Order isomorphisms, order antimorphisms and their interplay with Jordan algebra structures

Imhoff, H.B. van

Author: Imhoff, H.B. van
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Promotor:
Prof.dr. A. Doelman — Universiteit Leiden

Copromotores:
Dr.ir. O.W. van Gaans — Universiteit Leiden
Dr. B. Lemmens — University of Kent

Promotiecomissie:
Prof.dr. A.W. van der Vaart *(voorzitter)* — Universiteit Leiden
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Order isomorphisms, order antimorphisms and their interplay with Jordan algebra structures

Dedicated to my parents. In loving memory,

Evert van Imhoff and Annelies de Graaf.

Contents

Introduction

A real vector space paired with a partial order that respects the linear structure is called a partially ordered vector space, and in this case the set of positive elements forms a cone. Conversely, any cone in a real vector space gives rise to a unique partial order respecting the linear structure that has this cone as its positive elements. The duality between an analytic structure of a partial order and the convex geometric structure of a cone yields an interesting branch of mathematics. Ordered vector spaces were developed parallel to functional analysis and operator theory from the start of the twentieth century, but appear more sparsely in the literature than normed spaces and topological vector spaces. The main focus of study in the area has been on Riesz spaces. An excellent monograph by Jameson [Jam70] establishes partially ordered vector spaces as a separate theory. More recently, Aliprantis and Tourky [AT07] outlined the theory of partially ordered vector spaces from a contemporary perspective.

Motivating examples

Partially ordered vector spaces and their related cones appear naturally in various fields. For instance, a space consisting of continuous functions on some topological space appears frequently as state space in mathematical models. In the case that these functions are real-valued, the pointwise order leads to a partially ordered vector space. Likewise, the cone of positive definite matrices is often considered, for instance in optimization theory and computational science. Even in Relativity, where the future light cone is modelled as the 3-dimensional Lorentz cone, order structure is present. The theory of partially ordered vector spaces is often used to derive results on monotone dynamical systems and game theory used to model biological, chemical and economic phenomena.

Having a linear partial order is a rather weak structure, and beyond elementary observations not many results hold for general partially ordered vector spaces. It is common to endow a partially ordered vector spaces with additional structure, for instance a lattice structure, geometric properties of the cone or even algebraic structure. In each of these cases it is interesting to understand the relation between this additional structure and to that of the order. More precisely, we consider maps that preserve order related properties and derive what additional structure on the spaces they induce. Below we describe the different settings and the corresponding order preserving maps that we consider. Rigorous definitions and examples of the objects discussed below are supplied in Chapter 1.

Pre-Riesz spaces and their homomorphisms

Among partially ordered vector spaces a widely studied class are the Riesz spaces, or vector lattices. The additional structure they carry is that any pair of vectors has a least upper bound. A rich theory of Riesz spaces and operators between Riesz spaces has been developed in the past century, see [LZ71, dJvR77, Zaa83, Zaa97, AA02]. There are, however, many natural partially ordered vector spaces in functional analysis that are not Riesz space. For instance, a space of operators between Riesz spaces or a tensor product of Riesz spaces generally fails to be a Riesz space itself. The more general concept of a pre-Riesz space has been developed by van Haandel in [vH93]. These pre-Riesz spaces are those partially ordered vector spaces that allow an order dense embedding into a Riesz space. In this case, the smallest Riesz subspace containing the embedded pre-Riesz space is considered the *Riesz completion*. Preceeding the work of van Haandel, a theory for Archimedean spaces is due to Buskes and van Rooij [BvR93]. Many Riesz space concepts, such as disjointness, bands and ideals, have been generalised to the setting of pre-Riesz spaces by Kalauch and van Gaans. A comprehensive overview of the theory on pre-Riesz spaces is given in [vGK18]. Even though pre-Riesz spaces are well understood, their class of corresponding homomorphisms has seen relatively little research. First introduced by van Haandel, the Riesz* homomorphisms are those linear maps between pre-Riesz spaces that extend to a Riesz homomorphism between their Riesz completions. A natural family of questions arises. Given any result concerning Riesz homomorphisms between Riesz spaces, one can ask whether an analogous result holds for Riesz* homomorphisms between pre-Riesz spaces. In Chapter 2, we answer a variety of such questions, mainly in the context of spaces of continuous functions. Examples of pre-Riesz spaces consisting of continuous functions include the space of differentiable functions on a smooth manifold or a Sobolev space on a sufficiently regular domain. Our first notable result, Theorem 2.5, states that a Riesz* homomorphism $f\colon X \to Y$, between order dense subspaces $X \subseteq C(S)$ and $Y \subseteq C(T)$ that separate the points, is given by

$$f(x)(t) = w(t)x(\pi(t)), \quad x \in X, t \in T,$$

for suitable $w\colon T \to \mathbb{R}_+$ and $\pi\colon T \to S$. A similar description of Riesz* homomorphisms is given in Theorem 2.12 for the case where S and T are locally compact, $X \subseteq C_0(S)$ and $Y \subseteq C_0(T)$, by adapting the properties of w and π appropriately. Afterwards, we investigate to what extent the following result generalises; a bijective Riesz homomorphism between Riesz spaces is necessarily a linear order isomorphism and its inverse is again a Riesz homomorphism. We argue that a similar statement holds for bijective Riesz* homomorphisms between pre-Riesz space that are pervasive, see Theorem 2.17, and provide a counterexample for the general case. Finally, we consider the question whether a linear positive disjointness preserving map between pre-Riesz spaces is necessarily a Riesz* homomorphism, another assertion that holds in the Riesz space case.

We provide a counterexample to this statement and exhibit a sufficient condition on the pre-Riesz space under which it is valid.

Automatic linearity of order isomorphisms

After concluding our exploration of lattice structure preserving maps in pre-Riesz spaces, we return to the setting of general partially ordered vector spaces, in Chapter 3. A fundamental problem is to understand the structure of order isomorphims. Here an order isomorphism is an order preserving bijection whose inverse is also order preserving. There are spaces on which all order isomorphisms are linear up to translation. It is an intriguing problem to understand between which partially ordered vector spaces it is the case that all order isomorphism are linear up to translation. Stated in different words, when is the linear structure of a space fully determined by that of the order? Research on this quesiton originates from Special Relativity. During the 1950s and 1960s various results in this area appeared dealing with finite dimensional cones, [AO53, Zee64, Ale67, Rot66]. In the 1970s Noll and Schäffer made numerous contributions to this area in a series of papers, [NS77, NS78, Sch77, Sch78]. Most notably, they showed that an order isomorphism between infinite dimensional cones that are the sum of their engaged extreme rays are linear. In many natural settings, however, such as in operator algebras, their result in not applicable. Molnár considered order isomorphisms on the cone $B(H)_{sa}^{+}$, of positive semidefinite self-adjoint operators on a Hilbert space, and showed in [Mol01] among other things, that all such order isomorphism are linear. We provide a generalisation of Noll and Schäffer's result in Theorem 3.15, that provides a condition on infinite dimensional cones that guarantees that order isomorphism are affine. This condition is sufficiently mild to include Molnár's result. Our approach to obtain this result is to use the fact that order isomorphisms preserve infima and suprema. This allows us to weaken the necessary condition imposed on the cone by Noll and Schäffer to the cone being merely equal to the inf-sup hull of the span of its engaged extreme rays. From the results stated in [NS77] it is not clear, however, how they can be extended with this method. Restricting an order isomorphism to the span of the engaged extreme rays is, a priori, not possible. So we carefully rework the ideas of their proofs to obtain a more general alternative to their result. We generalise their assertion that all order isomorphism are affine provided that the cone is the sum of its engaged extreme rays, to order isomorphims being affine on the sum of the engaged extreme rays of the cone. From here we can extend the set on which the order isomorphism is affine to all elements in the domain that we can reach by taking infima and suprema of positive sums of engaged extreme vectors. In this manner we create a framework to study the linearity of order isomorphisms that encompasses the existing theories. In fact, in Chapter 5 we encounter that the classes of operator algebras called the JB-algebras and JBW-algebras, naturally fit this framework. We elaborate on this after we have introduced the terminology to do so.

