow us to easily compare it with action (2.9) and obtain a general recipe to map an ADM action into the EFT language. To this purpose, an important step is to connect the δg^{00} used in this formalism with δN used in the ADM formalism:

$$g^{00} = -\frac{1}{N^2} = -1 + 2\delta N - 3(\delta N)^2 + ... \equiv -1 + \delta g^{00}\,, \tag{2.13}$$

from which follows that $(\delta g^{00})^2 = 4(\delta N)^2$ at second order. Considering the Eqs. (2.7) and (2.13), it is very easy to write the EFT action in terms of ADM quantities, the only term which requires a bit of manipulation is $(1 + \Omega(t))R$, which we will show in the following. First, let us use the Gauss-Codazzi relation [19] which allows one to express the four dimensional Ricci scalar in terms of three dimensional quantities typical of ADM formalism:

$$R = \mathcal{R} + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\nu(n^\nu\nabla_\mu n^\mu - n^\mu\nabla_\mu n^\nu)\,. \tag{2.14}$$

Then, we can write:

$$\begin{aligned} \int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)R = & \int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)\left[\mathcal{R} + K_{\mu\nu}K^{\mu\nu} - K^2 \right. \\ & \left. +2\nabla_\nu\left(n^\nu\nabla_\mu n^\mu - n^\mu\nabla_\mu n^\nu\right)\right]\,, \\ = & \int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)\left[\mathcal{R} + \mathcal{S} - K^2 + 2\nabla_\nu\left(n^\nu K - a^\nu\right)\right]\,, \\ = & \int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}(1+\Omega)\left(\mathcal{R} + \mathcal{S} - K^2\right) + m_0^2\dot{\Omega}\frac{K}{N}\right]\,, \end{aligned} \tag{2.15}$$

where in the last line we have used that $\nabla^\nu a_\nu = 0$. Proceeding as usual and employing the relation (2.139), we obtain:

$$
\begin{aligned}
\int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)R = \int d^4x\sqrt{-g}m_0^2 \Big\{ &\frac{1}{2}(1+\Omega)\mathcal{R} + 3H^2(1+\Omega) \\
&+2\dot{H}(1+\Omega) + 2H\dot{\Omega} + \ddot{\Omega} + \Big[H\dot{\Omega} - 2\dot{H}(1+\Omega) - \ddot{\Omega}\Big]\delta N \\
&-\dot{\Omega}\delta K\delta N + \frac{(1+\Omega)}{2}\delta K^\mu_\nu \delta K^\nu_\mu - \frac{(1+\Omega)}{2}(\delta K)^2 \\
&+ \Big[2\dot{H}(1+\Omega) + 2H\dot{\Omega} + \ddot{\Omega} - 3H\dot{\Omega}\Big](\delta N)^2 \Big\}.
\end{aligned}
\tag{2.16}
$$

Finally, after combining terms correctly, we obtain the final form of the EFT action in the ADM notation, up to second order in perturbations:

$$
\begin{aligned}
\mathcal{S}_{EFT} = \int d^4x\sqrt{-g}\Big\{ &\frac{m_0^2}{2}(1+\Omega)\mathcal{R} + 3H^2m_0^2(1+\Omega) + 2\dot{H}m_0^2(1+\Omega) \\
&+2m_0^2H\dot{\Omega} + m_0^2\ddot{\Omega} + \Lambda + \Big[H\dot{\Omega}m_0^2 - 2\dot{H}m_0^2(1+\Omega) - \ddot{\Omega}m_0^2 - 2c\Big]\delta N \\
&-(m_0^2\dot{\Omega} + \bar{M}_1^3)\delta K\delta N + \frac{1}{2}\left[m_0^2(1+\Omega) - \bar{M}_3^2\right]\delta K^\mu_\nu\delta K^\nu_\mu - \frac{1}{2}\left[m_0^2(1+\Omega)\right. \\
&\left.+\bar{M}_2^2\right](\delta K)^2 + \hat{M}^2\delta N\delta\mathcal{R} + \Big[2\dot{H}m_0^2(1+\Omega) + \ddot{\Omega}m_0^2 - Hm_0^2\dot{\Omega} + 3c \\
&+2M_2^4\Big](\delta N)^2 + 4m_2^2h^{\mu\nu}\partial_\mu\delta N\partial_\nu\delta N + \frac{\bar{m}_5}{2}\delta\mathcal{R}\delta K + \lambda_1(\delta\mathcal{R})^2 \\
&+\lambda_2\delta\mathcal{R}^\mu_\nu\delta\mathcal{R}^\nu_\mu + 2\lambda_3\delta\mathcal{R}h^{\mu\nu}\nabla_\mu\partial_\nu\delta N + 4\lambda_4h^{\mu\nu}\partial_\mu\delta N\nabla^2\partial_\nu\delta N \\
&+\lambda_5h^{\mu\nu}\nabla_\mu\mathcal{R}\nabla_\nu\mathcal{R} + \lambda_6h^{\mu\nu}\nabla_\mu\mathcal{R}_{ij}\nabla_\nu\mathcal{R}^{ij} + 4\lambda_7h^{\mu\nu}\partial_\mu\delta N\nabla^4\partial_\nu\delta N \\
&+2\lambda_8h^{\mu\nu}\nabla^2\mathcal{R}\nabla_\mu\partial_\nu\delta N\Big\}\ .
\end{aligned}
\tag{2.17}
$$

This final form of the action will be the starting point from which we will construct a general mapping between the EFT and ADM formalisms.

2.3.3 The Mapping

We now proceed to explicitly work out the mapping between the EFT action (2.17) and the ADM one (2.9). The result will be a very convenient recipe in order to quickly map any model written in the ADM notation into the EFT formalism. In the next Section we will apply it to most of the interesting candidate models of DE/MG, providing a complete guide on how to go from covariant formulations all the way to the EFT formalism at the basis of the Einstein-Boltzmann solver *EFTCAMB* [22, 23].

A direct comparison between actions (2.9) and (2.17) allows us to straightforwardly identify the following:

$$
\begin{aligned}
&\frac{m_0^2}{2}(1+\Omega) = \mathcal{E}\,, \qquad -2c + m_0^2\left[-2\dot{H}(1+\Omega) - \ddot{\Omega} + H\dot{\Omega}\right] = L_N - \dot{\mathcal{F}}\,,\\
&\Lambda + m_0^2\left[3H^2(1+\Omega) + 2\dot{H}(1+\Omega) + 2H\dot{\Omega} + \ddot{\Omega}\right] = \bar{L} + 3H\mathcal{F} + \dot{\mathcal{F}}\,,\\
&m_0^2\left[2\dot{H}(1+\Omega) - H\dot{\Omega} + \ddot{\Omega}\right] + 2M_2^4 + 3c = \dot{\mathcal{F}} + \frac{L_{NN}}{2}\,,\\
&-m_0^2(1+\Omega) - \bar{M}_2^2 = \mathcal{A}, \quad \lambda_1 = \frac{\mathcal{G}}{2}, \quad -m_0^2\dot{\Omega} - \bar{M}_1^3 = \mathcal{B}, \quad \frac{\bar{m}_5}{2} = \mathcal{C},\\
&\quad \hat{M}^2 = \mathcal{D}, \quad \frac{m_0^2}{2}(1+\Omega) - \frac{\bar{M}_3^2}{2} = L_{\mathcal{S}}, \quad 4m_2^2 = L_{\alpha_1}, \quad \lambda_5 = L_{\mathcal{Z}_1}, \quad 4\lambda_4 = L_{\alpha_2},\\
&\quad 2\lambda_3 = L_{\alpha_3}, \qquad 4\lambda_7 = L_{\alpha_4}, \quad 2\lambda_8 = L_{\alpha_5}, \quad \lambda_2 = L_{\mathcal{Z}}, \quad \lambda_6 = L_{\mathcal{Z}_2}.
\end{aligned}
\tag{2.18}
$$

It is now simply a matter of inverting these relations in order to obtain the desired general mapping results:

$$
\begin{aligned}
&\Omega(t) = \frac{2}{m_0^2}\mathcal{E} - 1, \qquad c(t) = \frac{1}{2}(\dot{\mathcal{F}} - L_N) + (H\dot{\mathcal{E}} - \ddot{\mathcal{E}} - 2\mathcal{E}\dot{H}),\\
&\Lambda(t) = \bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} - (6H^2\mathcal{E} + 2\ddot{\mathcal{E}} + 4H\dot{\mathcal{E}} + 4\dot{H}\mathcal{E})\,, \qquad \bar{M}_2^2(t) = -\mathcal{A} - 2\mathcal{E},\\
&M_2^4(t) = \frac{1}{2}\left(L_N + \frac{L_{NN}}{2}\right) - \frac{c}{2}, \qquad \bar{M}_1^3(t) = -\mathcal{B} - 2\dot{\mathcal{E}}, \qquad \bar{M}_3^2(t) = -2L_{\mathcal{S}} + 2\mathcal{E},\\
&m_2^2(t) = \frac{L_{\alpha_1}}{4}, \qquad \bar{m}_5(t) = 2\mathcal{C}, \qquad \hat{M}^2(t) = \mathcal{D}, \qquad \lambda_1(t) = \frac{\mathcal{G}}{2},\\
&\lambda_2(t) = L_{\mathcal{Z}}, \qquad \lambda_3(t) = \frac{L_{\alpha_3}}{2}, \qquad \lambda_4(t) = \frac{L_{\alpha_2}}{4}, \qquad \lambda_5(t) = L_{\mathcal{Z}_1},\\
&\lambda_6(t) = L_{\mathcal{Z}_2}, \qquad \lambda_7(t) = \frac{L_{\alpha_4}}{4}, \qquad \lambda_8(t) = \frac{L_{\alpha_5}}{2}.
\end{aligned}
\tag{2.19}
$$

Let us stress that the above definitions of the EFT functions are very useful if one is interested in writing a specific action in EFT language. Indeed the only step required before applying (2.19), is to write the action which specifies the chosen theory in ADM form, without the need of perturbing the theory and its action up to quadratic order.

The expressions of the EFT functions corresponding to a given model, and their time-dependence, are all that is needed in order to implement a specific model of DE/MG in *EFTCAMB* and have it solve for the dynamics of perturbations, outputting observable quantities of interest. Since *EFTCAMB* uses the scale factor as the time variable and the Hubble parameter expressed w.r.t conformal time, one needs to convert the cosmic time t in the argument of the functions in Eq. (2.19) into the scale factor, a, their time derivatives into derivatives w.r.t. the scale factor and transform the Hubble parameter into the one in conformal time τ, while considering it a function of a, see Ref. [80]. This is a straightforward step and we will give some examples in Appendix 2.10.

Let us conclude this Section looking at the equations for the background. Working with the EFT action, and expanding it to first order while using the ADM notation, one obtains:

$$\begin{aligned}
\mathcal{S}^{(1)}_{EFT} &= \int d^4x \left\{ a^3 \frac{m_0^2}{2} (1+\Omega)\, \delta_1 \mathcal{R} + \left[3H^2 m_0^2 (1+\Omega) + 2\dot{H} m_0^2 (1+\Omega) \right.\right. \\
&+ \left. 2m_0^2 H\dot{\Omega} + m_0^2 \ddot{\Omega} + \Lambda \right] \delta\sqrt{h} + a^3 \left[3H\dot{\Omega} m_0^2 - 2c + 3H^2 m_0^2 (1+\Omega) \right. \\
&+ \left.\left. \Lambda \right] \delta N \right\} , \qquad (2.20)
\end{aligned}$$

therefore the variation w.r.t. δN and $\delta\sqrt{h}$ yields:

$$\begin{aligned}
&3H\dot{\Omega} m_0^2 - 2c + 3H^2 m_0^2 (1+\Omega) + \Lambda = 0 \, , \\
&3H^2 m_0^2 (1+\Omega) + 2\dot{H} m_0^2 (1+\Omega) + 2m_0^2 H \dot{\Omega} + m_0^2 \ddot{\Omega} + \Lambda = 0 \, .
\end{aligned} \qquad (2.21)$$

Using the mapping (2.19), it is easy to verify that these equations correspond to those in the ADM formalism (2.12). Once the mapping (2.19) has been worked out, it is straightforward to obtain the Friedmann equations without having to vary the action for each specific model.

2.4 Model mapping examples

Having derived the precise mapping between the ADM formalism and the EFT approach in Section 2.3.3, we proceed to apply it to some specific cases which are of cosmological interest, i.e. minimally coupled quintessence [71], $f(R)$ theory [11], Horndeski/GG [78, 79], GLPV [66] and Hořava gravity [64]. The mapping of some of these theories is already present in the literature (see Refs. [14, 15, 39, 65, 74, 75] for more details). However, since one of the main purposes of this work is to provide a self-contained and general recipe that can be used to easily implement a specific theory in *EFTCAMB*, we will present all the mapping of interest, including those that are already in the literature due to the aforementioned differences in the definition of the normal vector and some of the EFT functions. Let us notice that the mapping of the GLPV Lagrangians in particular, is one of the new results obtained in this work.

2.4.1 Minimally coupled quintessence

As illustrated in Refs. [14, 15, 75], the mapping of minimally coupled quintessence [71] into EFT functions is very straightforward. The typical action for such a model is of the following form:

$$\mathcal{S}_\phi = \int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}R - \frac{1}{2}\partial^\nu\phi\partial_\nu\phi - V(\phi)\right] , \tag{2.22}$$

where $\phi(t, x^i)$ is a scalar field and $V(\phi)$ is its potential. Let us proceed by rewriting the second term in unitary gauge and in ADM quantities:

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \;\rightarrow\; -\frac{\dot{\phi}_0^2(t)}{2}g^{00} \equiv \frac{\dot{\phi}_0^2(t)}{2N^2} , \tag{2.23}$$

where $\phi_0(t)$ is the field background value. Substituting back into the action we get, in the ADM formalism, the following action:

$$\mathcal{S}_\phi = \int d^4x\sqrt{-g}\left\{\frac{m_0^2}{2}\left[\mathcal{R} + \mathcal{S} - K^2\right] + \frac{1}{N^2}\frac{\dot{\phi}_0^2(t)}{2} - V(\phi_0)\right\} , \tag{2.24}$$

where we have used the Gauss-Codazzi relation (2.14) to express the four dimensional Ricci scalar in terms of three dimensional quantities. Now, since the initial covariant action has been written in terms of ADM quantities, we can finally apply the results in Eqs. (2.19) to get the EFT functions:

$$\Omega(t) = 0, \qquad c(t) = \frac{\dot{\phi}_0^2}{2}, \qquad \Lambda(t) = \frac{\dot{\phi}_0^2}{2} - V(\phi_0). \tag{2.25}$$

Notice that the other EFT functions are zero. In Refs. [14, 15] the above mapping has been obtained directly from the covariant action while our approach follows more strictly the one adopted in Ref. [75]. However, let us notice that w.r.t. it, our results differ due to a different definition of the background EFT functions.*

*The background EFT functions adopted here are related to the ones in Ref. [75], by the following relations:

$$1 + \Omega(t) = f(t) , \qquad \Lambda(t) = -\tilde{\Lambda}(t) + c(t) , \qquad c(t) = \tilde{c}(t) . \tag{2.26}$$

where f and tildes quantities correspond to the EFT functions in Ref. [75]. These differences are due to the fact that in our formalism we have in the EFT action the term $-c\delta g^{00}$ while in the other formalism the authors use $-\tilde{c}g^{00}$, therefore an extra contribution to $\tilde{\Lambda}$ from this operator comes when using $g^{00} = -1 + \delta g^{00}$. Instead the different definition of the conformal coupling function, Ω, is due to numerical reasons related to the implementation of the EFT approach in CAMB.

Moreover, in order to use them in *EFTCAMB* one need to convert them in conformal time τ, therefore one has:

$$c(\tau) = \mathcal{H}^2 \frac{\phi_0'^2}{2}\,, \qquad \Lambda(\tau) = \mathcal{H}^2 \frac{\phi_0'^2}{2} - V(\phi_0)\,, \tag{2.27}$$

where the prime indicates the derivative w.r.t. the scale factor, $a(\tau)$, and $\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\tau}$ is the Hubble parameter in conformal time. Minimally coupled quintessence models are already implemented in the public versions of *EFTCAMB* [80].

2.4.2 $f(R)$ gravity

The second example we shall illustrate is that of $f(R)$ gravity [10, 11]. The mapping of the latter into the EFT language was derived in Refs. [14, 75]. Here, we present an analogous approach which uses the ADM formalism. Let us start with the action :

$$\mathcal{S}_f = \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left[R + f(R)\right], \tag{2.28}$$

where $f(R)$ is a general function of the four dimensional Ricci scalar.

In order to map it into our EFT approach, we will proceed to expand this action around the background value of the Ricci scalar, $R^{(0)}$. Therefore, we choose a specific time slicing where the constant time hypersurfaces coincide with uniform R hypersurfaces. This allows us to truncate the expansion at the linear order because higher orders will always contribute one power or more of δR to the equations of motion, which vanishes. For a more complete analysis we refer the reader to Ref. [14] . After the expansion we obtain the following Lagrangian:

$$\mathcal{S}_f = \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left\{\left[1 + f_R(R^{(0)})\right] R + f(R^{(0)}) - R^{(0)} f_R(R^{(0)})\right\}, \tag{2.29}$$

where $f_R \equiv \frac{df}{dR}$. In the ADM formalism the above action reads:

$$\begin{aligned} \mathcal{S}_f &= \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left\{\left[1 + f_R(R^{(0)})\right]\left[\mathcal{R} + \mathcal{S} - K^2\right] + \frac{2}{N}\dot{f}_R K \right. \\ &+ \left. f(R^{(0)}) - R^{(0)} f_R(R^{(0)})\right\}, \end{aligned} \tag{2.30}$$

where we have used as usual the Gauss Codazzi relation (2.14). Using Eqs. (2.19), it is easy to calculate that the only non zero EFT functions for $f(R)$ gravity are:

$$\Omega(t) = f_R(R^{(0)})\,, \qquad \Lambda(t) = \frac{m_0^2}{2} f(R^{(0)}) - R^{(0)} f_R(R^{(0)})\,. \tag{2.31}$$

The public version of *EFTCAMB* already contains the designer $f(R)$ models [12, 80, 81], while the specific Hu-Sawicki model has been implemented through the full mapping procedure [82].

2.4.3 The Galileon Lagrangians

The Galileon class of theories were derived in Ref. [83], by studying the decoupling limit of the five dimensional model of modified gravity known as DGP [84]. In this limit, the dynamics of the scalar d.o.f., corresponding to the longitudinal mode of the massive graviton, decouple from gravity and enjoy a galilean shift symmetry around Minkowski background, as a remnant of the five dimensional Poincare' invariance [85]. Requiring the scalar field to obey this symmetry and to have second order equations of motion allows one to identify a finite amount of terms that can enter the action. These terms are typically organized into a set of Lagrangians which, subsequently, have been covariantized [86] and the final form is what is known as the Generalized Galileon (GG) model [79]. This set of models represent the most general theory of gravity with a scalar d.o.f. and second order field equations in four dimensions and has been shown to coincide with the class of theories derived by Horndeski in Ref. [78]. It is therefore common to refer to these models with the terms GG and Horndeski gravity, alternatively. GG models have been deeply investigated in the cosmological context, since they display self accelerated solutions which can be used to realize both a single field inflationary scenario at early times [87–96] and a late time accelerated expansion [97–101]. Moreover, on small scales these models naturally display the Vainshtein screening mechanism [102, 103], which can efficiently hide the extra d.o.f. from local tests of gravity [83, 85, 104–108].

GG models include most of the interesting and viable theories of DE/MG that we aim to test against cosmological data. To this extent, the Einstein-Boltzmann solver *EFTCAMB* can be readily used to explore these theories both in a model-independent way, through a subset of the EFT functions, and in a model-specific way [22, 80]. In the latter case, the first step consists of mapping a given GG model into the EFT language. In the following we derive the general mapping between GG and EFT functions, in order to provide an instructive and self-consistent compendium to easily map any given GG model into the formalism at the basis of *EFTCAMB*.

