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## Order isomorphisms, order antimorphisms and their interplay with Jordan algebra structures

Imhoff, H.B. van

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**Author:** Imhoff, H.B. van

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## *Chapter 7*

# *Symmetric cones and order antimorphisms*

In a finite dimensional vector space the interior of a closed cone is considered *symmetric* if it is homogeneous and self-dual. A variety of characterisations are given for this special class of cones, see Section 1.6 for more details. The famous Koecher-Vinberg theorem ([Koe57] and [Vin60]) shows that a symmetric cone arises precisely as the interior of the cone of squares for a formally real Jordan algebra. An infinite dimensional analogue of this result for JB-algebras does not exist, since the notion of a symmetric cone is not well-defined in a Banach space, which in general cannot be realised as a Hilbert space. An alternative characterisation of symmetric cones in finite dimensions is given, due to Walsh [Wal13], in terms of the existence of an antihomogeneous order antimorphism on the interior of the cone. In Chapter 6, we considered strictly convex cones in arbitrary dimensions where an antihomogeneous order antimorphism exists on the interior, and we obtained precisely the spin factors. With the techniques developed there, we further investigate the relation between the existence of an antihomogeneous order antimorphism on the interior of the cone and the symmetric property of that cone.

Let  $(H, (\cdot | \cdot))$  be a Hilbert space and  $C \subseteq H$  be a cone. Then  $C^\circ$  is considered a *symmetric* cone if  $C^\circ$  is *homogeneous*, meaning for  $x, y \in C^\circ$  there exists a linear order isomorphism  $S: C^\circ \rightarrow C^\circ$  such that  $S(x) = y$ , and *self-dual* with respect to the inner product  $(\cdot | \cdot)$ , meaning

$$C^\circ = \{x \in H: (x | y) > 0 \text{ for all } y \in C \setminus \{0\}\}.$$

In the sequel, when we have an order unit space  $(V, C, u)$  we say that  $C^\circ$  is symmetric whenever it is homogeneous and there exists an inner product on  $V$  that turns it into a Hilbert space and with respect to which  $C^\circ$  is self-dual. Our aim is to characterise symmetric cones in complete order unit spaces in an order theoretic way without a priori imposing Hilbert space structure. We remark that the interior of the cone of a spin factor is symmetric. Therefore, characterising properties for symmetric cones should be weaker than those imposed in Theorem 6.1 for spin factors. It turns out that one should replace the condition on  $C$  of being strictly convex with being the sum of

its extreme rays. In other words, instead of every element of the cone being the sum of two positive extreme vectors, we merely require that they are a finite positive linear combination of positive extreme vectors.

Our general strategy to obtain this characterisation is to reduce problems to finite dimensional subcones. A key observation is that, the results of [NS77] concerning order isomorphisms and the way they interact with line segments that are parallel to extreme rays, which we outlined in Section 3.1, for the most part also apply to order antimorphisms. This will yield that any subcone of  $C^\circ$  that is spanned by finitely many extreme rays will be mapped by the antimorphism  $g: C^\circ \rightarrow C^\circ$  onto a subcone that is again spanned by finitely many extreme rays. Before we make the ideas more rigorous, we summarise the results obtained in [Wal13], that are relevant for our purpose.

**Theorem 7.1** (Walsh). *Suppose  $K$  is a finite dimensional closed cone with non-empty interior and  $g: K^\circ \rightarrow K^\circ$  is an antihomogeneous order antimorphism. Then  $K$  is a symmetric cone that is self-dual for an inner product  $(\cdot | \cdot)$  which satisfies*

$$(y | x) = M(x/g(y)), \quad (7.1)$$

for all  $y \in K^\circ$  and  $x \in K$  extreme.

We recall that by definition for  $x \in K$  and  $y \in K^\circ$  we have

$$M(x/y) = \inf\{\beta > 0: x \leq \beta y\}.$$

In Proposition 3.1 and Proposition 3.3 and their subsequent corollaries, we have shown that for an order isomorphism  $f: C \rightarrow K$  and extreme vector  $r \in C$ , the element  $f(x+r) - f(x)$  is an extreme vector of  $K$ . Furthermore, if the difference of  $x, y \in C$  is a linear combination of extreme vectors, then  $f(x+r) - f(x)$  is a scalar multiple of  $f(y+r) - f(y)$ . In our setting, where  $C$  is assumed to be the sum of its extreme rays, we obtain that  $f(x+r) - f(x)$  and  $f(y+r) - f(y)$  lies on the same extreme ray for all  $x, y \in C$  and  $r \in C$  extreme. We draw a similar conclusion for order antimorphisms. Recall that for a cone  $C$  the set  $\mathcal{E}(C)$  denotes the collection of its extreme rays.

**Proposition 7.2.** *Let  $(X, C)$  and  $(Y, K)$  be Archimedean partially ordered vector spaces, where  $C$  is the sum of its extreme rays and  $U \subseteq X$  and  $V \subseteq Y$  are upper sets. For any order anti- or isomorphism  $f: U \rightarrow V$ , there exists a bijection  $\varphi: \mathcal{E}(C) \rightarrow \mathcal{E}(K)$  such that any line segment in  $U$  parallel to some  $R \in \mathcal{E}(C)$  is mapped by  $f$  onto a line segment in  $V$  parallel to  $\varphi(R) \in \mathcal{E}(K)$ . In symbols, for every  $x \in U$  and  $R \in \mathcal{E}(C)$*

$$f((x \pm R) \cap U) = (f(x) \pm \varphi(R)) \cap V. \quad (7.2)$$

*Proof.* Consider the case where  $f$  is an order isomorphism. Equation (7.2) follows from Theorem 3.10 for  $R \in \mathcal{E}(C)$  that are engaged and from Proposition 3.17 for  $R \in \mathcal{E}(C)$  that are disengaged.

Suppose that  $f$  is an order antimorphism. It follows from Proposition 3.1, that a subset  $L$  of  $U$  is of the form  $(x \pm R) \cap U$  for some extreme ray  $R \in \mathcal{E}(C)$  if and only

if  $L$  is maximal among those subset of  $U$  that are directed and whose subintervals are totally ordered. These properties are not only preserved by order isomorphisms, but also by order antimorphisms. The only difference being that an order antimorphism reverses the order within an extreme half-line. In symbols, for  $x \in U$ ,  $r \in X$  an extreme vector and  $\lambda \in \mathbb{R}$  with  $x + r, x + \lambda r \in U$  we have that

$$f(x + \lambda r) - f(x) = c(f(x + r) - f(x)), \quad (7.3)$$

for some  $c \in \mathbb{R}$ . Let  $R$  and  $S$  be different extreme rays of  $C$ . Suppose  $x \in U$ ,  $r \in -R$  and  $s \in -S$  are given such that  $x, x + r, x + s, x + r + s \in U$ . We construct  $R_j := (x + js - R) \cap U$ , for  $j \in \{0, 1, 2\}$ . Their images  $f(R_j)$  are distinct half-lines with apexes  $f(x + js)$ , for  $j \in \{0, 1, 2\}$  respectively, and they are unbounded in the direction of a positive extreme vector. The half-lines  $f(R_0)$ ,  $f(R_1)$  and  $f(R_2)$  satisfy the same conditions as their namesakes in the proof of Proposition 3.3. So we deduce that  $f(x), f(x + s), f(x + r), f(x + r + s)$  are the consecutive corners of a parallelogram, and hence

$$f(x + r + s) - f(x + s) = f(x + r) - f(x). \quad (7.4)$$

We considered negative extreme vectors  $r \in -R$  and  $s \in -S$  to guarantee that the lines  $R_j$  were flipped by the antimorphism  $f$  to a positive direction. Suppose now that  $r \in R$ , then we can apply the above arguments to  $x + r \in U$ ,  $-r \in -R$  and  $s \in -S$  and obtain (7.4). Similarly the sign of  $s$  is irrelevant for the conclusion.