Monotone dynamical systems

Order structure also plays a role in a special class of dynamical systems. Economic and biological models often describe competitive or cooperative relations between parties and species. A prototypical example is a Kolmogorov model of cooperating species:

$$\dot{x}_i(t) = x_i(t)F_i\big(x(t)\big), \quad x \in [0,\infty)^n,\ i \in \{1,\dots,n\},$$

where $F\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ is continuously differentiable. The cooperation between the different species is then modelled by the assumption that $\partial F_i/\partial x_j \geq 0$, for all $i \neq j$. Let us assume that solutions of this model exist for all positive time. In this case, the corresponding semiflow $\Phi\colon \mathbb{R}\times\mathbb{R}^n_+ \to \mathbb{R}^n_+$, which for each initial value x describes the solution by $t \mapsto \Phi(t,x)$, is *monotone*, in the sense that $\Phi(t,\cdot)$ is monotone for any $t \geq 0$. These monotone semiflows are the subject of pioneering studies by Hirsch [Hir82, Hir85, Hir88] and numerous subsequent works, see [DH91, HS06, LN12, PT92, Smi95] and references therein. Under suitable additional conditions the generic behaviour of dynamical systems corresponding to monotone semiflows cannot be very complex. The behaviour of their discrete-time counterpart, where the evolution is described by an order preserving map, is not well understood, without further assumption on the map. Recently, however, Hirsch [Hir17] showed that if the order is induced by a polyhedral cone, then the system cannot display chaotic behaviour in the following sense. He showed that if such a monotone dynamical system has a dense set of periodic points, then the whole system is periodic. Furthermore, he conjectured that this result holds for general closed cones in finite dimensional vector spaces. We confirm this conjecture in Chapter 4. During our analysis, we point out a connection between monotone dynamical systems with dense periodic points and the structure of order isomorphisms on intervals.

Symmetric cones and Jordan algebras

An open cone C in a finite dimensional inner product space is considered a *symmetric* cone if it is self-dual ($C = C^*$) and homogeneous (the automorphism group $\mathrm{Aut}(C)$ acts transitively on C). Typical examples of symmetric cones include the cone of real positive definite $n\times n$ matrices and Lorentz cones. A detailed overview of the theory on symmetric cones is given in the book of Faraut and Korányi [FK94]. Symmetric cones have various connections with other fields of mathematics. A famous result, discovered independently by Koecher [Koe57] and Vinberg [Vin60], states that symmetric cones arise precisely as the interior of the cone of squares of a formally real Jordan algebra. Originally introduced by Pascual Jordan (1936), as an attempt to find alternative formal settings for quantum mechanics, a *Jordan algebra* is a real vector space with a bilinear product that is commutative and satisfies the so-called Jordan identity. With the aid of the characterisation of Koecher and Vinberg, one can endow a symmetric cone with a Riemannian metric, making it a prime example of a Riemannian symmetric space. This connection between symmetric cones, formally real Jordan algebras and Riemannian symmetric spaces is outlined in more depth in Section 1.6.

Koecher-Vinberg for JB-algebras

The notion of a formally real Jordan algebra has been generalised to the infinite dimensional setting, by Alfsen, Schultz and Størmer [ASS78], to a Jordan Banach algebra, or JB-algebra for short. One naturally wonders if an analogue of the Koecher-Vinberg characterisation exists in inifite dimensions to describe the geometry of the interior of the cone of squares in a JB-algebra. In general a JB-algebra cannot be realised as an inner-product space, so there is no natural notion of self-duality, nor can one define a Riemannian metric on the interior of the cone of squares. We start by exploring an order theoretic approach to characterise the cone of a JB-algebra. In a JB-algebra $\mathcal{A}$, the inverse map $\iota\colon a \mapsto a^{-1}$ on $\mathcal{A}_+^\circ$, is an antihomogeneous order antimorphism. Recent work by Walsh [Wal13], has shown that a finite dimensional open cone is symmetric if and only if it admits an antihomogeneous order antimorphism. We make the first steps towards extending this order theoretic charcterisation to classes of infinite dimensional JB-algebras. A special class of JB-algebras are spin factors, an infinite dimensional analogue of the Lorentz cone. In Chapter 6, we show that a complete order unit space (V, C, u) is a spin factor if and only if C is strictly convex and C° admits an antihomogeneous order antimorphism.

Infinite dimensional symmetric cones

Alternatively, besides characterising the JB-algebras among the complete order unit spaces in terms of the geometric structure of the cone, we aim to describe the infinite dimensional analogue of a symmetric cone in order theoretic terms. In the setting of a Hilbert space the notion of a symmetric cone of being homogeneous and self-dual is valid. The connection with Jordan algebras is still valid in this setting. Indeed, symmetric cones in a Hilbert space arise as the interior of the cone of squares of a so-called JH-algebra, as in shown in [Chu17]. Here a JH-algebra is both a JB-algebra and a Hilbert space such that the inner product is associative for the product. A JH-algebra is a finite direct sum of factors that are either a formally real Jordan algebra or a spin factor of arbitrary dimension, and in particular the cone of a JH-algebra is the sum of its extreme rays. This last property is what characterises the infinite dimensional symmetric cones among the interiors of cones in a complete order unit space that admit an antihomogeneous order antimorphism, as we show in Theorem 7.16. A key observation, that is used to obtain this characterisation, is that an antihomogeneous order antimorphism on the interior of a cone maps any subcone spanned by finitely many extreme rays surjectively onto another subcone that is spanned by finitely many extreme rays, which allows us to apply the rich structure of finite dimensional symmetric cones. This restriction property is based on the ideas of Noll and Schäffer that we further develop in Chapter 3. Convenient properties that order isomorphisms have concerning their relation to lines that are parallel to extreme rays, are equally shared with order antimorphisms. The interplay between symmetric cones and order antimorphisms are studied in Chapter 7.

Order isomorphism in JB-algebras

Apart from special classes of JB-algebras being able to be characterised with order theoretic terms, the order structure of a JB-algebra is closely related to its algebraic structure. In [H-OS84] much of the theory on C*-algebras and von Neumann algebras has been lifted to the setting of JB-algebras and, their von Neumann analogues, JBW-algebras. Later, Alfsen and Schultz [AS01, AS03] studied JB- and JBW-algebras mainly in the perspective of the geometry of the state spaces. A classic result by Kadison [Kad52, Corollary 5] states that any linear order isomorphisms between C*-algebras, that carries the unit of one algebra onto the unit of the other algebra, is a C^*-isomorphism and, in particular, a Jordan isomorphism on the self-adjoint part. Based on the work of Isidro and Rodrígues-Palacios in [IR-P95], concerning linear isometries between unital JB-algebras, it is shown [LRW] that if unital JB-algebras are linearly isomorphic then they are also Jordan isomorphic. This leads us to the interesting problem of classifying those JB-algebras for which any order isomorphism between cones is linear. Surprisingly, the machinery we develop in Chapter 3, concerning the existence of sufficiently many engaged extreme rays, is of use here. Indeed, a JB-algebra has a linear isometric bipositive embedding into the atomic part of its bidual as a JB-algebra subalgebra. The bidual of a JB-algebra is an example of a JBW-algebra. In a JBW-algebra atoms, or minimal projections, are precisely the extreme vectors of the cone of squares up to scaling. Furthermore, any element in the cone of an atomic JBW-algebra is the supremum of positive linear combinations of pairwise orthogonal atoms, due to the spectral theorem [AS03, Theorem 2.20]. In Proposition 5.9 we show that an atomic JBW-algebra has an algebraic decomposition in a part containing the engaged atoms and a part containing the disengaged atoms, and proceed to describe all order isomorphism between cones of atomic JBW-algebras, in Theorem 5.12. Afterwards, we investigate when an order isomorphism between cones of JB-algebras extends to an order isomorphism between the cones of the atomic parts of the biduals. By a result in [H-OS84], the Jordan analogue of Pedersen's result [Ped72], we show in Proposition 5.16 that it is sufficient to extend to a homeomorphism with respect to a suitable topology on the bidual. In order to subsequently apply our result concerning order isomorphisms between cones of atomic JBW-algebras, we need to guarantee that the bidual of a JB-algebra does not contain a disengaged atom. We classify that this holds exactly for the JB-algebras that do not have a norm closed ideal of codimension one. This leads to the result, Theorem 5.19, that an order isomorphism between cones of JB-algebras, that do not contain any ideals of codimension one, is linear if and only if it extends to a homeomorphism between the atomic parts of the biduals.

