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Order isomorphisms, order antimorphisms and their interplay with Jordan algebra structures

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Chapter 2

Lattice structure preservers in $C(S)$

For a compact Hausdorff space S the space $C(S)$ of real valued continuous functions on S endowed with pointwise order is an example of a Riesz space. Indeed, any pair of functions $x, y \in C(S)$ has a supremum $x \vee y \in C(S)$ given by

$$x \vee y(s) = \max\{x(s), y(s)\}, \quad \text{for } s \in S.$$

The other lattice operations of $C(S)$ are also determined pointwise. A classic result of Kaplansky [Kap47] states that compact Hausdorff spaces S and T are homeomorphic if and only if $C(S)$ and $C(T)$ are lattice isomorphic, here a bijection $f: C(S) \rightarrow C(T)$ is considered a lattice isomorphism whenever $f(x \vee y) = f(x) \vee f(y)$ holds for all $x, y \in C(S)$. This reinforces the idea that the lattice structure of $C(S)$ has a strong connection with the underlying topological space. It is therefore unsurprising that Riesz homomorphisms on $C(S)$, as defined in 1.9, which preserve the lattice structure, have a simple description.

Theorem 2.1 (Theorem 4.25 in [AA02]). *Let S and T be compact Hausdorff spaces. A positive linear map $f: C(S) \rightarrow C(T)$ is a Riesz homomorphism if and only if there exist a map $\pi: T \rightarrow S$ and a weight $w \in C(T)^+$ such that we have*

$$f(x)(t) = w(t)x(\pi(t)), \quad x \in C(S), t \in T. \tag{2.1}$$

Moreover, in this case, $w = f(\mathbb{1}_S)$ and the map π is uniquely determined and continuous on the set $\{w > 0\}$.

One naturally wonders whether such a result can be extended to a variety of subspaces of $C(S)$ and $C(T)$. However, many natural subspaces of $C(S)$ are themselves not a Riesz space. For example, in the case $S = [0, 1]$, the spaces $C^1[0, 1]$ and $P[0, 1]$ consisting of the continuously differentiable functions and polynomial functions on $[0, 1]$, respectively, are not Riesz spaces. However, $C^1[0, 1]$ and $P[0, 1]$ are examples of pre-Riesz spaces. We can therefore attempt to generalise Theorem 2.1 to an analogous statement concerning Riesz* homomorphism between pre-Riesz subspaces of $C(S)$ and $C(T)$.

An overview of pre-Riesz spaces is given in Section 1.2. We briefly recall some terminology for the reader's convenience. A partially ordered vector space X is a pre-Riesz space if it admits a vector lattice cover, here a pair (E, i) of a Riesz space E and a linear order embedding $i: X \rightarrow E$ is a *vector lattice cover* of X if $i[X]$ is order dense in E . If, in addition, $i[X]$ generates E as a Riesz space then the pair (E, i) is unique up to isomorphism for these properties and is called the *Riesz completion* of X . For short, we denote the Riesz completion of a pre-Riesz space X by X^ρ . Recall that a linear map $f: X \rightarrow Y$ between pre-Riesz spaces is a Riesz* homomorphism if and only if f extends to a Riesz homomorphism $f^\rho: X^\rho \rightarrow Y^\rho$.

The aim of this chapter, which is based on [vI18], is to generalise classic results concerning Riesz homomorphisms to the setting of Riesz* homomorphisms between pre-Riesz spaces. We start, as mentioned earlier, by generalising Theorem 2.1 to the setting of Riesz* homomorphisms between order dense subspace of $C(S)$ and $C(T)$. In doing so, we also highlight differences between Riesz* homomorphisms and other classes of operators, which appear in literature, that extend the notion of a Riesz homomorphism. After that we consider the case where S and T are locally compact spaces and our pre-Riesz spaces are order dense in $C_0(S)$ and $C_0(T)$, respectively. We find that any Riesz homomorphism is still of the form (2.1), but with more restriction on the maps w and π . The second vector lattice result we consider is the following.

Theorem 2.2 (Theorem 2.15 in [AB06]). *Let E and F be Riesz spaces and $f: E \rightarrow F$ a bijective Riesz homomorphism. Then f^{-1} is a Riesz homomorphism.*

An analogous statement for Riesz* homomorphisms fails to hold in general. We provide a counterexample in 2.19. However, on a pervasive pre-Riesz space it is true that the inverse of a bijective Riesz* homomorphism is again a Riesz* homomorphism. We then continue by studying the linear order isomorphisms of again order dense subspaces of $C_0(S)$ and $C_0(T)$, using our knowledge on Riesz* homomorphism obtained earlier. We then turn our attention to the last result we examine.

Theorem 2.3. *Let E and F be Riesz spaces. A linear map $f: E \rightarrow F$ is a Riesz homomorphism if and only if f is positive and disjointness preserving.*

We illustrate with an example, that it is not true in general, that a positive linear disjointness preserving map between pre-Riesz spaces is necessarily a Riesz* homomorphism. This example is based on a pre-Riesz space that does not contain a pair of non-trivial positive disjoint elements. It is reasonable to expect that on a pervasive pre-Riesz space any positive linear disjoint preserving operator is a pre-Riesz space. However, we are only able to prove this assertion under stronger conditions.

Weighted composition maps

Let S and T be compact Hausdorff spaces and $X \subseteq C(S)$ and $Y \subseteq C(T)$ linear subspaces. A map $f: X \rightarrow Y$ that satisfies (2.1) for some w and π is called a *weighted composition map*. We argue that if X and Y are order dense that any Riesz* homo-

morphism from X to Y is such a weighted composition map. We elaborate on this setting. Let $X \subseteq C(S)$ and $Y \subseteq C(T)$ be order dense. By Proposition 1.4, both X and Y are pre-Riesz spaces, as they are directed and Archimedean. Moreover, we can describe their Riesz completions as follows. Consider $L(X)$ to be the Riesz subspace of $C(S)$ generated by X , in other words, $x \in L(X)$ if and only if there exist $x_1, \dots, x_n, y_1, \dots, y_m \in X$ such that

$$x = \bigvee_{i=1}^n x_i - \bigvee_{j=1}^m y_j.$$

By construction $L(X)$ is a Riesz subspace of $C(S)$. Remark that X is order dense in $L(X)$, since X is order dense in $C(S)$. Therefore, we can identify the Riesz completion (X^ρ, i) of X as the pair $(L(X), i)$, where $i: X \rightarrow L(X)$ is the canonical embedding. In conclusion, we can view the Riesz completion X^ρ of X as a majorizing Riesz subspace of $C(S)$. Analogously, we identify the Riesz completion Y^ρ as a majorizing Riesz subspace of $C(T)$. We use this observation freely throughout.

Riesz homomorphisms*

Let $f: X \rightarrow Y$ be a Riesz* homomorphism and $f^\rho: X^\rho \rightarrow Y^\rho$ denote the Riesz homomorphism that extends f . An intuitive approach to show that f is a weighted composition operator is to extend f^ρ further to a Riesz homomorphism between $C(S)$ and $C(T)$ and apply the general theory, namely Theorem 2.1. However, generally not every Riesz* homomorphism on X is the restriction of a Riesz homomorphism on $C(S)$, which we illustrate with the following example.

Example 2.4. Consider the subspace X of $C[0, 1]$ consisting of functions $x \in X$ that satisfy $x(0) = x(1)$. Straightforward verification yields that X is an order dense Riesz subspace of $C[0, 1]$. We consider the *inside-out operator* $f: X \rightarrow X$ defined by $f(x)(t) = x(\pi(t))$ for all $x \in X$ and $t \in [0, 1]$, where

$$\pi(t) = \frac{1}{2} - t + \mathbb{1}_{[t > \frac{1}{2}]}$$

For all $x \in X$ the defining property $x(0) = x(1)$ guarantees that $f(x)$ is continuous. Moreover, we obtain from $f(x)(0) = x(\frac{1}{2}) = f(x)(1)$ that f maps into X . Since the lattice structure is determined pointwise, it is clear that f is a Riesz homomorphism. Suppose that $g: C[0, 1] \rightarrow C[0, 1]$ is a Riesz homomorphism that extends f . By Theorem 2.1, there exist $w: [0, 1] \rightarrow \mathbb{R}_+$ and $\tau: [0, 1] \rightarrow [0, 1]$ such that $g(x)(t) = w(t)x(\tau(t))$ for all $x \in C[0, 1]$ and $t \in T$. As X contains the constant functions we get $w(t) = 1$ for all $t \in T$. We get $\tau = \pi$, which contradicts that τ is continuous. We conclude that f does not extend to a Riesz homomorphism on $C[0, 1]$.

As we will see in the following result, requiring our subspaces to separate the points of the underlying topological space, guarantees that the weight and composition map are automatically continuous. This leads to the following characterisation.

Theorem 2.5. *Let X and Y be order dense subspaces of $C(S)$ and $C(T)$, respectively.*

(i) *For every Riesz* homomorphism $f: X \rightarrow Y$ there exist $w: T \rightarrow \mathbb{R}_+$ and $\pi: T \rightarrow S$ such that*

$$f(x)(t) = w(t)x(\pi(t)), \quad x \in X, t \in T. \quad (2.2)$$

Moreover, if in addition X separates the points of S , then w is continuous on T and π is continuous and uniquely determined on $\{w > 0\}$.

(ii) *A linear map $f: X \rightarrow Y$ that satisfies (2.2) for some $w \in C(T)^+$ and $\pi: T \rightarrow S$ continuous on $\{w > 0\}$ is a Riesz* homomorphism.*

Proof. Suppose X and Y are given as in the first statement of (i) and $f: X \rightarrow Y$ is a Riesz* homomorphism. Let $f^\rho: X^\rho \rightarrow Y^\rho$ be the Riesz homomorphism that extends f . We fix $t \in T$. Consider the Riesz homomorphism $f_t: X^\rho \rightarrow \mathbb{R}$ as the composition of f^ρ with the point evaluation at t , i.e., $f_t(x) = f^\rho(x)(t)$ for all $x \in X^\rho$. We apply the Lipecki-Luxemburg-Schep Theorem [AA02, Theorem 4.36] to f_t to obtain a Riesz homomorphism $\hat{f}_t: C(S) \rightarrow \mathbb{R}$ that extends f_t . The conditions of this theorem are satisfied as X^ρ is a majorizing Riesz subspace of $C(S)$ and \mathbb{R} is Dedekind complete. Riesz homomorphisms from $C(S)$ to \mathbb{R} are characterized as scalar multiples of point evaluations, see for example [AA02, Lemma 4.23]. In other words, there exist $w(t) \in \mathbb{R}_+$ and $\pi(t) \in S$ such that $\hat{f}_t(x) = w(t)x(\pi(t))$ holds for all $x \in C(S)$. As $t \in T$ was chosen arbitrarily we obtain that f satisfies (2.2).