By repeated application of (7.4), we obtain that for  $x \in U$  and  $s_1, \dots, s_n, r \in X$  extreme vectors such that  $r \neq \lambda s_i$  for all  $\lambda \in \mathbb{R}$  and  $i = 1, \dots, n$  with  $x, x + r, x + \sum_{i=1}^n s_i, x + r + \sum_{i=1}^n s_i \in U$  that

$$f(x + r + \sum_{i=1}^n s_i) - f(x + \sum_{i=1}^n s_i) = f(x + r) - f(x). \quad (7.5)$$

Let  $x \in U$ ,  $R \in \mathcal{E}(C)$  and  $r \in (R \cup -R) \setminus \{0\}$  be such that  $x + r \in U$ . Now  $s := f(x + r) - f(x)$  is an extreme vector of  $K$ . Let  $y \in U$  with  $y + r \in U$ . By our assumption we can write  $y - x = \sum_{i=1}^n r_i$ , where  $r_i \in X$  is an extreme vector, for  $i = 1, \dots, n$ . Remark that one of these  $r_i$  might be a multiple of  $r$ . By relabelling we assume that  $r_1, \dots, r_k < 0$  and  $r_{k+1}, \dots, r_n > 0$  and, moreover, that if  $r = \lambda r_i$  for some  $\lambda \in \mathbb{R}$  then in the new labelling  $r_i$  becomes  $r_1$  if  $\lambda < 0$  and  $r_{k+1}$  if  $\lambda > 0$ . We assume the latter to be the case, since the other cases follow in fewer steps. Combining (7.3)

and (7.5) yields

$$\begin{aligned}
f(y+r) - f(y) &= f\left(x+r + \sum_{i=1}^n r_i\right) - f\left(x + \sum_{i=1}^n r_i\right) \\
&= f\left(x+r + \sum_{i=k+1}^n r_i\right) - f\left(x + \sum_{i=k+1}^n r_i\right) \\
&= c \left( f\left(x+r + \sum_{i=k+2}^n r_i\right) - f\left(x + \sum_{i=k+2}^n r_i\right) \right) \\
&= c(f(x+r) - f(x)),
\end{aligned}$$

for some non-zero  $c \in \mathbb{R}$ . This shows that  $f(y+r) \in f(y) + S$ , where  $S$  denotes the extreme ray in  $K$  spanned by  $s$ . Defining  $\varphi(R) := S$  yields a map  $\varphi: \mathcal{E}(C) \rightarrow \mathcal{E}(K)$  that satisfies (7.2). That  $\varphi$  is bijective follows from the fact that  $f$  is bijective.  $\square$

**The standing hypotheses in the sequel are as follows.**

Let  $(V, C, u)$  be a complete order unit space. The cone  $C$  equals *the sum of its extreme rays*, that is, any element of  $C$  can be written as a positive linear combination of extreme vectors of  $C$ . Furthermore, let  $g: C^\circ \rightarrow C^\circ$  be an antihomogeneous order antimorphism. Lastly, we denote by  $\mathcal{E}(C)$  the collection of extreme rays of  $C$  and  $\varphi: \mathcal{E}(C) \rightarrow \mathcal{E}(C)$  for the bijection corresponding to  $g$ , that satisfies (7.2), as obtain in Proposition 7.2.

We argue in steps that under these conditions  $C^\circ$  is a symmetric cone.

### *Homogeneous cone*

It is convenient to introduce some notation. For a finite subset  $F \subseteq \mathcal{E}(C)$  we let  $C(F) = \text{span}(F) \cap C$  and  $C^\circ(F) = \text{span}(F) \cap C^\circ$ . We remark that as the finite dimensional subspace  $\text{span}(F)$  is closed, that the relative interior of  $C(F)$  equals  $C^\circ(F)$  if  $\text{span } F \cap C^\circ$  is non-empty. Henceforth, *any* finite subset  $F \subseteq \mathcal{E}(C)$  is assumed to yields a non-empty  $C^\circ(F)$ .

In the following result our approach is similar to that of [Wal18, Lemma 3.9].

**Lemma 7.3.** *For any finite subset  $F \subseteq \mathcal{E}(C)$  with  $\text{span } F \cap C^\circ$  non-empty, we have  $g[C^\circ(F)] = C^\circ(\varphi[F])$ .*

*Proof.* Suppose  $F \subseteq \mathcal{E}(C)$  is finite. Fix  $x \in C^\circ(F)$ . We define  $W = g(x) + \text{span}(\varphi[F])$ , an affine subspace of  $V$ . Let  $y \in C^\circ(F)$  and write  $y - x = \sum_{i=1}^n r_i$  with  $r_i \in R_i \cup -R_i$  and  $R_i \in F$ , for  $i = 1, \dots, n$ . We reorder the indices if necessary so that all  $r_i$  are positive for  $i \leq m$  and negative for  $i > m$  for some  $m$ . Now we define  $x_0 = x$  and  $x_k = x + \sum_{i=1}^k r_i$ , for  $k = 1, \dots, n$ . We remark that our reordering guarantees that all  $x_k \in C^\circ(F)$ . Also, by construction  $x_n = y$ . Note that  $g(x_0) \in W$ . By Proposition 7.2, we now iteratively obtain that all subsequent  $g(x_i)$ , and in particular  $g(x_n) = g(y)$ , are contained in  $W$ . As  $y$  was chosen arbitrarily, we conclude  $g[C^\circ(F)] \subseteq W$ . We remark

that  $g^{-1}$  satisfies (7.2) for the bijection  $\varphi^{-1}: \mathcal{E}(C) \rightarrow \mathcal{E}(C)$ . Therefore, we obtain the reverse inclusion  $W \cap C^\circ \subseteq g[C^\circ(F)]$ .

It remains to argue that  $W$  is a linear subspace of  $V$ . Let  $x \in C^\circ(F)$ . Then  $x \in C^\circ$  and hence  $x$  is an order unit. For all  $z \in C^\circ$  there exists an  $n \in \mathbb{N}$  such that  $nx > g^{-1}(z)$ , so for all  $m \geq n$  we have  $0 \leq g(mx) < z$ . In particular,  $\|g(nx) - 0\|_u \rightarrow 0$  as  $n \rightarrow \infty$ .  $W$  is a finite dimensional affine subspace of  $V$  and, hence, is closed. We conclude  $0 \in W$  and that  $W$  is a linear subspace.  $\square$

In the sequel we will denote the restriction  $g|_{C^\circ(F)}: C^\circ(F) \rightarrow C^\circ(\varphi[F])$  simply by  $g_F$ . Similar as in Chapter 6, our first step is to construct a point symmetry for each  $x \in C^\circ$ .

**Proposition 7.4.** *Let  $(V, C, u)$  be an order unit space, such that  $C$  equals the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  an antihomogeneous order antimorphism. For  $x \in C^\circ$  and  $y \in V$  the following limit exists*

$$\Delta_x^y g(x) := \lim_{t \rightarrow 0} \frac{g(x + ty) - g(x)}{t}.$$

Moreover, for any finite  $F \subseteq \mathcal{E}(C)$  with  $\text{span } F \cap C^\circ$  non-empty the restriction  $g_F: C^\circ(F) \rightarrow C^\circ(\varphi[F])$  is Fréchet differentiable.