List of chapters and related works

- Chapter 1 – *Preliminaries.*
- Chapter 2 – *Lattice structure preserves in $C(S)$*, adaptation of the work
  H. van Imhoff, Riesz* homomorphisms on pre-Riesz spaces consisting of continuous functions, *Positivity.* **22**(2), (2018), 425–447.
- Chapter 3 – *Linearity of order isomorphism*, based on the work
  B. Lemmens, O. van Gaans and H. van Imhoff, On the linearity of order-isomorphisms, arXiv:1904.06393.
- Chapter 4 – *Monotone dynamical systems*, based on the work
  B. Lemmens, O. van Gaans and H. van Imhoff, Monotone dynamical systems with dense periodic points, J. Differ. Equ. **265**(11), 5709–5715.
- Chapter 5 – *Order isomorphisms in JB-algebras*, based on the work
  H. van Imhoff and M. Roelands, Order isomorphisms between cones of JB-algebras, arXiv:1904.09278.
- Chapter 6 – *Order theoretic characterisation of spin factors*, based on the work
  B. Lemmens, M. Roelands, H. van Imhoff, An order theoretic characterization of spin factors, *Q. J. Math.* **68**(3), (2017), 1001–1017.
- Chapter 7 – *Symmetric cones and order antimorphisms.*

Chapter 1

Preliminaries

Partially ordered vector spaces

Recall that a partial ordering $\leq$ on a set X is a binary relation which is reflexive, anti-symmetric, and transitive. That is, the relation $\leq$ satisfies the properties

$$\begin{aligned} &\text{(reflexive)} && x \leq x \\ &\text{(anti-symmetric)} && x \leq y \text{ and } y \leq x \text{ imply } x = y \\ &\text{(transitive)} && x \leq y \text{ and } y \leq z \text{ imply } x \leq z, \end{aligned}$$

for all $x, y, z \in X$. A pair $(X, \leq)$ of a real vector space X and a partial order $\leq$ on X is called a *partially ordered vector space* if the order is stable under *addition* ($x \leq y$ implies $x + z \leq y + z$) and *positive scalar multiplication* ($x \leq y$ and $\alpha \in \mathbb{R}_{\geq 0}$ imply $\alpha x \leq \alpha y$). A subset $C \subseteq X$ is called a *cone* whenever it is closed under addition and positive scalar multiplication ($x, y \in C$ and $\alpha, \beta \geq 0$ imply $\alpha x + \beta y \in C$) and does not contain full lines ($C \cap (-C) = \{0\}$). In a partially ordered vector space $(X, \leq)$ the collection of all *positive elements* $X_+ = \{x \in X : 0 \leq x\}$ is a cone, and is aptly called the *positive cone*. Conversely, a cone C in a real vector space X gives rise to a partial order by defining for $x, y \in X$ that $x \leq_C y$ if and only if $y - x \in C$. The pair $(X, \leq_C)$ is a partially ordered vector space whose positive cone is exactly C. We often denote the order $\leq_C$ induced by a cone simply by $\leq$ if no confusion can arise and (X, C) for the pair $(X, \leq_C)$.

Let (X, C) be a partially ordered vector space. For $x, y \in X$ with $x \leq y$ the set

$$[x, y] := \{z \in X \colon x \leq z \leq y\},$$

is called the *order interval* of x and y. A set $\Omega \subseteq X$ has an *upper bound* $x \in X$ if for all $\omega \in \Omega$ one has $\omega \leq x$. The collection of all upper bounds of Ω is denoted by Ω^u. A vector $x \in X$ is the *least upper bound* or *supremum* of Ω whenever $y \in \Omega^u$ implies $x \leq y$, in this case we write $x = \sup \Omega$. In completely analogous fashion we define to notion of *lower bounds*, the set of all lower bounds of Ω is denoted by Ω^l and the *greatest lower bound* or *infimum* of Ω is denoted by $\inf \Omega$.

A set $\Omega \subseteq X$ is called *majorizing* if for all $x \in X$ there is a $\omega \in \Omega$ with $x \leq \omega$, called *bounded* if there exist $x, y \in X$ such that $x \leq \omega \leq y$ for all $\omega \in \Omega$ and called *upward(downward) directed* whenever for every pair $x, y \in \Omega$ has an upper(lower) bound in Ω. For a linear subspace the notions of upward and downward directed coincide and will simply be called directed. The set X is directed in (X, C) precisely whenever C is *generating* ($X = C - C$). (X, C) is called *Archimedean* if for every $x, y \in C$ with $nx \leq y$ for all $n \in \mathbb{N}$ one has $x \leq 0$.

Example 1.1. In $\mathbb{R}^n$ a cone C is called *polyhedral* if it is the intersection of finitely many half-spaces, that is, there are linear functionals $\varphi_1, \ldots, \varphi_m$ on $\mathbb{R}^n$ such that

$$C = \bigcap_{k=1}^{m} \{x \in \mathbb{R}^n \colon \varphi_k(x) \geq 0\}.$$

The functional $\varphi := \sum_{k=1}^{m} \varphi_k$ is strictly positive, meaning $\varphi(x) > 0$ for all $x \in C \setminus \{0\}$, and the corresponding cross-section $\Lambda := \{x \in C : \varphi(x) = 1\}$ is a polyhedron. Let $\{x_1, \ldots, x_l\}$ be the extreme points of Λ. Then the Krein-Milman theorem implies that Λ is the closed convex hull of its extreme points, and therefore

$$C = \left\{ \sum_{k=1}^{l} \lambda_k x_k \colon \lambda_k \in \mathbb{R}_+ \text{ for all } 1 \leq k \leq l \right\}.$$

In the case that $l = n$ and $x_1, \ldots, x_n$ are linearly independent, we say C is a *simplicial cone* and C is linearly order isomorphic to the n-dimensional standard cone $\mathbb{R}^n_+$ consisting of vectors with all non-negative entries.

Let (X, C) and (Y, K) be partially ordered vector spaces, $U \subseteq X$ and $V \subseteq Y$. A map $f \colon U \to V$ is said to be *order preserving* or *monotone* if $f(x) \leq f(y)$ holds whenever $x \leq y$ and $x, y \in U$. If for all $x, y \in U$ one has $x \leq y$ if and only if $f(x) \leq f(y)$, then f is called an *order embedding*. Remark that an order embedding is necessarily injective; suppose $f(x) = f(y)$ then both $f(x) \leq f(y)$ and $f(x) \geq f(y)$, hence we get $x \leq y$ and $y \leq x$ and the anti-symmetry of the partial order yields $x = y$. An *order isomorphism* is a surjective order embedding. In this case, U and V are called *order isomorphic*. Note that in the sequel, order isomorphism are *never* implicitly assumed to be linear, and we explicitly speak of a linear order isomorphism if we do consider the linear case. In a similar fashion we say that $f \colon U \to V$ is *order reversing* or *antitone* if $f(x) \leq f(y)$ holds whenever $x \geq y$ and $x, y \in U$, and we call f an *order antimorphism* if f is an order reversing bijection whose inverse f^{-1} is also order reversing.

Let X and Y be vector spaces. A map $f \colon X \to Y$ is called *linear* if $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ holds, for all $\lambda, \mu \in \mathbb{R}$ and $x, y \in X$. A map $g \colon X \to Y$ is called *affine* if it is a translation of a linear map, that is, there is an $a \in X$ such that $f \colon X \to Y$ defined by $f(x) := g(x + a) - g(a)$, for $x \in X$, is linear. We similarly define an affine map $f \colon U \to Y$, where U is an affine subspace of X. For a subset $U \subseteq X$, we say that a map $f \colon U \to Y$ is linear or affine if it is the restriction of a linear map $F \colon \mathrm{span}(U) \to Y$ or an affine map $F \colon \mathrm{aff}(U) \to Y$, respectively. We remark that in a directed partially ordered vector space (X, C), one has that $\operatorname{aff}(C) = X$, and so the linear maps on the cone C are precisely those that are restriction from linear map on the whole space.

It is straightforward to verify that a map $f: U \to Y$ is affine if and only if

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i f(x_i),$$

for all $x_1, \ldots, x_n \in U$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \ldots + \lambda_n = 1$ such that $\lambda_1 x_1 + \ldots + \lambda_n x_n$. Moreover, if U is a convex set, then $f: U \to Y$ is affine if and only if f is convex linear, that is, for $x, y \in U$ and $0 \leq \lambda \leq 1$ we have $f(\lambda x + (1-\lambda) y) = \lambda f(x) + (1-\lambda) f(y)$.

Pre-Riesz spaces

A partially ordered vector space $(X, \leq)$ is called a *Riesz space* or a *vector lattice* if for every pair $x, y \in X$ the least upper bound or *supremum* of $\{x, y\}$, denoted by $x \vee y$, exists. This latter condition is equivalent to requiring that each pair $x, y \in X$ has an infimum $x \wedge y$. Let X be a Riesz space. For a vector $x \in X$ we define its positive and negative part by $x_+ = x \vee 0$ and $x_- = (-x) \vee 0$. Both x_+ and x_- are positive and we have $x = x_+ - x_-$. The modulus of $x \in X$ is now defined as $|x| = x_+ + x_-$. A pair of elements $x, y \in X$ is called *disjoint* whenever $|x| \wedge |y| = 0$.

Let X be a linear subspace of a Riesz space Y. Then X is said to be *order dense* in Y whenever

$$y = \inf\{x \in X : y \leq x\}, \quad \text{for all } y \in Y. \tag{1.1}$$

For a majorizing Riesz subspace X of an Archimedean space Y, this property is equivalent to

$$\text{for all } y \in Y_+ \backslash \{0\} \text{ there exists a } x \in X_+ \backslash \{0\} \text{ with } x \leq y. \tag{1.2}$$

In some literature [AB06, p.31], the latter is taken as the definition of being order dense for Riesz subspace. Generally, in the setting of pre-Riesz spaces, which we will introduce next, these notions do not coincide as we will see later in Example 1.8, even for majorizing subspaces. For a partially ordered vector space X a pair (Y, i), consisting of a Riesz space Y and a linear order embedding $i: X \to Y$, is called a *vector lattice cover* of X if $i[X]$ is an order dense subspace of Y. In this case, we say that $i[X]$ *generates* Y *as a Riesz space* if all $y \in Y$ are of the form

$$\bigvee_{i=1}^{n} x_i - \bigvee_{j=1}^{m} y_j,$$

with all x_i and y_j in X.