Suppose now that X separates the points of S . We redefine if necessary $w(t)$ to equal zero whenever $f(x)(t) = 0$ holds for all $x \in X$. Equation (2.2) remains satisfied. Let $x \in X$ be greater than the constant one function. Then $f(x)$ is a bounded function as element of $C(T)$, which implies by (2.2) that w is a bounded map. Fix $t \in T$. We argue that w is continuous at t and that π is continuous at t whenever $w(t)$ is non-zero.

Let (t_α) in T be a net that converges to t . We show that (t_α) has a subnet (t_β) such that $\lim_\beta w(t_\beta) = w(t)$ and that $\lim_\beta \pi(t_\beta) = \pi(t)$ whenever $w(t)$ is non-zero. For $x \in X$ we get by continuity of $f(x)$ and application of (2.2) that

$$w(t)f(\pi(t)) = f(x)(t) = \lim_\alpha f(x)(t_\alpha) = \lim_\alpha w(t_\alpha)x(\pi(t_\alpha)). \quad (2.3)$$

As we have shown that w is bounded, the fact that S is compact yields the existence of a subnet (t_β) of (t_α) such that $(w(t_\beta))$ converges to some $a \in \mathbb{R}$ and $(\pi(t_\beta))$ to $s \in S$. Therefore, for any $x \in X$ equation (2.3) yields $f(x)(t) = a \lim_\beta x(\pi(t_\beta))$. Moreover, by continuity of the functions $x \in X$ we obtain $f(x)(t) = ax(\lim_\beta \pi(t_\beta)) = ax(s)$ and, in particular, that $w(t)x(\pi(t)) = ax(s)$.

We remark that $a = 0$ whenever $w(t) = 0$ as X contains an element for which $x(s) \neq 0$ and, hence, we are done in that case as then $w(t) = a = \lim_\beta w(t_\beta)$. We consider the remaining case where $w(t) > 0$ holds. It is evident that $a > 0$. Consequently, for any $x \in X$ the equation $x(s) = cx(\pi(t))$ is satisfied, where $c = w(t)/a$ is non-zero and independent of f . As X separates the points of S we obtain the equalities $s = \pi(t)$ and $c = 1$. In other words, π is continuous at t . Plugging this into (2.3) yields

$w(t)x(\pi(t)) = \lim_{\beta} w(t_{\beta})f(\pi(t))$ and, hence, applying that to a $x \in X$ with $x(\pi(t)) \neq 0$ yields that w is continuous at t . We conclude that w and π are continuous on T and $\{w > 0\}$, respectively. Additionally, π is uniquely determined on $\{w > 0\}$ due to separating property of X .

Lastly, suppose that $f: X \rightarrow Y$ satisfies (2.2) for suitable $w \in C(T)^+$ and $\pi: T \rightarrow S$ continuous on $\{w > 0\}$. The weighted composition operator between the Riesz completions $X^{\rho} \subseteq C(S)$ and $Y^{\rho} \subseteq C(T)$ defined by w and π is a well-defined Riesz homomorphism that extends T , hence T is a Riesz* homomorphism. \square

Henceforth, for notational convenience let $f_{w,\pi}: X \rightarrow Y$ denote the weighted composition operator between X and Y with weight map $w: T \rightarrow \mathbb{R}_+$ and composition map $\pi: T \rightarrow S$ that satisfies (2.2).

Riesz homomorphisms

A predecessor of the Riesz* homomorphism is the Riesz homomorphism, an alternative class of operators on a pre-Riesz space X that extend to Riesz homomorphisms on X^{ρ} . A linear operator $f: X \rightarrow Y$ between pre-Riesz spaces is called a *Riesz homomorphism* whenever $f(\{x, y\}^u)^l \subseteq \{f(x), f(y)\}^{ul}$ holds for all $x, y \in X$. Similarly to Riesz* homomorphisms these operators extend to Riesz homomorphisms. However, not all Riesz homomorphisms between the completions are obtained as such extensions. Another disadvantage of the class of Riesz homomorphisms is that it is not stable under composition, see [vH93]. We use our knowledge on the weighted composition structure of Riesz* homomorphisms to investigate similarities and differences between these two classes of operators.

Suppose $f: X \rightarrow Y$ is a positive linear operator. Positivity of f immediately yields that for any finite $A \subseteq X$ we have $f[A^{ul}] \subseteq f[A^u]^l$. We infer that any Riesz homomorphism is a Riesz* homomorphism. A converse statement does not generally hold on pre-Riesz spaces, which will be illustrated later by a counterexample in Example 2.8. However, we show that on a wide class of subspaces of $C(S)$, which is contained in the class of separating order dense subspaces, the notions of a Riesz homomorphism and a Riesz* homomorphism coincide.

A linear subspace X of $C(S)$ is called *pointwise order dense* if it satisfies

$$y(s) = \inf\{x(s) : x \in X, x \geq y\}$$

for all $y \in C(S)$ and $s \in S$. Straightforward verification yields that any pointwise order dense subspace X of $C(S)$ is separating and order dense. Moreover, it is routine to show that a norm dense subspace of $C(S)$ containing the constant functions is pointwise order dense.

Example 2.6. Consider the so-called *Namioka space* defined as $\mathcal{N} = \{x \in C([0, 1]) : x(0) + x(1) = 2x(\frac{1}{2})\}$. \mathcal{N} is a pervasive and order dense subspace of $C([0, 1])$, which is not pointwise order dense.

We consider an equivalent defining property of a Riesz homomorphism. A linear map $f: X \rightarrow Y$ is a Riesz homomorphism if and only if

$$\inf\{f(z): z \in X, z \geq x, y\} = f(x) \vee f(y) \text{ in } Y^\rho, \quad x, y \in X. \quad (2.4)$$

Here the infimum and supremum are taken within the Riesz space Y^ρ . We use this characterisation and Theorem 2.5 to prove that any Riesz* homomorphism on a pointwise order dense subspace of $C(S)$ is automatically a Riesz homomorphism.

Theorem 2.7. *Let X be a pointwise order dense subspace of $C(S)$, Y an order dense subspace of $C(T)$ and $f: X \rightarrow Y$ a linear operator. Then f is a Riesz* homomorphism if and only if f is a Riesz homomorphism.*

Proof. Let X and Y be as in the statement and suppose $f: X \rightarrow Y$ is a Riesz* homomorphism. Due to Theorem 2.5 there exist suitable w and π such that $f = f_{w,\pi}: X \rightarrow Y$. Suppose $x, y \in X$ are given. We argue that (2.4) is satisfied. As f is positive $f(x) \vee f(y)$ is a lower bound of $\{f(z): z \in X, z \geq x, y\}$ in Y^ρ . Suppose w is another lower bound of $\{f(z): z \in X, z \geq x, y\}$ in Y^ρ . We compute for all $t \in T$

$$\begin{aligned} w(t) &\leq \inf\{f(z)(t): z \in X, z \geq x, y\} \\ &= \inf\{w(t)z(\pi(t)): z \in X, z \geq x, y\} \\ &= w(t)[\inf\{z(\pi(t)): z \in X, z \geq x, y\}] \\ &= w(t)(x \vee y)(\pi(t)) = (f(x) \vee f(y))(t). \end{aligned}$$

Here we used that X is pointwise order dense in $C(S)$ in the second last equality on $x \vee y \in C(S)$ and $\pi(t) \in S$. This shows that $w \leq f(x) \vee f(y)$ holds in the order induced by $C(T)$, which shows that f satisfies condition (2.4). Recall that the other implication holds for general pre-Riesz spaces as discussed earlier. \square

We consider an example that shows that the above theorem fails to hold generally for separating order dense subspaces of $C(S)$.

Example 2.8. Let \mathcal{N} be the Namioka space as considered in Example 2.6 and recall that \mathcal{N} is indeed separating and order dense, however, not pointwise order dense in $C(S)$. Let $\varphi: \mathcal{N} \rightarrow \mathbb{R}$ be the functional that composes any $x \in \mathcal{N}$ with the point evaluation at $s = \frac{1}{2}$. Evidently φ is a Riesz* homomorphism by Theorem 2.5. However, letting $x, y \in \mathcal{N}$ be defined by $x(s) = s$ and $y(s) = 1 - s$, we obtain $\varphi(x) \vee \varphi(y) = \frac{1}{2} \vee \frac{1}{2} = \frac{1}{2}$, while for any $z \in \mathcal{N}$ with $z \geq x, y$ we get $z(0) \geq y(0) \geq 1$ and $z(1) \geq x(1) \geq 1$, hence $\varphi(z) = z(\frac{1}{2}) \geq 1$. Therefore, φ does not satisfy condition (2.4) and, hence, is not a Riesz homomorphism.

Complete Riesz Homomorphisms

First introduced and studied by Buskes and van Rooij in [BvR93] is the class of complete Riesz homomorphisms. Between Riesz spaces these complete Riesz homomorphisms are

exactly the order continuous Riesz homomorphisms. Between pre-Riesz spaces the complete Riesz homomorphisms are exactly the operators that extend to order continuous Riesz homomorphisms between the completions [vH93, Theorem 5.12]. Our aim is to characterize the complete Riesz homomorphisms between order dense subspaces of $C(S)$ and, in doing so, characterize the order continuous Riesz homomorphisms between Riesz subspaces of $C(S)$. More specifically, our aim is to determine a necessary condition imposed on w and π that when imposed guarantees $f_{w,\pi}: X \rightarrow Y$ to be a complete Riesz homomorphism.