Let us introduce the GG action:

$$\mathcal{S}_{GG} = \int d^4x\sqrt{-g}\left(L_2 + L_3 + L_4 + L_5\right), \tag{2.32}$$

where the Lagrangians have the following structure:

$$
\begin{aligned}
L_2 &= \mathcal{K}(\phi, X)\,,\\
L_3 &= G_3(\phi, X)\Box\phi\,,\\
L_4 &= G_4(\phi, X)R - 2G_{4X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right]\,,\\
L_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu}\right.\\
&\quad \left. +2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}\right]\,,
\end{aligned}
\tag{2.33}
$$

here $G_{\mu\nu}$ is the Einstein tensor, $X \equiv \phi^{;\mu}\phi_{;\mu}$ is the kinetic term and $\{\mathcal{K}, G_i\}$ $(i = 3, 4, 5)$ are general functions of the scalar field ϕ and X, and $G_{iX} \equiv \partial G_i/\partial X$. Moreover, $\Box = \nabla^2$ and ; stand for the covariant derivative w.r.t. the metric $g_{\mu\nu}$. The mapping of GG is already present in the literature. For instance in Ref. [74] the mapping is obtained directly from the covariant Lagrangians, while in Refs. [65, 75] the authors start from the ADM version of the action. In this Chapter we present in details all the steps from the covariant Lagrangians (2.33) to their expressions in ADM quantities; we then use the mapping (2.19) to obtain the EFT functions corresponding to GG. This allows us to give an instructive presentation of the method, while providing a final result consistent with the EFT conventions at the basis of *EFTCAMB*. Throughout these steps, we will highlight the differences w.r.t. Refs. [65, 74, 75] which arise because of different conventions. Finally, in Appendix 2.10 we rewrite the results of this Section with the scale factor as the independent variable and the Hubble parameter defined w.r.t. the conformal time, making them readily implementable in *EFTCAMB*.

Since the GG action is formulated in covariant form, we shall use the following relations to rewrite the GG Lagrangians in ADM form:

$$
n_\mu = \gamma\phi_{;\mu}, \qquad \gamma = \frac{1}{\sqrt{-X}}, \qquad \dot{n}_\mu = n^\nu n_{\mu;\nu}\,, \tag{2.34}
$$

where we have, as usual, assumed that constant time hypersurfaces correspond to uniform field ones. We notice that the acceleration, $\dot{n}_\mu$, and the extrinsic curvature $K^{\mu\nu}$ are orthogonal to the normal vector. This allows us to decompose the covariant derivative of the normal vector as follows:

$$
n_{\nu;\mu} = K_{\mu\nu} - n_\mu\dot{n}_\nu. \tag{2.35}
$$

With these definitions it can be easily verified that:

$$
\phi_{;\mu\nu} = \gamma^{-1}(K_{\mu\nu} - n_\mu\dot{n}_\nu - n_\nu\dot{n}_\mu) + \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}n_\mu n_\nu, \tag{2.36}
$$

$$
\Box\phi = \gamma^{-1}K - \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}. \tag{2.37}
$$

- **$\mathbf{L_2}$- Lagrangian**

Let us start with the simplest of the Lagrangians which can be Taylor expanded in the kinetic term X, around its background value X_0, as follows:

$$\mathcal{K}(\phi, X) = \mathcal{K}(\phi_0, X_0) + \mathcal{K}_X(\phi_0, X_0)(X - X_0) + \frac{1}{2}\mathcal{K}_{XX}(X - X_0)^2, \quad (2.38)$$

where in terms of ADM quantities we have:

$$X = -\frac{\dot{\phi}_0(t)^2}{N^2} = \frac{X_0}{N^2}\,. \quad (2.39)$$

Now by applying the results in Eqs. (2.19), the corresponding EFT functions can be written as:

$$\Lambda(t) = \mathcal{K}(\phi_0, X_0), \qquad c(t) = \mathcal{K}_X(\phi_0, X_0)X_0 \qquad M_2^4(t) = \mathcal{K}_{XX}(\phi_0, X_0)X_0^2. \quad (2.40)$$

The differences with previous works in this case are the ones listed in Eq. (2.26).

- **$\mathbf{L_3}$- Lagrangian**

In order to rewrite this Lagrangian into the desired form, which depends only on ADM quantities, we introduce an auxiliary function:

$$G_3 \equiv F_3 + 2XF_{3X}\,. \quad (2.41)$$

We proceed to plug this in the L_3-Lagrangian (2.33) and using Eq. (2.37) we obtain, up to a total derivative:

$$L_3 = -F_{3\phi}X - 2(-X)^{3/2}F_{3X}K\,. \quad (2.42)$$

Now going to unitary gauge and considering Eq. (2.39), we can directly use (2.19). Let us start with $c(t)$:

$$\begin{aligned} c(t) \;&= \tfrac{1}{2}(\mathcal{F} - L_N) = -3\dot{\phi}_0^2\ddot{\phi}_0 F_{3X} + 2\ddot{\phi}_0 F_{3XX}\dot{\phi}_0^4 - \dot{\phi}_0^4 F_{3X\phi} + F_{3\phi}\dot{\phi}_0^2 \\ &\quad - F_{3\phi X}\dot{\phi}_0^4 - 6H\dot{\phi}_0^5 F_{3XX} + 9HF_{3X}\dot{\phi}_0^3\,. \end{aligned} \quad (2.43)$$

Now we want to eliminate the dependence on the auxiliary function F_3. In order to do this, we need to recombine terms by using the following:

$$\begin{aligned} &G_3 = F_3 + 2XF_{3X}, \quad G_{3\phi} = F_{3\phi} - 2\dot{\phi}_0^2 F_{3X\phi}, \quad G_{3X} = 3F_{3X} - 2\dot{\phi}_0^2 F_{3XX}\,, \\ &G_{3XX} = 3F_{3XX} - 2\dot{\phi}_0^2 F_{3XXX} + 2F_{3XX}, \quad G_{3\phi X} = 3F_{3X\phi} - 2\dot{\phi}_0^2 F_{3\phi XX}\,, \end{aligned} \quad (2.44)$$

which gives the final expression:

$$c(t) = \dot{\phi}_0^2 G_{3X}(3H\dot{\phi}_0 - \ddot{\phi}_0) + G_{3\phi}\dot{\phi}_0^2. \tag{2.45}$$

Now let us move on to the remaining non zero EFT functions corresponding to the L_3 Lagrangian:

$$\begin{aligned}
&\Lambda(t) = \bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} = G_{3\phi}\dot{\phi}_0^2 - 2\ddot{\phi}_0\dot{\phi}_0^2 G_{3X}\,, \\
&\bar{M}_1^3(t) = -L_{KN} = -2G_{3X}\dot{\phi}_0^3\,, \\
&M_2^4(t) = \frac{1}{2}\left(L_N + \frac{L_{NN}}{2}\right) - \frac{c}{2} = G_{3X}\frac{\dot{\phi}_0^2}{2}(\ddot{\phi}_0 + 3H\dot{\phi}_0) - 3HG_{3XX}\dot{\phi}_0^5 \\
&\qquad -G_{3\phi X}\frac{\dot{\phi}_0^4}{2}\,,
\end{aligned} \tag{2.46}$$

where we have used the relations (2.44). In the definitions of the EFT functions, G_3 and its derivatives are evaluated on the background. We suppressed the dependence on (ϕ_0, X_0) to simplify the final expressions. Before proceeding to map the remaining GG Lagrangians, let us comment on the differences w.r.t. the results in literature [65, 74, 75]. The results coincide up to two notable exceptions. The background functions are redefined as presented in Eq. (2.26) and $\bar{M}_1^3 = -\bar{m}_1^3$. In the latter term, the minus sign is not a simple redefinition but rather comes from the fact that our extrinsic curvature has an overall minus sign difference due to the definition of the normal vector. Therefore, the term proportional to $\delta K \delta g^{00}$ will always differ by a minus sign.

- **$\mathbf{L_4}$- Lagrangian**

Let us now consider the L_4 Lagrangian:

$$L_4 = G_4 R - 2G_{4X}\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right]. \tag{2.47}$$

After some preliminary manipulations of the Lagrangian, we get:

$$L_4 = G_4\mathcal{R} + 2G_{4X}(K^2 - K_{\mu\nu}K^{\mu\nu}) + 2G_{4X}X_{;\lambda}(Kn^\lambda - \dot{n}^\lambda)\,. \tag{2.48}$$

We proceed by using the relation:

$$\partial_\mu G_4 = G_{4X}X_{;\mu} + G_{4\phi}\phi_{;\mu}\,, \tag{2.49}$$

which we substitute in the last term of the Lagrangian (2.48) and, using integration by parts, we get:

$$L_4 = G_4\mathcal{R} + (2G_{4X}X - G_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) + 2G_{4\phi}\sqrt{-X}K\,, \tag{2.50}$$

where we have used the Gauss-Codazzi relation (2.14). Let us recall that we can relate $\phi_{;\mu}$ to X by using Eq. (2.39).

Finally, in the same spirit as for L_3, we derive from the Lagrangian (2.50) the corresponding non zero EFT functions by using the results (2.19):

$$
\begin{aligned}
\Omega(t) &= -1 + \frac{2}{m_0^2} G_4 \,, \\
c(t) &= -\frac{1}{2}\Big(- \dot{L}_K + 2\dot{H} L_{\mathcal{S}} + 2H\dot{L}_{\mathcal{S}} \Big) + H\dot{L}_{\mathcal{R}} - \ddot{L}_{\mathcal{R}} - 2\dot{H} L_{\mathcal{R}} \\
&= G_{4X}(2\ddot{\phi}_0^2 + 2\dot{\phi}_0 \dddot{\phi}_0 + 4\dot{H}\dot{\phi}_0^2 + 2H\dot{\phi}_0\ddot{\phi}_0 - 6H^2\dot{\phi}_0^2) \\
&\quad + G_{4X\phi}(2\dot{\phi}_0^2\ddot{\phi}_0 + 10H\dot{\phi}_0^3) + G_{4XX}(12H^2\dot{\phi}_0^4 - 8H\dot{\phi}_0^3\ddot{\phi}_0 - 4\dot{\phi}_0^2\ddot{\phi}_0^2) \,, \\
\Lambda(t) &= \bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} - (6H^2 L_{\mathcal{R}} + 2\ddot{L}_{\mathcal{R}} + 4H\dot{L}_{\mathcal{R}} + 4\dot{H} L_{\mathcal{R}}), \\
&= G_{4X}\left[12H^2\dot{\phi}_0^2 + 8\dot{H}\dot{\phi}_0^2 + 16H\dot{\phi}_0\ddot{\phi}_0 + 4(\ddot{\phi}_0^2 + \dot{\phi}_0\dddot{\phi}_0) \right] \\
&\quad - G_{4XX}\big(16H\dot{\phi}_0^3\ddot{\phi}_0 + 8\dot{\phi}_0^2\ddot{\phi}_0^2\big) + 8HG_{4X\phi}\dot{\phi}_0^3 \,, \\
M_2^4(t) &= \frac{1}{2}(L_N + L_{NN}/2) - \frac{c}{2} = G_{4\phi X}\big(4H\dot{\phi}_0^3 - \ddot{\phi}_0\dot{\phi}_0^2\big) - 6H\dot{\phi}_0^5 G_{4\phi XX} \\
&\quad - G_{4X}\left(2\dot{H}\dot{\phi}_0^2 + H\dot{\phi}_0\ddot{\phi}_0 + \dot{\phi}_0\dddot{\phi}_0 + \ddot{\phi}_0^2 \right) \\
&\quad + G_{4XX}\big(18H^2\dot{\phi}_0^4 + 2\dot{\phi}_0^2\ddot{\phi}_0^2 + 4H\ddot{\phi}_0\dot{\phi}_0^3\big) - 12H^2 G_{4XXX}\dot{\phi}_0^6 \,, \\
\bar{M}_2^2(t) &= -L_{KK} - 2L_{\mathcal{R}} = 4G_{4X}\dot{\phi}_0^2 \,, \\
\bar{M}_3^2(t) &= -2L_{\mathcal{S}} + 2L_{\mathcal{R}} = -4G_{4X}\dot{\phi}_0^2 \equiv -\bar{M}_2^2(t) \,, \\
\hat{M}^2(t) &= L_{N\mathcal{R}} = 2\dot{\phi}_0^2 G_{4X} \,, \\
\bar{M}_1^3(t) &= 2HL_{\mathcal{S}N} - 2\dot{L}_{\mathcal{R}} - L_{KN} = G_{4X}(4\dot{\phi}_0\ddot{\phi}_0 + 8H\dot{\phi}_0^2) \\
&\quad - 16HG_{4XX}\dot{\phi}_0^4 - 4G_{4\phi X}\dot{\phi}_0^3 \,,
\end{aligned} \tag{2.51}
$$

where also in this case G_4 and its derivative are evaluated on the background. Let us notice that the above relations satisfy the conditions which define Horndeski/GG theories, i.e.:

$$
\bar{M}_2^2 = -\bar{M}_3^2(t) = 2\hat{M}^2(t), \tag{2.52}
$$

as found in Refs. [65, 74]. Finally, besides the differences mentioned previously for the L_2 and L_3 Lagrangians which also apply here, we notice that $\hat{M}^2 = \mu_1^2$ when comparing with Ref. [65].

- **L_5- Lagrangian**

Finally, let us conclude with the L_5 Lagrangian. This Lagrangian contains cubic terms which makes it more complicated to express it in the ADM form:

$$L_5 = G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}\right]. \tag{2.53}$$

In order to rewrite L_5, we have to enlist once again the help of an auxiliary function, F_5, which is defined as follows:

$$G_{5X} \equiv F_{5X} + \frac{F_5}{2X}. \tag{2.54}$$

Then, using this definition, we get the following relation:

$$G_{5X}X_{;\rho} = \gamma\nabla_\rho(\gamma^{-1}F_5) - F_{5\phi}\gamma^{-1}n_\rho. \tag{2.55}$$

Let us start with the first term of the Lagrangian, which can be written as:

$$G_5G_{\mu\nu}\phi^{;\mu\nu} = F_5\phi^{;\mu\nu}G_{\mu\nu} - \frac{\gamma}{2}X^{;\nu}n^\mu G_{\mu\nu}F_5 + (F_{5\phi} - G_{5\phi})\gamma^{-2}n^\mu n^\nu G_{\mu\nu}, \tag{2.56}$$

hence we need to rewrite $F_5\phi^{;\mu\nu}G_{\mu\nu}$ in terms of ADM quantities which can be achieved by employing the following relation:

$$\begin{aligned} K^{\mu\nu}G_{\mu\nu} &= KK^{\mu\nu}K_{\mu\nu} - K^3_{\mu\nu} + \mathcal{R}_{\mu\nu}K - K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} - \frac{1}{2}K\big(\mathcal{R} - K^2 \\ &+ K_{\mu\nu}K^{\mu\nu} - 2R_{\mu\nu}n^\mu n^\nu\big). \end{aligned} \tag{2.57}$$

This leads to the following:

$$\begin{aligned} F_5\phi^{;\mu\nu}G_{\mu\nu} = {} & F_5(\gamma^{-1}(-2R_{\mu\nu}n^\mu\dot{n}^\nu) + \frac{\gamma^2}{2}n^\mu n^\nu\phi^{;\lambda}X_{;\lambda}G_{\mu\nu}) \\ & + F_5\gamma^{-1}\big[KK^{\mu\nu}K_{\mu\nu} - K^3_{\mu\nu} + \mathcal{R}_{\mu\nu}K^{\mu\nu} - K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} \\ & - \frac{1}{2}K\big(\mathcal{R} - K^2 + K_{\mu\nu}K^{\mu\nu} - 2R_{\mu\nu}n^\mu n^\nu\big)\big]. \end{aligned} \tag{2.58}$$

The second term of the Lagrangian can be computed by considering Eqs. (2.36)-(2.37), which yields:

$$\begin{aligned} &\frac{1}{3}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}\right] = \\ &= \frac{G_{5X}}{3}\gamma^{-3}\big(K^3 - 3KS + 2K_{\mu\nu}K^{\mu\sigma}K^\nu_\sigma\big) + G_{5X}\big(-\frac{1}{2}K^2\phi_{;\lambda}X^{;\lambda} - 2\dot{n}_\sigma\dot{n}_\nu K^{\nu\sigma} \\ &+ \frac{S}{2}\phi_{;\lambda}X^{;\lambda} + 2\gamma^{-3}K\dot{n}^\nu\dot{n}_\nu\big) \\ &= \frac{G_{5X}}{3}\gamma^{-3}\tilde{\mathcal{K}} + G_{5X}\mathcal{J}, \end{aligned} \tag{2.59}$$

where the definitions of $\tilde{\mathcal{K}}$ and $\mathcal{J}$ come directly from the second line of the above expression. In Appendix 2.11 we treat in detail the $G_{5X}\mathcal{J}$ term but for now we simply state the final result:

$$G_{5X}\mathcal{J} = F_5\gamma^{-1}\Big[\frac{\tilde{\mathcal{K}}}{2}+K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho}+\dot{n}^\sigma n^\rho R_{\sigma\rho}-Kn^\sigma n^\rho R_{\sigma\rho}\Big]-\frac{F_{5\phi}}{2}(K^2-S). \tag{2.60}$$

Hence, after collecting all the terms, we get:

$$L_5 = F_5\sqrt{-X}\Big(K^{\mu\nu}\mathcal{R}_{\mu\nu} - \frac{1}{2}K\mathcal{R}\Big) + (G_{5\phi} - F_{5\phi})X\frac{\mathcal{R}}{2} + \frac{(-X)^{3/2}}{3}G_{5X}\tilde{\mathcal{K}} + \frac{G_{5\phi}}{2}X(K^2 - K_{\mu\nu}K^{\mu\nu})\,. \tag{2.61}$$

Now, in order to proceed with the mapping, we need to analyse $\tilde{\mathcal{K}}$ and $\mathcal{U} = K^{\mu\nu}\mathcal{R}_{\mu\nu}$ terms. The latter will be treated as in Appendix 2.9, while the former can be written up to third order as follows:

$$\tilde{\mathcal{K}} = -6H^3 - 6H^2K - 3HK^2 + 3HK_{\mu\nu}K^{\mu\nu} + \mathcal{O}(3). \tag{2.62}$$

Finally, the ultimate Lagrangian is:

$$L_5 = F_5\sqrt{-X}\Big(\mathcal{U} - \frac{1}{2}K\mathcal{R}\Big) + (G_{5\phi} - F_{5\phi})X\frac{\mathcal{R}}{2} + \frac{(-X)^{3/2}}{3}G_{5X}(-6H^3 - 6H^2K - 3HK^2 + 3H\mathcal{S}) + \frac{G_{5\phi}}{2}X(K^2 - \mathcal{S})\,. \tag{2.63}$$

Although F_5 is present in the above Lagrangian, it will disappear when computing the EFT functions as was the case for L_3. At this point we can write down the non zero EFT functions as follows:

$$\begin{aligned}
\Omega(t) &= \frac{2}{m_0^2}\left(G_{5X}\ddot{\phi}_0\dot{\phi}_0^2 - G_{5\phi}\frac{\dot{\phi}_0^2}{2}\right) - 1\,,\\
c(t) &= \frac{1}{2}\dot{\tilde{\mathcal{F}}} + \frac{3}{2}Hm_0^2\dot{\Omega} - 3H^2\dot{\phi}_0^2G_{5\phi} + 3H^2\dot{\phi}_0^4G_{5\phi X} - 3H^3\dot{\phi}_0^3G_{5X}\\
&\quad + 2H^3\dot{\phi}_0^5G_{5XX}\,,\\
\Lambda(t) &= \dot{\tilde{\mathcal{F}}} - 3m_0^2H^2(1+\Omega) + 4G_{5X}H^3\dot{\phi}_0^3 + 3HG_{5\phi}\dot{\phi}_0^2\,,\\
M_2^4(t) &= -\frac{\dot{\tilde{\mathcal{F}}}}{4} - \frac{3}{4}Hm_0^2\dot{\Omega} - 2H^3G_{5XXX}\dot{\phi}_0^7 - 3H^2\dot{\phi}_0^6G_{5\phi XX} + 6G_{5XX}H^3\dot{\phi}_0^5\\
&\quad + 6H^2G_{5\phi X}\dot{\phi}_0^4 - \frac{3}{2}H^3G_{5X}\dot{\phi}_0^3\,,\\
\hat{M}^2(t) &= -G_{5X}\dot{\phi}_0^2\ddot{\phi}_0 + HG_{5X}\dot{\phi}_0^3 + G_{5\phi}\dot{\phi}_0^2\,,
\end{aligned}$$

$$\bar{M}_2^2(t) = -\bar{M}_3^2(t) = 2\hat{M}^2(t)\,,$$
$$\bar{M}_1^3(t) = -m_0^2\dot{\Omega} + 4H\dot{\phi}_0^2 G_{5\phi} - 4H\dot{\phi}_0^4 G_{5\phi X} - 4H^2\dot{\phi}_0^5 G_{5XX} + 6H^2\dot{\phi}_0^3 G_{5X}\,, \quad (2.64)$$

with $\tilde{\mathcal{F}} = \mathcal{F} - m_0^2\dot{\Omega} - 2Hm_0^2(1+\Omega) = 2H^2 G_{5X}\dot{\phi}_0^3 + 2HG_{5\phi}\dot{\phi}_0^2 - m_0^2\dot{\Omega} - 2Hm_0^2(1+\Omega)$. We have omitted, in the EFT functions, the dependence on the background quantities ϕ_0 and X_0 of G_5 and its derivatives. Finally we recover, as expected, the relation (2.52).