First introduced by van Haandel in [vH93], the class of partially ordered vector spaces that admit a vector lattice cover that they generated as Riesz space, is defined as follows.

Definition 1.2. A partially ordered vector space (X, C) is called a *pre-Riesz space* if for every $x, y, z \in X$ the inclusion $\{x + z, y + z\}^u \subseteq \{x, y\}^u$ implies $z \in C$.

Note that every Riesz space is a pre-Riesz space. The main property of pre-Riesz spaces is the way they are embedded in a Riesz space, as shown in the following result [vH93, Corollary 4.9].

Theorem 1.3. *A partially ordered vector space X is a pre-Riesz space if and only if there exists a Riesz space Y and a linear order embedding $i : X \to Y$ such that $i[X]$ is order dense in Y and generates Y as a Riesz space. Moreover, all Riesz spaces Y with these properties are linearly order isomorphic.*

The pair (Y, i) is called the *Riesz completion* of X and denoted by (X^ρ, i) or simply X^ρ. Here we consider *the* Riesz completion as though it is unique while it is only unique up to isomorphism. So when we speak of the Riesz completion we actually mean a realization of it. The subsequent result [vH93, Theorem 1.7(ii)] ensures that pre-Riesz space cover a wide class of partially ordered vector spaces.

Proposition 1.4. *Every directed Archimedean partially ordered vector space is a pre-Riesz space, and every pre-Riesz space is directed.*

In a partially ordered vector space X, two elements $x, y \in X$ are defined to be *disjoint*, denoted by $x \perp y$, whenever

$$\{x + y, -x - y\}^u = \{x - y, -x + y\}^u.$$

The intuition of this definition is that the left- and right-hand side of this equality replace the moduli of $|x + y|$ and $|x - y|$. In [vGK18, Proposition 4.1.4] it is shown that two elements in a pre-Riesz space X are disjoint according to this definition if and only if they are disjoint in any vector lattice cover of X.

Let Y be a partially ordered vector space and $X \subseteq Y$ a linear subspace. We say that X is *order dense* in Y if (1.1) is satisfied, and that X is *pervasive* in Y if (1.2) is satisfied. In the case that X is a pre-Riesz space, we say that X is pervasive, if $i[X]$ is pervasive in X^ρ where (X^ρ, i) is the Riesz completion of X. We state the following results [vGK18, Lemma 2.84], [vGK18, Proposition 2.8.5] and [vGK18, Proposition 2.8.8] regarding the pervasive property that we will use in the sequel, for convenience of reference.

Lemma 1.5. *Let E be an Archimedean Riesz space and $X \subseteq E$ a linear subspace. Then X is pervasive in E if and only if for all $y \in E_+$ one has $y = \sup\{x \in X : 0 \leq x \leq y\}$.*

Lemma 1.6. *Let E be an Archimedean Riesz space and $X \subseteq E$ a linear subspace. If X is majorizing and pervasive, then X is order dense in E.*

Proposition 1.7. *A pre-Riesz space X is pervasive if and only if X is pervasive in any vector lattice cover.*

As we will shall see later, pervasive pre-Riesz spaces share many properties with their Riesz completion and, therefore, are an interesting subclass of pre-Riesz spaces when generalising results from Riesz space theory.

Example 1.8. Let S be a compact Hausdorff space and $C(S)$ be the space of all real-valued continuous functions on S. Consider the partial order $C(S)$ defined for $x, y \in C(S)$ by $x \leq y$ if and only if $x(s) \leq y(s)$ for all $s \in S$. Equipped with this order $C(S)$ is an Archimedean Riesz space. Next we consider two natural subspaces of $C(S)$ in the case $S = [0,1]$.

For $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C^k[0,1]$ the subspace of $C[0,1]$ consisting of k-times differentiable functions. Remark that the functions $x, y \in C^k[0,1]$ given by $x(s) = s$ and $y(s) = 1 - s$ for $s \in [0,1]$ do not have a least upper bound in $C^k[0,1]$. By Proposition 1.4, $C^k[0,1]$ is a pre-Riesz space. We will show later, in Theorem 2.25, that $C^k[0,1]$ is pervasive and order dense in $C[0,1]$. Consider the space $P[0,1]$ consisting of all polynomial functions on $[0,1]$. Since $P[0,1]$ is directed and Archimedean, it follows by Proposition 1.4 that $P[0,1]$ is a pre-Riesz space. The Riesz completion of $P[0,1]$ is the space of piece-wise polynomial functions on $[0,1]$. Remark that $P[0,1]$ is not pervasive as any polynomial that vanish on an open set must vanish everywhere. However, $P[0,1]$ is order dense in $C[0,1]$, see for example [vGK18, Example 1.7.1].

We now discuss the natural class of homomorphisms in the setting of pre-Riesz spaces.

Definition 1.9. Let X and Y be Riesz space. A linear map $f\colon X \to Y$ is called a *Riesz (or lattice) homomorphism* if for all $x, y \in X$ one has

$$f(x \vee y) = f(x) \vee f(y).$$

One easily verifies that a Riesz homomorphism also preserves the other lattice operations. In order to generalise results concerning Riesz homomorphisms between Riesz spaces to the setting of pre-Riesz spaces, we are interested in those linear maps between pre-Riesz spaces that extend to a Riesz homomorphism between the Riesz completions of those pre-Riesz spaces. The following notion is first introduced by van Haandel in [vH93, Definition 5.1].

Definition 1.10. Let X and Y be partially ordered vector spaces. A linear map $f\colon X \to Y$ is called a *Riesz* homomorphism* if for every non-empty finite subset $A \subseteq X$ one has

$$f[A^{ul}] \subseteq f[A]^{ul}.$$

Here A^{ul} is the set $(A^u)^l$ of lower bounds of the set of upper bounds of A. In the case that X and Y are Riesz spaces, this definition coincides with f being a Riesz homomorphism. The core property of Riesz* homomorphism between pre-Riesz spaces is described below, and is the content of [vH93, Theorem 5.6].

Theorem 1.11. *A linear map $f\colon X \to Y$ between pre-Riesz spaces extends to a Riesz homomorphism $f^\rho : X^\rho \to Y^\rho$ if and only if f is a Riesz* homomorphism.*

There are many natural questions concerning Riesz* homomorphisms. Indeed, for any statement about Riesz homomorphisms one might wonder whether that same statement is true when replacing all instances of Riesz spaces for pre-Riesz spaces and all instances of Riesz homomorphisms by Riesz* homomorphisms. We shall generalise a variety of classical results concerning Riesz homomorphism between Riesz spaces to Riesz* homormorphism between pre-Riesz spaces consisting of continuous functions in Chapter 2.

Extreme rays of a cone

Let (X, C) be a partially ordered vector space. For any $x \in C\backslash\{0\}$ the *ray* through x or spanned by x is the set $\{\alpha x \colon \alpha \geq 0\}$. A vector $x \in C$ is called an *extreme vector* of C whenever $x \neq 0$ and $0 \leq y \leq x$ implies $y = \alpha x$ for some $\alpha \geq 0$. A vector $x \in X\backslash\{0\}$ is an *extreme vector* if x or $-x$ is an extreme vector of C. We remark that if $x \in C$ is an extreme vector, then for any $\alpha > 0$ the vector $\alpha x \in C$ is also extreme. A ray $R \subseteq C$ is called an *extreme ray* if it is spanned by a positive extreme vector. Since extreme rays play an important role in the sequel, we give alternative descriptions of them in terms of one-dimensional faces and extreme points of a base for the cone.

A non-empty convex subset $F \subseteq C$ is called a *face* of C if $\alpha x + (1-\alpha)y \in F$ and $0 < \alpha < 1$ imply $x, y \in F$. A face $F \subseteq C$ is a subcone. The faces $\{0\}$ and C are considered trivial. A linear functional $\varphi \colon X \to \mathbb{R}$ is called *positive* if $\varphi[C] \subseteq [0, \infty)$. For any positive linear functional $\varphi \colon X \to \mathbb{R}$, $\ker(\varphi) \cap C$ is a face of C. In this terminology the extreme rays of C are exactly the one-dimensional faces, see [AT07, Lemma 1.43].