A linear map $f: X \rightarrow Y$ between partially ordered vector spaces is called a *complete Riesz homomorphism* whenever $Z \subseteq X$ with $\inf Z = 0$ implies $\inf f(Z) = 0$. As not all Riesz homomorphisms are order continuous we easily construct an example of a Riesz* homomorphism that is not a complete Riesz homomorphism. Consider a weighted composition operator $f_{w,\pi}: C[0,1] \rightarrow C[0,1]$ where w is positive and non-vanishing and π is constant. There exists a sequence in $C[0,1]$ that descends to zero and is constantly one on the singleton $\pi[[0,1]]$. Therefore, f is indeed not a complete Riesz homomorphism. It holds generally, however, that for $w \geq 0$ non-vanishing and π an open map that $f_{w,\pi}$ is a complete Riesz homomorphism. It turns out that π being an open map is not a necessary condition, as will be shown in Theorem 2.10.

A function $\pi: T \rightarrow S$ is called *weak-open* if for all non-empty open $U \subseteq T$ the image $\pi[U]$ is dense somewhere, i.e., there exists a non-empty $V \subseteq S$ open such that $\pi[U] \cap V$ is dense in V , and π is called *nowhere constant* if for all non-empty $U \subseteq T$ open the image $\pi[U]$ is not a singleton. One easily verifies that the former implies the latter and π being open implies both properties.

We characterise subsets in $C(S)$ whose infimum exist and equal zero.

Lemma 2.9. *Let X be an order dense subspace of $C(S)$ and let $Z \subseteq X_+$ be given. Then $\inf Z = 0$ holds in X if and only if Z satisfies the following property*

$$\forall \epsilon > 0, U \subseteq S \setminus \{\emptyset\} \text{ open } \exists z \in Z, s \in U \text{ such that } z(s) \leq \epsilon. \quad (2.5)$$

Proof. Let $Z \subseteq X$ with $\inf Z = 0$. Suppose that the converse of (2.5) holds. Let $\epsilon > 0$ and $U \subseteq S$ be non-empty and open such that for all $z \in Z$ and $s \in U$ we have $z(s) > \epsilon$. An application of Urysohn's Lemma yields a non-zero positive $y \in C(S)$ whose support is contained in U . After rescaling if necessary y is a lower bound of Z in $C(S)$. As X is order dense in $C(S)$ there exists a $x \in X$ with $x \not\leq 0$ and $x \leq y$. This yields a contradiction with the assumption that $\inf Z = 0$.

Suppose $Z \subseteq X_+$ does not satisfy $\inf Z = 0$. Then there exists a lower bound $x \in X$ of Z such that $x \not\leq 0$. Remark that the positive part x^+ of x is a non-zero positive element of $C(S)$ and a lower bound of Z . By continuity there exists an $\epsilon > 0$ and $U \subseteq S$ non-empty and open such that $z(s) \geq x^+(s) > \epsilon$ holds for all $s \in U, z \in Z$. \square

We are now ready to characterize complete Riesz homomorphisms on order dense subspaces of $C(S)$ and note that no additional conditions are imposed on the subspace Y of $C(T)$.

Theorem 2.10. *Let X be an order dense subspace of $C(S)$ and Y a subspace of $C(T)$. Let $w \in C(T)_+$ and $\pi: T \rightarrow S$ be such that $f_{w,\pi}: X \rightarrow C(T)$ maps into Y . Then $f_{w,\pi}: X \rightarrow Y$ is a complete Riesz homomorphism if and only if π is weak-open on $\{w > 0\}$.*

Proof. Let w and π be as in the statement and let us denote $f_{w,\pi}$ by f . Suppose π is weak open on $\{w > 0\}$. Let $Z \subseteq X$ with $\inf Z = 0$. Fix $\delta > 0$ and $U \subseteq T$ non-empty and open. Due to Lemma 2.9 it suffices to show existence of an $z \in Z$ and $t \in U$ such that $f(z)(t) \leq \delta$ holds. Suppose there exists a $t \in U \cap \{w = 0\}$. Then for all $z \in Z$ we have $f(z)(t) = 0 < \delta$ and, hence, we are done. Therefore, we assume that $U \subseteq \{w > 0\}$ holds. In particular, there exists a non-empty open $V \subseteq S$ with $\pi[U] \cap V$ dense in V , as π is weak-open on $\{w > 0\}$. Put $\epsilon := \delta(m+1)^{-1} > 0$, where $m := \sup\{w(t) : t \in T\}$. As $\inf W = 0$ holds Lemma 2.9 yields a $z \in Z$ and $s \in V$ such that $z(s) \leq \frac{\epsilon}{2}$. Therefore, by continuity there exists an $s_0 \in \pi[U] \cap V$ with $f(s_0) \leq \epsilon$. Let $t_0 \in \pi^{-1}(\{s_0\}) \cap U$. We compute

$$f(z)(t_0) = w(t_0)z(\pi(t_0)) \leq m \cdot z(\pi(t_0)) = m \cdot z(s_0) \leq M\epsilon \leq \delta,$$

and conclude that due to Lemma 2.9 f is a complete Riesz homomorphism.

Conversely, suppose that π is not weak open on $\{w > 0\}$. In other words, there exist $\delta > 0$ and non-empty and open $U \subseteq T$ with $U \subseteq \{w \geq \delta\}$ and $\pi[U]$ is nowhere dense in S . We recall that generally a complete Riesz homomorphism between pre-Riesz spaces extends to a order continuous Riesz homomorphism between the Riesz completions, which, therefore, is itself is a complete Riesz homomorphism. Hence, it suffices to show that $\hat{f} = f_{w,\pi}: E^\rho \rightarrow F^\rho$ is not a complete Riesz homomorphism. We define

$$Z := \{z \in X^\rho : z \geq 0 \text{ and } z \geq 1 \text{ on } \pi[U]\}.$$

We argue by contradiction that $\inf Z = 0$. Suppose there exists a lower bound $z \in X_+^\rho$ of Z not smaller than zero. As X^ρ is a Riesz space we replace z by z^+ if necessary to obtain a non-zero positive lower bound z of Z . In particular, there exist $\epsilon > 0$ and $W \subseteq S$ non-empty and open such that $z \geq \epsilon$ on W . Recall that $\pi[U]$ is nowhere dense, so $\pi[U] \cap W$ is not dense in W . Therefore, the closure V of $\pi[U] \cap W$ in W is a closed strict subset of W . Let $W_0 \subseteq W$ be non-empty and open with $\overline{W_0} \cap V = \emptyset$. By Urysohn's lemma there is a $z_0 \in C(S)^+$ with $z_0 = 1$ on $V \supseteq \pi[U]$ and $z_0 = 0$ on W_0 . As X^ρ is order dense in $C(S)$ there exists an $x \in X^\rho$ with $x \geq z_0$ and $x(s) < \epsilon$ for some $s \in W_0$. By construction $x \geq z_0$ yields $x \in Z$ and from $W_0 \subseteq W$ we infer that $z \not\leq x$. This yields a contradiction with z being a lower bound of Z and we conclude that $\inf Z = 0$.

On the other hand, however, we argue that $f[Z]$ has a lower bound that is not negative. By construction of U any $y \in f[Z]$ satisfies $y \geq \delta$ on U . Since Y^ρ is an order dense Riesz subspace of $C(T)$, there exists a lower of $f[Z]$ in Y^ρ that is strictly positive on U . \square

Let us remark that Theorem 2.10 shows, in particular, that the order continuous Riesz homomorphisms between order dense Riesz subspaces of $C(S)$ and $C(T)$ are exactly the composition multiplication operators where the composition map is weak open on the set where the multiplication map is non-zero.

Concluding this section we remark that, in the special case that S and T are compact intervals of \mathbb{R} , the condition that π is weak-open on $\{w > 0\}$ in the above theorem can be relaxed to π being nowhere constant on $\{w > 0\}$. The proof follows immediately from the following lemma.

Lemma 2.11. *Let I and J be intervals in \mathbb{R} and $\pi: I \rightarrow J$ a map. For all $U \subseteq I$ which are open in \mathbb{R} , π is weak-open on U if and only if π is nowhere constant on U .*

Proof. As mentioned earlier, it is evident that the former implies the latter. Let $U \subseteq I$ is open in \mathbb{R} . Without loss of generality we assume that U is non-empty. Suppose that π is nowhere constant on U . Then we can find distinct points $s, t \in J$ contained in $\pi[U]$ and say $s < t$. Consequently, there are $a, b \in U$ with $a < b$ such $\pi(a) = s$ and $\pi(b) = t$ or vice versa. We restrict π to the continuous map $\hat{\pi}: [a, b] \rightarrow J$. For any $r \in (s, t)$ we can find, by the Intermediate Value Theorem, a $c \in (a, b)$ with $\pi(c) = r$. Therefore, (s, t) is contained in $\pi[U]$ and we conclude that π is weak open on U . In the case that $\pi(a) = t$ and $\pi(b) = s$ we can interchange the roles of s and t and obtain similarly that (t, s) is contained in $\pi[U]$. \square

Locally compact spaces

We have investigated Riesz* homomorphisms on separating order dense subspaces of the space of continuous functions on some compact Hausdorff space. With similar techniques we characterize the Riesz* homomorphisms between pre-Riesz spaces of continuous functions as weighted composition maps with some additional conditions on the weight and composition map.

In this section S and T are locally compact Hausdorff spaces. Consider the subspace $C_0(S)$ of $C(S)$ consisting of all functions $x \in C(S)$ that *vanish at infinity*, i.e., for all $\epsilon > 0$ the set $\{x \geq \epsilon\}$ is compact in S . Evidently, $C_0(S)$ is a Riesz space and it coincides with $C(S)$, whenever S is compact. We generalise the results in Theorem 2.5 to the setting of vanishing functions on locally compact spaces. We use the following result from [Fol84, 7.3]: the positive norm-bounded linear functionals on $C_0(S)$ are exactly those functionals that are given by integration against a finite Radon measure.

Theorem 2.12. *Let X and Y be separating order dense subspaces of $C_0(S)$ and $C_0(T)$, respectively. A linear map $f: X \rightarrow Y$ is a Riesz* homomorphism if and only if $T = T_{w,\pi}$ for some $w \in C_b(Y)^+$ and $\pi: Y \rightarrow X$ continuous on $\{w > 0\}$. Moreover, in this case π is proper on $\{w \geq \epsilon\}$ for each $\epsilon > 0$, i.e.,*

$$K \subseteq S \text{ is compact, } \epsilon > 0 \Rightarrow \pi^{-1}(K) \cap \{w \geq \epsilon\} \text{ is compact.} \quad (2.6)$$

Proof. Let $f: X \rightarrow Y$ be a Riesz* homomorphism and let $f^\rho: X^\rho \rightarrow Y^\rho$ be the Riesz homomorphism that extends f . Fix $t \in T$. We define a positive linear functional

$f_t: X^\rho \rightarrow \mathbb{R}$ by $f_t(x) := f^\rho(x)(t)$ for $x \in X^\rho$. We extend f_t to a Riesz homomorphism $\hat{f}_t: C_0(S) \rightarrow \mathbb{R}$ using the Lipecki-Luxemburg-Schep extension theorem. By [Fol84, 7.3] the functional f_t is given by integration against a finite Radon measure, say μ_t . This result applies since positive operators between Banach lattices are norm continuous by [AB06, Theorem 4.3], and hence, f_t is norm-bounded.