2.4.4 GLPV Lagrangians

We shall now move on to the beyond Hordenski models derived by Gleyzes *et al.* [66, 67], known as GLPV. These build on the premises of the Galileon models and include some extra terms in the Lagrangians that, while contributing higher order spatial derivatives in the field equations, maintain second order equations of motion for the true propagating d.o.f.. Specifically, the GLPV action assumes the following form:

$$\mathcal{S}_{\rm GLPV} = \int d^4x\sqrt{-g}\left[L_2^{GG} + L_3^{GG} + L_4^{GG} + L_5^{GG} + L_4^{\rm GLPV} + L_5^{\rm GLPV}\right]\,, \quad (2.65)$$

where L_i^{GG}(i=2,3,4,5) are the GG Lagrangians listed in Eq.(2.33) and the new terms to be added to the GG Lagrangians are the following:

$$L_4^{\rm GLPV} = \tilde{F}_4(\phi,X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\phi_{;\mu}\phi_{;\mu'}\phi_{;\nu\nu'}\phi_{\rho\rho'}\,,$$
$$L_5^{\rm GLPV} = \tilde{F}_5(\phi,X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_{;\mu}\phi_{;\mu'}\phi_{;\nu\nu'}\phi_{;\rho\rho'}\phi_{;\sigma\sigma'}\,, \quad (2.66)$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor and $\tilde{F}_4, \tilde{F}_5$ are two new arbitrary functions of (ϕ, X).

As usual, we will first express the new Lagrangians in terms of ADM quantities using, among others, relations (2.36)-(2.37), and we get:

$$L_4^{\rm GLPV} = -X^2\tilde{F}_4(\phi,X)(K^2 - K_{ij}K^{ij})\,,$$
$$L_5^{\rm GLPV} = \tilde{F}_5(\phi,X)(-X)^{5/2}\tilde{\mathcal{K}}$$
$$= \tilde{F}_5(\phi,X)(-X)^{5/2}(-6H^3 - 6H^2K - 3HK^2 + 3HK_{\mu\nu}K^{\mu\nu})\,. \quad (2.67)$$

The last equality holds up to second order in perturbations. It is now easy to apply the familiar procedure. Moreover, since different Lagrangians contribute separately to the EFT functions, we can simply calculate the EFT functions corresponding to the new Lagrangians (2.67) and add those to the results previously derived for the GG Lagrangians.

- $\mathbf{L}_4^{\mathrm{GLPV}}$**- Lagrangian**

Let us start with the operators included in the L_4^{GLPV} Lagrangian:

$$L_4^{\mathrm{GLPV}} = -X^2\tilde{F}_4(K^2 - \mathcal{S}). \tag{2.68}$$

We can easily derive the following quantities that are useful for the mapping:

$$\begin{aligned}
L_K &= 6H\dot{\phi}_0^4\tilde{F}_4, \quad L_{\mathcal{S}} = \dot{\phi}_0^4\tilde{F}_4, \quad L_{KK} = -2\dot{\phi}_0^4\tilde{F}_4, \\
L_N &= 4\frac{\dot{\phi}_0^4}{N^5}\tilde{F}_4(K^2 - \mathcal{S}) = 24H^2\dot{\phi}_0^4\tilde{F}_4 \quad L_{NN} = -120\dot{\phi}_0^4\tilde{F}_4H^2, \\
L_{NK} &= -24H\dot{\phi}_0^4\tilde{F}_4, \quad L_{N\mathcal{S}} = -4\dot{\phi}_0^4\tilde{F}_4, \quad \mathcal{F} = 4H\dot{\phi}_0^4\tilde{F}_4\,, \\
\dot{\mathcal{F}} &= 4\dot{H}\dot{\phi}_0^4\tilde{F}_4 + 16H\tilde{F}_4\dot{\phi}_0^3\ddot{\phi}_0 - 8H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X} + 4H\dot{\phi}_0^5\tilde{F}_{4\phi}\,.
\end{aligned} \tag{2.69}$$

Using the relations (2.19), we obtain the non-zero EFT functions corresponding to L_4^{GLPV}:

$$\begin{aligned}
c(t) &= 2\dot{H}\dot{\phi}_0^4\tilde{F}_4 + 8H\dot{\phi}_0^3\ddot{\phi}_0\tilde{F}_4 - 4H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X} + 2H\tilde{F}_{4\phi}\dot{\phi}_0^5 - 12H^2\dot{\phi}_0^4\tilde{F}_4\,, \\
\Lambda(t) &= 6H^2\dot{\phi}_0^4\tilde{F}_4 + 4\dot{H}\dot{\phi}_0^4\tilde{F}_4 + 16H\dot{\phi}_0^3\ddot{\phi}_0\tilde{F}_4 + 4H\dot{\phi}_0^5\tilde{F}_{4\phi} - 8H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X}\,, \\
M_2^4(t) &= -18\dot{\phi}_0^4\tilde{F}_4H^2 - \dot{H}\dot{\phi}_0^4\tilde{F}_4 - 4H\dot{\phi}_0^3\ddot{\phi}_0\tilde{F}_4 + 2H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X} - H\tilde{F}_{4\phi}\dot{\phi}_0^5 + 6H^2\dot{\phi}_0^4\tilde{F}_4\,, \\
\bar{M}_2^2(t) &= 2\dot{\phi}_0^4\tilde{F}_4, \\
\bar{M}_1^3(t) &= 16H\dot{\phi}_0^4\tilde{F}_4, \\
\bar{M}_3^2(t) &= -\bar{M}_2^2(t)\,.
\end{aligned} \tag{2.70}$$

As before, $\tilde{F}_4$ and its derivatives are evaluated on the background, therefore they only depend on time.

- $\mathbf{L}_5^{\mathrm{GLPV}}$**- Lagrangian**

Let us now consider the last Lagrangian:

$$L_5^{\mathrm{GLPV}} = -(-X)^{5/2}\tilde{F}_5(-6H^3 - 6H^2K - 3HK^2 + 3H\mathcal{S})\,, \tag{2.71}$$

which gives the derivatives, w.r.t. ADM quantities, one needs to obtain the mapping:

$$\begin{aligned}
L_K &= -12H^2\dot{\phi}_0^5\tilde{F}_5, \quad L_{\mathcal{S}} = -3H\dot{\phi}_0^5\tilde{F}_5, \quad L_N = 5\frac{\dot{\phi}_0^5}{N^6}\tilde{F}_5\tilde{\mathcal{K}} = -30\dot{\phi}_0^5H^3\tilde{F}_5\,, \\
L_{KK} &= 6H\dot{\phi}_0^5\tilde{F}_5, \quad L_{NN} = 180H^3\dot{\phi}_0^5\tilde{F}_5, \quad L_{NK} = 60\dot{\phi}_0^5\tilde{F}_5H^2, \\
L_{N\mathcal{S}} &= 15H\dot{\phi}_0^5\tilde{F}_5, \quad \mathcal{F} = -6H^2\dot{\phi}_0^5\tilde{F}_5, \\
\dot{\mathcal{F}} &= 12H^2\dot{\phi}_0^6\tilde{F}_{5X}\ddot{\phi}_0 - 12H\dot{H}\dot{\phi}_0^5\tilde{F}_5 - 30H^2\dot{\phi}_0^4\tilde{F}_5\ddot{\phi}_0 - 6H^2\dot{\phi}_0^6\tilde{F}_{5\phi}\,.
\end{aligned} \tag{2.72}$$

Employing these, allows us to obtain the non-zero EFT functions:

$$\begin{aligned}
\Lambda(t) &= -3H^3\dot{\phi}_0^5\tilde{F}_5 - 12H\dot{H}\dot{\phi}_0^5\tilde{F}_5 - 30H^2\dot{\phi}_0^4\tilde{F}_5\ddot{\phi}_0 + 12H^2\dot{\phi}_0^6\tilde{F}_{5X}\ddot{\phi}_0 - 6H^2\dot{\phi}_0^6\tilde{F}_{5\phi}\,, \\
c(t) &= 6H^2\dot{\phi}_0^6\ddot{\phi}_0\tilde{F}_{5X} - 6H\dot{H}\dot{\phi}_0^5\tilde{F}_5 - 15H^2\dot{\phi}_0^4\tilde{F}_5\ddot{\phi}_0 - 3H^2\dot{\phi}_0^6\tilde{F}_{5\phi} + 15\dot{\phi}_0^5H^3\tilde{F}_5\,, \\
M_2^4(t) &= \frac{45}{2}\dot{\phi}_0^5H^3\tilde{F}_5 + 3H\dot{H}\dot{\phi}_0^5\tilde{F}_5 + \frac{15}{2}H^2\dot{\phi}_0^4\ddot{\phi}_0\tilde{F}_5 - 3H^2\dot{\phi}_0^6\ddot{\phi}_0\tilde{F}_{5X} + \frac{3}{2}H^2\dot{\phi}_0^6\tilde{F}_{5\phi}\,, \\
\bar{M}_2^2(t) &= -6H\dot{\phi}_0^5\tilde{F}_5, \\
\bar{M}_1^3(t) &= -30H^2\dot{\phi}_0^5\tilde{F}_5, \\
\bar{M}_3^2(t) &= -\bar{M}_2^2(t)\,. \qquad (2.73)
\end{aligned}$$

As usual the functions $\tilde{F}_5$ and its derivatives are functions of time. Their expressions in terms of the scale factor and the Hubble parameter w.r.t. conformal time can be found in Appendix 2.10. Let us notice that GLPV models correspond to:

$$\bar{M}_2^2 = -\bar{M}_3^2, \qquad (2.74)$$

which is a less restrictive condition than the one defining GG theories (2.52) as $\bar{M}_2^2 \neq 2\hat{M}^2$.

Let us conclude this Section by working out the mapping between the EFT functions and a common way to write the GLPV action. This action is built directly in terms of geometrical quantities, hence guaranteeing the unitary gauge since the scalar d.o.f. has been eaten by the metric [66]. Therefore now we will consider the following GLPV Lagrangian instead of the one defined previously:

$$\begin{aligned}
L_{\text{GLPV}} &= A_2(t,N) + A_3(t,N)K + A_4(t,N)(K^2 - K_{ij}K^{ij}) + B_4(t,N)\mathcal{R} \\
&+ A_5(t,N)\left(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j_k\right) \\
&+ B_5(t,N)K^{ij}\left(\mathcal{R}_{ij} - h_{ij}\frac{\mathcal{R}}{2}\right)\,, \qquad (2.75)
\end{aligned}$$

where A_i, B_i are general functions of t and N, and can be expressed in terms of the scalar field, ϕ, , as shown in Ref. [66], effectively creating the equivalence between the above Lagrangian and the one introduced in Eq. (2.65).

It is very easy to write the above Lagrangian in terms of the quantities introduced in Section 2.3.1:

$$\begin{aligned}
L_{\text{GLPV}} &= A_2(t,N) + A_3(t,N)K + A_4(t,N)(K^2 - \mathcal{S}) + B_4(t,N)\mathcal{R} \\
&+ A_5(t,N)\left(-6H^3 - 6H^2K - 3HK^2 + 3H\mathcal{S}\right) \\
&+ B_5(t,N)\left(\mathcal{U} - \frac{\mathcal{R}K}{2}\right)\,.
\end{aligned} \qquad (2.76)$$

Now, we can compute the quantities that we need for the mapping (2.19):

$$
\begin{aligned}
&\bar{L} = \bar{A}_2 - 3H\bar{A}_3 + 6H^2\bar{A}_4 - 6H^3\bar{A}_5\,, \quad \mathcal{E} = \bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\,,\\
&\mathcal{F} = \bar{A}_3 - 4H\bar{A}_4 + 6H^2\bar{A}_5\,, \quad L_{\mathcal{S}} = -\bar{A}_4 + 3H\bar{A}_5\,,\\
&L_K = \bar{A}_3 - 6H\bar{A}_4 + 12H^2\bar{A}_5, \quad L_{KK} = 2\bar{A}_4 - 6H\bar{A}_5,\\
&L_N = \bar{A}_{2N} - 3H\bar{A}_{3N} + 6H^2\bar{A}_{4N} - 6H^3\bar{A}_{5N}\,, \quad L_{\mathcal{U}} = \bar{B}_5\,,\\
&L_{NN} = \bar{A}_{2NN} - 3H\bar{A}_{3N} + 6H^2\bar{A}_{4NN} - 6H^3\bar{A}_{5NN}\,,\\
&L_{\mathcal{S}N} = -\bar{A}_{4N} + 3H\bar{A}_{5N}, \quad L_{KN} = \bar{A}_{3N} - 6H\bar{A}_{4N} + 12H^2\bar{A}_{5N}\,,\\
&L_{K\mathcal{R}} = -\frac{1}{2}\bar{B}_5\,, \quad L_{N\mathcal{U}} = \bar{B}_{5N}\,, \quad L_{N\mathcal{R}} = \bar{B}_{4N} + \frac{3}{2}H\bar{B}_{5N}\,,
\end{aligned} \tag{2.77}
$$

where the quantities with the bar are evaluated in the background and A_{iY} means derivative of A_i w.r.t. Y. Then the EFT functions follow from Eq. (2.19):

$$
\begin{aligned}
\Omega(t) &= \frac{2}{m_0^2}\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right) - 1\,,\\
\Lambda(t) &= \bar{A}_2 - 6H^2\bar{A}_4 + 12H^3\bar{A}_5 + \dot{\bar{A}}_3 - 4\dot{H}\bar{A}_4 - 4H\dot{\bar{A}}_4 + 6H^2\dot{\bar{A}}_5 + 12H\dot{H}\bar{A}_5\\
&\quad - \left[2(3H^2 + 2\dot{H})\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right) + 2\ddot{\bar{B}}_4 - \bar{B}_5^{(3)} + 4H\left(\dot{\bar{B}}_4 - \frac{1}{2}\ddot{\bar{B}}_5\right)\right],\\
c(t) &= \frac{1}{2}\Big(\dot{\bar{A}}_3 - 4\dot{H}\bar{A}_4 - 4H\dot{\bar{A}}_4 + 6H^2\dot{\bar{A}}_5 + 12H\dot{H}\bar{A}_5 - \bar{A}_{2N} + 3H\bar{A}_3N\Big)\\
&\quad -6H^2\bar{A}_{4N} + 6H^3\bar{A}_{5N} + H\left(\dot{\bar{B}}_4 - \frac{1}{2}\ddot{\bar{B}}_5\right) - \ddot{\bar{B}}_4 + \frac{1}{2}\bar{B}_5^{(3)}\\
&\quad -2\dot{H}\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right),\\
\bar{M}_2^2(t) &= = -2\bar{A}_4 + 6H\bar{A}_5 - 2\bar{B}_4 + \dot{\bar{B}}_5\,,\\
\bar{M}_1^3(t) &= -\bar{A}_{3N} + 4H\bar{A}_{4N} - 6H^2\bar{A}_{5N} - 2\dot{\bar{B}}_4 + \ddot{\bar{B}}_5\,,\\
\bar{M}_3^2(t) &\equiv -\bar{M}_2^2(t)\,,\\
M_2^4(t) &= \frac{1}{4}\left(\bar{A}_{2NN} - 3H\bar{A}_{3NN} + 6H^2\bar{A}_{4NN} - 6H^3\bar{A}_{5NN}\right) - \frac{1}{4}\Big(\dot{\bar{A}}_3 - 4\dot{H}\bar{A}_4\\
&\quad -4H\dot{\bar{A}}_4 + 6H^2\dot{\bar{A}}_5 + 12H\dot{H}\bar{A}_5\Big) + \frac{3}{4}\big(\bar{A}_{2N} - 3H\bar{A}_{3N} + 6H^2\bar{A}_{4N}\\
&\quad -6H^3\bar{A}_{5N}\big) - \frac{1}{2}\Bigg[H\left(\dot{\bar{B}}_4 - \frac{1}{2}\ddot{\bar{B}}_5\right) - \ddot{\bar{B}}_4 + \frac{1}{2}\bar{B}_5^{(3)}\\
&\quad -2\dot{H}\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right)\Bigg],\\
\hat{M}^2(t) &= \bar{B}_{4N} + \frac{1}{2}H\bar{B}_{5N} + \frac{1}{2}\dot{\bar{B}}_5\,.
\end{aligned} \tag{2.78}
$$

The condition (2.74) is satisfied as desired and one can focus on the GG subset of theories by enforcing the condition $\bar{M}_2^2(t) = 2\hat{M}^2(t)$.

2.4.5 Hořava Gravity

One of the main aspects of our paper is the inclusion of operators with higher order spatial derivatives in the EFT action. Thus, it is natural to proceed with the mapping of the most popular theory containing such operators, i.e. Hořava gravity [35, 36]. This theory is a recent proposed candidate to describe the gravitational interaction in the ultra-violet regime (UV). This is done by breaking the Lorentz symmetry resulting in a modification of the graviton propagator. Practically, this amounts to adding higher-order spatial derivatives to the action while keeping the time derivatives at most second order, in order to avoid Ostrogradsky instabilities [25]. As a result, time and space are treated on a different footing, therefore the natural formulation in which to construct the action is the ADM one. It has been shown that, in order to obtain a power-counting renormalizable theory, the action needs to contain terms with up to sixth-order spatial derivatives [62–64]. The resulting action does not demonstrate full diffeomorphism invariance but is rather invariant under a restricted symmetry, the foliation preserving diffeomorphisms (for a review see [55, 59] and references therein). Besides the UV regime, Hořava gravity has taken hold on the cosmological side as well as it exhibits a rich phenomenology [40–47, 49–51, 53] and very recently it has started to be constrained in that context [39, 48, 52, 54, 56–58].

Here, we will consider the following action which contains up to six order spatial derivatives, (and is therefore included in the extended EFT action):

$$\begin{aligned}\mathcal{S}_H &= \frac{1}{16\pi G_H}\int d^4x\sqrt{-g}\left[K_{ij}K^{ij} - \lambda K^2 - 2\xi\bar{\Lambda} + \xi\mathcal{R} + \eta a_i a^i + g_1\mathcal{R}^2\right. \\ &+ g_2\mathcal{R}_{ij}\mathcal{R}^{ij} + g_3\mathcal{R}\nabla_i a^i + g_4 a_i\Delta a^i + g_5\mathcal{R}\Delta\mathcal{R} + g_6\nabla_i\mathcal{R}_{jk}\nabla^i\mathcal{R}^{jk} \\ &+ \left. g_7 a_i\Delta^2 a^i + g_8\Delta\mathcal{R}\nabla_i a^i\right] , \end{aligned} \tag{2.79}$$

where the coefficients λ, η, ξ and g_i are running coupling constants, $\bar{\Lambda}$ is the "bare" cosmological constant and G_H is the coupling constant [39, 64]:

$$\frac{1}{16\pi G_H} = \frac{m_0^2}{(2\xi - \eta)} . \tag{2.80}$$

The above action is already in unitary gauge and ADM form, then we just need few steps to write it in terms of the quantities introduced in

Section 2.3.1:

$$\begin{aligned}\mathcal{S}_H &= \frac{1}{16\pi G_H}\int d^4x\sqrt{-g}\left[S-\lambda K^2-2\xi\bar{\Lambda}+\xi\mathcal{R}+\eta\alpha_1+g_1\mathcal{R}^2+g_2\mathcal{Z}\right.\\ &\left.+g_3\alpha_3+g_4\alpha_2-g_5\mathcal{Z}_1+g_6\mathcal{Z}_2+g_7\alpha_4+g_8\alpha_5\right], \qquad (2.81)\end{aligned}$$

then by using the results (2.19) it is easy to show that the EFT functions read:

$$\begin{aligned}
&m_0^2(1+\Omega)=\frac{2m_0^2\xi}{(2\xi-\eta)}, \qquad c(t)=-\frac{m_0^2}{(2\xi-\eta)}(1+2\xi-3\lambda)\dot{H},\\
&\Lambda(t)=\frac{2m_0^2}{(2\xi-\eta)}\left[-\xi\bar{\Lambda}-(1-3\lambda+2\xi)\left(\frac{3}{2}H^2+\dot{H}\right)\right],\\
&\bar{M}_3^2=-\frac{2m_0^2}{(2\xi-\eta)}(1-\xi), \qquad \bar{M}_2^2=-2\frac{m_0^2}{(2\xi-\eta)}(\xi-\lambda),\\
&m_2^2=\frac{m_0^2}{4(2\xi-\eta)}\eta, \qquad M_2^4(t)=\frac{m_0^2}{2(2\xi-\eta)}(1+2\xi-3\lambda)\dot{H},\\
&\lambda_1=g_1\frac{m_0^2}{(2\xi-\eta)}, \qquad \lambda_2=g_2\frac{m_0^2}{(2\xi-\eta)},\\
&\lambda_3=g_3\frac{m_0^2}{2(2\xi-\eta)}, \qquad \lambda_4=g_4\frac{m_0^2}{4(2\xi-\eta)}, \qquad \lambda_5=-g_5\frac{m_0^2}{(2\xi-\eta)}\\
&\lambda_6=g_6\frac{m_0^2}{(2\xi-\eta)}, \qquad \lambda_7=g_7\frac{m_0^2}{4(2\xi-\eta)}, \qquad \lambda_8=g_8\frac{m_0^2}{2(2\xi-\eta)}, \qquad (2.82)
\end{aligned}$$

and the remaining EFT functions are zero. The mapping of Hořava gravity has been worked out in details in Ref. [39], by some of the authors of this paper. Subsequently, the low-energy part of Hořava action, which is described by $\{\Omega, c, \Lambda, \bar{M}_3^2, \bar{M}_2^2, M_2^4, m_2^2\}$, has been implemented in *EFTCAMB* [80] and constraints on the low-energy parameters $\{\xi, \eta, \lambda\}$ have been obtained in Ref. [39].