*Proof.* Let  $F \subseteq \mathcal{E}(C)$  with  $\text{span } F \cap C^\circ$  non-empty. As the restriction  $g_F$  of  $g$  is an antihomogeneous order antimorphism from  $C^\circ(F)$  onto  $C^\circ(\varphi[F])$ , [Wal13, Corollary 1.2] yields that  $C^\circ(F)$  and  $C^\circ(\varphi[F])$  are linearly isomorphic. Let  $h: C^\circ(F) \rightarrow C^\circ(\varphi[F])$  be a linear order isomorphism and  $f = h^{-1} \circ g_F: C^\circ(F) \rightarrow C^\circ(F)$ . Then  $C^\circ(F)$  is a symmetric cone by Theorem 7.1, since  $f$  is an antihomogeneous order antimorphism. Therefore,  $\text{span } F$  is a Euclidean Jordan algebra with  $C^\circ(F)$  as the interior of its cone of squares, by the Koecher-Vinberg theorem. In [LRW, Theorem 3.2] the isometries for Thompson's metric on the interior of the cone of a JB-algebra are characterised. This yields in our case that  $f$  is the composition of a linear bijection and the inversion map with respect to the Jordan product. In general, the inversion map on the interior of a cone in a JB-algebra is smooth, whose derivative at  $x$  is given by  $-Q_x^{-1}$ . We conclude that  $f$ , and hence  $g_F$ , is Fréchet differentiable. Let  $Dg_F(x): \text{span } F \rightarrow \text{span } F$  denote the Fréchet derivative of  $g_F$  at  $x$ .

Now let  $x \in C^\circ$  and  $y \in V$  be given. Let  $F \subseteq \mathcal{E}(C)$  be finite with  $x, y \in \text{span } F$ . Then  $\Delta_x^y g(x)$  exists and is given by  $Dg_F(x)(y)$ .  $\square$

Results in Chapter 6, from Proposition 6.5 up to and including Theorem 6.9, now directly follow in our case with the following modification. For any pair  $x, y \in C^\circ$  we consider the restriction  $g_F$  of  $g$  to a subcone  $C^\circ(F)$ , for some  $F \subseteq \mathcal{E}(C)$  finite with  $x, y \in \text{span}(F)$ , instead of the restriction  $g_{xy}$  of  $g$  to the 2-dimensional subcone  $C^\circ(x, y)$ . We summarise these results here for convenience.

Let  $x \in C^\circ$ . The map  $G_{g,x}: V \rightarrow V$  defined by  $G_{g,x}(y) = -\Delta_x^y g(x)$  is a linear order isomorphism whose inverse is given by  $G_{g^{-1},g(x)}$ . The *symmetry at  $x$*  defined by

$$S_x = G_{g,x}^{-1} \circ g, \quad (7.6)$$

is an antihomogeneous order antimorphism, whose Gateaux derivative satisfies  $\mathcal{D}S_x = -\text{Id}$  and satisfies both  $S_x(x) = x$  and  $S_x \circ S_x = \text{Id}$  on  $C^\circ$ .

Even though unnecessary for showing that  $C$  is homogeneous, we argue that the point symmetries  $S_x$  have  $x$  as their unique fixed point in our setting, giving rise to a globally symmetric Banach-Finsler manifold similar as in Remark 6.12. We need to employ arguments different to those used in the strictly convex cone case. First we make the following observation on closed balls of Hilbert's metric.

**Lemma 7.5.** *Let  $(V, C, u)$  be a complete order unit space. If  $y \in C^\circ$  and  $r > 0$ , then  $B_r(y) \cup \{0\}$  is a  $\|\cdot\|_u$ -closed subcone of  $C$ , where  $B_r(x)$  is the closed ball for Hilbert's metric centered at  $y$  with radius  $r$ .*

*Proof.* Suppose  $y \in C^\circ$  and  $r > 0$  are given. Let  $B := B_r(y) \cup \{0\}$ . In [LN12, Lemma 2.6.1] it is shown that a closed ball for Hilbert's metric in the interior of a finite dimensional closed cone is projectively convex. From the arguments, however, it follows that any closed ball for Hilbert's metric is convex. Since Hilbert's metric is constant on rays it follows that  $B$  is a cone. We verify that  $B$  is closed for  $\|\cdot\|_u$ . Suppose  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $B$  that converges to  $x \in C$ . We consider several cases. If  $x \in C^\circ$  holds, then eventually all  $x_n \in B$  and as  $(a, b) \mapsto M(a/b)$  is a continuous map from  $V \times C^\circ$  to  $\mathbb{R}$  by [LLNW18, Lemma 2.2], and  $x \in B$  follows. In the case  $x \in \partial C \setminus \{0\}$ , we obtain a contradiction as eventually all  $x_n \in B$  and  $d_H(x_n, y)$  tends to infinity. The last case to consider is  $x = 0$ , which follows from  $0 \in B$ .  $\square$

Now we can show that the map  $S_x$ , for some  $x \in C^\circ$ , has  $x$  as a *unique* fixed point, by using that  $S_x$  is an isometry of Hilbert's metric.

**Lemma 7.6.** *For each  $x \in C^\circ$  we have that  $S_x$  has  $x$  as a unique fixed point.*

*Proof.* Suppose  $y \in C^\circ$  is a fixed point of  $S_x$  and  $y \neq x$ . Due to  $S_x$  being antihomogeneous it follows that  $y$  is not a scalar multiple of  $x$ . Consider the closed  $d_H$ -ball  $B_r(y)$  centered at  $y$  with radius  $r = d_H(x, y)$ . Then  $B := B_r(y) \cup \{0\}$  is a  $\|\cdot\|_u$ -norm closed subcone of  $C$  by Lemma 7.5 with  $y \in B^\circ$  and  $x \in \partial B$ . By the Hahn-Banach theorem, let  $\psi: V \rightarrow \mathbb{R}$  be a  $\|\cdot\|_u$ -norm continuous functional that supports  $B$  at  $x$ . Then  $\psi(x) = 0$ ,  $\psi(y) > 0$  and  $\psi(v) \geq 0$  for all  $v \in B$ . Consider the  $d_H$ -geodesic defined by  $\gamma(t) = tx + (1-t)y$ , for  $t \in [0, 1]$ , which is fully contained in  $B_r(y)$ . We remark that  $S_x[B_r(y)] \subseteq B_r(y)$  holds, as  $y$  is a fixed point of the  $d_H$ -isometry  $S_x$ . Therefore, the  $d_H$ -geodesic  $t \mapsto \hat{\gamma}(t) := S_x(\gamma(t))$  is also contained in  $B_r(y)$ .

Consider the composition  $\psi \circ S_x: C^\circ \rightarrow \mathbb{R}$  and remark that it is Gateaux differentiable. We compute the Gateaux derivative of  $\psi \circ S_x$  at  $x$  in the direction of  $y - x$ . As  $(\psi \circ S_x)(x) = \psi(S_x(x)) = \psi(x) = 0$ , we get

$$\lim_{t>0} \frac{(\psi \circ S_x)(x + t(y - x)) - (\psi \circ S_x)(x)}{t} = \lim_{t>0} \frac{\psi(\hat{\gamma}(t))}{t} \geq 0.$$

However, as  $\psi$  is linear we can compute the same derivative as follows

$$\psi(\mathcal{D}S_x(x)(y-x)) = \psi(x-y) = -\psi(y) < 0.$$

Here we used that  $\mathcal{D}S_x = -\text{Id}$ . This yields the desired contradiction and we conclude that indeed  $x$  is the unique fixed point of  $S_x$ .  $\square$

We continue our analysis of the point symmetries  $S_x$  induced by  $g$  and how their existence guarantees that  $C^\circ$  is a homogeneous cone. First off, we study the interaction between such symmetries and unique geodesics with respect to Thompson's metric. A point symmetry  $S_x$  maps a unique  $d_T$ -geodesic through  $x$  onto itself and reverses its orientation, as we show below. A similar statement is made in Lemma 6.10 with the difference being that there the cone is strictly convex, hence uniqueness of the geodesic is automatic. Here we have to restrict the scope to unique geodesics. For an overview of geodesics and their properties, see Section 1.5.