We argue that μ_t is supported in at most a single point. Suppose that s and t are distinct points in the support of μ_t . Here the support of μ_t , denoted by $\text{supp}(\mu_t)$, consists of all $s \in S$ such that for all open $U \subseteq S$ that contain s we have $\mu_t(U) > 0$. By the Hausdorff property of S we obtain disjoint open sets $U, V \subseteq S$ with $s \in U, t \in V$. In particular, we get $\mu_t(U), \mu_t(V) > 0$. Furthermore, since the Radon measure μ_t is inner regular, we can assume without loss of generality, that U and V are contained in some compact set $K \subseteq S$. Applying Urysohn's lemma yields continuous $x, y: S \rightarrow [0, 1]$ with $x(s) = 1$ and $x = 0$ on $S \setminus U$, and $y(t) = 1$ and $y = 0$ on $S \setminus V$. We remark that $x, y \in C_0(S)$ as both x and y are zero outside the compact set K . Moreover, by construction x and y are disjoint. As f is a Riesz homomorphism, we infer $f_t(x) \perp f_t(y)$. However, for $\epsilon \in (0, 1]$, we have

$$f_t(x) = \int_S x d\mu_t \geq \int_{\{x \geq \epsilon\}} \epsilon d\mu_t = \epsilon \cdot \mu_t(\{x \geq \epsilon\}) > 0,$$

since the set $\{x \geq \epsilon\}$ contains s and, therefore, has strictly positive measure, as s is in the support of μ_t . Analogously, $f_t(y) > 0$. This contradicts our earlier conclusion that $f_t(x) \wedge f_t(y) = 0$. Therefore, the support of μ_t is either a singleton or the functional f_t is identically zero.

Suppose $t \in T$ is given such that $f_t \neq 0$. Let $\pi(t) \in S$ be the unique element in the support of μ_t and $w(t) := \mu_t(\{\pi(t)\}) > 0$. For $x \in C_0(S)$ we obtain the desired formula

$$f_t(x) = \int_S x d\mu_t = \int_{\{\pi(t)\}} x d\mu_t = w(t)x(\pi(t)).$$

We put $w(t) := 0$ whenever the corresponding functional f_t equals zero. We conclude that f is a weighted composition map and $f = f_{w,\pi}$.

In the proof of Theorem 2.5, where we considered the $C(S)$ case, we showed that w and π are automatically continuous on T and $\{w > 0\}$, respectively. Our arguments made there only used that any bounded net in S has a convergent subnet, as property of the topological space S . As any locally compact topological space has this property we infer that w and π are continuous on Y and $\{w > 0\}$, respectively. Straightforward verification yields that w inherits the positive and bounded property from f .

We argue that any weighted composition map $f_{w,\pi}: X \rightarrow Y$ satisfies (2.6). Let $K \subseteq S$ be compact and $\epsilon > 0$. Let $x \in C_0(S)$ be equal to one on K ; such an x as K is compact. For $t \in \pi^{-1}(K) \cap \{w \geq \epsilon\}$ we have

$$f(x)(t) = w(t)x(\pi(t)) = w(t) \geq \epsilon.$$

Therefore, $\pi^{-1}(K) \cap \{w \geq \epsilon\}$ is compact since $f(x) \in C_0(T)$.

Let $f = f_{w,\pi}: X \rightarrow Y$ for suitable w and π . By the previous paragraph equation (2.6) is satisfied. Therefore, the weighted composition map $f^\rho = f_{w,\pi}: X^\rho \rightarrow Y^\rho$ is well-defined, as any $f^\rho(x)$ vanishes at infinity for $x \in X^\rho$ by (2.6). Moreover, f^ρ is clearly a Riesz homomorphism that extends f . We conclude that f is a Riesz* homomorphism. \square

An application: Sobolev spaces

A result by Biegert [Bie10, Theorem 4.4] states that any Riesz homomorphism on the Sobolev space $W_0^{1,p}(\Omega)$ is a weighted composition map, where $\Omega \subseteq \mathbb{R}^d$ is open. A higher order Sobolev space $W_0^{m,p}(\Omega)$ with $m > 1$ is generally not a Riesz space. Our aim is to extend Biergert's characterization to Riesz* homomorphisms on the pre-Riesz space $W_0^{m,p}(\Omega)$. Our strategy is to use the classical Sobolev Embedding Theorem to embed the Sobolev space into $C_0(\Omega)$ and show that this embedding satisfies the conditions of Theorem 2.12.

We start by describing the setting and giving the necessary definitions. All definitions and terminology we introduce here are taken from Adams [Ada75]. Let $d \in \mathbb{N}$ be given and Ω a *domain* in \mathbb{R}^d , i.e., $\Omega \subseteq \mathbb{R}^d$ is open. For any $m \in \mathbb{N}$ and $1 \leq p < \infty$ we define the Sobolev space $W^{m,p}(\Omega)$ as the space consisting of L^p -functions x on Ω for which all distributional partial derivatives $D^\alpha x$, with $1 \leq |\alpha| \leq m$, are in L^p . Equipped with the norm $\|\cdot\|_{m,p}$ defined by

$$\|x\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha x\|_p^p \right)^{\frac{1}{p}},$$

the space $W^{m,p}(\Omega)$ is a Banach space. For smooth functions the distributional and classical partial derivatives coincide, hence, we infer $C^\infty(\Omega) \subseteq W^{m,p}(\Omega)$.

Under some regularity conditions imposed on the domain Ω every equivalence class $u \in W^{m,p}(\Omega)$ contains a unique continuous function. In this case we define $W_0^{m,p}(\Omega)$ as the norm-closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For the reader's convenience we include these conditions here. They can be found on page 66 of [Ada75].

Definition 2.13. Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ open be given.

- (i) Let $x \in \mathbb{R}^d$ and B_1 and B_2 open balls in \mathbb{R}^d with $x \in B_1$ and $x \notin B_2$. The set $C_x := B_1 \cap \{x + \lambda(y - x) : y \in B_2, \lambda > 0\}$ is a *finite cone* with vertex x .
- (ii) Every domain Ω for which there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C is said to have the *cone property*.

The classical Sobolev Embedding Theorem [Ada75, Theorem 5.4 part III(C)] states that if Ω is a domain in \mathbb{R}^d which has the cone property and $mp > d$ holds, then $W_0^{m,p}(\Omega) \subseteq C_0(\Omega)$ holds. In this case, we therefore get

$$C_0^\infty(\Omega) \subseteq W_0^{m,p}(\Omega) \subseteq C_0(\Omega).$$

In particular, $W_0^{m,p}(\Omega)$ is separating and order dense in $C_0(\Omega)$. Therefore, Theorem 2.12 yields the following.

Theorem 2.14. *Suppose Ω_1 and Ω_2 are domains in \mathbb{R}^d having the cone property, $1 \leq p, q < \infty$ and $m, n \in \mathbb{N}$ be such that $pm > d$ and $qn > d$ hold. Any Riesz* homomorphism $f: W_0^{m,p}(\Omega_1) \rightarrow W_0^{n,q}(\Omega_2)$ is a weighted composition map.*

In his proof, Biegert does not use the order structure of the space $W^{1,p}(\Omega)$ nor the Sobolev Embedding Theorem. Due to the latter he does not need to impose the cone property on Ω or any condition on p and d . However, Theorem 2.14 can deal with Sobolev spaces up to arbitrary order. Remark that $W^{m,p}(\Omega)$ is a Riesz space exactly when $m = 1$ holds. In conclusion, under additional regularity conditions on the domain, we extend the result of Biegert to higher order Sobolev spaces by considering Riesz* homomorphisms.

Linear order isomorphisms

We have considered three types of homomorphisms defined on pre-Riesz spaces. In order from weak to strong, Riesz* homomorphisms, Riesz homomorphisms and complete Riesz homomorphisms. Analogous to Theorem 2.2, we consider whether the inverse of a bijective homomorphism of one of these types is again of the same type. Since the results that we obtain hold in general partially ordered vector spaces, we shall consider them in that setting.

Secondly, returning to the setting of order dense subspaces in spaces of continuous functions, we study linear order isomorphisms. Combining our observations with Theorem 2.5 we obtain a method of describing the automorphism group for any order dense subspace of $C(S)$. We then apply this method to the space of differentiable functions on a smooth manifold.

Inverses of Riesz homomorphisms*

We start with an elementary observation on linear order isomorphisms.

Proposition 2.15. *Suppose X and Y are partially ordered vector spaces and $f: X \rightarrow Y$ is a linear map. Then f is an order isomorphism if and only if f is bijective and both f and f^{-1} are complete Riesz homomorphisms.*

Proof. Suppose f is a linear order isomorphism. It suffices to show that f is a complete Riesz homomorphism, as f^{-1} is also a linear order isomorphism. Let $Z \subset X$ with $\inf Z = 0$. Suppose $y \in Y$ is a lower bound of $f[Z]$, then $f^{-1}(y)$ is a lower bound of Z . Therefore, $f^{-1}y \leq 0$ holds as the infimum of Z equals zero. Applying the positivity of f again yields $f(f^{-1}(x)) = x \leq 0$, which proves that $\inf f[Z] = 0$.

For the converse, it suffices to observe that complete Riesz homomorphisms are positive. Let $x \in X$ be positive. We put $Z := \{0, x\}$. Evidently $\inf Z = 0$ holds and,

hence, if $f: X \rightarrow Y$ is a complete Riesz homomorphism, this yields $\inf f[Z] = 0$, which shows, in particular, $f(x) \geq 0$. \square

In the following result we highlight the usefulness of the pervasive property.