2.5 Stability

Along with its unifying aspect, a very important advantage of the EFToDE/MG formalism is that of being formulated at the level of the action in a model independent way. By inspecting the EFT action expanded to quadratic order in the perturbations, it is possible to impose conditions on the EFT functions to ensure that none of the undesired instabilities develop. It has been preliminary shown in Ref. [23], that the impact of such conditions can be quite significant as they can efficiently reduce the parameter space that one needs to explore when performing a fit to data. In some cases they have been shown to dominate over the constraining power of current data [23].

The study of the theoretical viability of the EFT action has already been performed to some extent in the literature [14, 15, 65, 67, 68], however here we will include in the analysis, for the first time, higher order operators and consider also the instabilities related to a negative squared mass of the scalar d.o.f.. Specifically, we will consider three possible instabilities: ghost and gradient instabilities both in the scalar and tensor sector, and tachyonic scalar modes (for a review see Ref. [109]). Starting from the general action (2.17), we expand it up to quadratic order in tensor and scalar perturbations of the metric around a flat FLRW background. Our focus is on the stability of the gravity sector, hence we will not consider matter fluids. The complete stability analysis of the general action (2.17) in the presence of a matter sector is the main topic of the next Chapter.

Let us consider the following metric perturbations for the scalar components:

$$ds^2 = -(1+2\delta N)dt^2 + 2\partial_i\psi dt dx^i + a^2(1+2\zeta)\delta_{ij}dx^i dx^j\,, \qquad (2.83)$$

where as usual $\delta N(t,x^i)$ is the perturbation of the lapse function, $\partial_i\psi(t,x^i)$ and $\zeta(t,x^i)$ are the scalar perturbations respectively of the shift function and the three dimensional metric. Then, the scalar perturbations of the quantities involved in the action (2.17) are:

$$\begin{aligned}
\delta K &= -3\dot{\zeta} + 3H\delta N + \frac{1}{a^2}\partial^2\psi\,,\\
\delta K_{ij} &= a^2\delta_{ij}(H\delta N - 2H\zeta - \dot{\zeta}) + \partial_i\partial_j\psi\,,\\
\delta K^i_j &= (H\delta N - \dot{\zeta})\delta^i_j + \frac{1}{a^2}\partial^i\partial_j\psi\,,\\
\delta\mathcal{R}_{ij} &= -(\delta_{ij}\partial^2\zeta + \partial_i\partial_j\zeta)\,,\\
\delta_1\mathcal{R} &= -\frac{4}{a^2}\partial^2\zeta\,,\\
\delta_2\mathcal{R} &= -\frac{2}{a^2}[(\partial_i\zeta)^2 - 4\zeta\partial^2\zeta]. \qquad (2.84)
\end{aligned}$$

Now, we can expand action (2.17) to quadratic order in metric perturbations. In the following we will Fourier transform the spatial part* and

*More properly, in Fourier space we should write $(\zeta(t,k))^2 \to \zeta_{\mathbf{k}}\zeta_{-\mathbf{k}}$, however in the following we prefer to drop the indices in order to simplify the notation.

after regrouping terms, we obtain:

$$
\begin{aligned}
\mathcal{S}_{EFT}^{(2)} \; = \; & \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \Big\{-\left(\mathcal{W}_0+\mathcal{W}_3k^2+\mathcal{W}_2k^4\right)k^2\zeta^2-3a^2\mathcal{W}_4\dot{\zeta}\delta N \\
& -\frac{3}{2}a^2\mathcal{W}_5(\dot{\zeta})^2-\left(\mathcal{W}_4\delta N+\mathcal{W}_5\dot{\zeta}-\mathcal{W}_7k^2\psi+\frac{2}{a^4}\bar{m}_5k^2\zeta\right)k^2\psi \\
& +\left(\mathcal{W}_1+4m_2^2\frac{k^2}{a^2}-4\frac{\lambda_4}{a^4}k^4+4\frac{\lambda_7}{a^6}k^6\right)(\delta N)^2 \\
& -\left(\mathcal{W}_6+8\lambda_3\frac{k^2}{a^4}+8\frac{\lambda_8}{a^6}k^4\right)\delta N k^2\zeta\Big\}\,,
\end{aligned}
\tag{2.85}
$$

where:

$$
\begin{aligned}
\mathcal{W}_0 &= -\frac{1}{a^2}\left[m_0^2(1+\Omega)+3H\bar{m}_5+3\dot{\bar{m}}_5\right]\,, \\
\mathcal{W}_1 &= c+2M_2^4-3m_0^2H^2(1+\Omega)-3m_0^2H\dot{\Omega}-\frac{3}{2}H^2\bar{M}_3^2-\frac{9}{2}H^2\bar{M}_2^2-3H\bar{M}_1^3\,, \\
\mathcal{W}_2 &= -16\frac{\lambda_5}{a^6}-6\frac{\lambda_6}{a^6}\,, \\
\mathcal{W}_3 &= -16\frac{\lambda_1}{a^4}-6\frac{\lambda_2}{a^4}\,, \\
\mathcal{W}_4 &= \frac{1}{a^2}\left(-2m_0^2H(1+\Omega)-m_0^2\dot{\Omega}-H\bar{M}_3^2-\bar{M}_1^3-3H\bar{M}_2^2\right)\,, \\
\mathcal{W}_5 &= \frac{1}{a^2}\left(2m_0^2(1+\Omega)+\bar{M}_3^2+3\bar{M}_2^2\right)\,, \\
\mathcal{W}_6 &= -\frac{4}{a^2}\left(\frac{1}{2}m_0^2(1+\Omega)+\hat{M}^2\right)-6H\frac{\bar{m}_5}{a^2}\,, \\
\mathcal{W}_7 &= -\frac{1}{2a^4}\left(\bar{M}_3^2+\bar{M}_2^2\right)\,.
\end{aligned}
\tag{2.86}
$$

In this action we have three d.o.f. $\{\zeta, \delta N, \psi\}$, but in reality only one, ζ, is dynamical, while the other two, $\{\delta N, \psi\}$, are auxiliary fields. This implies that they can be eliminated through the constraint equations obtained by varying the above action w.r.t. them. We will leave for the next Sections the details of such a calculation, here we want to outline the general procedure we are adopting. After replacing back in the action the general expression for δN and ψ , we end up with an action of the form:

$$
\mathcal{S}_{EFT}^{(2)}=\frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2-\left[k^2G(t,k)+\bar{M}(t,k)\right]\zeta^2\right\}\,.
\tag{2.87}
$$

where $\bar{M}(t,k)$ depends on inverse powers of k. $\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)$ is usually called the kinetic term and its positivity guarantees that the theory is free from ghost in the scalar sector. The variation of the above action w.r.t. ζ gives:

$$\ddot{\zeta} + \left(3H + \frac{\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\right)\dot{\zeta} + \left(k^2\frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}} + \frac{\bar{M}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\right)\zeta = 0\,, \tag{2.88}$$

where the coefficient of $\dot{\zeta}$ is called the friction term and its sign will damp or enhance the amplitude of the field fluctuations. $\bar{M}/\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$ is called the dispersion coefficient which, in principle, can be both negative and positive. Finally, we define the propagation speed as:

$$c_s^2 \equiv \frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}. \tag{2.89}$$

Let us note that the speed of propagation and the dispersion coefficient (or "mass" term) and their effective counterparts have non-local expressions. Therefore, their interpretation as the actual physical entities might be ambiguous at first glance because usually these quantities are defined in some specific limit, where they assume local expressions. In this work, we still retain the labeling of speed of propagation and mass term for the non-local expressions, because they reduce to the corresponding local and physical quantity when the proper limit is considered. Moreover, the non-local definitions are the ones which serve to our purpose, since they represent the proper quantities on which the stability conditions have to be imposed in order to guarantee a viable theory at all times and scales.

Now, let us perform a field redefinition in order to have a canonical action. This step is important in order to identify the correct conditions to avoid the gradient and tachyonic instabilities, in particular the last one which is related to the condition of boundedness from below of the corresponding canonical Hamiltonian. We will show that not only the mass is sensitive to this normalization, as it is known, but that in the general case in which the kinetic term is scale-dependent also the speed of propagation, is affected by the field redefinition. In general, we can use:

$$\zeta(t,k) = \frac{\phi(t,k)}{\sqrt{2\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)}}, \tag{2.90}$$

which, once applied to the action (2.87), gives:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \left[\frac{1}{2}\dot{\phi}^2 - c^2_{s,\text{eff}}(t,k)\frac{k^2}{2}\phi^2 - m^2_{\text{eff}}(t,k)\phi^2\right], \tag{2.91}$$

where $m_{\rm eff}(t,k)$ is an effective mass and depends on inverse powers of k, while $c^2_{s,\rm eff}(t,k)$ is the effective speed of propagation.

When $\mathcal{L}_{\dot\zeta\dot\zeta}$ is only a function of time, the field redefinition (2.90) will give time-dependent contributions only to $\bar{M}$ thus generating $m^2_{\rm eff}$ and leaving G unaffected. In this case we have:

$$c^2_{s,\rm eff}(t,k) = c^2_s(t,k)\,,$$
$$m^2_{\rm eff}(t) = \frac{\mathcal{L}_{\dot\zeta\dot\zeta}\left(4\bar{M}(t) - 2\ddot{\mathcal{L}}_{\dot\zeta\dot\zeta}\right) + \dot{\mathcal{L}}^2_{\dot\zeta\dot\zeta} - 6H\mathcal{L}_{\dot\zeta\dot\zeta}\dot{\mathcal{L}}_{\dot\zeta\dot\zeta}}{8\mathcal{L}^2_{\dot\zeta\dot\zeta}}\,. \quad (2.92)$$

Let us notice that in case in which the kinetic term depends only on time, the term $\bar{M}$ usually turns out to be zero or at most a function of time.

On the contrary, when $\mathcal{L}_{\dot\zeta\dot\zeta}$ exhibits a k-dependence, the field redefinition will affect both $\bar{M}$ and G and in general $c^2_{s,\rm eff} \neq c^2_s$ and the above expression for the effective mass does not hold anymore. In Section 2.5.2 we will discuss the general expressions for these two quantities. In general, the GLPV class of theories belongs to the case in which $\mathcal{L}_{\dot\zeta\dot\zeta}$ is only a function of time. When one starts including operators like $\{m^2_2, \bar{m}_5, \lambda_i, \bar{M}^2_3 \neq -\bar{M}^2_2\}$, k-dependence will be generated in the kinetic term. In the following sections we will analyse these cases in details.

Finally, in order to study the stability, one has to analyse the evolution of the field equation obtained by varying the action (2.91) w.r.t. ϕ, i.e.:

$$\ddot\phi + 3H\dot\phi + \left(k^2 c^2_{s,\rm eff} + m^2_{\rm eff}\right)\phi = 0, \quad (2.93)$$

In this case H represents a friction term, which is always positive, and $m^2_{\rm eff}$ is the dispersion coefficient. A negative value of the effective mass squared generates a tachyonic instability, however requiring $m^2_{\rm eff}$ to be positive is a stringent condition. It is sufficient to guarantee that the time scale on which the instability evolves is longer then the time evolution of the system [109] in order to be free of said instability. Therefore, we can require that, when $m^2_{\rm eff} < 0$, the typical evolution scale is of the same order as the Hubble time, H_0.

Continuing, in order to avoid gradient instabilities one enforces a positive value of the effective speed of propagation. In the simpler cases in which the kinetic term depends only on time (e.g. Horndeski and GLPV theories), the normalization of the field leaves the speed of sound unchanged, i.e. $c^2_s = c^2_{\rm eff}$, thus the condition to impose is $c^2_{\rm eff} = c^2_s > 0$. For the more general case in which the kinetic term depends on scale in a non trivial way, $c^2_{\rm eff} = c^2_s + f(t,k)$ (see Section 2.5.2 for the full expression of $f(t,k)$); however, in the high k-limit, where the gradient instability shows up, $f(t,k)$ is maximally of order $\mathcal{O}(1/k^2)$ which can be neglected in this limit. Therefore, the condition on the effective speed of propagation reduces indeed to the original condition on the speed of propagation, i.e. $c_s^2 > 0$. In summary, in order to guarantee the stability of the scalar sector the combination of $c_{\rm eff}^2 > 0$ and $m_{\rm eff}^2 > 0$, along with the no-ghost condition, i.e. $\mathcal{L}_{\dot\zeta\dot\zeta} > 0$, provides the full set of stability conditions.

We conclude with the stability analysis on the tensor modes. The perturbed metric components which contribute to tensor modes are:

$$g_{ij}^T(t,x^i) = a^2 h_{ij}^T(t,x^i)\,, \tag{2.94}$$

therefore, the terms containing tensor perturbations in (2.17), are the following:

$$\delta\mathcal{R}_{ij} = -\frac{\delta^{lk}}{a^2}\partial_l\partial_k h_{ij}^T, \quad \delta_2\mathcal{R} = \frac{1}{a^2}\Big(\frac{3}{4}\partial_k h_{ij}^T\partial^k h^{ij\,T} + h_{ij}^T\partial^2 h^{ij\,T} \\ -\frac{1}{2}\partial_k h_{ij}^T\partial^j h^{ik\,T}\Big), \quad \delta K^i_j = -\frac{\dot h^{i\,T}_j}{2} \tag{2.95}$$

where $\delta_2\mathcal{R}$ is the second order perturbation of the Ricci scalar, $\mathcal{R}$. Then, the EFT action for tensor perturbations up to second order reads:

$$\begin{aligned}\mathcal{S}^{T\,(2)}_{EFT} \;&=\; \int d^4x\, a^3\left\{\frac{m_0^2}{2}(1+\Omega)\delta_2\mathcal{R} + \left(\frac{m_0^2}{2}(1+\Omega) - \frac{\bar M_3^2}{2}\right)\delta K^i_j\delta K^j_i\right. \\ &\left. +\lambda_2\delta\mathcal{R}_{ij}\delta\mathcal{R}^{ij} + \lambda_6\frac{\tilde g^{kl}}{a^2}\partial_k\mathcal{R}_{ij}\partial_l\mathcal{R}^{ij}\right\},\end{aligned} \tag{2.96}$$

from which we can notice that only four EFT functions describe the dynamics of tensors, i.e. $\{\Omega, \bar M_3^2, \lambda_2, \lambda_8\}$. Among the extra operators that we added in action (2.17), only two contribute to tensor modes $\{\lambda_2, \lambda_8\}$. Now, using (2.95), the action becomes:

$$\begin{aligned}\mathcal{S}^{T\,(2)}_{EFT} \;&=\; \int d^4x\, a^3\left\{-\frac{m_0^2}{2}(1+\Omega)\frac{1}{4a^2}(\partial_k h_{ij}^T)^2 + \left(\frac{m_0^2}{2}(1+\Omega) - \frac{\bar M_3^2}{2}\right)\frac{(\dot h_{ij}^T)^2}{4}\right. \\ &+\; \left.\lambda_2\left(\frac{\delta^{lk}}{a^2}\partial_l\partial_k h_{ij}^T\right)^2 + \lambda_6\frac{1}{a^6}(\partial_k\partial_l\partial^l h_{ij}^T)^2\right\}.\end{aligned} \tag{2.97}$$

It is clear that the additional operators associated to higher spatial derivatives do not affect the kinetic term. However, they affect the speed of propagation of the tensor modes, as we will show in the following. Indeed, action (2.97) can be written in the compact form:

$$\mathcal{S}^{T\,(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\frac{A_T(t)}{8}\left[(\dot h_{ij}^T)^2 - \frac{c_T^2(t,k)}{a^2}k^2(h_{ij}^T)^2\right], \tag{2.98}$$

with

$$\begin{aligned} A_T(t) &= m_0^2(1+\Omega) - \bar{M}_3^2\,, \\ c_T^2(t,k) &= \bar{c}_T^2(t) - 8\frac{\lambda_2\frac{k^2}{a^2}+\lambda_6\frac{k^4}{a^4}}{m_0^2(1+\Omega)-\bar{M}_3^2}\,, \\ \bar{c}_T^2(t) &= \frac{m_0^2(1+\Omega)}{m_0^2(1+\Omega)-\bar{M}_3^2}, \end{aligned} \tag{2.99}$$

where we have Fourier transformed the spatial part. $\bar{c}_T^2$ is the tensor speed of propagation for all the theories belonging to the GLPV class, as shown in Refs. [67, 69]. However, GLPV theories are characterized by the condition $\bar{M}_3^2(t) = -\bar{M}_2^2(t)$, while the present definition of the tensor speed does not rely on this constraint as it holds for a wider class of theories. In order to avoid the development of instabilities in the tensorial sector, one generally demands the kinetic term to be positive, i.e. $A_T > 0$, and to have a positive speed of propagation $c_T^2 > 0$. From Eqs. (2.99) it is easy to identify the corresponding conditions on the EFT functions.

2.5.1 Stability conditions for the GLPV class of theories

Let us focus on the GLPV class of theories by considering the appropriate set of operators:

$$\begin{aligned} \mathcal{S}^{(2)}_{\rm GLPV} &= \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\Big[-\mathcal{W}_6\delta N k^2\zeta - \mathcal{W}_4\delta N k^2\psi - \mathcal{W}_5 k^2\psi\dot{\zeta} \\ &- \mathcal{W}_0 k^2\zeta^2 + \mathcal{W}_1(\delta N)^2 - 3a^2\mathcal{W}_4\delta N\dot{\zeta} - \frac{3}{2}a^2\mathcal{W}_5\dot{\zeta}^2\Big]\,, \end{aligned} \tag{2.100}$$

which is obtained from action (2.85) by imposing the following constraints:

$$\mathcal{W}_7 = 0\,, \qquad \left\{m_2^2, \bar{m}_5, \lambda_i\right\} = 0. \tag{2.101}$$

By varying the above action w.r.t. δN and ψ we get, respectively,:

$$\begin{aligned} &-\mathcal{W}_6 k^2\zeta - \mathcal{W}_4 k^2\psi + 2\mathcal{W}_1\delta N - 3a^2\mathcal{W}_4\dot{\zeta} = 0\,, \\ &-\mathcal{W}_4\delta N - \mathcal{W}_5\dot{\zeta} = 0. \end{aligned} \tag{2.102}$$

Inverting these relations gives:

$$\begin{aligned} \delta N &= -\frac{\mathcal{W}_5}{\mathcal{W}_4}\dot{\zeta}\,, \\ k^2\psi &= -\frac{1}{\mathcal{W}_4^2}\left[\left(3a^2\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5\right)\dot{\zeta} + \mathcal{W}_4\mathcal{W}_6 k^2\zeta\right]\,, \end{aligned} \tag{2.103}$$

which, once substituted back in the action (2.100), yields:

$$\mathcal{S}^{(2)}_{\rm GLPV} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \left\{ \left(\frac{3}{2}a^2\mathcal{W}_5 + \frac{\mathcal{W}_1\mathcal{W}_5^2}{\mathcal{W}_4^2}\right)\dot\zeta^2 - k^2\left[\frac{3}{2}H\frac{\mathcal{W}_5\mathcal{W}_6}{\mathcal{W}_4} + \frac{1}{2}\frac{d}{dt}\left(\frac{\mathcal{W}_5\mathcal{W}_6}{\mathcal{W}_4}\right) + \mathcal{W}_0\right]\zeta^2\right\}. \quad (2.104)$$

This particular form has been obtained after integrating by parts the term containing $\dot\zeta\zeta$. The above action has the same form of (2.87), where $\bar M = 0$. Therefore, it is easy to read the no-ghost condition:

$$\mathcal{L}_{\dot\zeta\dot\zeta}(t) \equiv \frac{3}{2}a^2\mathcal{W}_5 + \frac{\mathcal{W}_1\mathcal{W}_5^2}{\mathcal{W}_4^2} > 0\,, \quad (2.105)$$

and the condition on the speed of propagation ($c_s^2 > 0$):

$$c_s^2(t) = \frac{3H\mathcal{W}_5\mathcal{W}_6\mathcal{W}_4 + \mathcal{W}_6\mathcal{W}_4\dot{\mathcal{W}}_5 + \mathcal{W}_5\mathcal{W}_4\dot{\mathcal{W}}_6 - \mathcal{W}_5\mathcal{W}_6\dot{\mathcal{W}}_4 + 2\mathcal{W}_0\mathcal{W}_4^2}{3a^2\mathcal{W}_5\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5^2}\,. \quad (2.106)$$

The speed of propagation coincides with the phase velocity due to the lack of k-dependence in the kinetic term, as discussed at earlier stage. Additionally, this implies that only the mass term will be sensitive to the field redefinition which, in this case, reads:

$$\zeta(t,k) = \frac{\phi(t,k)}{\sqrt{2\left(\frac{3}{2}a^2\mathcal{W}_5 + \frac{\mathcal{W}_1\mathcal{W}_5^2}{\mathcal{W}_4^2}\right)}}\,. \quad (2.107)$$

After this transformation the effective mass follows directly form Eq. (2.92), i.e.:

$$m^2_{\rm eff}(t) = \frac{-2\mathcal{L}_{\dot\zeta\dot\zeta}\ddot{\mathcal{L}}_{\dot\zeta\dot\zeta} + \dot{\mathcal{L}}^2_{\dot\zeta\dot\zeta} - 6H\mathcal{L}_{\dot\zeta\dot\zeta}\dot{\mathcal{L}}_{\dot\zeta\dot\zeta}}{8\mathcal{L}^2_{\dot\zeta\dot\zeta}}\,, \quad (2.108)$$

where the kinetic term is given by Eq. (2.105).