**Lemma 7.7.** *Let  $x \in C^\circ$ . For a unique  $d_T$ -geodesic line  $\gamma: \mathbb{R} \rightarrow C^\circ$  with  $\gamma(0) = x$  we have  $S_x(\gamma(t)) = \gamma(-t)$ , for all  $t \in \mathbb{R}$ .*

*Proof.* Let  $x \in C^\circ$  and  $\gamma: \mathbb{R} \rightarrow C^\circ$  a unique geodesic line with  $\gamma(0) = x$ . We remark that  $\hat{\gamma}: \mathbb{R} \rightarrow C^\circ$  defined by  $\hat{\gamma}(t) = S_x(\gamma(t))$ , for  $t \in \mathbb{R}$ , is a unique geodesic as  $S_x$  is an isometry under  $d_T$ , and satisfies  $\hat{\gamma}(0) = x$ . Suppose that  $\gamma$  is of type II. Then  $\gamma(t) = e^t x$  for all  $t \in \mathbb{R}$ . In this case, the antihomogeneity of  $S_x$  immediately yields

$$\hat{\gamma}(t) = S_x(e^t x) = e^{-t} S_x(x) = e^{-t} x = \gamma(-t).$$

In particular, this yields that as  $S_x$  is an involution that  $\gamma$  and  $\hat{\gamma}$  are necessarily of the same type, as each unique geodesic is either of type I or of type II.

Suppose now that  $\gamma$  is a unique geodesic of type I. Then there two pairs  $r, s \in \partial C$  and  $u, v \in \partial C$  with  $r + s = x$  and  $u + v = x$  such that  $\gamma(t) = e^t r + e^{-t} s$ , and  $\hat{\gamma}(t) = e^t u + e^{-t} v$ , for all  $t \in \mathbb{R}$ . Recall that  $S_x$  is Gateaux differentiable and satisfies  $\mathcal{D}S_x = -\text{Id}$ , and hence

$$\hat{\gamma}'(0) = \mathcal{D}S_x(\gamma(0))(\gamma'(0)) = \mathcal{D}S_x(x)(r-s) = -r+s.$$

Computing the same derivative directly yields  $\hat{\gamma}'(0) = u - v$ . In combination with  $r + s = x = u + v$  this yields  $r = v$  and  $s = u$ . Therefore, we conclude

$$S_x(\gamma(t)) = \hat{\gamma}(t) = e^t u + e^{-t} v = e^{-t} r + e^t s = \gamma(-t).$$

$\square$

We verify that our cone has sufficiently many unique geodesics for Thompson's metric. Even though this is a direct consequence of [LR15, Theorem 4.3], we provide a proof for the reader's convenience.

**Lemma 7.8.** *Let  $x \in C^\circ$  and  $r, s \in \partial C$  with  $x = r + s$ . The type I geodesic  $\gamma: \mathbb{R} \rightarrow C^\circ$  through  $x$  defined by  $\gamma(t) = e^t r + e^{-t} s$ , for all  $t \in \mathbb{R}$ , is unique whenever  $r$  or  $s$  is an extreme vector of  $C$ .*

*Proof.* Let  $x \in C^\circ$  and  $r, s \in \partial C$  be given with  $x = r + s$ . We remark that  $r \neq s$ . Let  $\gamma$  denote the type I geodesic given by  $\gamma(t) = e^t r + e^{-t} s$ , for all  $t \in \mathbb{R}$ . Without loss of generality we assume that  $r$  is an extreme vector.

By [LR15, Theorem 4.3] the geodesic  $\gamma$  is unique if no  $y \in V \setminus \{0\}$  and  $\epsilon > 0$  exist such that  $r + \lambda y$  and  $s + \lambda y$  are elements of  $\partial C(r, s, y)$  for all  $|\lambda| < \epsilon$ , where  $C(r, s, y) = C \cap \text{span}(r, s, y)$ . Suppose the converse holds. Let  $\lambda \in \mathbb{R}$  with  $|\lambda| < \epsilon$ . Then both  $r - \lambda y$  and  $r + \lambda y$  are in  $C$  and  $r$  lies on the straight line segment connecting them. Therefore, as the extreme ray spanned by  $r$  is a face of the cone, we infer that  $r + \lambda y$  is a scalar multiple of  $r$ . In particular,  $y$  is a multiple of  $r$ . Now  $s - \mu r \in \partial C(r, s)$  for some  $\mu > 0$ , which yields a contradiction.  $\square$

Combining the existence of unique type I geodesics for Thompson's metric in the direction of extreme rays, as given by Lemma 7.8, with the fact that a point symmetry  $S_x$  mirrors a unique geodesic through  $x$ , as shown in Lemma 7.7, we obtain information on the automorphism group  $\text{Aut}(C^\circ)$ .

**Lemma 7.9.** *For any  $x \in C^\circ$  and  $r \in V$  an extreme vector of  $C$  with  $x + r \in C^\circ$ , there exists an  $S \in \text{Aut}(C^\circ)$  such that  $S(x) = x + r$ .*

*Proof.* Let  $\lambda \in \mathbb{R}$  and  $s \in \partial C$  such that  $x = r' + s$  with  $r = \lambda r'$  and  $r' \in C$ . We remark that  $x + \lambda r' = x + r \in C^\circ$ , so  $\lambda > -1$ . Indeed, if  $\lambda \leq -1$  then  $x + r \leq x - r' = s$  holds and  $x + r \notin C^\circ$  follows. Consider the geodesic line  $\gamma: \mathbb{R} \rightarrow C^\circ$  for Thompson's metric defined by  $\gamma(t) = e^t r' + e^{-t} s$ , for  $t \in \mathbb{R}$ . By Lemma 7.8,  $\gamma$  is unique. Let  $\alpha = \sqrt{1 + \lambda} > 0$ . Now  $\gamma_\alpha(t) := \alpha \gamma(t)$  is a unique geodesic through  $\alpha x$  such that

$$\begin{aligned} \gamma_\alpha(\ln \alpha) &= \alpha e^{\ln \alpha} r' + \alpha e^{-\ln \alpha} s \\ &= \alpha^2 r' + s \\ &= (r' + s) + \lambda r' = x + r. \end{aligned}$$

Let  $y = \gamma_\alpha(\frac{1}{2} \ln \alpha)$ . Then  $t \mapsto \gamma_\alpha(t + \frac{1}{2} \ln \alpha)$  is a unique  $d_T$ -geodesic line and by Lemma 7.7 we get

$$S_y(\alpha x) = S_y(\gamma_\alpha(-\frac{1}{2} \ln \alpha + \frac{1}{2} \ln \alpha)) = \gamma_\alpha(\ln \alpha) = x + r.$$

Now consider the type II geodesic  $\mu(t) = e^t x$ , which is necessarily unique. Let  $z = \mu(\frac{1}{2} \ln \alpha)$ . Then

$$S_z(x) = S_z(\mu(-\frac{1}{2} \ln \alpha + \frac{1}{2} \ln \alpha)) = \mu \ln \alpha = \alpha x.$$

Consider the composition  $S = S_y \circ S_z$ . We get  $S(x) = S_y(\alpha x) = x + r$ . Moreover,  $S$  is a composition of two antihomogeneous order antimorphisms and is, therefore, a homogeneous order isomorphism. By [NS77, Theorem B],  $S$  is linear and we conclude  $S \in \text{Aut}(C^\circ)$ .  $\square$

We are in position to show that under the standing hypotheses  $C^\circ$  is a homogeneous cone.