Lemma 2.16. *Let X and Y be pre-Riesz spaces, $f: X \rightarrow Y$ a Riesz* homomorphism and $f^\rho: X^\rho \rightarrow Y^\rho$ the Riesz homomorphism that extends f . The following assertions hold:*

- (i) *If f is surjective, then f^ρ is surjective;*
- (ii) *If X is pervasive and f is injective, then f^ρ is injective.*

Proof. Suppose f is surjective. Let $y \in Y^\rho$ be given. Recall that Y^ρ is generated as Riesz space by Y . Let $x_1, \dots, x_n, y_1, \dots, y_m \in Y$ be such that $y = \bigvee_{i=1}^n x_i - \bigvee_{j=1}^m y_j$ in Y^ρ . Using that f is surjective we let $a_1, \dots, a_n, b_1, \dots, b_m \in X$ with $x_i = f(a_i)$ and $y_j = f(b_j)$, $i = 1, \dots, n$ and $j = 1, \dots, m$. We define $x := \bigvee_{i=1}^n a_i - \bigvee_{j=1}^m b_i \in X^\rho$. The image of x under f^ρ is computed as follows

$$\begin{aligned} f^\rho(x) &= f^\rho\left(\bigvee_{i=1}^n a_i - \bigvee_{j=1}^m b_j\right) \\ &= \bigvee_{i=1}^n f^\rho(a_i) - \bigvee_{j=1}^m f^\rho(b_j) \\ &= \bigvee_{i=1}^n x_i - \bigvee_{j=1}^m y_j = y, \end{aligned}$$

and, hence, f is surjective.

Suppose X is pervasive and f is injective. Let $x \in X^\rho$ be non-zero and positive. As X is pervasive there exists a $y \in X_+$ with $0 < y \leq x$. Since f^ρ is positive this yields $0 \leq f(y) = f^\rho(y) \leq f^\rho(x)$. The injectivity of f yields $f^\rho(x) \neq 0$. From this we conclude that for any $x \in X^\rho$ with $f^\rho(x) = 0$ that $x = 0$, since $f^\rho(x_+) = f(x)_+ = 0$ and $f^\rho(x_-) = f(x)_- = 0$ hold and both x_+ and x_- are positive. \square

In particular, Lemma 2.16 shows that a bijective Riesz* homomorphism on a pervasive pre-Riesz space extends to a bijective Riesz homomorphism on the Riesz completion. This fact is useful in studying properties of the inverse of bijective homomorphisms.

Theorem 2.17. *Suppose X and Y are pre-Riesz spaces, X is pervasive and that $f: X \rightarrow Y$ is a bijective Riesz* homomorphism. Then f^{-1} is a Riesz* homomorphism and, hence, f is an order isomorphism.*

Proof. Suppose f is a bijective Riesz* homomorphism. Lemma 2.16 yields that f extends to a bijective Riesz homomorphism $f^\rho: X^\rho \rightarrow Y^\rho$. The inverse $(f^\rho)^{-1}: Y^\rho \rightarrow X^\rho$ is a Riesz homomorphism by Theorem 2.2 that extends $f^{-1}: Y \rightarrow X$ and, hence, f^{-1} is a Riesz* homomorphism. \square

In conjunction with Proposition 2.15 this yields the following list of equivalent statements.

Corollary 2.18. *Let X and Y be pre-Riesz spaces and X pervasive. For a linear bijection $f: X \rightarrow Y$ the following statements are equivalent:*

- (i) f is a Riesz* homomorphism.
- (ii) f is a Riesz homomorphism.
- (iii) f is a complete Riesz homomorphism.
- (iv) f^{-1} is a Riesz* homomorphism.
- (v) f^{-1} is a Riesz homomorphism.
- (vi) f^{-1} is a complete Riesz homomorphism.
- (vii) f is an order isomorphism.

The main implication of Corollary 2.18 is that a bijective Riesz* homomorphism between pervasive pre-Riesz spaces is an order isomorphism, just as in the vector lattice case. We consider an example of a non-pervasive pre-Riesz space for which the statement in Theorem 2.17 fails to hold.

Example 2.19. Let X be the subspace of $C([0, 1])$ consisting of all polynomials. Then X is a pre-Riesz space and its Riesz completion X^ρ is the Riesz subspace of $C([0, 1])$ consisting of all piecewise polynomial functions. Since non-constant polynomials can only be zero in finitely many points, one easily verifies that X is not pervasive. We consider $f = f_{w,\pi}: X \rightarrow X$, where $w = \mathbb{1}$ and $\pi(s) = \frac{1}{2}s$ for $s \in [0, 1]$. Theorem 2.5 yields that f is a Riesz* homomorphism. Moreover, since π is a weak-open map, f is even a complete Riesz homomorphism by Theorem 2.10. We argue that f is a bijective map. By definition it is evident that f is injective. Let $y \in X$ be of the form $y(s) = \alpha_n s^n + \dots + \alpha_1 s + \alpha_0$, for $s \in [0, 1]$ with $\alpha_0, \dots, \alpha_n \in \mathbb{R}$. The pre-image $f^{-1}(y)$ is given by $x(s) = \sum_{i=0}^n \beta_i s^i \in X$, where $\beta_i = 2^i \alpha_i$ for $0 \leq i \leq n$. We conclude that T is a bijective Riesz* homomorphism.

Suppose there exist $v: [0, 1] \rightarrow \mathbb{R}$ continuous and $\tau: [0, 1] \rightarrow [0, 1]$ continuous on $\{\theta > 0\}$ such that $f^{-1} = f_{v,\tau}$ on X . The equality $f(\mathbb{1}) = \mathbb{1}$, where $\mathbb{1}$ denote the constant one functions, immediately yields that v is identically zero. Let $x \in X$ satisfy $x(s) = s$ for all $s \in [0, 1]$. We compute for all $s \in [0, 1]$ that

$$s = f^{-1}(f(x))(s) = f(x)(\tau(s)) = x\left(\frac{1}{2}(\tau(s))\right) = \frac{1}{2}\tau(s).$$

However, the equality $\tau(s) = 2s$ can not be satisfied on all of $[0, 1]$, therefore, f^{-1} is not a weighted composition operator. In particular, Theorem 2.5 yields that f^{-1} is not a Riesz* homomorphism.

Automorphism groups

Combining our characterisation of Riesz* homomorphisms on spaces of continuous functions, obtained in Theorem 2.12, and the general observation made in the previous section concerning linear order isomorphisms, we can describe the linear order isomorphisms between order dense subspaces $X \subseteq C_0(S)$ and $Y \subseteq C_0(T)$ that separate the points, where S and T are locally compact Hausdorff spaces.

Theorem 2.20. *A linear map $f: X \rightarrow Y$ is an order isomorphism if and only if $f = f_{w,\pi}$, where $w \in C(T)$ satisfies $0 < \delta \mathbb{1} \leq w \leq D \mathbb{1}$ for some $\delta, D > 0$ and $\pi: T \rightarrow S$ is a homeomorphism. In this case, w and π are uniquely determined by f .*

We remark that a homeomorphism is proper and, therefore, (2.6) is satisfied in this case.

Proof. Let $f: X \rightarrow Y$ be a linear order isomorphism. Due to Proposition 2.15, f is a complete Riesz homomorphism and, in particular, f is a Riesz* homomorphism. Therefore, Theorem 2.12 yields the existence of maps $w \in C_b(T)_+$ and $\pi: T \rightarrow S$ continuous on $\{w > 0\}$ such that $f = f_{w,\pi}$.

Suppose that $w(t) = 0$ holds for some $t \in T$. In that case, we get $y(t) = 0$ for all $y \in f(X)$. This yields an immediate contradiction with the fact that f is surjective and Y is majorizing. We conclude that w is indeed non-vanishing. We remark that, since $\{w > 0\} = T$ holds, π is continuous and uniquely determined everywhere.

Suppose π is not injective. Let $t_1, t_2 \in T$ be such that $t_1 \neq t_2$ and $\pi(t_1) = \pi(t_2)$. We obtain for all $x \in X$ that

$$f(x)(t_1) = w(t_1)x(\pi(t_1)) = w(t_2)\frac{w(t_1)}{w(t_2)}x(\pi(t_2)) = \frac{w(t_1)}{w(t_2)}f(x)(t_2).$$

Therefore, any $g \in f(X)$ satisfies $g(t_1) = \lambda g(t_2)$, where $\lambda = w(t_1)/w(t_2)$. Since Y separates the points of T this contradicts the surjectivity of f and we conclude that π is injective.

We argue that π is surjective. Recall that π is continuous. Hence, $\pi(T)$ is compact and, hence, $\pi(T)$ is closed in S . Therefore, supposing that π is not surjective yields a non-empty open $U \subseteq S \setminus \pi(T)$. Due to Urysohn's lemma we find a non-zero negative $y \in C(S)$ with $\text{supp}(y) \subseteq U$. We infer the existence of an $x \in X$ that satisfies $x \geq y$ and $x \not\geq 0$, from the fact X is order dense in $C(S)$. We remark that $x \geq y$ yields $x(s) \geq 0$ for all $s \in \pi(T)$. Therefore, for all $t \in T$ we get $f(x)(t) = w(t)x(\pi(t)) \geq 0$ as w is positive. This contradicts that f^{-1} is positive as $f(x) \geq 0$ and $x \not\geq 0$.

Consider the weighted composition map $g: Y \rightarrow X$ with weight map η the reciprocal of $w \circ \pi^{-1}$ and composition map π^{-1} . This is well-defined as we have shown that w does not vanish and π is bijective. We verify that g is the inverse of f . For all $y \in Y$ and $t \in T$ we compute

$$f \circ g(y)(t) = w(t)g(y)(\pi(t)) = \frac{w(t)}{\eta(\pi(t))}y(\pi^{-1}(\pi(t))) = y(t).$$

Completely analogously one verifies that $g \circ f$ equals the identity on X . We conclude that $g = f^{-1}$. Theorem 2.12 applied to the linear order isomorphism g yields that η is bounded and π^{-1} is continuous. In particular, $0 < \delta \mathbb{1} \leq w \leq D \mathbb{1}$ holds for some $\delta, D > 0$ and π is a homeomorphism.