2.5.2 Stability conditions for the class of theories beyond GLPV

To go beyond the GLPV class of theories we start by naively considering the general action (2.85) with all the higher order operators. We proceed to integrate out the auxiliary fields δN and ψ by solving the following field equations:

$$
\begin{aligned}
&-2\bar{m}_5\frac{k^2}{a^4}\zeta + 2\mathcal{W}_7 k^2\psi - \mathcal{W}_4\delta N - \mathcal{W}_5\dot{\zeta} = 0\,,\\
&8\left(m_2^2 - \frac{\lambda_4}{a^2}k^2 + \frac{\lambda_7}{a^4}k^4\right)\frac{k^2}{a^2}\delta N - \left(\mathcal{W}_6 + 8\lambda_3\frac{k^2}{a^4} + 8\frac{\lambda_8}{a^6}k^4\right)k^2\zeta\\
&-\mathcal{W}_4 k^2\psi + 2\mathcal{W}_1\delta N - 3a^2\mathcal{W}_4\dot{\zeta} = 0\,,
\end{aligned}
\tag{2.109}
$$

and we finally end up with an action of the form:

$$
\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2 - k^2\bar{\mathcal{B}}(t,k)\zeta^2 - k^2\bar{\mathcal{V}}(t,k)\dot{\zeta}\zeta\right\}\,,
\tag{2.110}
$$

where:

$$
\begin{aligned}
&\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) = \frac{\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\left[3a^4{\mathcal{W}_4}^2 + 2a^2\mathcal{W}_1\mathcal{W}_5 + 8k^2\mathcal{W}_5\left(m_2^2 - \frac{\lambda_4}{a^2}k^2 + \frac{\lambda_7}{a^4}k^4\right)\right]}{2a^2\left({\mathcal{W}_4}^2 - 4\mathcal{W}_1\mathcal{W}_7\right) - 32k^2\mathcal{W}_7\left(m_2^2 - \frac{\lambda_4}{a^2}k^2 + \frac{\lambda_7}{a^4}k^4\right)}\,,\\
&\bar{\mathcal{B}}(t,k) = \Bigg\{a^2\mathcal{W}_0\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right) + k^2\Bigg[\frac{1}{a^6}\left(-a^6\mathcal{W}_7\left(a^2\mathcal{W}_6^2 + 16m_2^2\mathcal{W}_0\right)\right.\\
&\left.-2a^4\bar{m}_5\mathcal{W}_4\mathcal{W}_6 + a^8\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)\mathcal{W}_3 - 4\bar{m}_5^2\mathcal{W}_1\right)\\
&+k^2\Bigg(\frac{1}{a^8}\left(a^{10}\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)\mathcal{W}_2 - 16\left(a^6\mathcal{W}_7\left(a^2m_2^2\mathcal{W}_3 + \lambda_3\mathcal{W}_6 - \lambda_4\mathcal{W}_0\right)\right.\right.\\
&\left.\left.+a^2\bar{m}_5\lambda_3\mathcal{W}_4 + \bar{m}_5^2m_2^2\right)\right)\Bigg)\\
&+k^4\Bigg(-\frac{16}{a^{10}}\left(a^4\mathcal{W}_7\left(a^6m_2^2\mathcal{W}_2 - a^4\lambda_4\mathcal{W}_3 + a^2\lambda_7\mathcal{W}_0 + 4{\lambda_3}^2\right)\right.\\
&\left.+a^2\lambda_8\left(a^4\mathcal{W}_6\mathcal{W}_7 + \bar{m}_5\mathcal{W}_4\right) - {\bar{m}_5}^2\lambda_4\right)\Bigg)\\
&+k^6\left(\frac{16}{a^{12}}\left(a^4\mathcal{W}_7\left(a^6\lambda_4\mathcal{W}_2 - a^4\lambda_7\mathcal{W}_3 - 8\lambda_3\lambda_8\right) - {\bar{m}_5}^2\lambda_7\right)\right)\\
&+k^8\left(-\frac{16}{a^{10}}\mathcal{W}_7\left(a^6\lambda_7\mathcal{W}_2 + 4{\lambda_8}^2\right)\right)\Bigg]\Bigg\}\Big/\Bigg\{a^2\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)\\
&-16k^2\mathcal{W}_7\left(m_2^2 - \frac{\lambda_4}{a^2}k^2 + \frac{\lambda_7}{a^4}k^4\right)\Bigg\}\,,\\
&\bar{\mathcal{V}}(t,k) = -\Bigg\{\frac{k^2}{a^2}\Bigg[8\mathcal{W}_4\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\left(\lambda_3 + \lambda_8\frac{k^2}{a^2}\right) + 16\frac{\bar{m}_5\mathcal{W}_5}{a^2}\Bigg(m_2^2 - \frac{\lambda_4}{a^2}k^2\\
&+\frac{\lambda_7}{a^4}k^4\Bigg)\Bigg] + 6a^4\mathcal{W}_4\mathcal{W}_7\mathcal{W}_6 + a^2\mathcal{W}_4\mathcal{W}_5\mathcal{W}_6 + 6\bar{m}_5\mathcal{W}_4^2 + 4\frac{\bar{m}_5}{a^2}\mathcal{W}_1\mathcal{W}_5\Bigg\}\Big/\Big\{a^2\\
&\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right) - 16k^2\mathcal{W}_7\left(m_2^2 - \frac{\lambda_4}{a^2}k^2 + \frac{\lambda_7}{a^4}k^4\right)\Big\}\,.
\end{aligned}
\tag{2.111}
$$

It is easy to notice that the above expressions can be written in a more compact form as:

$$\begin{aligned}\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) &= \frac{k^2\mathcal{A}_4(t,k)+\mathcal{A}_1(t)}{k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)}\,,\\ \bar{\mathcal{B}}(t,k) &= \frac{k^2\mathcal{B}_2(t,k)+\mathcal{B}_1(t)}{k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)}\,,\\ \bar{\mathcal{V}}(t,k) &= \frac{k^2\mathcal{V}_2(t,k)+\mathcal{V}_1(t)}{k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)}\,.\end{aligned} \tag{2.112}$$

By considering the above definitions the action can be written in the same form of (2.87), i.e.:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2 - k^2 G(t,k)\zeta^2\right\}\,, \tag{2.113}$$

where we have identified the “gradient” term as:

$$\begin{aligned}G(t,k) &= \Big\{k^2\Big[\mathcal{V}_2\left(k^2\dot{\mathcal{A}}_2+\dot{\mathcal{A}}_3-3H\left(k^2\mathcal{A}_2+\mathcal{A}_3\right)\right)+\mathcal{A}_2\mathcal{A}_3\,(2\mathcal{B}_1\\ &- \dot{\mathcal{V}}_1-k^2\dot{\mathcal{V}}_2+2k^2\mathcal{B}_2\Big)+\mathcal{V}_1\left(\dot{\mathcal{A}}_2-3H\mathcal{A}_2\right)\Big]\\ &+ \mathcal{V}_1\left(\dot{\mathcal{A}}_3-3H\mathcal{A}_3\right)\Big\}\,/\,\{2\left(k^2\mathcal{A}_2+\mathcal{A}_3\right)^2\}\\ &\equiv \frac{k^2\mathcal{G}_2(t,k)+\mathcal{G}_1(t)}{\left(k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)\right)^2}\,.\end{aligned} \tag{2.114}$$

Then the speed of propagation is $c_s^2(t,k) = G/\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$ and the friction term in the field equation of ζ turn out to be a function of both t and k. Let us notice that when considering the most general case, at least one of the functions $\{m_2^2, \lambda_i\}$ is not zero and none of the $\mathcal{A}_i$ functions are nil. Additionally the action does not contain the term $\bar{M}$. We will show in the next Section some particular cases of the action (2.85) for which such a term is present.

Let us now normalize the field by means of (2.90) with the kinetic term given by Eq. (2.111). Since the kinetic term is a function of k, the normalization will affect both the effective mass and speed of propagation. Thus we have:

$$\begin{aligned}m^2_{\rm eff}(t,k) &= \Big(\mathcal{A}_1{}^2\left[2\mathcal{A}_3\left(3H\dot{\mathcal{A}}_3+\ddot{\mathcal{A}}_3\right)-3\dot{\mathcal{A}}_3^2\right]-2\mathcal{A}_3\mathcal{A}_1\Big[\mathcal{A}_3\left(3H\dot{\mathcal{A}}_1+\ddot{\mathcal{A}}_1\right)\\ &- \dot{\mathcal{A}}_1\dot{\mathcal{A}}_3\Big]+\mathcal{A}_3^2\dot{\mathcal{A}}_1^2\Big)\left(8\left(k^2\mathcal{A}_4+\mathcal{A}_1\right)^2\left(k^2\mathcal{A}_2+\mathcal{A}_3\right)^2\right)\,,\\ c^2_{s,\rm eff}(t,k) &= \Big\{6H\Big[\Big[k^2\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)\mathcal{A}_2^2+2\Big[\mathcal{A}_3\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)-k^2\mathcal{A}_4\Big(\dot{\mathcal{A}}_2k^2\end{aligned}$$

$$
\begin{aligned}
+ &\quad \dot{\mathcal{A}}_3\Big)\Big]\mathcal{A}_2+\mathcal{A}_3\left(\mathcal{A}_3\dot{\mathcal{A}}_4 2\mathcal{A}_4\left(\dot{\mathcal{A}}_2k^2+\dot{\mathcal{A}}_3\right)\right)\Big]\mathcal{A}_1\\
- &\quad \mathcal{A}_1^2\left(\mathcal{A}_3\dot{\mathcal{A}}_2+\mathcal{A}_2\left(\dot{\mathcal{A}}_2k^2+\dot{\mathcal{A}}_3\right)\right)+\left(\mathcal{A}_2k^2+\mathcal{A}_3\right)\mathcal{A}_4\Big[\mathcal{A}_2\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)k^2\\
- &\quad k^2\mathcal{A}_4\left(\dot{\mathcal{A}}_2k^2+\dot{\mathcal{A}}_3\right)+\mathcal{A}_3\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)\Big]\Big]+\Big[3\mathcal{A}_4^2\dot{\mathcal{A}}_2^2k^6-4\mathcal{A}_3\mathcal{A}_4\mathcal{G}_2k^4\\
+ &\quad 6\mathcal{A}_4^2\dot{\mathcal{A}}_2\dot{\mathcal{A}}_3k^4-2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4k^4-2\mathcal{A}_3\mathcal{A}_4^2\ddot{\mathcal{A}}_2k^4+3\mathcal{A}_4^2\dot{\mathcal{A}}_3^2k^2-\mathcal{A}_3^2\dot{\mathcal{A}}_4^2k^2\\
- &\quad 4\mathcal{A}_3\mathcal{A}_4\mathcal{G}_1k^2-2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_1\dot{\mathcal{A}}_2k^2-2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_3\dot{\mathcal{A}}_4k^2-2\mathcal{A}_3\mathcal{A}_4^2\ddot{\mathcal{A}}_3k^2\\
+ &\quad 2\mathcal{A}_3^2\mathcal{A}_4\ddot{\mathcal{A}}_4k^2-\mathcal{A}_2^2\left(\dot{\mathcal{A}}_4^2k^4+2\dot{\mathcal{A}}_1\dot{\mathcal{A}}_4k^2-2\mathcal{A}_4\left(\ddot{\mathcal{A}}_4k^2+\ddot{\mathcal{A}}_1\right)k^2+\dot{\mathcal{A}}_1^2\right)k^2\\
- &\quad 2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_1\dot{\mathcal{A}}_3-2\mathcal{A}_3{}^2\dot{\mathcal{A}}_1\dot{\mathcal{A}}_4+2\mathcal{A}_3^2\mathcal{A}_4\ddot{\mathcal{A}}_1+\mathcal{A}_1^2\Big[3k^2\dot{\mathcal{A}}_2{}^2+6\dot{\mathcal{A}}_3\dot{\mathcal{A}}_2\\
- &\quad 2\left(\mathcal{A}_3\ddot{\mathcal{A}}_2+\mathcal{A}_2\left(\ddot{\mathcal{A}}_2k^2+\ddot{\mathcal{A}}_3\right)\right)\Big]-2\mathcal{A}_2\Big[\mathcal{A}_4{}^2\left(\ddot{\mathcal{A}}_2k^2+\ddot{\mathcal{A}}_3\right)k^4\\
+ &\quad \mathcal{A}_4\Big(2\mathcal{G}_2k^4+\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4k^4+2\mathcal{G}_1k^2+\dot{\mathcal{A}}_1\dot{\mathcal{A}}_2k^2+\dot{\mathcal{A}}_3\dot{\mathcal{A}}_4k^2-2\mathcal{A}_3\ddot{\mathcal{A}}_4k^2\\
+ &\quad +\dot{\mathcal{A}}_1\dot{\mathcal{A}}_3-2\mathcal{A}_3\ddot{\mathcal{A}}_1\Big)k^2+\mathcal{A}_3\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)^2\Big]+2\mathcal{A}_1\Big[k^2\left(\ddot{\mathcal{A}}_4k^2+\ddot{\mathcal{A}}_1\right)\mathcal{A}_2^2\\
- &\quad \Big(2\mathcal{G}_2k^4+\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4k^4+2\mathcal{A}_4\ddot{\mathcal{A}}_2k^4+2\mathcal{G}_1k^2+\dot{\mathcal{A}}_1\dot{\mathcal{A}}_2k^2\\
+ &\quad \dot{\mathcal{A}}_3\dot{\mathcal{A}}_4k^2+2\mathcal{A}_4\ddot{\mathcal{A}}_3k^2-2\mathcal{A}_3\ddot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\dot{\mathcal{A}}_3-2\mathcal{A}_3\dot{\mathcal{A}}_1\Big)\mathcal{A}_2\\
+ &\quad 3\mathcal{A}_4\left(\dot{\mathcal{A}}_2k^2+\dot{\mathcal{A}}_3\right)^2-\mathcal{A}_3\big(2\mathcal{G}_2k^2\\
+ &\quad \dot{\mathcal{A}}_2\dot{\mathcal{A}}_4k^2+2\mathcal{A}_4\ddot{\mathcal{A}}_2k^2+2\mathcal{G}_1+\dot{\mathcal{A}}_1\dot{\mathcal{A}}_2+\dot{\mathcal{A}}_3\dot{\mathcal{A}}_4+2\mathcal{A}_4\ddot{\mathcal{A}}_3\Big)\\
+ &\quad \mathcal{A}_3^2\ddot{\mathcal{A}}_4\Big]\Big]\Big\}/[8\left(\mathcal{A}_2k^2+\mathcal{A}_3\right)^2\left(\mathcal{A}_4k^2+\mathcal{A}_1\right)^2]\\
\equiv &\quad c_s^2+f(t,k).
\end{aligned}
\tag{2.115}
$$

As said before the effective mass is a function of inverse powers of k. For sufficiently high k, the effective mass is negligible while in the low k limit, which is the one of interest in linear cosmology, it is solely a function of time. Let us notice that the effective mass in this case has been obtained directly from action (2.113), not from Eq. (2.92) which is valid only for cases when the kinetic term does not depend on k.

2.5.3 Special cases

Although the subset of theories with higher than second order spatial derivatives treated in the previous Section is very general, there are some special cases for which the action assumes some particular forms due to specific combinations of the EFT functions in the kinetic term. In order to illustrate said cases, we will consider the following action for practical examples:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \Big[4m_2^2\frac{k^2}{a^2}(\delta N)^2 - \mathcal{W}_6\delta N k^2\zeta - \mathcal{W}_4\delta N k^2\psi - \mathcal{W}_5 k^2\psi\dot{\zeta}$$
$$-\mathcal{W}_0 k^2\zeta^2 + \mathcal{W}_7(k^2\psi)^2 + \mathcal{W}_1(\delta N)^2 - 3a^2\mathcal{W}_4\delta N\dot{\zeta} - \frac{3}{2}a^2\mathcal{W}_5\dot{\zeta}^2\Big] \quad (2.116)$$

for which the following conditions hold:

$$\mathcal{W}_7 \neq 0 \qquad \{\bar{m}_5, \lambda_i\} = 0\,. \quad (2.117)$$

By solving the Eqs. (2.109) for δN and ψ we get:

$$\delta N = \frac{\mathcal{W}_4\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\dot{\zeta} + 2\mathcal{W}_6\mathcal{W}_7 k^2\zeta}{16m_2^2\mathcal{W}_7\frac{k^2}{a^2} - \mathcal{W}_4^2 + 4\mathcal{W}_1\mathcal{W}_7}\,,$$
$$k^2\psi = \frac{\mathcal{W}_4\mathcal{W}_6 k^2\zeta + \left(2\mathcal{W}_1\mathcal{W}_5 + 3a^2\mathcal{W}_4^2 + 8m_2^2\mathcal{W}_5\frac{k^2}{a^2}\right)\dot{\zeta}}{16m_2^2\mathcal{W}_7\frac{k^2}{a^2} - \mathcal{W}_4^2 + 4\mathcal{W}_1\mathcal{W}_7} \quad (2.118)$$

which allow us to eliminate the two auxiliary fields in the action. Substituting back in the action we get:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \left\{\left[\frac{\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\left(3a^4\mathcal{W}_4^2 + 2a^2\mathcal{W}_1\mathcal{W}_5 + 8m_2^2\mathcal{W}_5 k^2\right)}{2a^2\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right) - 32m_2^2\mathcal{W}_7 k^2}\right]\dot{\zeta}^2\right.$$
$$+ k^2\Big[\Big(\left(a^2\left(\mathcal{W}_0\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right) - k^2\mathcal{W}_6^2\mathcal{W}_7\right) - 16m_2^2\mathcal{W}_0\mathcal{W}_7 k^2\right)\zeta^2$$
$$\left. - \left(a^2\mathcal{W}_4\mathcal{W}_6\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\right)\dot{\zeta}\zeta\Big) / \left(16m_2^2\mathcal{W}_7 k^2 - a^2\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)\right)\Big]\right\}\,, \quad (2.119)$$

where the kinetic term reads:

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) \equiv \frac{\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\left(3a^4\mathcal{W}_4^2 + 2a^2\mathcal{W}_1\mathcal{W}_5 + 8k^2m_2^2\mathcal{W}_5\right)}{2a^2\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right) - 32k^2m_2^2\mathcal{W}_7}\,. \quad (2.120)$$

In the following we will consider two special cases in which 1) the kinetic term depends only on time; 2) the kinetic term has a particular k-dependence, which needs to be studied carefully in order to correctly identify the speed of propagation.