**Theorem 7.10.** *Let  $(V, C, u)$  be an order unit space, such that  $C$  equals the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  be an antihomogeneous order antimorphism. Then  $C^\circ$  is homogeneous.*

*Proof.* Let  $x, y \in C^\circ$  be given. By assumption we can write  $y - x = \sum_{i=1}^n \sigma_i r_i$  with all  $r_i \in C$  extreme vectors and  $\sigma_i \in \{-1, 1\}$ . We reorder the indices if necessary such that the  $\sigma_i$  form a sequence of exclusively positive signs followed by negative signs. Let  $x_0 = x$  and for  $k \in \{1, \dots, n\}$  we let  $x_k = x + \sum_{i=1}^k \sigma_i r_i$ . For each  $k \in \{1, \dots, n\}$ , we denote by  $S_k$  the automorphism of  $C^\circ$ , obtained by Lemma 7.9, that maps  $x_{k-1}$  to  $x_k$ . The automorphism defined as the composition  $S := S_n \circ \dots \circ S_1$  satisfies  $S(x) = x_n = y$ .  $\square$

### Self-dual cone in a Hilbert space

We show that under the standing hypotheses, our vector space  $V$  can be endowed with an inner product that makes  $V$  a Hilbert space, and that  $C^\circ$  is self-dual for this inner product.

#### Construction of an inner product

Our strategy in constructing a bilinear form on  $V \oplus V$ , is to build it up from inner products induced by the finite dimensional subcones of  $C^\circ$  that are symmetric. From Lemma 7.3 we know that for any finite  $F \subseteq C^\circ$  with  $\text{span } F \cap C^\circ$  non-empty we have  $g[C^\circ(F)] = C^\circ(\varphi[F])$ . In the situation  $F = \varphi[F]$ , the restriction  $g_F$  is an antihomogeneous order antimorphism from the interior of a finite dimensional closed cone to itself and, due to Theorem 7.1(Walsh),  $C^\circ(F)$  is a symmetric cone. A priori, it is not apparent that there exists a finite subset  $F \subseteq \mathcal{E}(C)$  with  $\varphi[F] = F$ .

We remark, however, that the point symmetry  $S_u: C^\circ \rightarrow C^\circ$  given by (7.6) is an antihomogeneous order antimorphism that, in addition, is an involution. Henceforth, we assume without loss of generality that  $g$  is an involution. This means that the corresponding  $\varphi$  is also an involution. For a finite subset  $F \subseteq \mathcal{E}(C)$  with  $\text{span } F \cap C^\circ$  non-empty we now define

$$F^* := F \cup \varphi[F],$$

and remark that  $\varphi[F^*] = F^*$ . Now  $g_{F^*}: C^\circ(F^*) \rightarrow C^\circ(F^*)$  is an antihomogeneous order antimorphism. By Theorem 7.1 there exists an inner product  $(\cdot | \cdot)_{F^*}$  on  $\text{span}(F^*) \oplus \text{span}(F^*)$ , for which  $C^\circ(F^*)$  is self-dual and that satisfies (7.1).

It is convenient to introduce some notation. Recall that the  $M$ -function is defined by

$$M(x/y) = \inf\{\alpha \geq 0: x \leq \alpha y\},$$

for  $x \in C$  and  $y \in C^\circ$ . If multiple cones are under consideration, we denote the same functions by  $M(\cdot/\cdot; C)$  to emphasize the dependence on the cone  $C$ . An obvious fact is that for any finite  $F \subseteq \mathcal{E}(C)$  with  $\text{span } F \cap C^\circ$  non-empty we have  $M(x/y; C(F)) = M(x/y; C)$ , for all  $x \in C(F)$  and  $y \in C^\circ(F)$ .

**Lemma 7.11.** *Let  $F$  and  $G$  be finite subsets of  $\mathcal{E}(C)$ . Then we have*

$$(x | y)_{F^*} = (x | y)_{G^*},$$

for all  $x, y \in \text{span}(F^*) \cap \text{span}(G^*)$ .

*Proof.* It is sufficient to consider the case  $F \subseteq G$ , since for any  $F, G \subseteq \mathcal{E}(C)$  finite, the union  $F \cup G$  is also finite and contains both  $F$  and  $G$ . Let  $x \in C(F^*)$  and  $y \in C^\circ(F^*)$  be given. By (7.1) we get for all  $y \in C^\circ(F^*)$  and  $x \in R$ , for some  $R \in F^*$ , that

$$(x | y)_{F^*} = M(x/g(y); C^\circ(F^*)) = M(x/g(y); C^\circ(G^*)) = (x | y)_{G^*}.$$

As  $(\cdot | \cdot)_{F^*}$  and  $(\cdot | \cdot)_{G^*}$  are bilinear, and both  $C^\circ(F^*)$  and the union of rays in  $F^*$  generate  $\text{span}(F^*)$ , this yields the assertion.  $\square$

We are now in position to construct an inner product on  $V \oplus V$  for which  $C^\circ$  is a domain of positivity.

**Theorem 7.12.** *Let  $(V, C, u)$  be an order unit space, such that  $C$  equals the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  be an antihomogeneous order antimorphism. Then there exists an inner product  $(\cdot | \cdot)$  on  $V$  such that*

$$C^\circ = \{v \in V : (v | x) > 0 \text{ for all } x \in C \setminus \{0\}\}. \quad (7.7)$$

*Proof.* As before, we assume without loss of generality that  $g$  is an involution, by replacing it with  $S_u$  if necessary. For any pair  $(x, y) \in V \oplus V$  there exists a finite  $F \subseteq \mathcal{E}(C)$  with  $x, y \in \text{span}(F)$ . The quantity

$$B(x, y) = (x | y)_{F^*},$$

is well-defined according to Lemma 7.11. The properties of  $B: V \oplus V \rightarrow \mathbb{R}$  of being bilinear, symmetric and positive-definite follow, as they are verified on a finite set of vectors. Indeed, any such finite set of vectors is contained in a subcone  $C^\circ(F^*)$ , whose corresponding inner product  $(\cdot | \cdot)_{F^*}$  has these listed properties by Theorem 7.1 and determines  $B$ . Thus  $(\cdot | \cdot) := B(\cdot, \cdot)$  defines an inner product on  $V \oplus V$ .

We prove (7.7). Let  $v \in C^\circ$ . For  $x \in C \setminus \{0\}$  there exists a finite  $F \subseteq \mathcal{E}(C)$  with  $v, x \in \text{span}(F)$ . Then  $(v | x) = (v | x)_{F^*} > 0$ , as  $C^\circ(F^*)$  is self-dual for  $(\cdot | \cdot)_{F^*}$  due to Theorem 7.1. For the reverse inclusion, suppose  $v \in V$  is contained in the right-hand side of (7.7). Let  $F \subseteq \mathcal{E}(C)$  be finite with  $v \in \text{span}(F)$ . Then for all  $x \in C(F^*) \setminus \{0\}$  we have  $(v | x)_{F^*} = (v | x) > 0$ . Hence, the self-duality of  $C^\circ(F^*)$  for  $(\cdot | \cdot)_{F^*}$  yields that  $v \in C^\circ(F^*) \subseteq C^\circ$ .  $\square$

*Finite rank and completeness*

We argue that the inner product constructed in Proposition 7.12 induces a complete norm, by showing that the induced inner product norm is equivalent to the order unit norm. For this purpose, we introduce a concept of rank in our space. For  $x \in V$  we define its *rank*, denoted by  $\rho(x)$ , as the smallest number of extreme vectors needed to linearly span  $x$ . Remarkably, there exists a global bound on the rank of elements in  $V$ , as a consequence of  $C^\circ$  being a homogeneous cone.