Conversely, suppose $f = f_{w,\pi}: X \rightarrow Y$ with w and π as in the assertion. The weighted composition map $g = g_{\eta,\pi^{-1}}: Y \rightarrow X$, where η is the reciprocal of $w \circ \pi^{-1}$, is a well-defined inverse of f by the assumptions on w and π . Therefore, f is bijective. Moreover, as both w and η are positive we get that f and $f^{-1} = g$ are positive. Indeed f is an order isomorphism. \square

Remark 2.21. A class result [Kap47] by Kaplansky states that you can recover a locally compact Hausdorff space S from the lattice structure of $C(S)$, or in other words, that if there exists a lattice isomorphism from $C(S)$ to $C(T)$ then S and T must be homeomorphic. We remark that a lattice isomorphism here is not necessarily linear. One could ask whether a compact Hausdorff space S is also fully determined by the order structure of subspaces of $C(S)$. Theorem 2.20 yields a partial answer, namely we can determine S by the *linear* order structure of any order dense separating subspace of $C(S)$. This question is also studied in [LL13]. Their result [LL13, Theorem 1], for the case where S is compact, states that if $X \subseteq C(S)$ and $Y \subseteq C(T)$ contain the constant functions and precisely separates points from closed sets, then any linear order isomorphism $f: X \rightarrow Y$ is a weighted composition map, where the composition map is a homeomorphism from S to T . Here X *precisely separates points from closed sets* whenever for any closed $F \subseteq S$ and $s \notin F$ there exists an $x \in X$ with $x[S] \subseteq [0, 1]$, $x[F] \subseteq \{0\}$ and $x(s) = 1$. We remark that these conditions in [LL13, Theorem 1] are more restrictive than ours in Theorem 2.20. Suppose $X \subseteq C(S)$ contains the constant functions and precisely separates the points from closed sets. The former conditions implies that X is majorizing in $C(S)$, and the latter yields both that X separates the points of S and that X is pervasive. Now Lemma 1.5 yields that X is order dense in $C(S)$.

Let (X, C) be a partially ordered vector space. The *automorphism group of X* , denoted by $\text{Aut}(X)$, is the set consisting of all linear order isomorphisms from X onto itself equipped with the group action of composition. Due to Theorem 2.20 we can describe the automorphism group of $C_0(S)$.

Consider the group $C_b(S)_+$ consisting of all positive bounded continuous functions on S equipped with pointwise multiplication. We denote the interior of $C_b(S)_+$ relative to the maximum norm by $C_b(S)_+^\circ$. Note that $w \in C_b(S)_+^\circ$ exactly when $w \in C(S)$ satisfies $0 < \delta \mathbb{1} \leq w \leq D \mathbb{1}$ for some $\delta, D > 0$. We denote the group consisting of all homeomorphisms from S to itself equipped with the group action composition by $\text{Hom}(S)$.

Theorem 2.22. $\text{Aut}(C_0(S))$ is isomorphic to $C_b(S)_+^\circ \times \text{Hom}(S)$ endowed with the group action $(w, \pi) \bullet (\eta, \rho) = (\eta(w \circ \rho), \pi \circ \rho)$. Here the group isomorphism is given by $(w, \pi) \mapsto f_{w,\pi}$.

Proof. We verify that $(C_b(S)_+^\circ \times \text{Hom}(S), \bullet)$ is a group. The operation \bullet is closed as composition with a homeomorphism $\rho: S \rightarrow S$ leaves $C_b(S)_+^\circ$ invariant. Straightforward verification yields that \bullet is associative. For any pair (w, π) the inverse with respect to \bullet is given by $((w \circ \pi^{-1})^{-1}, \pi^{-1})$. Lastly, the identity element is $(\mathbb{1}, \text{Id}_S)$.

Theorem 2.20 yields that the map $(w, \pi) \mapsto f_{w,\pi}$ is a bijection. The group action \bullet is constructed to make $(w, \pi) \mapsto f_{w,\pi}$ a group homomorphism. \square

Let $X \subseteq C_0(S)$ be a separating and order dense subspace. Due to Theorem 2.20 any $f \in \text{Aut}(X)$ is of the form $f = f_{w,\pi}$ with $(w, \pi) \in C_b(S)_+^\circ \times \text{Hom}(S)$. Therefore, $\text{Aut}(X)$ is a subgroup of $\text{Aut}(C_0(S))$. In particular, this shows that automorphisms on X extend to automorphisms on $C_0(S)$. The observation to use Theorem 2.20 to extend linear order isomorphisms, leads to a general statement in the setting of order unit spaces.

Theorem 2.23. *Let (V, C, u) and (W, K, u') be order unit spaces. A linear order isomorphism $f: V \rightarrow W$ extends to a linear order isomorphism $f^\rho: V^\rho \rightarrow W^\rho$ between their Riesz completions.*

Proof. By Proposition 1.23 it suffices to verify the statement for separating order dense subspaces $X \subseteq C(S)$ and $Y \subseteq C(T)$, with S and T compact Hausdorff spaces. Let $f: X \rightarrow Y$ be an order isomorphism. Due to Theorem 2.20 we have $f = f_{w,\pi}$ with $w \in C(T)_+^\circ$ and $\pi: T \rightarrow S$ a homeomorphism. Therefore, $\hat{f} = f_{w,\pi}: C(S) \rightarrow C(T)$ is a well-defined linear order isomorphism that extends f . The restriction of \hat{f} to $X^\rho \subseteq C(S)$ now maps into Y^ρ , as a linear order isomorphism preserves infima, suprema and linear combinations and X^ρ is generated as a Riesz space by X . Similarly, the inverse \hat{f}^{-1} restricted to Y^ρ maps into X^ρ . We conclude that $f^\rho = \hat{f}|_{X^\rho}$ is the desired linear order isomorphism. \square

Generally, automorphisms of $C_0(S)$ do not restrict to separating order dense subspace. For example, an automorphism $f_{w,\pi}$ on $C[0, 1]$ with either w or π not differentiable does not restrict to an automorphism on $C^k[0, 1]$. It is possible, however, to fully describe the automorphism group of the differentiable functions up to arbitrary order on a locally compact space. We do so in the general context of smooth manifolds.

Smooth Manifolds

We recall several elementary definitions concerning smooth manifolds (see [Lee03]). Let (M, τ) be a second countable Hausdorff space. M is called a d -dimensional *topological manifold* if there exists an open cover $(U_i)_{i \in I}$ of M such that for all $i \in I$, U_i is homeomorphic to an open subset V_i of \mathbb{R}^d . In that case, the collection of triplets $\mathcal{A} = \{(U_i, h_i, V_i) : i \in I\}$ is called an *atlas* of M , where $h_i: U_i \rightarrow V_i$ are homeomorphisms. One such a triplet is then called a *chart* of the atlas \mathcal{A} . $M = (X, \mathcal{A})$ is an *m -smooth manifold* if in addition for all $i, j \in I$ the gluing map $(h_i \circ h_j^{-1})|_{h_j(U_i \cap U_j)}: h_j(U_i \cap U_j) \rightarrow h_i(U_i \cap U_j)$ is m -times differentiable as a map on \mathbb{R}^d , or simply a smooth manifold whenever $m = \infty$.

Let M be an m -smooth d -dimensional manifold. A continuous map $x: M \rightarrow \mathbb{R}$ is called m -times *differentiable* if for all charts (U, h, V) of M the map $(x \circ h^{-1}): V \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ is m -times differentiable. Let $C_0^\infty(M)$ be the space consisting of continuous functions from M to \mathbb{R} that vanish at infinity and are infinitely many times differentiable. A useful tool when dealing with the space $C_0^\infty(M)$ is the notion of a partition of unity. Suppose $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ is an open cover of M . A *partition of unity subordinate to \mathcal{U}* is a collection of continuous functions $\varphi_\alpha: M \rightarrow [0, 1]$, $\alpha \in A$, such that $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$, $\{\text{supp}(\varphi_\alpha): \alpha \in A\}$ is a locally finite cover and $\sum_{\alpha \in A} \varphi_\alpha = 1$. Since the supports of the φ_α form a locally finite cover, $\sum_{\alpha} \varphi_\alpha$ has only finitely many non-zero terms in a neighborhood around every point and we encounter no convergence problems. Such a partition of unity is called *m -smooth* if every φ_α is a m -smooth function. An important result in the study of m -smooth manifolds is the existence of a m -smooth partition of unity subordinate to any given open cover (see [Lee03, Theorem 2.25, p.54]). A useful consequence of the existence of an m -smooth partition of unity is the existence of m -smooth *bump functions* on M . Let U and V be open subsets of M such that $\overline{V} \subseteq U$ holds. Letting $U_1 = U$ and $U_2 = M \setminus \overline{V}$ we get an open cover $\{U_1, U_2\}$ of M , hence there exists a subordinated m -smooth partition of unity $\{\varphi_1, \varphi_2\}$. Observe that φ_1 is an m -smooth map on M with values in $[0, 1]$, supported in U and constantly one on \overline{V} . A map φ_1 satisfying these properties is called an *m -smooth bump function* of V supported in U .

For the remainder of this section let M be an n -dimensional locally compact m -smooth manifold with $m \in \mathbb{N} \cup \{\infty\}$ and $k \leq m$ an integer or $k = \infty$. We argue that our results concerning Riesz* homomorphisms and order isomorphisms apply to $C_0^k(M)$ in Proposition 2.25. Before proving this it is convenient to understand the pervasive property of spaces of continuous functions.

Lemma 2.24. *Let S be a compact Hausdorff space and $X \subseteq C_0(S)$ a pre-Riesz space. Then X is pervasive in $C_0(S)$ if and only if for every non-empty open $U \subseteq S$ there exists a positive non-zero $x \in X$ with $\text{supp}(x) \subseteq U$.*

Proof. Suppose X is pervasive in $C_0(S)$ and $U \subseteq S$ is non-empty and open. By Urysohn's lemma there exists a non-zero $y \in C_0(S)^+$ with $\text{supp}(y) \subseteq U$. Due to the pervasive assumption on X , there exists a non-zero $x \in X$ with $0 \leq x \leq y$. In particular, we get $\text{supp}(x) \subseteq \text{supp}(y) \subseteq U$.