- First case: $3a^2\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5 \neq 0$ and $m_2^2 = 0$. The kinetic term is only a function of time:

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t) = \frac{\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\left(3a^4\mathcal{W}_4^2 + 2a^2\mathcal{W}_1\mathcal{W}_5\right)}{2a^2\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)}\,, \quad (2.121)$$

which corresponds to the case $\mathcal{A}_2 = \mathcal{A}_4 = 0$. The above expression must be positive in order to guarantee that the theory does not exhibit ghost instabilities. Then, the speed of propagation can be easily obtained from action (2.119) once the terms proportional to $\dot{\zeta}\zeta$ have been integrated by parts and it reads:

$$
\begin{aligned}
c_s^2(t,k) &= \frac{1}{(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7)\,(3a^2\mathcal{W}_4{}^2 + 2\mathcal{W}_1\mathcal{W}_5)\,(6a^2\mathcal{W}_7 + \mathcal{W}_5)} \\
&\times \ \left\{30a^2\mathcal{W}_4\mathcal{W}_6\mathcal{W}_7\left(\mathcal{W}_4{}^2 - 4\mathcal{W}_1\mathcal{W}_7\right)H + 3\mathcal{W}_4\mathcal{W}_5\mathcal{W}_6\left(\mathcal{W}_4^2\right.\right. \\
&- \ \left.4\mathcal{W}_1\mathcal{W}_7\right)H - \mathcal{W}_6\mathcal{W}_4^2\mathcal{W}_5\dot{\mathcal{W}}_4 - 4\mathcal{W}_1\mathcal{W}_6\mathcal{W}_7\mathcal{W}_5\dot{\mathcal{W}}_4 \\
&+ \ \mathcal{W}_4^3\left(\mathcal{W}_6\dot{\mathcal{W}}_5 + \mathcal{W}_5\dot{\mathcal{W}}_6\right) + 4\mathcal{W}_4\left[\mathcal{W}_5\left(\mathcal{W}_6\left(\mathcal{W}_7\dot{\mathcal{W}}_1\right.\right.\right. \\
&+ \ \left.\left.\left.\mathcal{W}_1\dot{\mathcal{W}}_7\right) - \mathcal{W}_1\mathcal{W}_7\dot{\mathcal{W}}_6\right) - \mathcal{W}_1\mathcal{W}_6\mathcal{W}_7\dot{\mathcal{W}}_5\right] \\
&+ \ 2\mathcal{W}_0\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)^2 + 6a^2\left[\mathcal{W}_4^3\left(\mathcal{W}_7\dot{\mathcal{W}}_6 + \mathcal{W}_6\dot{\mathcal{W}}_7\right)\right. \\
&+ \ 4\mathcal{W}_7^2\mathcal{W}_4\left(\mathcal{W}_6\dot{\mathcal{W}}_1 - \mathcal{W}_1\dot{\mathcal{W}}_6\right) - 4\mathcal{W}_1\mathcal{W}_6\mathcal{W}_7^2\dot{\mathcal{W}}_4 \\
&- \ \left.\left.\mathcal{W}_4^2\mathcal{W}_6\mathcal{W}_7\dot{\mathcal{W}}_4\right] - 2k^2a\mathcal{W}_6^2\mathcal{W}_7(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7)\right\}, \qquad (2.122)
\end{aligned}
$$

where the k-dependence of the speed is due to $W_7 \neq 0$. Moreover, in this case, the final action is of the form (2.87) with $\bar{M} = 0$. Since the kinetic terms is free from any k-dependence there is no ambiguity in defining the mass term which, after the normalization (2.90), ends up being of the same form as in Eq. (2.92) where, in this case, $\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$ is given by Eq. (2.121). Finally, the effective speed of propagation remains invariant under the field redefinition.

- Second case: $3a^2\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5 = 0$ and $m_2^2 \neq 0$. In this case the kinetic term reduces to:

$$
\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) = \frac{4m_2^2\mathcal{W}_5^2\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\frac{k^2}{a^2}}{\mathcal{W}_4^2\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right) - 16\frac{k^2}{a^2}m_2^2\mathcal{W}_5\mathcal{W}_7}, \qquad (2.123)
$$

which corresponds to $\mathcal{A}_1 = 0$ and $\mathcal{A}_2(t)$, $\mathcal{A}_4(t)$ both being functions of time. From the action (2.119) it follows that there is an overall factor k^2 in front of the Lagrangian which can be reabsorbed by redefining the field as $\tilde{\zeta} = k\zeta$. As a result we obtain an action of the form (2.110). Let us notice that, in this case, $\mathcal{V}_2 = 0$. After integrating by parts the term $\sim \dot{\zeta}\zeta$, we end up with an action as in (2.87) where $\bar{M} \neq 0$, and both the friction and dispersive coefficients in the field equation are functions of time and k. Now we can compute the speed of propagation which is:

$$c_s^2(t,k) = \frac{\mathcal{V}_1\dot{\mathcal{A}}_2 + \mathcal{A}_2(2k^2\mathcal{B}_2 - \dot{\mathcal{V}}_1 + 2\mathcal{B}_1) + 2\mathcal{A}_3\mathcal{B}_2 - 3H\mathcal{A}_2\mathcal{V}_1}{2\mathcal{A}_4\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)}\,. \tag{2.124}$$

In conclusion, we give the expressions for the effective mass and speed of propagation:

$$\begin{aligned}
m_{\rm eff}^2(t,k) &= \Big\{6\mathcal{A}_2\mathcal{A}_3H(\mathcal{A}_2\dot{\mathcal{A}}_3 - \mathcal{A}_3\dot{\mathcal{A}}_2) + \mathcal{A}_3\mathcal{A}_2(2\dot{\mathcal{A}}_2\dot{\mathcal{A}}_3 - 2\mathcal{A}_3\ddot{\mathcal{A}}_2 \\
&+ \mathcal{G}_1) + {\mathcal{A}_2}^2(2\mathcal{A}_3\ddot{\mathcal{A}}_3 - 3{\dot{\mathcal{A}}_3}^2) + {\mathcal{A}_3}^2{\dot{\mathcal{A}}_2}^2\Big\}\left\{8{\mathcal{A}_4}^2\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)^2\right\} \\
c_{s,\rm eff}^2(t,k) &= \Big\{6\mathcal{A}_4H\Big[\mathcal{A}_2\Big(\mathcal{A}_4\Big(k^2\dot{\mathcal{A}}_2 + \dot{\mathcal{A}}_3\Big) - 2\mathcal{A}_3\dot{\mathcal{A}}_4\Big) + \mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_2 \\
&- k^2\mathcal{A}_2^2\dot{\mathcal{A}}_4\Big] + 2\mathcal{A}_2\Big[\mathcal{A}_4\Big(k^2\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 + 2k^2\mathcal{G}_2 + \dot{\mathcal{A}}_3\dot{\mathcal{A}}_4 \\
&- 2\mathcal{A}_3\ddot{\mathcal{A}}_4 + 2\mathcal{G}_1\Big) + {\mathcal{A}_4}^2\Big(k^2\ddot{\mathcal{A}}_2 + \ddot{\mathcal{A}}_3\Big) + \mathcal{A}_3\dot{\mathcal{A}}_4^2\Big] \\
&+ \mathcal{A}_4\Big[2\mathcal{A}_3\Big(\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 + \mathcal{A}_4\ddot{\mathcal{A}}_2 + 2\mathcal{G}_2\Big) \\
&- 3\mathcal{A}_4\dot{\mathcal{A}}_2\Big(k^2\dot{\mathcal{A}}_2 + 2\dot{\mathcal{A}}_3\Big)\Big] + k^2\mathcal{A}_2^2\Big(\dot{\mathcal{A}}_4^2 - 2\mathcal{A}_4\ddot{\mathcal{A}}_4\Big)\Big\} \\
&/ \;\Big[8\mathcal{A}_4^2\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)^2\Big]\,,
\end{aligned} \tag{2.125}$$

where the function $\mathcal{G}_i(i=1,2)$ can be read from:

$$G(t,k) = \frac{\mathcal{V}_1\dot{\mathcal{A}}_2 + \mathcal{A}_2(-\dot{\mathcal{V}}_1 + 2\mathcal{B}_1) + 2\mathcal{A}_3\mathcal{B}_2 - 3H\mathcal{A}_2\mathcal{V}_1 + 2k^2\mathcal{A}_2\mathcal{B}_2}{2\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)^2}. \tag{2.126}$$

Finally, let us notice that in the case $\bar{M} \neq 0$, one may wonder if the conservation of the curvature perturbation is preserved on super-horizon scales. It is not so trivial to draw a general conclusion about the behaviour of ζ in such limit, because the EFT functions involved in the $\bar{M}$ term are all unknown functions of time. Therefore, we can conclude that in the general field equation for ζ on super-horizon scales such term might be non zero, possibly leading to a non conserved curvature perturbation. However, we expect that well behaved DE/MG models will have either $\bar{M} = 0$ or that such term will contribute a decaying mode, thus leaving the conservation of ζ unaffected. In this regard, we will argue our last statement by using an explicit example, which is not conclusive but can give an insight on how $\bar{M}$ can behave in the low k regime when theoretical models are considered. Considering the mapping (2.82), it is easy to verify that the low energy Hořava gravity falls in the special case under analysis and that the corresponding $\bar{M} \neq 0$. However, when considering the super-horizon limit the $\bar{M}$ term goes to zero and the equation for ζ reduces to

$$\ddot{\zeta} + H\dot{\zeta} = 0, \tag{2.127}$$

which solution is $\zeta \to \zeta_c - \frac{c_1 e^{-\sqrt{2}t\sqrt{\frac{\xi\Lambda}{9\lambda-3}}}}{\sqrt{2}\sqrt{\frac{\xi\Lambda}{9\lambda-3}}}$. ζ_c, c_1 are constant and the second term is a decaying mode. Hence, the conservation of ζ is preserved.

Let us conclude by saying that the cases treated in this Section are only few examples of “special” cancellations that might happen.

2.6 An extended basis for theories with higher spatial derivatives

In Ref. [69], the authors proposed a new basis to describe Horndeski theories, in terms of four free functions of time which parametrize the departure from GR. Specifically, these functions are: $\{\alpha_B, \alpha_M, \alpha_K, \alpha_T\}$, hereafter referred to as ReParametrized Horndeski (RPH). They are equivalent and an alternative to the EFT functions needed to describe the dynamics of perturbations in the Horndeski class, i.e. $\{\Omega, M_2^4, \bar{M}_2^2, \bar{M}_1^3\}$. In both cases one needs to supply also the Hubble parameter, $H(a)$. The latest publicly released version of *EFTCAMB* contains also the RPH basis as a built-in alternative [80]. RPH is also the building block at the basis of HiCLASS [110].

The RPH basis was constructed in order to encode departures from GR in terms of some key properties of the (effective) DE component. As discussed in details in Ref. [69], the braiding function α_B is connected to the clustering of DE, α_M parametrizes the time-dependence of the Planck mass and, along with α_T, is related to the anisotropic stress while large values of the kinetic function, α_K correspond to suppressed values of the speed of propagation of the scalar mode. In Ref. [67], the RPH basis has been extended to include the GLPV class of theories by adding the function α_H, which parametrizes the deviation from the Horndeski class.

In this Section we introduce an extended version of the RPH basis which generalizes the original one [69], as well as its extension to GLPV [67], by encompassing the higher order spatial derivatives terms appearing in action (2.1). We also present the explicit mapping between this new basis and the EFT functions in the extended action (2.1), in order to facilitate the link between phenomenological properties and the theory which is responsible for them.

Let us start with tensor perturbations of the EFT action (2.17) analysed in Section 2.5. Here, for completeness we rewrite its compact form:

$$\mathcal{S}^{T\,(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \frac{A_T(t)}{8}\left[(\dot{h}^T_{ij})^2 - \frac{c_T^2(t,k)}{a^2}k^2(h^T_{ij})^2\right]. \quad (2.128)$$

Now, following Ref. [69], we define the deviation from GR of the tensor speed of propagation as:

$$c_T^2(t,k) = 1 + \tilde{\alpha}_T(t,k), \quad (2.129)$$

where:

$$\tilde{\alpha}_T(t,k) = \alpha_T(t) + \alpha_{T_2}(t)\frac{k^2}{a^2} + \alpha_{T_6}(t)\frac{k^4}{a^4}\,, \quad (2.130)$$

with:

$$\alpha_T(t) = \frac{\bar{M}_3^2}{m_0^2(1+\Omega)-\bar{M}_3^2} \equiv \bar{c}_T^2 - 1\,, \quad \alpha_{T_2}(t) = -8\frac{\lambda_2}{m_0^2(1+\Omega)-\bar{M}_3^2}\,,$$
$$\alpha_{T_6}(t) = -8\frac{\lambda_6}{m_0^2(1+\Omega)-\bar{M}_3^2}. \quad (2.131)$$

As expected, the additional higher order operators will contribute by adding a k-dependence in the original definition of the α_T function introduced in Ref. [69]. Moreover, we can define the rate of evolution of the mass function $M^2(t) \equiv A_T(t)$ (defined in Eq. (2.99)) as:

$$\alpha_M(t) = \frac{1}{H(t)}\frac{d}{dt}\left(\ln M^2(t)\right). \quad (2.132)$$

It is clear that α_T and α_M differ from the ones in Ref. [69] since, in general, $\bar{M}_3^2(t) \neq -\bar{M}_2^2(t)$ for theories with higher spatial derivatives. It is important to notice that the EFT functions which are involved in the definition of α_M and α_T are $\{\Omega, \bar{M}_3^2\}$. Therefore, the class of theories which can contribute to a time dependent Planck mass and modify the tensor speed of propagation, are the ones which are non-minimally coupled with gravity and/or contain the $\mathcal{S}$-term in the action; specifically, Horndeski models with non zero L_4^{GG}, L_5^{GG}, GLPV models with non zero L_4^{GLPV}, L_5^{GLPV} and Hořava gravity. Moreover, the k-dependence in the speed of propagation is related to the α_{T2}, α_{T6} functions which are present in Hořava gravity. Finally, let us notice that, since M^2 appears in the denominator of c_T^2, high values of M^2 will generally suppress the speed of propagation and in case only background EFT functions are at play or theories for which $\{\bar{M}_3^2(t), \lambda_{2,6}\} = 0$ are considered, c_T^2 is identically one. Therefore, it would be not possible to discriminate between minimally and non-minimally coupled models.

Let us now focus on the scalar perturbations. Collecting terms with the same perturbations, the second order action (2.17) can be written as follows:

$$
\begin{aligned}
\mathcal{S}^{(2)}_{EFT} &= \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\,\frac{M^2}{2}\Big\{(1+\tilde{\alpha}_H)\,\delta N\delta_1\tilde{\mathcal{R}} - 4H\alpha_B\delta N\delta\tilde{K} \\
&+ \delta\tilde{K}^{\mu}_{\nu}\delta\tilde{K}^{\nu}_{\mu} - (\alpha_B^{GLPV}+1)(\delta\tilde{K})^2 + \tilde{\alpha}_K H^2(\delta N)^2 \\
&- \frac{1}{4}\left(\alpha_{T_2}+\alpha_{T_6}\frac{k^2}{a^2}\right)\delta\tilde{\mathcal{R}}_{ij}\delta\tilde{\mathcal{R}}^{ij} + (1+\alpha_T)\delta_2\tilde{\mathcal{R}} + (1+\alpha_T)\delta_1\tilde{\mathcal{R}}\delta(\tilde{\sqrt{h}}) \\
&+ \left(\alpha_1+\alpha_5\frac{k^2}{a^2}\right)(\delta\tilde{\mathcal{R}})^2 + \bar{\alpha}_5\delta_1\tilde{\mathcal{R}}\delta\tilde{K}\Big\}\,,
\end{aligned}
\tag{2.133}
$$

where the geometrical quantities with tildes are the Fourier transform of the corresponding quantities in Eq. (2.84), moreover we have identified the following functions:

$$
\begin{aligned}
&\alpha_B(t) = \frac{m_0^2\dot{\Omega}+\bar{M}_1^3}{2HM^2}\,, \qquad \alpha_B^{GLPV}(t) = \frac{\bar{M}_3^2+\bar{M}_2^2}{M^2}\,, \\
&\tilde{\alpha}_K(t,k) = \alpha_K(t) + \alpha_{K_2}(t)\frac{k^2}{a^2} + \alpha_{K_4}(t)\frac{k^4}{a^4} + \alpha_{K_7}(t)\frac{k^6}{a^6}\,, \\
&\text{where}\quad \alpha_K(t) = \frac{2c+4M_2^4}{H^2M^2}\,, \quad \alpha_{K_2}(t) = \frac{8m_2^2}{M^2H^2}\,, \quad \alpha_{K_4}(t) = -\frac{8\lambda_4}{M^2H^2}\,, \\
&\alpha_{K_7}(t) = \frac{8\lambda_7}{H^2M^2}\,, \quad \tilde{\alpha}_H(t,K) = \alpha_H(t) + \alpha_{H_3}(t)\frac{k^2}{a^2} + \alpha_{H_8}(t)\frac{k^4}{a^4}\,, \\
&\text{where}\quad \alpha_H(t) = \frac{2\hat{M}^2+\bar{M}_3^2}{M^2}\,, \quad \alpha_{H_3}(t) = -\frac{4\lambda_3}{M^2}\,, \quad \alpha_{H_8}(t) = \frac{4\lambda_8}{M^2}\,, \\
&\alpha_1(t) = \frac{2\lambda_1}{M^2}\,, \qquad \alpha_5(t) = \frac{2\lambda_5}{M^2}\,, \qquad \bar{\alpha}_5(t) = \frac{\bar{m}_5}{M^2}.
\end{aligned}
\tag{2.134}
$$

The relations between the $\mathcal{W}$-functions introduced in Section 2.5 and the above α-functions are the following:

$$
\begin{aligned}
&\mathcal{W}_0 \equiv -\frac{M^2}{a^2}\left(\alpha_T + 1 + 3H\bar{\alpha}_5 + 3\dot{\bar{\alpha}}_5 + 3\bar{\alpha}_5 H\alpha_M\right)\,, \\
&\mathcal{W}_1 \equiv \frac{M^2H^2}{2}\alpha_K + \frac{3}{2}a^2H\mathcal{W}_4 - 3H^2M^2\alpha_B\,, \\
&\mathcal{W}_2 \equiv \frac{M^2}{a^6}\left(-8\alpha_5 + \frac{3}{4}\alpha_{T_6}\right)\,, \qquad \mathcal{W}_3 \equiv \frac{M^2}{a^4}\left(-8\alpha_1 + \frac{3}{4}\alpha_{T_2}\right)\,, \\
&\mathcal{W}_4 \equiv -\frac{HM^2}{a^2}\left(2 + 2\alpha_B + 3\alpha_B^{GLPV}\right)\,, \quad \mathcal{W}_5 \equiv \frac{M^2}{a^2}\left(2 + 3\alpha_B^{GLPV}\right)\,, \\
&\mathcal{W}_6 \equiv -\frac{2M^2}{a^2}\left(1 + \alpha_H + 3H\bar{\alpha}_5\right)\,, \quad \mathcal{W}_7 \equiv -\frac{M^2}{2a^4}\alpha_B^{GLPV}\,.
\end{aligned}
\tag{2.135}
$$

Before discussing in details the meaning of the α-functions and how they contribute to the evolution of the propagating d.o.f., we introduce the perturbed linear equations which will help us in the discussion. The variation of the action (2.133) w.r.t to ψ and δN gives:

$$H\left[2(1+\alpha_B)+3\alpha_B^{GLPV}\right]\delta N-(2+3\alpha_B^{GLPV})\dot{\zeta}-\alpha_B^{GLPV}\frac{k^2\psi}{a^2}-2\bar{\alpha}_5\frac{k^2\zeta}{a^2}=0\,,$$
$$\Big[3H^2\left(2-4\alpha_B-3\alpha_B^{GLPV}\right)+H^2\tilde{\alpha}_K\Big]\delta N+2H\Big[2\alpha_B+3\alpha_B^{GLPV}+2\Big]\frac{k^2}{a^2}\psi$$
$$+\Big[3H\left(2+2\alpha_B+3\alpha_B^{GLPV}\right)\Big]\dot{\zeta}+2\Big[1+H\bar{\alpha}_5+\tilde{\alpha}_H\Big]\frac{k^2}{a^2}\zeta=0. \qquad (2.136)$$

These equations allow us to eliminate the auxiliary fields δN and ψ from the action, yielding an action solely in terms of the dynamical field ζ. A detailed description of how to eliminate the auxiliary fields was the subject of the previous Section 2.5, indeed the above equations are equivalent to Eqs. (2.109), once the relations (2.135) have been considered. At this point, we can describe the meaning of the different α-functions in terms of the phenomenology of ζ.