A convex subset $B \subseteq C\backslash\{0\}$ is called a *base* for C if for each $x \in C\backslash\{0\}$ there exist unique $\alpha \geq 0$ and $y \in B$ such that $x = \alpha y$. In general not every cone has a base. Its existence depends on the presence of a strictly positive functional. A linear functional $\varphi \colon X \to \mathbb{R}$ is *strictly positive* if $\varphi[C\backslash\{0\}] \subseteq (0, \infty)$. It is shown in [AT07, Theorem 1.47] that a cone has a base if and only if it admits a strictly positive functional. Moreover, given a strictly positive $\varphi \colon X \to \mathbb{R}$ then for any $\alpha > 0$ the cross-section $\{x \in C \colon \varphi(x) = \alpha\}$ is a base for C. Suppose that a cone C has a base B. Then a point $x \in B$ is an extreme point of the convex set B if and only if x is an extreme vector of C, see [AT07, Theorem 1.48].

Example 1.12. In $\mathbb{R}^3$ the *Lorentz cone* Λ_3 is defined as

$$\Lambda_3 = \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq z^2\right\}.$$

We remark that Λ_3 has the circular base $B = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \leq 1,\ z = 1\}$. Any point $(x, y, 1) \in B$ with $x^2 + y^2 = 1$ is extreme and, hence all non-trivial faces of Λ_3 are extreme rays. Also, Λ_3 is not a Riesz cone as will follow from Corollaries 1.17 and 1.18.

In general, for a Hilbert space $(H, (\cdot \mid \cdot))$ we define the corresponding *spin factor* as the space $H \oplus \mathbb{R}$ equipped with the Lorentz cone

$$\Lambda_H = \left\{(x, \lambda) \in H \oplus \mathbb{R} \colon \sqrt{(x \mid x)} \leq \lambda\right\}.$$

The cone Λ_H is closed for the norm $\|(x, \lambda)\| = \sqrt{(x \mid x) + \lambda^2}$ on $H \oplus \mathbb{R}$ and, hence by [AT07, Lemma 2.4], Λ_H is an Archimedean cone.

The following elementary property of extreme vectors will be used frequently in the sequel.

Lemma 1.13 (Lemma 1.44 in [AT07]). *Any three nonzero extreme vectors of a cone in a vector space that generate three distinct extremal rays are linearly independent.*

In an Archimedean partially ordered vector space one has that a vector $r \in C$ is extreme if and only if the order interval $[0, r]$ is totally ordered. This observation follows from the following lemma.

Lemma 1.14. *Suppose (X, C) is an Archimedean partially ordered vector space. If $x, y \in X$ are such that $0 \le y \le x$, and for each $0 \le \lambda \le 1$ we have that $y \le \lambda x$ or $\lambda x \le y$, then there exists a $\mu \ge 0$ such that $y = \mu x$.*

Proof. Let $x, y \in X$ be as in the statement. We may assume without loss of generality that x and y are non-zero. Now define $\mu := \sup\{\lambda \in \mathbb{R} : \lambda x \le y\}$. By assumption μ is well-defined and $0 \le \mu \le 1$.

Note that $\mu x \le y$. Indeed, for $n \ge 1$ we have that $(\mu - 1/n)x \le y$, so that $n(\mu x - y) \le x$, which implies that $\mu x \le y$, as (X, C) is Archimedean.

To show that $y \le \mu x$ we distinguish two cases: $0 \le \mu < 1$ and $\mu = 1$. In the case $0 \le \mu < 1$ we have that $y \le (\mu + 1/n)x$ for all n sufficiently large. Thus, $n(y - \mu x) \le x$, which shows that $y \le \mu x$ as the space is Archimedean. If $\mu = 1$, then $x = y$, as $y \le x$ by assumption, and $x = \mu x \le y$ as showed above. □

Extreme rays play a central role in the study of order isomorphisms. The main reason for this is that line segments in a cone that are parallel to an extreme ray can be characterized in a purely order theoretic way [NS78, Proposition 1] in Archimedean spaces. Therefore, order isomorphisms preserve lines in the direction of extreme rays. This will be described in full detail in Chapter 3.

There we encounter a crucial difference between extreme rays that are linearly independent from the other extreme rays and those that are not. We introduce terminology for this property as was originally defined in [NS78].

Definition 1.15. Let $V \subseteq X$ be a collection of vectors. A vector $v \in V$ is called *engaged* in V whenever v lies in the linear span of $V \backslash \{v\}$, and is called *disengaged* otherwise.

We make some remarks on this definition. A ray R in a collection of rays $\mathcal{R}$ is called either engaged or disengaged whenever any $r \in R$ is engaged or disengaged, respectively, in any set of representatives of the rays in $\mathcal{R}$. As is apparent from the definition a ray being engaged depends on the ambient collection of rays it is considered in. Mostly, we need this property only for extreme rays and, hence, we drop the mention of the ambient collection for convenience. This leads us to calling an extreme ray engaged if it is engaged within the collection of all extreme rays of the cone.

Originally the concept of a pervasive pre-Riesz space was introduced in [vGK08] as a sufficient condition such that there is a one-to-one correspondence between the bands of the pre-Riesz space and the bands of its Riesz completion. Here we argue that this condition is also sufficient for a pre-Riesz space to have exactly the same extreme vectors as its Riesz completion.

Proposition 1.16. *Suppose X is a pervasive Archimedean pre-Riesz space with Riesz completion (X^ρ, i). A vector $s \in X^\rho$ is an extreme vector of X^ρ_+ if and only if there exists an extreme vector $r \in X_+$ such that $i(r) = s$.*

Proof. Let $r \in X_+$ be an extreme vector. Suppose $y \in X^\rho$ satisfies $0 < y \leq i(r)$. For every $x \in X$ with $0 \leq i(x) \leq y$ we have $0 \leq i(x) \leq i(r)$ so $0 \leq x \leq r$. Hence, there exists $\alpha_x \in \mathbb{R}$ such that $x = \alpha_x r$. By Lemma 1.5 we have

$$y = \sup\{i(x)\colon x \in X,\ 0 \leq i(x) \leq y\} = \sup\{\alpha_x i(r)\colon x \in X,\ 0 \leq i(x) \leq y\}.$$

Since X is Archimedean, we have that X^ρ is Archimedean by [vGK18, Proposition 2.4.12], and it follows that y is a scalar multiple of $i(r)$. Thus, $i(r)$ is an extreme vector of X^ρ_+.

Conversely, suppose that $s \in X^\rho_+$ is an extreme vector. Since X is pervasive, there exists an $x \in X_+$ with $0 < i(x) \leq s$. Then $i(x) = \alpha s$ for some $\alpha > 0$. Take $r := \alpha^{-1}x$. Then $i(r) = s$ and if $0 \leq v \leq r$, then $0 \leq \alpha i(v) \leq s$ so there is a $\beta \geq 0$ with $\alpha i(v) = \beta s$, and hence $v = \beta\alpha^{-1} r$. This shows that r is an extreme vector of X_+. □

We exhibit interesting corollaries for pervasive pre-Riesz spaces concerning their extreme vectors.

Corollary 1.17. *The extreme rays of the cone in a pervasive pre-Riesz space are pairwise disjoint and disengaged.*

Proof. Let X be a pervasive pre-Riesz space and $r, s \in X$ be linearly independent extreme vectors. By Proposition 1.16 we have that $i(r)$ and $i(s)$ are extreme vectors of X^ρ_+. There the infimum $i(r) \wedge i(s)$ exists and must equal zero. Hence $i(r)$ and $i(s)$ are disjoint and by [vGK18, Proposition 4.1.4], the vectors r and s are disjoint in X. Suppose now that $r \in X$ is an engaged extreme vector and let $r_1, \ldots, r_n \in X$ be extreme with $r = \sum_{i=1}^n r_i$. From $r \perp r_i$ for $i = 1, \ldots, n$ we get

$$r = r \wedge r = r \wedge \sum_{i=1}^n r_i = \sum_{i=1}^n r \wedge r_i = 0.$$

We conclude that extreme rays in a pervasive pre-Riesz space are disengaged. □

Corollary 1.18. *A finite dimensional Archimedean pre-Riesz space X is pervasive if and only if it is a vector lattice.*

Proof. Let (X, C) be a finite dimensional Archimedean pre-Riesz space and $d = \dim\ X$. By the Krein-Milman theorem C equals the convex hull of its extreme rays. As C is generating in X it has at least d extreme rays. By Proposition 1.16 all extreme rays of C are disengaged. Therefore, C is spanned by exactly d extreme rays, say $v_1, ..., v_d$. The basis transformation from (v_i) to the standard basis of $\mathbb{R}^d$ is now a linear order isomorphism from (X, C) to $\mathbb{R}^d$ equipped with coordinate-wise ordering. We conclude that (X, C) is a vector lattice. Conversely, for a vector lattice X we have $X = X^\rho$. It immediately follows that X is pervasive in X^ρ and, hence, X is pervasive. $\square$

The statement of Proposition 1.16 fails to hold whenever X is not pervasive. We illustrate this we a counterexample.