Conversely, suppose that latter condition is satisfied. Let $y \in C_0(S)_+$ be non-zero. We fix $0 < \epsilon < \|y\|_\infty$. Now the set $U := \{s \in S: y(s) > \epsilon\}$ is a non-empty open subset of S . Let $z \in X$ with $\text{supp}(z) \subseteq U$. Then $x = \epsilon z / \|z\|_\infty \in X$ is non-zero and positive and is constructed to satisfy $x \leq y$. We conclude that X is pervasive in $C_0(S)$. \square

Proposition 2.25. *The space $C_0^k(M)$ is a separating, pervasive and order dense subspace of $C_0(M)$.*

Proof. It is sufficient to argue the case where $k = \infty$. The existence of smooth bump functions in $C_0^\infty(M)$ described above, immediately yields that $C_0^\infty(M)$ separates the

points of M , and is pervasive due to Lemma 2.24. Due to Lemma 1.6 it suffices to show that $C_0^\infty(M)$ is majorizing in $C_0(M)$ to obtain order denseness.

Lemma 2.23 in [Lee03] states that there exists a countable locally finite cover $(U_n)_{n=1}^\infty$ of M consisting of precompact open sets. Let $W_1 = U_1$ and observe that (U_n) covers the compact set \overline{W}_1 , hence there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $\overline{W}_1 \subseteq \bigcup_{j=1}^k U_{n_j} =: W_2$. Inductively, we obtain a cover (W_n) of M consisting of precompact open sets satisfying $\overline{W}_n \subseteq W_{n+1}$, for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $z_n \in C_0(M)$ be a bump function of W_n supported in W_{n+1} and let $z = \sum_n 2^{-n} z_n$. We conclude that $z \in C_0(M)$ is positive and vanishes nowhere.

Suppose $x \in C_0(M)$ is positive. We aim to construct a $y \in C_0^\infty(M)$ that dominates x . Remark that we may assume without loss of generality that f vanishes nowhere as we can replace x by $x \vee z$. For convenience sake we rescale x to have sup-norm equal to one. For $n \in \mathbb{N}$ we define the open set $V_n = \{p \in M : 2^{-(n+2)} < x(p) < 2^{-n}\}$ and $V_0 = \{p \in M : x(p) > 2^{-2}\}$. The collection $(V_n)_{n=0}^\infty$ is a locally finite countable open cover of $\{x > 0\}$, which equals M . Let $(\varphi_n : M \rightarrow \mathbb{R})_{n=0}^\infty$ be a smooth partition of unity subordinate to $(V_n)_{n=0}^\infty$ and define

$$y(p) := \sum_{j=0}^{\infty} 2^{-j} \varphi_j(p), \quad p \in M.$$

For any point $p \in M$ only finitely many terms are non-zero in a neighborhood of p , hence y is well-defined and smooth. Let $\epsilon > 0$ be given and let $j_0 \in \mathbb{N}$ be such that $\epsilon > \sum_{j=j_0}^{\infty} 2^{-j}$, then we get

$$\{y \geq \epsilon\} \subseteq \bigcup_{n=0}^{j_0} V_n \subseteq \{x \geq 2^{-(j_0+2)}\}. \quad (2.7)$$

Indeed, whenever $p \in M \setminus \bigcup_{n=0}^{j_0} V_n$ we have $y(p) = \sum_{j=j_0}^{\infty} 2^{-j} \varphi_j(p) \leq \sum_{j=j_0}^{\infty} 2^{-j} < \epsilon$, showing the first inclusion while the second inclusion follows from the construction of the set V_n . Since x vanishes at infinity, the set on the right hand side of (2.7) is compact. Therefore, the closed set $\{y \geq \epsilon\}$ is compact, showing that y vanishes at infinity. We are left to show that $y \geq x$ holds. Let $p \in M$ and $n \in \mathbb{N}$ the largest index such that $p \in V_n$. Then we have $y(p) = \sum_{j=0}^n 2^{-j} \varphi_j(p) \geq 2^{-n} \sum_{j=0}^n \varphi_j(p) = 2^{-n}$. On the other hand, we have $x(p) < 2^{-n} \leq y(p)$ as $p \in V_n$ holds. \square

We consider for a moment an open subset $S \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$. The following chain of inclusions is satisfied

$$C_0^k(S) \subseteq LC_0(S) \subseteq C_0^{k,\alpha}(S) \subseteq UC_0(S) \subseteq C_0(S).$$

Here $LC_0(S)$ denotes the Lipschitz continuous functions on S that vanish at infinity, $C_0^{k,\alpha}(S)$ is the subspace of $C_0^k(S)$ consisting of functions that are Hölder continuous with exponent $0 \leq \alpha \leq 1$, meaning that $|x(s) - x(t)| \leq C \|s - t\|^\alpha$ holds for all $s, t \in S$

and some constant $C > 0$, and $UC_0(S)$ denotes the space of all uniformly continuous functions on S that vanish at infinity. Proposition 2.25 yields that $LC_0(S)$, $C_0^{k,\alpha}(S)$ and $UC_0(S)$ are separating, pervasive and order dense subspaces of $C_0(K)$.

Suppose M and N are m - and n -smooth manifolds of independent dimension and let $k \leq n, m$ be an integer or $k = \infty$. Combining Theorem 2.12 and Proposition 2.25 yields that any Riesz* homomorphism $f: C_0^k(M) \rightarrow C_0^k(N)$ is a weighted composition map $f = f_{w,\pi}$, where $w \in C_b(N)^+$ and $\pi: N \rightarrow M$ is continuous and proper on $\{w > 0\}$. As discussed in the previous section $\text{Aut}(C_0^k(M))$ is a subgroup of $\text{Aut}(C_0(M))$. We aim to give a full description of this subgroup. To this end we show that any bijective weighted composition operator on $C_0^k(M)$ has automatically k -smooth weight and composition maps. This is the content of Lemma 2.27. We consider an intermediate result concerning the existence of k -smooth maps on M that locally behave like coordinate projections in \mathbb{R}^d .

Lemma 2.26. *Let $p \in M$ and (U, h, V) a chart of M with $p \in U$. For any index $1 \leq n \leq d$ there exists a k -smooth function $x \in C_0(M)$ and a neighborhood U_0 of p contained in U such that $x = x_n \circ h$ on U_0 , where $x_n(v_1, \dots, v_d) = v_n$, for all $(v_1, \dots, v_d) \in V$.*

Proof. Suppose $p \in M$ is given and (U, h, V) is a chart in M containing p . Let U_0 be a neighborhood of p with $\bar{U}_0 \subseteq U$ and $\varphi: M \rightarrow \mathbb{R}$ a k -smooth bump function of U_0 supported in U . Define $y: M \rightarrow \mathbb{R}$ by $y(q) = x_n(h(q))$ for all $q \in U$ and $y(q) = 0$ elsewhere, where x_n is the n -th coordinate projection in \mathbb{R}^d as in the statement. Since φ is supported in U the map x on M defined by $x = \varphi \cdot y$ is k -smooth. As φ is constantly equal to one on U_0 we conclude that $x = x_n \circ h$ on U_0 . \square

Lemma 2.27. *If $f = f_{w,\pi}: C_0^k(M) \rightarrow C_0^k(N)$ is a linear order isomorphism, then w and π are k -smooth.*

Proof. Recall from Theorem 2.20 and Proposition 2.25 that there exist $\delta, D > 0$ such that $\delta \mathbb{1} \leq w \leq D \mathbb{1}$ and that π is a homeomorphism. Let $q \in N$ be given and C be a compact neighborhood of q in N . Then $\pi(C)$ is compact in M . Let K be a compact neighborhood of $\pi(C)$ and $x \in C_0^k(M)$ be a bump function of K . For all $p \in N$ we have $f(x)(p) = w(p)x(\pi(p))$. The functions $f(x)$ and w coincide on $\pi^{-1}(K)$, which contains C . As $f(x) \in C_0^k(N)$ is k -smooth, we infer that w is k -times differentiable at q .

Fix $q \in N$. Let (U, h, V) a chart of N with $q \in U$ and (U', h', V') be a chart of M with $\pi(q) \in U'$. Let $1 \leq n \leq d$. Due to Lemma 2.26 we obtain an $x \in C_0^k(M)$ and some neighborhood U_n of $\pi(q)$ contained in U' such that $x = x_n \circ h'$ holds on U_n , where x_n is the n -th coordinate projection on \mathbb{R}^d . Since the reciprocal of w is well-defined and k -times differentiable on N , we get $w^{-1} \cdot f(x) = x \circ \pi$ and, hence, $(x \circ \pi)$ is k -times differentiable on N . Therefore, the map $(x_n \circ h' \circ \pi)$ is k -times differentiable on $\pi^{-1}(U_n)$ which is a neighborhood of q , since π is bijective. In particular, $(f_n \circ h' \circ \pi \circ h^{-1})$ is k -times differentiable on $h(\pi^{-1}(U_n))$.

Let $W := h(\pi^{-1}(U_1)) \cap \dots \cap h(\pi^{-1}(U_d))$. W is a neighborhood of q . We observe that the map $(h' \circ \pi \circ h^{-1})$ is k -times differentiable on W when composed with any of

the coordinate projection on \mathbb{R}^d . In conclusion, $(h' \circ \pi \circ h^{-1})$ is k -times differentiable at q and, hence, q is k -smooth. \square

We obtain the following description of the automorphism group of $C_0^k(M)$ as a corollary of Theorem 2.22 and Lemma 2.27.

Theorem 2.28. *Let M be an m -smooth manifold of arbitrary dimension and let $k \leq m$ be given, where $m, k \in \mathbb{N} \cup \{\infty\}$. The automorphism group of $C_0^k(M)$ can be described by*

$$\text{Aut}(C_0^k(M)) \simeq (C_0^k(M)_+^\circ \times \text{Diff}^k(M), \bullet),$$

where $\text{Diff}^k(M)$ denotes the space of all k -diffeomorphisms on M and the group action is given by $(w, \pi) \bullet (\eta, \rho) = (\eta(w \circ \rho), \pi \circ \rho)$.

Positive disjointness preserving operators

In a Riesz space X two elements $x, y \in X$ are called *disjoint*, denoted by $x \perp y$, whenever $|x| \wedge |y| = 0$. A linear map $f: X \rightarrow Y$ between Riesz spaces is disjointness preserving whenever $x \perp y$ implies $f(x) \perp f(y)$. Recall Theorem 2.3 that states that a linear map $f: X \rightarrow Y$ between Riesz spaces is a Riesz homomorphism if and only if f is positive and disjointness preserving. We investigate whether an analogous result holds in the setting of Riesz* homomorphisms between pre-Riesz spaces.