**Lemma 7.13.** *For  $x \in V$  we have  $\rho(x) \leq 2\rho(u)$ .*

*Proof.* Put  $m := \rho(u)$ . Let  $r_1, \dots, r_m \in C$  be extreme vectors such that  $u = \sum_{i=1}^m \lambda_i r_i$ , for some  $\lambda_i \in \mathbb{R}$ , for  $i = 1, \dots, m$ . Let  $y \in C^\circ$ . By Theorem 7.10 there exists a linear order isomorphism  $S: C \rightarrow C$  such that  $S(u) = y$ . We get  $y = S(u) = \sum_{i=1}^m \lambda_i S(r_i)$ . By Corollary 3.2, the vectors  $S(r_i)$  are extreme, for  $i = 1, \dots, m$ . Hence  $\rho(y) \leq m$ . As  $C^\circ$  generates  $V$ , we obtain for all  $x \in V$  that  $\rho(x) \leq 2m = 2\rho(u)$ .  $\square$

We briefly recall the spectral theory for finite dimensional formally real Jordan algebras, as also outlined in Section 1.6. The spectral theory allows us to express the norm induced by the inner product in terms of the eigenvalues of an element. Let  $(\mathcal{A}, \circ)$  be a finite dimensional formally real Jordan algebra with unit  $e$ . A  $c \in \mathcal{A}$  is said to be an *idempotent* if  $c^2 = c$ . An idempotent  $c \in \mathcal{A}$  is considered *primitive* if  $c$  is non-zero and cannot be written as the sum of two non-zero idempotents. A set  $\{c_1, \dots, c_k\} \subseteq \mathcal{A}$  of primitive idempotents is called a *Jordan frame* if  $c_i \circ c_j = 0$ , for all  $i \neq j$ , and  $\sum_{i=1}^k c_i = e$ . The Spectral Theorem [FK94, Theorem III.1.2] says that for each  $a \in \mathcal{A}$  there exists a Jordan frame  $\{c_1, \dots, c_k\}$  and unique real numbers  $\lambda_1 \leq \dots \leq \lambda_k$  such that  $a = \sum_{i=1}^k \lambda_i c_i$ . In fact,  $\sigma(a) = \{\lambda_1, \dots, \lambda_k\}$ . Here the number  $k \in \mathbb{N}$  is independent of  $a$  and satisfies  $k \leq \dim \mathcal{A}$ .

**Proposition 7.14.** *Suppose  $(V, C, u)$  be a complete order unit space, such that  $C$  equals the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  an antihomogeneous order antimorphism. Then  $V$  can be endowed with an inner product  $(\cdot | \cdot)$  such that (7.7) holds and  $(V, (\cdot | \cdot))$  is a Hilbert space.*

*Proof.* Let  $(\cdot | \cdot)$  be the inner product as obtained in Theorem 7.12. Then (7.7) is satisfied. Let  $\|\cdot\|_2$  denote the norm induced by  $(\cdot | \cdot)$ , i.e., for  $x \in V$  we have  $\|x\|_2 = \sqrt{(x | x)}$ . As  $(V, C, u)$  is assumed to be complete, it suffices to argue that  $\|\cdot\|_2$  and  $\|\cdot\|_u$  are equivalent. Let  $n := 6\rho(u)$  and  $m := \|u\|_2^2$ . We define

$$\tau(x, y) = \frac{n}{m}(x | y), \quad x, y \in V.$$

Remark that  $\tau$  is a positive definite symmetric bilinear form on  $V \oplus V$ .

Let  $x \in V$  be given. By Lemma 7.13 there exists a  $F \subseteq \mathcal{E}(C)$  such that  $x, u \in \text{span}(F)$  and  $\dim \text{span}(F) \leq 3\rho(u)$ . Let  $\mathcal{A} := \text{span}(F^*)$  and remark that  $\dim \mathcal{A} \leq 6\rho(u) = n$ . By construction  $\tau(u, u) = n$ . Since  $C^\circ(F^*)$  is a symmetric cone for  $(\cdot | \cdot)$ , by [Koe62, Theorem VI.15], there exists a bilinear product  $\circ: \mathcal{A} \oplus \mathcal{A} \rightarrow \mathcal{A}$  such that

$(\mathcal{A}, \circ)$  is a formally real Jordan algebra, such that  $\mathcal{A}_+^\circ = C^\circ(F^*)$ . Moreover, due to [Koe62, Theorem III.13] the unit of  $(\mathcal{A}, \circ)$  is  $u$  and  $\tau(a, b) = \text{Tr}L(a \circ b)$ , for all  $a, b \in \mathcal{A}$ . Here  $L(a \circ b)$  denotes left multiplication by  $a \circ b$ . By the Spectral Theorem there exists a Jordan frame  $\{c_1, \dots, c_k\}$  and unique real numbers  $\lambda_1 \leq \dots \leq \lambda_k$  such that  $x = \sum_{i=1}^k \lambda_i c_i$ . As the  $c_i$  are pairwise orthogonal idempotents we get  $x^2 = \sum_{i=1}^k \lambda_i^2 c_i$ . Now we compute

$$\|x\|_2 = \frac{m}{n} \sqrt{\tau(x, x)} = \frac{m}{n} \sqrt{\text{Tr}L(x^2)} = \frac{m}{n} \sqrt{\sum_{i=1}^k \lambda_i^2 \text{Tr}L(c_i)}. \quad (7.8)$$

The possible eigenvalues of  $L(c)$  for an idempotent  $c \in \mathcal{A}$  are  $0, \frac{1}{2}$  and  $1$  by [FK94, Proposition III.1.3] and, hence, for  $i = 1, \dots, k$  we have  $1 \leq \text{Tr}L(c_i) \leq n$ . Next we want to describe  $\|x\|_u$ . Remark that as  $u \in C(F^*) = \mathcal{A}_+$ , that computing the order unit norm of  $x$  in  $(V, C, u)$  yields the same as in  $(\mathcal{A}, \mathcal{A}_+, u)$ . So from  $u = \sum_{i=1}^k c_i$  we obtain  $\|x\|_u = \max\{|\lambda_1|, \dots, |\lambda_k|\}$ . Let  $j \in \{1, \dots, k\}$  be such that  $|\lambda_j| = \|x\|_u$ . We then get  $\|x\|_u \leq \sqrt{\sum_{i=1}^k \lambda_i^2} \leq k \|x\|_u$ . Combining this with  $k \leq \dim \mathcal{A} \leq n$  and (7.8) we get

$$\frac{m}{n} \|x\|_u \leq \|x\|_2 \leq m \sqrt{n} \|x\|_u.$$

Since  $n$  and  $m$  are defined independent on the choice of  $x$  we conclude that  $\|\cdot\|_2$  and  $\|\cdot\|_u$  are equivalent.  $\square$

### ***JH-algebras***

A real Jordan algebra  $\mathcal{H}$  that is a Hilbert space with an inner product  $(\cdot | \cdot)$  which is *associative*, that is,

$$(a \circ b | c) = (b | a \circ c) \quad a, b, c \in \mathcal{H},$$

is called a *JH-algebra*. It is shown in [Chu17, Theorem 3.1] that, as an infinite dimensional generalisation of the Koecher-Vinberg theorem, a symmetric cone in a Hilbert space arises precisely as the interior of the cone of squares of unital JH-algebra. We have shown that a complete order unit space  $(V, C, u)$ , such that  $C$  equals the sum of its extreme rays and  $C^\circ$  admits an antihomogeneous order antimorphism, can be endowed with an inner product turning  $V$  into a Hilbert space in which  $C^\circ$  is a symmetric cone. Therefore, by the result of Chu, under those assumptions  $V$  is a unital JH-algebra with  $C$  as its cone of squares. The converse of this last statement also holds. In order to prove this, it is convenient to consider the following characterisation of unital JH-algebras, which including its proof is due to Roelands and Wortel through personal communication.