Example 1.19. Consider the space $X = (c, K)$ where c is the space of all real-valued convergent sequences and K is the cone given by $K = \{x \in c^+ : \varphi(x) \geq 0\}$, where $\varphi : X \to \mathbb{R}$ is the functional given by

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n} - \lim_{n\to\infty} x_n, \qquad x = \{x_n\} \in X.$$

It is shown [vGK08, Example 2.2] that X is an Archimedean pre-Riesz space whose Riesz completion equals $X^\rho = c \times \mathbb{R}$ equipped with the pointwise order on c. Moreover, it is shown that X is not pervasive. We argue that X^ρ has strictly more extreme vectors than X.

Remark that for any standard unit vector $e_n \in X$ we have $\varphi(e_n) = 2^{-n}$ and, therefore, $e_n \in K$ holds for all $n \in \mathbb{N}$. Moreover, as K is contained in c^+, we find that e_n is an extreme vector in X. We argue that all extreme vectors of X arise in this way. Suppose that $a = \{a_n\} \in K$ is an extreme vector with $a_n, a_m \neq 0$ and $m > n$. Let us define $\alpha := a_n/(a_n + 2^{n-m}a_m)$ and construct a sequence $\tilde{a} \in c$ by

$$\tilde{a}_k := \begin{cases} \alpha a_k & \text{if} \quad k \neq m, n; \\ 0 & \text{if} \quad k = m; \\ a_k & \text{if} \quad k = n. \end{cases}$$

We observe that $0 < \alpha < 1$ since a_n and a_m are strictly positive. Therefore, $0 \leq_c \tilde{a} \leq_c a$ holds. We verify that $\varphi(\tilde{a}) = \alpha\varphi(a)$ holds. That conclusion yields $0 \leq_K \tilde{a} \leq_K a$ which contradicts our assumption that $a \in (c, K)$ is an extreme vector. We compute

$$\varphi(\tilde{a}) = \sum_{k=1}^{\infty} \frac{\tilde{a}_k}{2^k} - \lim_{k\to\infty} \tilde{a}_k = \frac{a_n}{2^n} + \sum_{k\neq n,m} \frac{\alpha a_k}{2^k} - \alpha \lim_{k\to\infty} a_k,$$

so that

$$\begin{aligned} \varphi(\tilde{a}) - \alpha\varphi(a) &= 2^{-n}a_n - \alpha(2^{-n}a_n + 2^{-m}a_m) \\ &= 2^{-n}\alpha(a_n + 2^{n-m}a_m)a_n - 2^{-n}\alpha(a_n + 2^{n-m}a_m) = 0. \end{aligned}$$

In conclusion, the extreme vectors of $X = (c, K)$ are exactly the standard unit vectors. However, the Riesz completion (X^ρ, i), which as mentioned earlier equals $c \times \mathbb{R}$ with pointwise order, has an extreme vector $(0, 1) \in c \times \mathbb{R}$, which is not contained in the image $i[X]$.

In an Archimedean Riesz space the notion of an extreme vector coincides with the notion of a discrete point, introduced in [dJvR77]. Here a vector $x \in C$ is called an *discrete point* if for all $0 \leq y, z \leq x$, $y \perp z$ implies $y = 0$ or $z = 0$. This definition is also valid in a general partially ordered vector space. Here any extreme vector is a discrete point, but the converse generally fails to hold. A counterexample of the latter is given in Example 1.21. We encounter that the situation is again more comparable to the Riesz space case when we consider a pervasive pre-Riesz space.

Proposition 1.20. *Let X be an Archimedean pervasive pre-Riesz space, then the following assertions hold:*

(i) $x \in X^\rho$ is a discrete point if and only if there exists a discrete point $y \in X$ such that $i(y) = x$;

(ii) $x \in X$ is an extreme vector if and only if x is a discrete point.

Proof. Let $x \in X$ be a discrete point and suppose $i(x) \in X^\rho$ is not a discrete point. Then there exist $y, z \in X^\rho$ with $0 \leq y, z \leq i(x)$, $y \perp z$ and $y, z \neq 0$. Due to X being pervasive there exist non-zero $v, w \in X$ such that $0 \leq i(v) \leq y$ and $0 \leq i(w) \leq z$. We remark that $i(v) \perp i(w)$. In particular, $v \perp w$ in X. Moreover, we get $0 \leq v, w \leq x$ and by the assumption that x is discrete this yields $v = 0$ or $w = 0$. This yields a contradiction and shows that $i(x) \in X^\rho$ is discrete. Conversely, suppose $y \in X^\rho$ is a discrete point. As X^ρ is an Archimedean Riesz space, y is an extreme vector of X^ρ. By Proposition 1.16 we get an extreme vector $x \in X$ such that $i(x) = y$. Invoking that generally extreme vectors are discrete points yields (i).

Combining (i) with Proposition 1.16 yields the chain of equivalences, $x \in X_+$ is extreme if and only if $i(x) \in X_+^\rho$ is extreme if and only if $i(x) \in X_+^\rho$ is discrete if and only if $x \in X_+$ is discrete, and assertion (ii) follows. □

Example 1.21. Consider the polyhedral cone C in $\mathbb{R}^3$ generated by the vectors $v_1 = (1, 0, 1)$, $v_2 = (0, 1, 1)$, $v_3(-1, 0, 1)$ and $v_4 = (0, -1, 1)$. Or equivalently, C is the cone with a square base whose corners are the points $v_1, \ldots, v_4$. The extreme rays of C are then exactly the rays spanned by the v_i. In particular, all extreme rays of C are engaged. It is shown [vGK06, Example 4.6] that the only pairs of distinct positive disjoint elements in C are $v_1 \perp v_3$ and $v_2 \perp v_4$ up to positive scaling. Consequently, one easily verifies that any vector in the boundary of C is a discrete point.

Order unit spaces

Let (X, C) be a partially ordered vector space. A vector $u \in C$ is called an *order unit* if for all $x \in X$ there exists a $\alpha \geq 0$ such that $-\alpha u \leq x \leq \alpha u$. In this case, the formula

$$\|x\|_u = \inf\{\alpha \geq 0 : -\alpha u \leq x \leq \alpha u\},$$

defines a semi-norm, which is a norm whenever (X, C) is Archimedean. A triple (X, C, u) of an Archimedean partially ordered vector space (X, C) and $u \in C$ an order unit is called an *order unit space*. The norm $\|.\|_u$ is called the *order unit norm*. With respect to the order unit norm C is closed by [AT07, Theorem 2.55(2)]. We denote the interior of C with respect to $\|.\|_u$ by C°. It follows from [AT07, Lemma 2.5] that the set of all order units of C coincides with C°. In particular, this shows that C° is non-empty. We remark that the order unit norm is monotone, in the sense that for $x, y \in C$ with $x \leq y$ we have $\|x\|_u \leq \|y\|_u$. Constructing a similar norm for another order unit $v \in C$ yields a norm $\|.\|_v$ equivalent to $\|.\|_u$. A cone C in an order unit space is called *strictly convex* if for every linearly independent $x, y \in \partial C$ the line segment $\{tx + (1-t)y \colon t \in (0,1)\}$ is contained in C°.

Let (X, C, u) be an order unit space. A linear functional $\varphi\colon X \to \mathbb{R}$ is called *positive* whenever $\varphi[C] \subseteq [0, \infty)$ and *strictly positive* whenever $\varphi[C \backslash \{0\}] \subseteq (0, \infty)$. A positive linear functional $\varphi\colon X \to \mathbb{R}$ is called a *state* on X if $\varphi(u) = 1$. The collection of all states on X is called the *state space* of X and is denoted by $S(X)$. We remark that any state is bounded and has norm $\|\varphi\| = 1$. In particular, the state space $S(X)$ is a weak*-closed subset of the closed unit ball B_{X^*} in the norm dual X^*. By the Banach-Alaoglu theorem $S(X)$ is weak*-compact. The Krein-Milman theorem implies that $S(X)$ is the closed convex hull of its extreme points. We refer to the extreme points of the state space as *pure states* of X. It is shown in [AS01, Lemma 1.18] that the state space $S(X)$ of an order unit space (X, C, u) determines both the order and norm in the following sense. For a vector $x \in X$ we have $x \in C$ if and only if $\varphi(x) \geq 0$ for all $\varphi \in S(X)$, and $\|x\|_u = \sup\{|\varphi(x)| \colon \varphi \in S(X)\}$.

Example 1.22. For a compact Hausdorff space S the partially ordered vector space $C(S)$ with pointwise order is an order unit space. Any strictly positive function in $C(S)_+$ is an order unit. The order unit norm corresponding to the constant one functions coincides with the supremum norm $\|.\|_\infty$ on $C(S)$. Now let S instead by a locally compact Hausdorff space and consider the vector space $C_0(S)$ of continuous functions on S that vanish at infinity, in other words, all functions $f \in C(S)$ for which $\{s \in S \colon |f(s)| \geq \alpha\}$ is compact for any $\alpha > 0$. Endowed with the pointwise order $C_0(S)$ has no order unit in general.