We recall the concept of disjointness in general partially ordered vector spaces. A pair of elements $x, y \in X$ are defined to be *disjoint*, denoted by $x \perp y$, whenever

$$\{x + y, -x - y\}^u = \{x - y, -x + y\}^u.$$

Here the intuition is that the left- and right-hand side of this equality replace the moduli $|x + y|$ and $|x - y|$, which are equal for disjoint elements in a Riesz space. In [vGK18, Proposition 4.1.4] it is shown that two elements in a pre-Riesz space X are disjoint if and only if they are disjoint in a vector lattice cover of X in the usual sense. Therefore, if X is a pre-Riesz space as an order dense subspace of some $C(S)$, then elements $x, y \in X$ are disjoint if and only if for all $s \in S$ either $x(s)$ or $y(s)$ equals zero, as $C(S)$ is a vector lattice cover.

We start by considering a counterexample that shows that generally in pre-Riesz space not all positive disjointness preserving maps are Riesz* homomorphisms.

Example 2.29. Consider the pre-Riesz space $P[0, 1]$ of all polynomials on the unit interval and the map $f: P[0, 1] \rightarrow P[0, 1]$ defined by $x \mapsto (s \mapsto \int_0^s x(t)dt)$. One easily verifies that f is positive. As $P[0, 1]$ does not contain a non-trivial pair of disjoint elements, f is disjointness preserving. Suppose that f is a Riesz* homomorphism, then by Theorem 2.5 there exist $w: [0, 1] \rightarrow \mathbb{R}_+$ and $\pi: [0, 1] \rightarrow [0, 1]$ such that for all $x \in P[0, 1]$ and $s \in [0, 1]$ we have $f(x)(s) = w(s)x(\pi(s))$. From the equality $f(t \mapsto 1) = (s \mapsto s)$ we obtain $w(s) = s$ for all $s \in [0, 1]$ and, moreover, the equality $f(t \mapsto t) = (s \mapsto \frac{1}{2}s^2)$ then yields $\pi(s) = \frac{1}{2}s$ for all $s \in [0, 1]$. Considering the

polynomial $(t \mapsto t^2)$ on the one hand yields $f(t \mapsto t^2) = (s \mapsto w(s)(\pi(s))^2 = \frac{1}{2}s^3)$, while integration of the same polynomial yields $(s \mapsto \frac{1}{3}s^3)$. This contradiction yields that f is not a Riesz* homomorphism.

Example 2.29 is based on the fact that the pre-Riesz space $P[0, 1]$ does not contain a non-trivial pair of disjoint elements. Therefore, it is natural to consider pervasive pre-Riesz spaces, since they contain pairs of positive disjoint elements below any pair of positive disjoint elements in the Riesz completion. We remark that $P[0, 1]$ is indeed not pervasive.

A potential generalisation of the classical result in vector lattice theory then becomes that any linear map between pervasive pre-Riesz spaces is positive and disjointness preserving if and only if it is a Riesz* homomorphism. We contribute to this, currently open, problem with a less general statement. We consider pre-Riesz subspaces of $C(S)$ that have satisfy a property stronger than being pervasive. Recall that $X \subseteq C(S)$ is pervasive whenever for all non-empty and open $U \subseteq S$ there exists an $x \in X_+$ non-zero with $\text{coz}(x) \subseteq U$, where $\text{coz}(x) = \{s \in S: x(s) \neq 0\}$ denotes the co-zero set. We say that X is *pointwise pervasive* in $C(S)$ if for every $s \in S$ and neighborhood U of s there exists an $x \in X_+$ such that $s \in \text{coz}(x) \subseteq U$.

Many ideas in the proof of the following theorem are inspired by [Jar90].

Theorem 2.30. *Suppose S and T are compact Hausdorff spaces and $X \subseteq C(S)$ and $Y \subseteq C(T)$ are pointwise pervasive order dense subspaces. Any linear positive disjointness preserving map $f: X \rightarrow Y$ is a weighted composition map and, in particular, a Riesz* homomorphism.*

Proof. Fix $t \in T$. We consider the functional $\varphi_t: X \rightarrow \mathbb{R}$ defined as the composition of f with the point evaluation at t :

$$\varphi_t(x) = f(x)(t), \quad \forall x \in X.$$

We remark that φ_t is linear, positive, disjointness preserving and, in particular, order bounded. We aim to show that φ_t is given by integration against a point measure to obtain the desired $w(t)$ and $\pi(t)$. Even though we can not immediately construct a measure, we introduce the notion of a support for our functional φ_t . The *support* of φ_t , denoted by $\text{supp}(\varphi_t)$, is the set of all $s \in S$ such that for all neighborhoods U of s there is an $x \in X$ with $\text{coz}(x) \subseteq U$ and $\varphi_t(x) \neq 0$.

Suppose that $t \in T$ is such that $\text{supp}(\varphi_t)$ contains at least two distinct points. Using that S is Hausdorff we obtain disjoint open set $U_1, U_2 \subseteq S$ and, hence, by the pervasive property of X this yields $x_1, x_2 \in X$ with $\text{coz}(x_i) \subseteq U_i$ and $\varphi_t(x_i) \neq 0$ for $i = 1, 2$. However, this yields that $x_1 \perp x_2$ and, hence, contradicts φ_t being disjointness preserving.

We argue that $\text{supp}(\varphi_t) = \emptyset$ if and only if $\varphi_t = 0$. Suppose the former. For all $s \in S$ there exists an open $U_s \subseteq S$ with $s \in U_s$ such that $x \in X$ with $\text{coz}(x) \subseteq U_s$ implies $\varphi_t(x) = 0$. Fix $\epsilon > 0$. As X is pointwise pervasive there exist $x_s \in X$ with $s \in \text{coz}(x_s) \subseteq U_s$. After rescaling if necessary we assume without loss of generality

that $f_s(s) > \epsilon$. We define $V_s := \{t \in S : f_s(t) \geq \epsilon\}$ for all $s \in S$. We remark that the V_s are non-empty open neighborhoods of the $s \in S$, respectively. By compactness of S there exists a finite set $\{s_1, \dots, s_n\} \subseteq S$ such that $S \subseteq V_{s_1} \cup \dots \cup V_{s_n}$. Therefore, $x := x_1 + \dots + x_n \in X$ satisfies $x \geq \epsilon \mathbb{1}_S$. Now for any $y \in X$, letting $\lambda = \|y\|_\infty / \epsilon$, we get

$$0 = -\lambda \sum_{i=1}^n \varphi_t(x_i) = -\lambda \varphi_t(x) \leq \varphi_t(y) \leq \lambda \varphi_t(x) = \lambda \sum_{i=1}^n \varphi_q(x_i) = 0.$$

This show that $\varphi_t = 0$. Conversely, the support of the zero functional is empty. We conclude for each $t \in T$ that $\text{supp}(\varphi_t)$ consists of exactly one point, or is empty in which case $\varphi_t = 0$.

Let $T_N := \{t \in T : \varphi_t = 0\}$ and $T_C := T \setminus T_N$. The map $\pi : T_C \rightarrow S$ that satisfies $\text{supp}(\varphi_t) = \{\pi(t)\}$ is well-defined. We argue that π is continuous. Supposing the converse, yields a net $(t_\alpha)_{\alpha \in A}$ in T_C converging to $t_0 \in T_C$ such that $s_\alpha := \pi(t_\alpha)$ converges to $s_1 \neq s_0 := \pi(t_0)$. By the Hausdorff property of S there exists disjoint open neighborhoods U_0 and U_1 of s_0 and s_1 , respectively. Let $x_0 \in X$ be such that $\text{coz}(x_0) \subseteq U_0$ and $\varphi_{t_0}(x_0) \neq 0$. There is an $\alpha_0 \in A$ such that $s_{\alpha_0} \in U_1$, as $s_\alpha \rightarrow s_1$, and simultaneously that $f(x_0)(s_{\alpha_0}) \neq 0$, as $t_\alpha \rightarrow t_0$ and $f(x_0) \in C(T)$. However, now we can find $x_1 \in X$ with $\text{coz}(x_1) \subseteq U_1$ and $f(x_1)(t_{\alpha_0}) \neq 0$, which by construction contradicts f from being disjointness preserving.

For any $t \in T$ we extend φ_t to a positive linear functional $\tilde{\varphi}_t : C(S) \rightarrow \mathbb{R}$, using the extension theorem of Kantorovich. The Riesz-Markov-Kakutani representation theorem now yields a Borel measure μ_t such that

$$\tilde{\varphi}_t(x) = \int_S x \, d\mu_t, \quad x \in C(S).$$

We aim to show for $t \in T_C$ that μ_t is the Dirac measure at $\pi(t)$. Let $t \in T_C$ be given. It suffices to argue that the support of the measure μ_t , which we denote by $\text{supp}(\mu_t)$, equals the singleton $\{\pi(t)\}$. By construction it is clear that $\pi(t) \in \text{supp}(\mu_t)$. In order to verify the other inclusion we let $s \in \text{supp}(\mu_t) \setminus \{\pi(t)\}$. Let $U \subseteq S$ open with $s \in U$ and $\pi(t) \notin \bar{U}$. Using the pointwise pervasive property of X yields an $x \in X$ such that $s \in \text{coz}(x) \subseteq U$. As $\{s > \frac{1}{2}\}$ is an open neighborhood of s it has strictly positive measure and, hence

$$\int_S x \, d\mu_t \geq \frac{1}{2} \mu_t(\{x > \frac{1}{2}\}) > 0.$$

In particular, this yields $\varphi_t(x) = \int_S x \, d\mu_t > 0$, which contradicts $\pi(t)$ being the unique supporting point of φ_t . We conclude that for all $t \in T_C$ and $x \in X$ we have

$$\int_S x \, d\mu_t = x(\pi(t)) \tilde{\varphi}_t(\mathbb{1}_S).$$

For $t \in T_C$ letting $w(t) := \tilde{\varphi}_t(\mathbb{1}_S) > 0$, we obtain for all $x \in X$

$$\varphi_t(x) = w(t)x(\pi(t)). \tag{2.8}$$

Now we extend π to a map from T to S and put $w(t) := 0$ for all $t \in T_N$. We remark that (2.8) is now satisfied for all $x \in X$ and $t \in T$. As X is majorizing in $C(S)$, X contains a strictly positive function $x \in X$. The fact that $f(x)$ is continuous on T , and $x \circ \pi$ does not vanish, ensures that w is continuous on T .

The last part of the assertion, which states that f is a Riesz* homomorphism, follows from Theorem 2.5(ii). \square