Let us now focus on the definition of the α-functions which characterize the new basis, $\{\alpha_M, \tilde{\alpha_T}, \alpha_B, \alpha_B^{GLPV}, \tilde{\alpha_H}, \tilde{\alpha}_K, \bar{\alpha}_5, \alpha_1, \alpha_5\}$, extending and generalizing the RPH one. A first difference that can be noticed w.r.t. the RPH parametrization, is the presence of $\{\tilde{\alpha}_H, \tilde{\alpha}_K\}$ which are now functions of k, since they contain the contributions from operators with higher spatial derivatives. Let us now describe the new basis in details with the help of the definitions (2.134) and Eqs. (2.136):

- $\{\alpha_B, \alpha_B^{GLPV}\}$: α_B is the braiding function as defined in Ref. [69]. * Its role is clear by looking at Eqs. (2.136), indeed α_B regulates the relation between the auxiliary field δN and the dynamical d.o.f. ζ. Analogously, we define α_B^{GLPV}, which contributes to the braiding since it mediates the relationship of ψ and δN with ζ. The effects of these braiding coefficients on the kinetic term and the speed of propagation is more involved. Indeed, by looking at the action (2.133) we can notice that α_B^{GLPV} has a direct contribution to the kinetic term since it is the pre-factor of $(\delta K)^2$, which contains $\dot{\zeta}^2$. Moreover, both α_B and α_B^{GLPV} affect indirectly the kinetic term: the δN term in Eq. (2.136), whose pre-factor contains the braiding functions, turns out to be proportional to $\dot{\zeta}$, then substituting it back to action (2.133), the term in $(\delta N)^2$ will generate a contribution to the kinetic term. Furthermore, their involvement in the speed of propagation of the scalar d.o.f. comes in two ways: 1) from the kinetic term as previously mentioned. Indeed through

*The definition of α_B presented here differs from the one in Ref. [69] by a minus sign and a factor 2.

Eq. (2.89) they enter in the denominator of the definition of the propagating speed; 2) because they multiply both the δN and ψ terms in Eq. (2.136) which result to be proportional to $k^2\zeta$ which contributes to G in Eq. (2.89). Moreover, analogously to the definition of α_H, which parametrizes the deviation w.r.t. Horndeski/GG theories, α_B^{GLPV} is defined such as to parametrize the deviation from GLPV theories; indeed the latter are characterized by the condition $\alpha_B^{GLPV} = 0$, hence the name. If $\alpha_B^{GLPV} \neq 0$, higher spatial derivatives appear in the ζ equation. Finally, α_B is different from zero for all the theories showing non-minimal coupling to gravity and/or possessing the $\delta N \delta K$ operator in the action, i.e. $f(R), L_3^{GG}, L_4^{GG}, L_5^{GG}, L_4^{GLPV}, L_5^{GLPV}$. This operator does not appear when one considers quintessence and k-essence models (L_2^{GG}) and Hořava gravity. α_B^{GLPV} is non zero for the low-energy Hořava gravity action.

- $\tilde{\alpha}_K(t,k)$: it is the generalization of the purely kinetic function $\alpha_K(t)$ and it describes the extension of the kinetic term to higher order spatial derivatives in the case of non zero $\{\alpha_{K2}, \alpha_{K4}, \alpha_{K7}\}$. It is easy to see that $\tilde{\alpha}_K(t,k)$ is related to the kinetic term of the scalar d.o.f. since it appears in action (2.133) as a coefficient of the operator $(\delta N)^2$ and, through the linear perturbed equations (2.136), $\delta N \sim \dot{\zeta}$. Since it describes the kinetic term, it will affect the speed of propagation of ζ as well as the condition for the absence of a scalar ghost. The last point is easy to understand because as we extensively discussed in Section 2.5 the kinetic terms goes in the denominator of the speed of propagation of scalar perturbation (see Eq. (2.89)). The α_K function is characteristic of theories belonging to GLPV, while for Hořava gravity it is identically zero. On the other hand, Hořava gravity contributes non zero $\{\alpha_{K2}, \alpha_{K4}, \alpha_{K7}\}$. Finally, let us note that when considering theories beyond GLPV the braiding coefficient discussed in the previous point, α_B^{GLPV}, gives a direct contribution to the kinetic term through the operator $(\delta K)^2$.

- $\{\alpha_1, \alpha_5, \bar{\alpha}_5, \tilde{\alpha}_H\}$: from the constraint equations (2.136), it can be noticed that $\tilde{\alpha}_H$ and $\bar{\alpha}_5$ contribute to the speed of propagation of the scalar d.o.f. since they multiply the term $k^2\zeta$. In particular, if $\bar{\alpha}_5$ and the k-dependent parts of $\tilde{\alpha}_H$ are different from zero, the dispersion relation of ζ will be modified and the speed of propagation will depend on k. The functions $\{\alpha_1, \alpha_5\}$ have a similar impact since they are the pre-factors of $\delta_1\mathcal{R}$ in the action which, once expressed in terms of the perturbations of the metric, gives a term proportional to $k^2\zeta$. In this case by looking at Eq. (2.89) these functions will enter in the definition of G. The theories where these functions are present are GLPV and Hořava gravity models. In particular, in the case of Hořava gravity the functions associated with higher order spatial derivatives terms are present.

The above represents an interesting extension and generalization of the original RPH parametrization [69], carefully built while considering the different phenomenological aspects of the dark energy fluid. However, let us notice that the desired correspondence between the α-functions and actual observables becomes weaker as we go beyond the Horndeski class. Indeed, due to the high number of α-functions involved, their dependence on many EFT functions and the way they enter in the actual physical quantities, such as the speed of sound and the kinetic term, identifying exactly the underlying theory of gravity responsible for a specific effect is a hard task.

2.7 Conclusions

We started this Chapter by generalizing the original EFToDE/MG action for DE/MG by including operators up to sixth order in spatial derivatives. This was motivated by the recent rise of theories containing a (sub)set of these operators with higher-order spatial derivatives, like Hořava gravity. As such, these theories were not covered by the operators included in the first proposal of the EFToDE/MG action as presented in Refs. [37, 39]. From there on, the extended Lagrangian (2.1) became the basis of the rest of the Chapter as the new operators play a central role.

Starting from the extended Lagrangian (2.1) we proceeded to show an efficient method to map theories of gravity, expressed in terms of geometrical quantities, into the EFToDE/MG language. This led to a general mapping between the ADM and the EFToDE/MG formalism for our new extended Lagrangian. Subsequently, we illustrated this procedure by mapping models of DE/MG, with an additional scalar d.o.f., into the EFToDE/MG formalism, resulting in a vast set of worked out examples. These include minimally coupled quintessence, f(R), Hornedski/GG, GLPV and Hořava gravity. The preliminary step of writing the theories in the ADM formalism has also been presented as it is an integral part of the procedure. Therefore we created a very useful guide for the theoretical steps necessary in order to implement a given model of DE/MG into *EFTCAMB* and a "dictionary" for many of the existing DE/MG models. To this extent, we have been very careful and explicit about the conventions which lie at the basis of the EFToDE/MG formalism and, by extension, *EFTCAMB*. These become obvious when comparing with the equivalent approaches in the literature as there are some clear differences.

Thus the take-home message is that the user should be careful with the conventions when implementing a given model into *EFTCAMB*.

An ongoing field of research regarding the EFToDE/MG is the determination of the parameter space corresponding to physically healthy theories, as we introduced at the beginning of this thesis. This is vital from a theoretical as well as from a numerical point of view. As such it was natural to subject our extended Lagrangian to a thorough stability analysis while considering only the gravity sector. In fact, since the EFToDE/MG formalism is based on an action, we were able to determine general conditions of theoretical viability which are model independent and can, a priori, greatly reduce the parameter space. The most common criteria would be the absence of ghosts and gradient instabilities in the scalar and tensor sector and the exclusion of tachyonic instabilities. Regarding the first two criteria, one can find results in the literature either with or without the inclusion of a matter sector [14, 15, 37, 38, 65, 67, 68, 111]. In this work the study of the physical stability is particularly interesting due to the appearance of operators with higher order spatial derivatives. We proceeded, without including a matter sector, to study the stability of different sets of theories, leaving the analysis of the matter backreactions to the next Chapter. After integrating out the auxiliary fields, we obtained an EFToDE/MG action describing only the dynamics of the propagating d.o.f.. From this action, we identified the kinetic term and the speed of propagation which have now become functions of scale and time, due to the presence of higher derivative operators. We required both to be positive in order to guarantee a viable theory free from ghost and gradient instabilities. Subsequently we identified, at the level of the equations of motion, the friction and dispersive coefficients. We did this both for the scalar and tensor d.o.f.. Finally, we normalized the scalar d.o.f. in order to obtain an action in the canonical form. This form allowed us to identify the effective mass term on which we imposed conditions in order to avoid the appearance of tachyonic instabilities in the scalar sector. As a result, we obtained a set of very general stability conditions which must be imposed in order to ensure theoretical viability of models with operators containing up to sixth order in spatial derivatives, in absence of matter. It is worth noting that due to the complicated nature of some classes of theories, when written in the EFToDE/MG formalism, we had to divide the treatment and the resulting conditions in different subsets.

In the final part of this Chapter, we have built an extended and generalized version of the phenomenological parametrization in terms of α functions introduced in Ref. [69], to which we refer as ReParametrized Horndeski (RPH). This parametrization was originally built to include all models in the Horndeski class, and was afterwards extended to encompass beyond Horndeski models known as GLPV, in Ref. [67]. This was achieved by introducing an additional function which parametrizes the deviation from Horndeski theories. From this point we proceeded to introduce new functions and generalize the definition of the original ones, in order to account for all the beyond GLPV models described by the higher order operators that we have included in our extended EFToDE/MG action (2.1). In particular, we have found a new function parametrizing the braiding, which also contributes to the kinetic term; we have generalized the definitions of the kinetic and tensor speed excess functions, the latter one now being both time and scale dependent; finally, we have identified four extra functions entering in the definition of the speed of propagation of the scalar d.o.f.. It is important to notice that the structure of this extended phenomenological basis in terms of α functions becomes quite cumbersome when higher order operators are considered and the correspondence between the different functions and cosmological observables becomes weaker.

2.8 Appendix A: On δK and δS perturbations

In this Section we explicitly work out the perturbations associated to δK and δS used in Section 2.3.1 and show the difference with previous approaches [37, 65]. For this purpose, we consider the following terms of the Lagrangian (2.8):

$$\delta L \supset L_K\delta K + L_S\delta S = \mathcal{F}\delta K + L_S\delta K^\mu_\nu\delta K^\nu_\mu \equiv \mathcal{F}(K+3H) + L_S\delta K^\mu_\nu\delta K^\nu_\mu\,, \tag{2.137}$$

where we have defined:

$$\mathcal{F} \equiv L_K - 2HL_S. \tag{2.138}$$

Now, let us prove a relation which is useful in order to obtain action (2.9):

$$\int d^4x\sqrt{-g}\mathcal{F}K = \int d^4x\sqrt{-g}\mathcal{F}\nabla_\mu n^\mu = -\int d^4x\sqrt{-g}\nabla_\mu\mathcal{F}n^\mu = \int d^4x\sqrt{-g}\frac{\dot{\mathcal{F}}}{N}\,. \tag{2.139}$$

Using the above relation and the expansion of the lapse function:

$$N = 1 + \delta N + \delta N^2 + \mathcal{O}(3), \tag{2.140}$$

finally, we obtain:

$$L_K\delta K + L_S\delta S = 3H\mathcal{F} + \dot{\mathcal{F}}\left(1 - \delta N + (\delta N)^2\right) + L_S\delta K^\mu_\nu\delta K^\nu_\mu. \tag{2.141}$$

The differences with previous works are due to the different convention on the normal vector, n^μ (see Eq. (2.3)), which is responsible of the different sign in Eq. (2.139) w.r.t. the definition used in Refs. [37, 65] and then in the final results (2.141). Moreover, the difference in the definition of the extrinsic curvature, see Eq. (2.3), which is a consequence of the convention adopted for the normal vector, leads to the minus sign in Eq. (2.138) because its background value is $K_j^{i(0)} = -H\delta_j^i$.

2.9 Appendix B: On $\delta\mathcal{U}$ perturbation

Due to the different convention for n^μ we adopted here (see Eq. (2.3)), the result obtained in Refs. [37, 65] concerning the perturbation associated to $\mathcal{U} = \mathcal{R}_{\mu\nu}K^{\mu\nu}$, can not be directly applied to our Lagrangian (2.8). Therefore, we need to derive again such result, which is crucial in order to obtain the coefficients of the action (2.9). Then, let us prove the following relation:

$$\int d^4x\sqrt{g}\lambda(t)\mathcal{R}_{\mu\nu}K^{\mu\nu} = \int d^4x\sqrt{g}\left(\frac{\lambda(t)}{2}\mathcal{R}K - \frac{\dot{\lambda}(t)}{2N}\mathcal{R}\right), \qquad (2.142)$$

where $\lambda(t)$ is a generic function of time. We notice that in Ref. [65] the above relation is defined with a plus in front of the second term in the last expression. Using the relation $K = \nabla^\mu n_\mu$ we obtain:

$$\int d^4x\sqrt{-g}\left(\lambda(t)\mathcal{R}_{\mu\nu}K^{\mu\nu} - \frac{\lambda(t)}{2}\mathcal{R}\nabla_\mu n^\mu + \frac{\dot{\lambda}(t)}{2N}\mathcal{R}\right) = 0\,. \qquad (2.143)$$

Now, after integration by parts of the second term and using $n^\mu = \big(-1/N, N^i/N\big)$, the last term cancels and we are left with:

$$\int d^4x\sqrt{-g}\left(\lambda(t)\mathcal{R}_{\mu\nu}K^{\mu\nu} + \frac{\lambda(t)}{2}n^\mu\nabla_\mu\mathcal{R}\right) = 0\,. \qquad (2.144)$$

The first term can be rewritten using the expression for the extrinsic curvature in the ADM formalism:

$$K_{ij} = -\frac{1}{2N}\big[\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i\big], \qquad (2.145)$$

where covariant derivative is w.r.t. the spatial metric h_{ij}. The overall minus sign which appears in the above definition makes the expression to differ from the one usually encountered that follows from the definition of n^μ we employed. After substituting this expression into Eq. (2.144) we get:

$$\int d^4x\sqrt{h}\lambda(t)\left[-\frac{1}{2}\left(\mathcal{R}_{ij}h^{il}h^{jk}\dot{h}_{lk}+\dot{\mathcal{R}}\right)+\nabla^i N^j\mathcal{R}_{ij}+\frac{1}{2}N^i\nabla_i\mathcal{R}\right]=0\,. \tag{2.146}$$

From here on the subsequent steps follows Ref. [65], indeed the last two terms vanish due to the Bianchi identity and the first two can be combined as a total divergence. Hence, the relation (2.142) holds.

Finally, using the above relation we can now compute the perturbations coming from $\mathcal{U}=\mathcal{R}_{\mu\nu}K^{\mu\nu}$. Indeed we have:

$$\begin{aligned}\int d^4x\sqrt{-g}L_{\mathcal{U}}\mathcal{R}_{\mu\nu}K^{\mu\nu} &= \int d^4x\sqrt{-g}\left[\frac{1}{2}L_{\mathcal{U}}\mathcal{R}K-\frac{1}{2N}\dot{L}_{\mathcal{U}}\mathcal{R}\right]\\ &= \int d^4x\sqrt{-g}\left[\frac{1}{2}L_{\mathcal{U}}\left(K^{(0)}\delta\mathcal{R}+\delta K\delta\mathcal{R}\right)\right.\\ &- \left.\frac{1}{2}\dot{L}_{\mathcal{U}}\mathcal{R}\left(1-\delta N\right)\right]\,,\end{aligned} \tag{2.147}$$

then we get:

$$L_{\mathcal{U}}\delta\mathcal{U}=-\frac{1}{2}\left(3L_{\mathcal{U}}+\frac{1}{2}\dot{L}_{\mathcal{U}}\right)\delta\mathcal{R}+\left(\frac{1}{2}L_{\mathcal{U}}\delta K+\frac{1}{2}\dot{L}_{\mathcal{U}}\delta N\right)\delta\mathcal{R}\,. \tag{2.148}$$

2.10 Appendix C: Conformal EFT functions for Generalized Galileon and GLPV

In this Appendix we collect the results of Sections 2.4.3 and 2.4.4, and convert them to functions of the scale factor; the Hubble parameter and its time derivative are defined in terms of the conformal time, still they need to be considered functions of the scale factor. This further step is important for a direct implementation in *EFTCAMB* of Horndeski/GG and GLPV theories. *In this Section only*, primes indicate derivatives w.r.t. the scale factor. Furthermore, $\mathcal{H}\equiv d\ln a/d\tau$ and $\dot{\mathcal{H}}\equiv d\mathcal{H}/d\tau$, where τ is the conformal time. In order to get the correct results $\{\mathcal{K}, G_i, \tilde{F}_i\}$ have to be considered functions of the scale factor.

First, we consider the EFToDE/MG functions derived in Section 2.4.3 for Horndeski/GG theories:

- L_2-Lagrangian

$$\begin{aligned}\Lambda(a) &= \mathcal{K},\\ c(a) &= \mathcal{K}_X X_0\,,\\ M_2^4(a) &= \mathcal{K}_{XX} X_0^2\,,\end{aligned} \tag{2.149}$$

where X_0 is:

$$X_0 = -\mathcal{H}^2\phi_0'^2\,. \tag{2.150}$$

- L_3-Lagrangian

$$\begin{aligned}\Lambda(a) &= \mathcal{H}^2\phi_0'^2\left[G_{3\phi} - 2G_{3X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\right]\,,\\ c(a) &= \mathcal{H}^2\phi_0'^2\left[G_{3X}\left(\left(3\mathcal{H}^2 - \dot{\mathcal{H}}\right)\frac{\phi_0'}{a} - \mathcal{H}^2\phi_0''\right) + G_{3\phi}\right]\,,\\ M_2^4(a) &= \frac{G_{3X}}{2}\mathcal{H}^2\phi_0'^2\left(\left(3\mathcal{H}^2 + \dot{\mathcal{H}}\right)\frac{\phi_0'}{a} + \mathcal{H}^2\phi_0''\right) - 3\frac{\mathcal{H}^6}{a}G_{3XX}\phi_0'^5 - \frac{G_{3\phi X}}{2}\mathcal{H}^4\phi_0'^4\,,\\ \bar{M}_1^3(a) &= -2\mathcal{H}^3 G_{3X}\phi_0'^3\,.\end{aligned} \tag{2.151}$$

- L_4-Lagrangian

$$\begin{aligned}\Omega(a) &= -1 + \frac{2}{m_0^2}G_4\,,\\ c(a) &= G_{4X}\left[2\left(\dot{\mathcal{H}}^2 + \mathcal{H}\ddot{\mathcal{H}} + 2\mathcal{H}^2\dot{\mathcal{H}} - 5\mathcal{H}^4\right)\frac{\phi_0'^2}{a^2} + 2\left(5\mathcal{H}^2\dot{\mathcal{H}} + \mathcal{H}^4\right)\frac{\phi_0'}{a}\phi_0'' + 2\mathcal{H}^4\phi_0''^2\right.\\ &\left.+2\mathcal{H}^4\phi_0'\phi_0'''\right] + G_{4X\phi}\left[2\mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) + 10\frac{\mathcal{H}^4}{a}\phi_0'^3\right]\\ &+ G_{4XX}\left[12\frac{\mathcal{H}^6}{a^2}\phi_0'^4 - 8\frac{\mathcal{H}^4}{a}\phi_0'^3\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\right.\\ &\left.-4\mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}^2}{a^2}\phi_0'^2 + 2\frac{\dot{\mathcal{H}}\mathcal{H}^2}{a}\phi_0'\phi_0'' + \mathcal{H}^4\phi_0''^2\right)\right]\,,\\ \Lambda(a) &= G_{4X}\left[4\left(\mathcal{H}^4 + 5\mathcal{H}^2\dot{\mathcal{H}} + \dot{\mathcal{H}}^2 + \mathcal{H}\ddot{\mathcal{H}}\right)\frac{\phi_0'^2}{a^2} + 4\left(4\mathcal{H}^4 + 5\mathcal{H}^2\dot{\mathcal{H}}\right)\frac{\phi_0'}{a}\phi_0'' + 4\mathcal{H}^4\phi_0''^2\right]\\ &+ 4\mathcal{H}^4\phi_0'\phi_0''' + 8\frac{\mathcal{H}^4}{a}G_{4X\phi}\phi_0'^3 - 8G_{4XX}\mathcal{H}^2\phi_0'^2\left(\dot{\mathcal{H}}\frac{\phi_0'}{a} + \mathcal{H}^2\phi_0''\right)\left(2\mathcal{H}^2\frac{\phi_0'}{a} + \dot{\mathcal{H}}\frac{\phi_0'}{a}\right.\\ &\left.+ \mathcal{H}^2\phi_0''\right)\,,\\ M_2^4(a) &= G_{4X\phi}\left[4\frac{\mathcal{H}^4}{a}\phi_0'^3 - \mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\right] - 6\frac{\mathcal{H}^6}{a}\phi_0'^5 G_{4\phi XX} - 12\frac{\mathcal{H}^8}{a^2}G_{4XXX}\phi_0'^6\end{aligned}$$

$$
\begin{aligned}
&+ G_{4XX}\mathcal{H}^2\phi_0'^2\left[2\left(9\mathcal{H}^4+\dot{\mathcal{H}}^2+2\mathcal{H}^2\dot{\mathcal{H}}\right)\frac{\phi_0'^2}{a^2}+2\left(2\mathcal{H}^2\dot{\mathcal{H}}+2\mathcal{H}^4\right)\frac{\phi_0'}{a}\phi_0''+2\mathcal{H}^4\phi''^2\right]\\
&+ G_{4X}\left[\left(-2\dot{\mathcal{H}}\mathcal{H}^2+2\mathcal{H}^4-\dot{\mathcal{H}}^2-\mathcal{H}\ddot{\mathcal{H}}\right)\frac{\phi_0'^2}{a^2}-\left(\mathcal{H}^4+5\mathcal{H}^2\dot{\mathcal{H}}\right)\frac{\phi_0'}{a}\phi_0''-\mathcal{H}^4\phi''^2\right.\\
&\left.-\mathcal{H}^4\phi_0'\phi_0'''\right],\\
\bar{M}_1^3(a) &= 4G_{4X}\mathcal{H}\phi_0'\left[\left(\dot{\mathcal{H}}+2\mathcal{H}^2\right)\frac{\phi_0'}{a}+\mathcal{H}^2\phi_0''\right]-16G_{4XX}\frac{\mathcal{H}^5}{a}\phi_0'^4-4G_{4X\phi}\mathcal{H}^3\phi_0'^3\,,\\
\bar{M}_2^2(a) &= 4\mathcal{H}^2G_{4X}\phi_0'^2=-\bar{M}_3^2(a)=2\hat{M}^2(a)\,. \qquad (2.152)
\end{aligned}
$$