**Lemma 7.15.** *A unital JH-algebra is a finite direct sum of formally real Jordan algebras and spin factors.*

*Proof.* Let  $\mathcal{H}$  be a unital JH-algebra. By [Chu17, Lemma 2.6] the order unit norm is equivalent with the norm from the inner product. Hence,  $\mathcal{H}$  is reflexive and has a predual. In particular,  $\mathcal{H}$  is a JBW-algebra. Let  $z \in \mathcal{H}$  be the central projection such that  $z\mathcal{H}$  is the nonatomic part of  $\mathcal{H}$ . Suppose  $z \neq 0$ . We split  $z$  into a sum of two non-trivial orthogonal projections. One of them we split again into the sum of two non-trivial orthogonal projections, and so on. This process does not terminate in finite steps as  $z\mathcal{H}$  is purely non-atomic. We obtain an infinite sequence  $(p_n)_{n=1}^\infty$  of pairwise orthogonal projections. The JB-subalgebra generated by these projections is associative, and is isometrically isomorphic to some  $C_0(S)$ , for a locally compact Hausdorff space  $S$ . Then the map

$$(\lambda_n) \mapsto \sum_{n=1}^\infty \lambda_n p_n,$$

is an isometric embedding of  $c_0$  into  $\mathcal{H}$ , contradicting the reflexivity of  $\mathcal{H}$ . Hence,  $z = 0$  and  $\mathcal{H}$  is an atomic JBW-algebra. Therefore, by [AS03, Proposition 3.45],  $\mathcal{H}$  is a direct summand of type I JBW-factors. Since by the above arguments  $\mathcal{H}$  cannot contain an infinite collection of orthogonal projections, this is a finite direct sum and each factor is of finite type. A finite type I JBW-factor is a spin factor or the self-adjoint matrices over  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  or the  $3 \times 3$ -matrices over  $\mathbb{O}$ .  $\square$

Using these results on JH-algebras we can now fully characterise the symmetric cones in infinite dimensions.

**Theorem 7.16.** *Let  $(V, C, u)$  be a complete order unit space. Then  $C$  equals the sum of its extreme rays and there exists an antihomogeneous order antimorphism  $g: C^\circ \rightarrow C^\circ$  if and only if  $C^\circ$  is a symmetric cone.*

*Proof.* Suppose that  $C$  equals the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  is an antihomogeneous order antimorphism. Then by Theorem 7.12 there exists an inner product  $(\cdot | \cdot)$  on  $V$ , which by Proposition 7.14 turns  $V$  into a Hilbert space, for which  $C^\circ$  is a self-dual cone by (7.7). Moreover,  $C^\circ$  is a homogeneous cone due to Theorem 7.10. We conclude that  $C^\circ$  is in fact a symmetric cone.

Conversely, suppose that  $(V, (\cdot | \cdot))$  is a Hilbert space such that  $C^\circ$  is a symmetric cone with respect to  $(\cdot | \cdot)$ . Then by [Chu17, Theorem 3.1],  $V$  can be endowed with a Jordan product turning into a unital JH-algebra such that  $C^\circ$  is the interior of the cones of squares. Now by our characterisation of unital JH-algebras in Lemma 7.15 we know that  $V$  is a finite direct summand of formally real Jordan algebras and spin factors. The cone in any formally real Jordan algebra or spin factor equals the sum of its extreme rays and, hence, as  $C$  is a finite direct sum of such cones it also equals the sum of its extreme rays. Lastly, the inversion map  $\iota: C^\circ \rightarrow C^\circ$  defined by  $x \mapsto x^{-1}$  is an antihomogeneous order antimorphism.  $\square$

### *Automatic antihomogeneity*

In this last section we argue that the antihomogeneous condition imposed on  $g$  in Theorem 7.16 is superfluous. Our strategy is to decompose our cone  $C$  into an engaged part and a disengaged part. Using Proposition 7.2 we can show that an antimorphism  $g: C^\circ \rightarrow C^\circ$  respects this decomposition. Restricted to the engaged part  $g$  will automatically become antihomogeneous, by results from [Wal18], and by Theorem 7.16 this part is symmetric. The disengaged part of  $C$  must be finite dimensional, and as all its extreme rays are linearly independent it is isometrically isomorphic to a standard Euclidean cone, which is symmetric.

Henceforth let  $(V, C, u)$  be a complete order unit space, such that  $C$  equals the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  an order antimorphism. As before let  $\varphi: \mathcal{E}(C) \rightarrow \mathcal{E}(C)$  be the bijection, as obtained in Proposition 7.2, corresponding to  $g$ . Furthermore, we introduce the notations  $\mathcal{R}_D$  and  $\mathcal{R}_E$  for the collection of disengaged extreme rays and engaged extreme rays of  $C$ , respectively. Let  $V_D := \text{span } \mathcal{R}_D$  and  $V_E := \text{span } \mathcal{R}_E$ .

**Lemma 7.17.** *Under the standing hypotheses  $(V, C) = (V_D, C(\mathcal{R}_D)) \oplus (V_E, C(\mathcal{R}_E))$ , as a direct sum of partially ordered vector spaces.*

*Proof.* Let  $x \in C$ . By assumption there exist extreme vectors  $x_1, \dots, x_n \in C$  such that  $x = \sum_{i=1}^n \lambda_i x_i$ . Let  $I$  be the subset of  $\{1, \dots, n\}$  consisting of those indices for which the corresponding  $x_i$  is a disengaged vector and let  $J$  be the complement of this set. Then by construction  $x_D = \sum_{i \in I} x_i \in C(\mathcal{R}_D)$  and  $x_E = \sum_{j \in J} x_j \in C(\mathcal{R}_E)$ .

Suppose now that  $x \in V_D \cap V_E$  and  $x \neq 0$ . We write

$$x = \sum_{i=1}^n x_i = \sum_{j=1}^m y_j,$$

with  $x_i \in V$  disengaged extreme vectors and  $y_j \in V$  engaged extreme vectors. It follows from

$$x_1 = \sum_{j=1}^m y_j - \sum_{i=2}^n x_i$$

that  $x_1$  is engaged, which yields a contradiction. We conclude  $V_D \cap V_E \subseteq \{0\}$ .  $\square$

Due to Lemma 7.17 we can write  $u = (u_D, u_E)$  with  $u_D \in C(\mathcal{R}_D)$  and  $u_E \in C(\mathcal{R}_E)$ . Then  $u_D$  and  $u_E$  are order units in  $C(\mathcal{R}_D)$  and  $C(\mathcal{R}_E)$ , respectively. Moreover, we get

$$C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E).$$

In what follows we argue that  $g$  factors of this direct sum.