Let Ω be the weak*-closure of the pure states of X. The canonical embedding $i\colon X \to C(\Omega)$ defined by $i(x)(\varphi) = \varphi(x)$ for all $x \in X$ and $\varphi \in \Omega$ is a linear order embedding. It is shown in [vGKL14, Theorem 10] that this embedding has nice properties.

Proposition 1.23. *Let (X, C, u) be an order unit space. There exists a compact Hausdorff space Ω and a linear order embedding $i\colon X \to C(\Omega)$ such that $i[X]$ separates the points of Ω, contains the constant functions and is order dense in $C(\Omega)$. In particular, we can take for Ω the weak*-closure of the pure states of X.*

Existence of an order unit in a partially ordered vector space implies that the space is directed. Therefore, as order unit spaces are defined to be Archimedean, Proposition 1.4 yields that order unit spaces are always pre-Riesz spaces. Proposition 1.23 tells us that we can obtain the Riesz completion of an order unit space by generating the Riesz subspace in $C(\Omega)$ generated by $i[X]$.

Most results in Chapter 2 apply to order dense subspace of $C(\Omega)$ for some compact

Hausdorff space Ω. Proposition 1.23 shows that, in particular, all those results apply to order unit spaces.

Metrics and geodesics

Useful tools in the analysis of cones in order unit space and maps between such cones are Hilbert's and Thompson's metrics. In an order unit space (X, C, u) they are defined on C° in terms of the following function. For $x \in C$ and $y \in C^\circ$ let

$$M(x/y) := \inf\{\beta > 0 \colon x \le \beta y\}.$$

In some literature this map is called a *gauge* of the open cone C°, see for example [NS77]. Note that $0 \le M(x/y) < \infty$ for all $x \in C$ and $y \in C^\circ$, with strict positivity whenever $x \in C^\circ$. Moreover, $M(\sigma x/\mu y) = \frac{\sigma}{\mu} M(x/y)$ for all $\sigma, \mu > 0$ and $x \in C$ and $y \in C^\circ$.

Recall that $S(X)$ denotes the state space of X and determines the order. Therefore, as $x \le \beta y$ is equivalent to $\varphi(x) \le \beta\varphi(y)$ for all $\varphi \in S(X)$, we get that

$$M(x/y) = \max_{\varphi \in S(X)} \frac{\varphi(x)}{\varphi(y)} \quad \text{for all } x, y \in C^\circ. \tag{1.3}$$

Example 1.24. Let $n \in \mathbb{N}$ and consider the partially ordered vector space $\mathrm{Sym}_n(\mathbb{R})$ of symmetric $n \times n$-matrices with real entries, ordered by the cone of positive semidefinite matrices

$$\begin{aligned} \mathrm{Sym}_n^+(\mathbb{R}) &= \{A \in \mathrm{Sym}_n(\mathbb{R}) \colon \langle Ax, x\rangle \ge 0, \text{ for all } x \in \mathbb{R}^n\} \\ &= \{A \in \mathrm{Sym}_n(\mathbb{R}) \colon \sigma(A) \subseteq [0, \infty)\}. \end{aligned}$$

For $n = 2$ this space is linearly order isomorphic to $(\mathbb{R}^3, \Lambda_3)$, see [vGK18, Example 1.7.4]. The identity matrix I is an order unit for $\mathrm{Sym}_n(\mathbb{R})$. A base for the cone is given by

$$\{A \in \mathrm{Sym}_n^+(\mathbb{R}) \colon \mathrm{tr}(A) = 1\}$$

and, in particular, the cone is Archimedean. The order unit norm induced by I coincides with the trace norm. Let $\mathrm{Sym}_n^+(\mathbb{R})^\circ$ denote the interior with respect to this norm, and remark that these are exactly the positive definite matrices. We can compute the M-function in this case directly. For all $A, B \in \mathrm{Sym}_n^+(\mathbb{R})^\circ$ we get

$$\begin{aligned} M(A/B) &= \inf\{\beta > 0 \colon A \le \beta B\} \\ &= \inf\{\beta > 0 \colon \sigma(\beta B - A) \subseteq [0, \infty)\} \\ &= \inf\{\beta > 0 \colon \sigma(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) \subseteq [0, \beta)\} \\ &= \max(\sigma(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})) \\ &= \max(\sigma(AB^{-1})). \end{aligned}$$

Let (X, C, u) be an order unit space. Now *Hilbert's metric* on C° is defined by

$$d_H(x, y) := \log M(x/y) + \log M(y/x),$$

and *Thompson's metric* on C° is given by

$$d_T(x, y) := \max\{\log M(x/y), \log M(y/x)\}$$

for $x, y \in C^\circ$. Note that $d_H(\sigma x, \mu y) = d_H(x, y)$ for all $x, y \in C^\circ$ and $\sigma, \mu > 0$. So, d_H is not a metric on C°. However, for cones in an order unit space it is known [LN12, Chapter 2] that d_H is a metric between pairs of rays in C°, as $d_H(x, y) = 0$ if and only if $x = \lambda y$ for some $\lambda > 0$ in that case. Thompson's metric is a metric on C° in an order unit space. Moreover, its topology coincides with the order unit norm topology on C°, see [LN12, Chapter 2].

A map $f\colon X \to Y$ between vector spaces is *homogeneous of degree* α if for all $x \in X$ and $\lambda \in \mathbb{R}$ one has $f(\lambda x) = \lambda^\alpha f(x)$. In the case $\alpha = 1$ we say that f is *homogeneous*, and in the case $\alpha = -1$ we say that f is *antihomogeneous*. Let (X, C, u) and (Y, K, v) be order unit spaces. A map $f\colon C^\circ \to K^\circ$ is called *gauge-preserving* if for all $x, y \in C^\circ$ one has

$$M(x/y) = M(f(x)/f(y)),$$

where the gauges are computed in their respective cones C° and K°. Similarly, we say that f is *gauge-reversing* whenever for all $x, y \in C^\circ$ one has

$$M(x/y) = M(f(y)/f(x)).$$

It is shown [NS77, Proposition 7.2] that a map $f\colon C^\circ \to K^\circ$ is gauge-preserving if and only if f is homogeneneous and monotone. Recall that f is monotone whenever $x \leq y$ implies $f(x) \leq f(y)$. Moreover, it is shown [NS77, Proposition 7.3] that f is gauge-reversing if and only if f is antihomogeneous and antitone. Here f is antitone whenever $x \leq y$ implies $f(y) \leq f(x)$. From these results it is easy to see that homogeneous order isomorphisms and antihomogeneous order antimorphisms between interiors of cones are always isometries for Thompson's metric. Another notable result [Sch78, Theorem B] states that any gauge-preserving bijection is linear.

Recall that given a metric space (X, d_X) a *geodesic path* $\gamma\colon I \to X$, where $I \subseteq \mathbb{R}$ is a possibly unbounded interval, is a map such that

$$d_X(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in I.$$

The image $\gamma[I]$ is simply called a *geodesic*, and $\gamma[\mathbb{R}]$ is said to be a *geodesic line* in (X, d_X).

We proceed to recall a few facts about geodesics for Thompson's metric from [LR15, Section 2]. If $x \in (C^\circ, d_T)$, then there are two special types of geodesic lines through x. There are the so-called *type I geodesic lines* γ, which are the images of the geodesic paths,

$$\gamma(t) := e^t r + e^{-t} s \quad \text{for } t \in \mathbb{R}, \tag{1.4}$$

with $r, s \in \partial C$ and $r + s = x$. The *type II geodesic line* μ through x is the image of the geodesic path $\mu(t) := e^t x$ with $t \in \mathbb{R}$. The type I geodesics γ have the property that $M(u/v) = M(v/u)$ for all u and v on γ, and the type II geodesics γ have the property that $M(u/v) = M(v/u)^{-1}$ for all u and v on γ.

A geodesic line γ is called *unique* if for each x and y on γ we have that γ is the only geodesic line through x and y in (X, d_X). Each unique geodesic line in (C°, d_T) is either of type I or type II. Moreover, the type II geodesic is always unique [LR15, Proposition 4.1], but the type I geodesics may not be unique. However, if C is strictly convex, then all type I geodesic lines are unique, see [LR15, Theorem 4.3].

Symmetric cones

Let $(V, (\cdot \mid \cdot))$ be a finite dimensional real inner product space and C a closed cone in V. Then the interior C° is called a *symmetric cone* if it is self-dual, that is

$$C^\circ = \{v \in V : (v \mid x) > 0 \text{ for all } x \in C \backslash \{0\}\},$$

and homogeneous, meaning the group $\mathrm{Aut}(C)$ of linear automorphisms acts transitively on C°. Symmetric cones arise precisely as the interiors of cones of squares in a formally real Jordan algebra. This celebrated result was independently discovered by Koecher [Koe57] and Vinberg [Vin60]. Euclidean Jordan algebras were originally introduced by Jordan, who gave a complete characterisation of them in work with Wigner and von Neumann [JvNW34]. Beyond quantum mechanics Jordan algebras turned out to have deep connections with diverse areas of mathematics including Lie theory, differential geometry and mathematical analysis, see for example [Hel62, Koe62, Jac71, McC78].