- L_5-Lagrangian

$$
\begin{aligned}
\Omega(a) &= \frac{2\mathcal{H}^2}{m_0^2}\phi_0'^2\left[G_{5X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0'+\mathcal{H}^2\phi_0''\right)-\frac{G_{5\phi}}{2}\right]-1\,,\\
c(a) &= \frac{\mathcal{H}}{2}\tilde{\mathcal{F}}'+\frac{3}{2}\frac{\mathcal{H}^2}{a}m_0^2\Omega'-3\frac{\mathcal{H}^4}{a^2}\phi_0'^2G_{5\phi}+\frac{3\mathcal{H}^6}{a^2}\phi_0'^4G_{5\phi X}-3\frac{\mathcal{H}^6}{a^3}\phi_0'^3G_{5X}\\
&+2\frac{\mathcal{H}^8}{a^3}\phi_0'^5G_{5XX}\,,\\
\Lambda(a) &= \tilde{\mathcal{F}}-3m_0^2\frac{\mathcal{H}^2}{a^2}(1+\Omega)+4G_{5X}\frac{\mathcal{H}^6}{a^3}\phi_0'^3+3\frac{\mathcal{H}^3}{a}G_{5\phi}\phi_0'^2\,,\\
M_4^2(a) &= -\mathcal{H}\frac{\tilde{\mathcal{F}}'}{4}-\frac{3}{4}\frac{\mathcal{H}^2}{a}m_0^2\Omega'-2\frac{\mathcal{H}^{10}}{a^3}\phi_0'^7G_{5XXX}-3\frac{\mathcal{H}^8}{a^2}\phi_0'^6G_{5\phi XX}+6G_{5XX}\frac{\mathcal{H}^8}{a^3}\phi_0'^5\\
&+6\frac{\mathcal{H}^6}{a^2}\phi_0'^4G_{5\phi X}-\frac{3}{2}\frac{\mathcal{H}^6}{a^3}\phi_0'^3G_{5X}\,,\\
\bar{M}_2^2(a) &= 2\left[\mathcal{H}^2\phi_0'^2G_{5\phi}-G_{5X}\left[-\frac{\mathcal{H}^4}{a}\phi_0'^3+\mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}}{a}\phi_0'+\mathcal{H}^2\phi_0''\right)\right]\right]\\
&= -\bar{M}_3^2(a)=2\hat{M}^2(a)\,,\\
\bar{M}_1^3(a) &= -\mathcal{H}m_0^2\Omega'+4\frac{\mathcal{H}^3}{a}\phi_0'^2G_{5\phi}-4\frac{\mathcal{H}^5}{a}\phi_0'^4G_{5\phi X}-4\frac{\mathcal{H}^7}{a^2}\phi_0'^5G_{5XX}\\
&+6\frac{\mathcal{H}^5}{a^2}\phi_0'^3G_{5X}\,, \qquad (2.153)
\end{aligned}
$$

where $\tilde{\mathcal{F}}(a)=\mathcal{F}-m_0^2\mathcal{H}\Omega'-2\frac{\mathcal{H}}{a}m_0^2(1+\Omega)$ and $\mathcal{F}(\tau)=2\frac{\mathcal{H}^5}{a^2}G_{5X}\phi_0'^3+2\frac{\mathcal{H}^3}{a}G_{5\phi}\phi_0'^2$.

Let us now consider the two Lagrangians which extend the Horndeski/GG theories to the GLPV ones introduced in Section 2.4.4:

- L_4^{GLPV}-Lagrangian

$$
\begin{aligned}
c(a) &= 2\frac{\mathcal{H}^4}{a^2}\phi_0'^4(\dot{\mathcal{H}}-\mathcal{H}^2)\tilde{F}_4 + 8\frac{\mathcal{H}^4}{a}\phi_0'^3\tilde{F}_4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&- 4\frac{\mathcal{H}^6}{a}\tilde{F}_{4X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^5 + 2\mathcal{H}^6\frac{\tilde{F}_{4\phi}}{a}\phi_0'^5 - 12\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_4\,, \\
\Lambda(a) &= 6\frac{\mathcal{H}^6}{a^2}\tilde{F}_4\phi_0'^4 + 4\frac{\mathcal{H}^4}{a^2}(\dot{\mathcal{H}}-\mathcal{H}^2)\phi_0'^4\tilde{F}_4 + 16\frac{\mathcal{H}^4}{a}\phi_0'^3\tilde{F}_4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&- 8\frac{\mathcal{H}^6}{a}\tilde{F}_{4X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^5 + 4\frac{\mathcal{H}^6}{a}\tilde{F}_{4\phi}\phi_0'^5\,, \\
M_2^4(a) &= -18\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_4 - \frac{\mathcal{H}^4}{a^2}\phi_0'^4(\dot{\mathcal{H}}-\mathcal{H}^2)\tilde{F}_4 - 4\frac{\mathcal{H}^4}{a}\phi_0'^3\tilde{F}_4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&+ 2\frac{\mathcal{H}^6}{a}\phi_0'^5\tilde{F}_{4X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}\phi_0''\right) - \frac{\mathcal{H}^6}{a}\phi_0'^5\tilde{F}_{4\phi} + 6\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_4\,, \\
\bar{M}_2^2(a) &= 2\mathcal{H}^4\phi_0'^4\tilde{F}_4 = -\bar{M}_3^2(a)\,, \\
\bar{M}_1^3(a) &= 16\frac{\mathcal{H}^5}{a}\phi_0'^4\tilde{F}_4\,.
\end{aligned}
\tag{2.154}
$$

- L_5^{GLPV}-Lagrangian

$$
\begin{aligned}
\Lambda(a) &= -3\frac{\mathcal{H}^8}{a^3}\phi_0'^5\tilde{F}_5 - 12\frac{\mathcal{H}^6}{a^3}\phi_0'^5(\dot{\mathcal{H}}-\mathcal{H}^2)\tilde{F}_5 - 30\frac{\mathcal{H}^6}{a^2}\tilde{F}_5\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^4 \\
&+ 12\frac{\mathcal{H}^8}{a^2}\tilde{F}_{5X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^6 - 6\frac{\mathcal{H}^8}{a^2}\tilde{F}_{5\phi}\phi_0'^6\,, \\
c(a) &= 6\frac{\mathcal{H}^8}{a^2}\phi_0'^6\tilde{F}_{5X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) - 6\frac{\mathcal{H}^6}{a^3}(\dot{\mathcal{H}}-\mathcal{H}^2)\phi_0'^5\tilde{F}_5 \\
&- 15\frac{\mathcal{H}^6}{a^2}\phi_0'^4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) - 3\frac{\mathcal{H}^8}{a^2}\phi_0'^6\tilde{F}_{5\phi} + 15\frac{\mathcal{H}^8}{a^3}\tilde{F}_5\phi_0'^5\,, \\
M_2^4(a) &= \frac{45}{2}\frac{\mathcal{H}^8}{a^3}\phi_0'^5\tilde{F}_5 + 3\frac{\mathcal{H}^6}{a^3}(\dot{\mathcal{H}}-\mathcal{H}^2)\phi_0'^5\tilde{F}_5 + \frac{15}{2}\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_5\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&- 3\frac{\mathcal{H}^8}{a^2}\phi_0'^6\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\tilde{F}_{5X} + \frac{3}{2}\frac{\mathcal{H}^8}{a^2}\phi_0'^6\tilde{F}_{5\phi}\,, \\
\bar{M}_2^2(a) &= -6\frac{\mathcal{H}^6}{a}\phi_0'^5\tilde{F}_5 = -\bar{M}_3^2(a)\,,
\end{aligned}
$$

$$\bar{M}_1^3(a) = -30\frac{\mathcal{H}^7}{a^2}\tilde{F}_5\phi_0'^5\,. \qquad (2.155)$$

Finally, we write the EFT functions obtained from the GLPV action (2.76) in Section 2.4.4 in the appropriate form adopted in *EFTCAMB* :

$$\begin{aligned}
\Omega(a) &= \frac{2}{m_0^2}\Big(\bar{B}_4 - \frac{\mathcal{H}}{2}\bar{B}_5'\Big) - 1\,,\\
\Lambda(a) &= \bar{A}_2 - 6\frac{\mathcal{H}^2}{a}\bar{A}_4 + 12\frac{\mathcal{H}^3}{a^3}\bar{A}_5 + \mathcal{H}\bar{A}_3' - \frac{4}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_4 - 4\frac{\mathcal{H}^2}{a}\bar{A}_4'\\
&+ 6\frac{\mathcal{H}^3}{a^2}\bar{A}_5' + 12\frac{\mathcal{H}}{a^3}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_5 - \left[\frac{2}{a^2}\left(\mathcal{H}^2 + 2\dot{\mathcal{H}}\right)\bar{B}_4 + \frac{2}{a}\left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right)\bar{B}_4'\right.\\
&\left.+2\mathcal{H}^2\bar{B}_4'' - \frac{\mathcal{H}}{a^2}\left(\mathcal{H}^2 + 3\dot{\mathcal{H}} + \frac{\ddot{\mathcal{H}}}{\mathcal{H}}\right)\bar{B}_5' - \frac{\mathcal{H}}{a}\left(3\dot{\mathcal{H}} + 2\mathcal{H}^2\right)\bar{B}_5'' - \mathcal{H}^3\bar{B}_5'''\right]\,,\\
c(a) &= \frac{1}{2}\Big(\mathcal{H}\bar{A}_3' - \frac{4}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_4 - 4\frac{\mathcal{H}^2}{a}\bar{A}_4' + 6\frac{\mathcal{H}^3}{a^2}\bar{A}_5' + 12\frac{\mathcal{H}}{a^3}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_5\\
&- \bar{A}_{2N} + 3\mathcal{H}\bar{A}_{3N} - 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{4N} + 6\frac{\mathcal{H}^3}{a^3}\bar{A}_{5N}\Big)\\
&+ \frac{1}{a}\left(\mathcal{H}^2 - \dot{\mathcal{H}}\right)\bar{B}_4' + \frac{\mathcal{H}}{2a}\left(3\dot{\mathcal{H}} - \mathcal{H}^2\right)\bar{B}_5'' - \mathcal{H}^2\bar{B}_4'' + \frac{\mathcal{H}^3}{2}\bar{B}_5'''\\
&+ \frac{1}{2a^2}\left(\ddot{\mathcal{H}} - 2\mathcal{H}^3\right)\bar{B}_5' - \frac{2}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{B}_4\,,\\
M_2^4(a) &= \frac{1}{4}\Big(\bar{A}_{2NN} - 3\frac{\mathcal{H}}{a}\bar{A}_{3NN} + 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{4NN} - 6\frac{\mathcal{H}^3}{a^3}\bar{A}_{5NN}\Big)\\
&- \frac{1}{4}\left[\mathcal{H}\bar{A}_3' - 4\frac{\bar{A}_4}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2) - 4\frac{\mathcal{H}^2}{a}\bar{A}_4' + 6\frac{\mathcal{H}^3}{a^2}\bar{A}_5'\right.\\
&\left.+12\bar{A}_5\frac{\mathcal{H}}{a^3}(\dot{\mathcal{H}} - \mathcal{H}^2)\right] + \frac{3}{4}\Big(\bar{A}_{2N} - 3\frac{\mathcal{H}}{a}\bar{A}_{3N} + 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{4N} - 6\frac{\mathcal{H}^3}{a^3}\bar{A}_{5N}\Big)\\
&- \frac{1}{2}\left[-\frac{2}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{B}_4 + \frac{1}{a}\left(\mathcal{H}^2 - \dot{\mathcal{H}}\right)\bar{B}_4' - \mathcal{H}^2\bar{B}_4''\right.\\
&\left.+\frac{1}{a^2}\left(\ddot{\mathcal{H}} - \mathcal{H}^3\right)\bar{B}_5' + \frac{\mathcal{H}}{2a}\left(3\dot{\mathcal{H}} - \mathcal{H}^2\right)\bar{B}_5'' + \frac{\mathcal{H}^3}{2}\bar{B}_5'''\right]\,,\\
\bar{M}_2^2(a) &= -2\bar{A}_4 + 6\frac{\mathcal{H}}{a}\bar{A}_5 - 2\bar{B}_4 + \mathcal{H}\bar{B}_5' = -\bar{M}_3^2(a)\,,\\
\bar{M}_1^3(a) &= -\bar{A}_{3N} + 4\frac{\mathcal{H}}{a}\bar{A}_{4N} - 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{5N} - 2\bar{B}_4'\mathcal{H} + \frac{\dot{\mathcal{H}}}{a}\bar{B}_5' + \mathcal{H}^2\bar{B}_5''\,,\\
\hat{M}^2(a) &= \bar{B}_{4N} + \frac{\mathcal{H}}{2a}\bar{B}_{5N} + \frac{\mathcal{H}}{2}\bar{B}_5'\,. \qquad (2.156)
\end{aligned}$$

2.11 Appendix D: On the $\mathcal{J}$ coefficient in the L_5 Lagrangian

In this Appendix we will show the details of the calculation regarding the J coefficient in the L_5 Lagrangian (2.53). Let us consider the following term:

$$\begin{aligned} G_{5X}\mathcal{J} &= G_{5X}\Big(-\frac{1}{2}\phi^{;\rho}X_{;\rho}(K^2-S)+2\gamma^{-3}(\gamma^2\frac{h^\rho_\mu}{2}X_{;\rho})(K\dot{n}_\mu-K^{\mu\nu}\dot{n}_\nu)\Big) \\ &= -\frac{1}{2}\Big(\gamma\nabla_\rho(\gamma^{-1}F_5)-F_{5\phi}\gamma^{-1}n_\rho\Big)(K^2-S)\phi^{;\rho} \\ &+\gamma^{-1}(K\dot{n}^\mu-K^{\mu\nu}\dot{n}_\nu)h^\mu_\rho\Big(\gamma\nabla_\rho(\gamma^{-1}F_5)+F_{5\phi}\gamma^{-1}n^\rho\Big)\,. \end{aligned} \tag{2.157}$$

The last parenthesis contains a quantity which is orthogonal to the quantities that multiply it, hence it vanishes. Therefore, we have:

$$\begin{aligned} G_{5X}\mathcal{J} &= \frac{F_{5\phi}}{2}n_\rho n^\rho(K^2-S)-\frac{1}{2}n^\rho\nabla_\rho(\gamma^{-1}F_5)(K^2-S)+h^\rho_\mu\nabla_\rho(\gamma^{-1}F_5)(K\dot{n}^\mu-K^{\mu\nu}\dot{n}_\nu) \\ &= -\frac{F_{5\phi}}{2}(K^2-S)+\frac{F_5}{\gamma}\left[\frac{1}{2}\nabla_\rho(n^\rho K^2-n^\rho K_{\mu\nu}K^{\mu\nu})-(K\dot{n}^\mu-K^{\mu\nu}\dot{n}_\nu)_{;\mu}\right] \\ &= \frac{F_5}{\gamma}\left(\frac{K^3}{2}+n^\rho K\nabla_\rho K-\frac{K}{2}K_{\mu\nu}K^{\mu\nu}-n^\rho K^{\mu\nu}\nabla_\rho K_{\mu\nu}-\dot{n}^\rho\nabla_\rho K\right. \\ &\left.-K\nabla_\rho\dot{n}^\rho+\dot{n}_\nu\nabla_\rho K^{\rho\nu}+K^{\rho\nu}\nabla_\rho\dot{n}_\nu\right)-\frac{F_{5\phi}}{2}(K^2-S), \end{aligned} \tag{2.158}$$

where in the second line we have used the fact that n_μ is orthogonal to $\dot{n}_\mu$ and $K^{\mu\nu}$. Now, employing the following geometrical quantities:

$$\begin{aligned} R_{\mu\nu}n^\mu n^\nu &= -n^\mu\nabla_\mu K+\nabla_\mu\dot{n}^\mu+n^\mu\nabla^\nu K_{\mu\nu}\,, \\ R_{\mu\nu}n^\nu\dot{n}^\mu &= \dot{n}^\mu\nabla^\nu K_{\mu\nu}-\dot{n}^\mu\dot{n}_\nu\nabla^\nu n_\mu-\dot{n}^\mu\nabla_\mu K\,, \\ K^{\mu\nu}n^\rho n^\sigma R_{\mu\sigma\nu\rho} &= K^{\gamma\alpha}n^\beta(\nabla_\alpha K_{\beta\gamma})-K^{\gamma\alpha}n^\beta(\nabla_\beta K_{\alpha\gamma})+K^{\gamma\alpha}(\nabla_\alpha\dot{n}_\gamma)+K^{\gamma\alpha}\dot{n}_\gamma\dot{n}_\alpha\,, \end{aligned} \tag{2.159}$$

we obtain:

$$\begin{aligned} G_{5X}\mathcal{J} &= \frac{F_5}{\gamma}\Big(\frac{K^3}{2}+n^\rho K\nabla_\rho K-\frac{K}{2}K_{\mu\nu}K^{\mu\nu}-n^\rho K^{\mu\nu}\nabla_\rho K_{\mu\nu}-\dot{n}^\rho\nabla_\rho K \\ &-K\nabla_\rho\dot{n}^\rho+\dot{n}_\nu\nabla_\rho K^{\rho\nu}+K^{\rho\nu}\nabla_\rho\dot{n}_\nu\Big)-\frac{F_{5\phi}}{2}(K^2-S) \\ &= \frac{F_5}{\gamma}\Big(\frac{K^3}{2}-\frac{K}{2}K_{\mu\nu}K^{\mu\nu}-KR_{\mu\nu}n^\mu n^\nu+n^\mu K(\nabla^\nu K_{\mu\nu})+K^{\mu\nu}n^\rho n^\sigma \\ &+K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho}-K^{\gamma\alpha}n^\beta(\nabla_\alpha K_{\beta\gamma}) \end{aligned}$$

$$
\begin{aligned}
&- K^{\gamma\alpha}\dot{n}_\gamma \dot{n}_\alpha + R_{\mu\nu}n^\mu \dot{n}^\nu + \dot{n}^\mu \dot{n}^\nu \nabla_\nu n_\mu\Big) - \frac{F_{5\phi}}{2}(K^2 - S) \\
&= \frac{F_5}{\gamma}\Big(\frac{K^3}{2} - \frac{K}{2}K_{\mu\nu}K^{\mu\nu} - KR_{\mu\nu}n^\mu n^\nu - KK^{\mu\nu}K_{\mu\nu} + K^{\mu\nu}n^\rho n^\sigma \\
&+ K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} + K^{\gamma\alpha}K^\beta_\alpha K_{\beta\gamma} \\
&- K^{\gamma\alpha}\dot{n}_\gamma \dot{n}_\alpha + R_{\mu\nu}n^\mu \dot{n}^\nu + \dot{n}^\mu \dot{n}^\nu \nabla_\nu n_\mu\Big) - \frac{F_{5\phi}}{2}(K^2 - S)\,,
\end{aligned}
\tag{2.160}
$$

where we have dropped a total derivative term. Finally, we use the definition $\tilde{\mathcal{K}}$ in Eq. (2.59) and we obtain the final result used in Section 2.4.3:

$$
G_{5X}\mathcal{J} = F_5\gamma^{-1}\Big[\frac{\tilde{\mathcal{K}}}{2} + K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} + \dot{n}^\sigma n^\rho R_{\sigma\rho} - Kn^\sigma n^\rho R_{\sigma\rho}\Big] - \frac{F_{5\phi}}{2}(K^2 - S)\,. \tag{2.161}
$$