**Lemma 7.18.** *Under the standing hypotheses,  $\varphi[\mathcal{R}_D] = \mathcal{R}_D$  and  $\varphi[\mathcal{R}_E] = \mathcal{R}_E$ .*

*Proof.* Let  $R \in \mathcal{R}_E$  be given. Then there exists a finite  $F \subseteq \mathcal{R}_E \setminus \{R\}$  with  $R \in \text{span } F$  and  $\text{span } F \cap C^\circ \neq \emptyset$ . By Lemma 7.2 we get

$$\varphi(R) \in g[C^\circ(F)] = C^\circ(\varphi[F]).$$

As  $\varphi$  is injective,  $\varphi(R) \notin \varphi[F]$  so  $\varphi(R)$  is engaged. We conclude  $\varphi[\mathcal{R}_E] \subseteq \mathcal{R}_E$ . Since  $g^{-1}$  is also an order antimorphism we get by Lemma 7.2 the reverse inclusion  $\mathcal{R}_E \subseteq \varphi[\mathcal{R}_E]$ . Due to  $\varphi$  being bijective we also obtain  $\varphi[\mathcal{R}_D] = \mathcal{R}_D$ .  $\square$

**Lemma 7.19.** *There exist order antimorphisms  $g_D: C^\circ(\mathcal{R}_D) \rightarrow C^\circ(\mathcal{R}_D)$  and  $g_E: C^\circ(\mathcal{R}_E) \rightarrow C^\circ(\mathcal{R}_E)$  such that for all  $(x_D, x_E) \in C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$  we have*

$$g((x_D, x_E)) = (g_D(x_D), g_E(x_E)).$$

*Proof.* Let  $x = (x_D, x_E)$  and  $y = (y_D, y_E)$  be given in  $C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$ . Suppose  $x_D = y_D$  holds. Then  $x = y + \sum_{i=1}^n r_i$  for some  $r_i \in R_i$  with  $R_i \in \mathcal{R}_E$ , for  $i = 1, \dots, n$ . Relabelling the indices such that  $x_1, \dots, x_j < 0$  and  $x_{j+1}, \dots, x_n > 0$  for some  $j \in \{1, \dots, n\}$ , guarantees that  $y + \sum_{i=1}^k r_i \in C^\circ$  for all  $k = 1, \dots, n$ . Due to Lemma 7.2 we obtain

$$g(x) - g(y) = g\left(y + \sum_{i=1}^n r_i\right) - g(y) \in \text{span } \varphi[\mathcal{R}_E].$$

As  $\varphi[\mathcal{R}_E] = \mathcal{R}_E$  by Lemma 7.18, we remark that  $g(x) - g(y) \in V_E$ . In other words,  $g(x)$  and  $g(y)$  coincide in their first argument with respect to the decomposition obtained in Lemma 7.17. Similarly, if  $g(x)$  and  $g(y)$  coincide in the first argument, then by applying the same arguments to  $g^{-1}$  also  $x$  and  $y$  coincide in the first argument. This shows that  $g_D(x_D) = g(x_D, x_E)$  is well-defined independent of  $x_E$ . Analogously, as Lemma 7.18 yields  $\varphi[\mathcal{R}_D] = \mathcal{R}_D$ , we obtain that  $g_E(x_E) = g(x_D, x_E)$  is well-defined independent of  $x_E$ . That both  $g_D$  and  $g_E$  are order antimorphisms now follows from the fact that  $g$  is an order antimorphism.  $\square$

On the engaged part of our cone the order antimorphism automatically becomes antihomogeneous, as a consequence of the finite dimensional result by Walsh, which states that an order antimorphism between the interiors of two closed cones is antihomogeneous whenever one of the cones does not contain a disengaged extreme vector.

**Lemma 7.20.** *The order antimorphism  $g_E$  as in Lemma 7.19 is antihomogeneous.*

*Proof.* Let  $x \in C^\circ(\mathcal{R}_E)$  and  $\lambda \in \mathbb{R}_+^\circ$  be given. Let  $F \subseteq \mathcal{R}_E$  be finite such that  $x \in C^\circ(F)$ . Then  $g_E$  restricts to an order antimorphism  $\tilde{g}_E: C^\circ(F) \rightarrow C^\circ(\varphi[F])$  by Lemma 7.3. We remark that  $C^\circ(F)$  does not contain a disengaged extreme ray. Now [Wal18, Theorem 1.1] yields that  $\tilde{g}_E$  is antihomogeneous and

$$g_E(\lambda x) = \tilde{g}_E(\lambda x) = \lambda^{-1} g_E(x).$$

$\square$

In contrast to the engaged part, the order antimorphism  $g_D$  need not be antihomogeneous. This, however, is not of importance as the disengaged part of the cone  $C^\circ(\mathcal{R}_D)$  is linearly isomorphic to a standard finite dimensional cone and, therefore, is symmetric.

**Lemma 7.21.** *Let  $(V, C, u)$  be an order unit space. If all extreme rays of  $C$  are disengaged and  $C$  equals the sum of its extreme rays, then  $C$  is linearly isomorphic to a standard Euclidean cone.*

*Proof.* Let  $\mathcal{R}$  denote the collection of extreme rays of  $C$ . By assumption we can write the order unit  $u$  as a linear combination of finitely many extreme vectors, say  $u = \sum_{i=1}^n r_i$  with  $r_i \in R_i \in \mathcal{R}$  for all  $i$ . Suppose there exists an  $R \in \mathcal{R}$  with  $R \neq R_i$  for  $i = 1, \dots, n$ . Let  $r \in R$  with  $r \leq u$ . Now write  $u - r = \sum_{j=1}^m s_j$  with all  $s_j$  extreme vectors. Then we compute

$$r = u - (u - r) = \sum_{i=1}^n r_i - \sum_{j=1}^m s_j.$$

In particular, this contradicts that  $r$  is a disengaged extreme vector by assumption. We have shown that  $\mathcal{R}$  does not contain additional rays besides  $R_1, \dots, R_n$ . Thus  $\mathcal{R}$  is finite. Since  $V$  is the linear span of the extreme rays,  $V$  is finite dimensional. As all extreme rays are disengaged, any collection of representatives form an algebraic basis for  $V$ . The basis transformation that maps this basis onto the standard coordinate basis is the desired linear order isomorphism from  $(V, C)$  onto  $(\mathbb{R}^n, \mathbb{R}_+^n)$ .  $\square$

We obtain a slight improvement for our characterisation of symmetric cones in infinite dimensions, Theorem 7.16, by dropping the antihomogeneous condition on  $g$ .

**Theorem 7.22.** *Let  $(V, C, u)$  be a complete order unit space. Then  $C$  equals the sum of its extreme rays and there exists an order antimorphism  $g: C^\circ \rightarrow C^\circ$  if and only if  $C^\circ$  is a symmetric cone.*

*Proof.* Let  $(V, C, u)$  be a complete order unit space,  $C$  the sum of its extreme rays and  $g: C^\circ \rightarrow C^\circ$  be an order antimorphism. By Lemma 7.17 we get  $C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$ . Let  $g_D$  and  $g_E$  be the order antimorphisms as obtained in Lemma 7.19. Since  $g_E: C^\circ(\mathcal{R}_E) \rightarrow C^\circ(\varphi[\mathcal{R}_E])$  is antihomogeneous by Lemma 7.20 the cones  $C^\circ(\mathcal{R}_E)$  and  $C^\circ(\varphi[\mathcal{R}_E])$  are linearly isomorphic. In particular, all extreme rays in  $C^\circ(\varphi[\mathcal{R}_E])$  are engaged. Thus we get  $C^\circ(\varphi[\mathcal{R}_E]) \subseteq C^\circ(\mathcal{R}_E)$  and hence  $\varphi[\mathcal{R}_E] \subseteq \mathcal{R}_E$ . As all arguments also apply to  $g^{-1}$  we get  $\varphi[\mathcal{R}_E] = \mathcal{R}_E$ . We conclude that  $g_E: C^\circ(\mathcal{R}_E) \rightarrow C^\circ(\mathcal{R}_E)$  is an antihomogeneous order antimorphism. By Theorem 7.16,  $C^\circ(\mathcal{R}_E)$  is a symmetric cone. Due to Lemma 7.21 the cone  $C^\circ(\mathcal{R}_D)$  is linearly isomorphic to a standard Euclidean cone and, in particular, is a symmetric cone. Hence, the product  $C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$  is also symmetric. The converse statement follows from Theorem 7.16.  $\square$