A *real Jordan algebra* is a real vector space $\mathcal{A}$ with a bilinear product $(a, b) \mapsto a \circ b$ which is commutative, that is $a \circ b = b \circ a$ for all $a, b \in \mathcal{A}$, and satisfies the so-called *Jordan identity*

$$a^2 \circ (a \circ b) = a \circ (a^2 \circ b).$$

In general, the Jordan product $\circ$ is not associative. Throughout this section we consider *finite dimensional* Jordan algebras with an *algebraic unit*, denoted by e. A Jordan algebra $\mathcal{A}$ is said to be *formally real* if $a^2 + b^2 = 0$ implies $a = 0$ and $b = 0$, for all $a, b \in \mathcal{A}$. The set of squares $\mathcal{A}_+ = \{a^2 : a \in \mathcal{A}\}$ is a closed cone in $\mathcal{A}$. The *spectrum* of $a \in \mathcal{A}$ is given by $\sigma(a) = \{\lambda \in \mathbb{R} : a - \lambda e \text{ is not invertible}\}$. An element $c \in \mathcal{A}$ is called an *idempotent* if $c^2 = c$ is satisfied, and is said to be a *primitive* idempotent if it cannot be written as the sum of two non-zero idempotents. A set $\{c_1, \ldots, c_n\} \subseteq \mathcal{A}$ of primitive idempotents is called a *Jordan frame* if $c_i \circ c_j = 0$ holds for all $i \neq j$, and $c_1 + \ldots + c_n = e$. The Spectral Theorem [FK94, Theorem III.1.2] says that for each $a \in \mathcal{A}$ there exists a Jordan frame $\{c_1, \ldots, c_n\}$ and uniquely determined real numbers $\lambda_1, \ldots, \lambda_n$ such that $a = \lambda_1 c_1 + \ldots + \lambda_n c_n$. In fact, $\sigma(a) = \{\lambda_1, \ldots, \lambda_n\}$. We remark that some of the λ_i may be equal. Using this fact it can be shown that the interior of the cone of squares $\mathcal{A}_+$ is given by

$$\mathcal{A}_+^\circ = \{a \in \mathcal{A} : \sigma(a) \subseteq (0, \infty)\} = \{a^2 : a \in \mathcal{A} \text{ invertible}\}.$$

By the Koecher-Vinberg theorem this cone is symmetric and all finite dimensional symmetric cones arise this way. We illustrate this connection with a prime example of a formally real Jordan algebra.

Example 1.25. Let $n \in \mathbb{N}$ be given. Consider the space $V = \mathrm{Sym}_n(\mathbb{R})$ as before, endowed with the inner product $(A \mid B) = \mathrm{tr}(AB)$. The product defined by

$$A \circ B = \frac{1}{2}(AB + BA),$$

for all $A, B \in V$, turns V into a formally real Jordan algebra with algebraic unit $e = I_n$. Remark that the cone of squares and its interior are exactly the positive semidefinite matrices and positive definite matrices, respectively, as in Example 1.24. We verify that indeed

$$V_+^\circ = \{A^2 \colon A \in \mathrm{Sym}_n(\mathbb{R}) \text{ invertible}\},$$

is a symmetric cone.

We remark that for $B \in V_+$, trace$(AB) > 0$ holds if and only if $\sigma(AB) \subseteq (0, \infty)$. This is guaranteed for all $B \in V_+ \backslash \{0\}$ exactly when $\sigma(A) \subseteq (0, \infty)$. Therefore,

$$V_+^\circ = \{A \in V \colon (A \mid B) > 0 \text{ for all } B \in V_+ \backslash \{0\}\},$$

and V_+° is self-dual. Next consider the *quadratic representation* $Q_B \colon V \to V$, for some $B \in V$, given by $Q_B(A) = BAB$. For $B \in V$ invertible Q_B is a linear order isomorphism, see for example [FK94, Proposition III.2.2]. Now suppose $A, B \in V_+^\circ$ are given. Then the composition $S := Q_{B^{1/2}} \circ Q_{A^{-1/2}}$ is an automorphism of V and satisfies

$$S(A) = Q_{B^{1/2}}(I) = B.$$

This shows that V_+° is a homogeneous cone.

We remark that, in general, the quadratic representation $Q_a \colon \mathcal{A} \to \mathcal{A}$ of an element $a \in \mathcal{A}$ is given by

$$Q_a(b) = 2(a \circ (a \circ b)) - a^2 \circ b,$$

for all $b \in \mathcal{A}$, and the result [FK94, Proposition III.2.2], which states that Q_a is a linear order isomorphism for all $a \in A$ invertible, is valid. In the case exhibited in Example 1.25 where $\mathcal{A} = \mathrm{Sym}_n(\mathbb{R})$, this yields $Q_A(B) = BAB$, for all $A, B \in \mathcal{A}$.

This characterisation of symmetric cones as the interior of the cone of squares in a formally real Jordan algebra, provides a connection with the geometry of real manifolds. Indeed, symmetric cones are prime examples of Riemannian symmetric spaces. To illustrate this, consider a finite dimensional symmetric cone C and let $\mathcal{A}$ be the formally real Jordan algebra such that $\mathcal{A}_+^\circ = C^\circ$. We can endow C° with a Riemannian metric,

$$\begin{aligned}\delta(a, b) &= \| \log Q_{a^{-1/2}}\, b \|_2 \\ &= \sqrt{\sum_{i=1}^{k} \log^2 \lambda_i(Q_{a^{-1/2}}\, b)},\end{aligned}$$

where the $\lambda_i(Q_{a^{-1/2}}\ b)$ are the eigenvalues of $Q_{a^{-1/2}}\ b$, including multiplicities. It can be shown that δ is a length metric. Indeed, for $a, b \in C^\circ$

$$\delta(a,b) = \inf_\gamma L(\gamma),$$

where the infimum is taken over all piece-wise smooth paths $\gamma\colon [s,t] \to C^\circ$ from a to b, and the length of γ is given by

$$L(\gamma) = \int_s^t \|Q_{\gamma(t)^{-1/2}}\gamma'(t)\|_2 \ \mathrm{d}t.$$

The Riemannian manifold (C°, δ) is a *symmetric space*, in the sense that, for all $a \in C^\circ$ the map $S_a\colon C^\circ \to C^\circ$ given by

$$S_a(b) = Q_a b^{-1}, \quad \text{for } b \in C^\circ,$$

is a *symmetry* at a, i.e., a δ-isometry that is an involution and has a as an isolated fixed point.

Yet another characterization of symmetric cones is given by Walsh. In a formally real Jordan algebra $\mathcal{A}$ the inversion map $\iota\colon A_+^\circ \to A_+^\circ$ given by $\iota(a) = a^{-1}$ is a gauge-reversing bijection, or in other words an antihomogeneous order antimorphism. In [Wal13] Walsh showed that the interior of a finite dimensional cone is symmetric if and only if it admits a gauge-reversing bijection.

Whether any of these characterisation results hold in setting of infinite dimensional space is presently largely unknown. In Chapters 6 and 7, we make pioneering steps towards developing infinite dimensional analogs of these characterisation, from the order theoretic perspective.

Jordan Banach algebras

An infinite dimensional generalisation of the Euclidean Jordan algebra is due to Alfsen, Schultz and Størmer [ASS78], which are called the Jordan Banach algebras, or JB-algebras for short. A *Jordan algebra* $(A, \circ)$ is a commutative, not necessarily associative algebra such that

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \quad \text{for all } x, y \in A.$$

A *JB-algebra* A is a normed, complete real Jordan algebra satisfying,

$$\begin{aligned} \|x \circ y\| &\leq \|x\|\,\|y\|\,, \\ \left\|x^2\right\| &= \|x\|^2\,, \\ \left\|x^2\right\| &\leq \left\|x^2 + y^2\right\| \end{aligned}$$

for all $x, y \in A$. In finite dimensions this definition coincides with the formally real Jordan algebras by the last property. An important class of examples of JB-algebras are the self-adjoint parts of C^*-algebras equipped with the Jordan product $x \circ y := (xy+yx)/2$. A triple (A, A_+, e) of a JB-algebra A, its cone of squares A_+ and a unit e is a complete order unit space, where e is an order unit and the JB-algebra norm on A coincides with the order unit norm induced by e.

Example 1.26. Recall from Example 1.12 that for a Hilbert space $(H, (\cdot \mid \cdot))$ we define the corresponding *spin factor* as the space $H \oplus \mathbb{R}$ equipped with the Lorentz cone

$$\Lambda_H = \left\{ (x, \lambda) \in H \oplus \mathbb{R} \colon \sqrt{(x \mid x)} \leq \lambda \right\}.$$

The cone Λ_H arises as the cone of squares for the Jordan product defined by

$$(x, \lambda) \circ (y, \mu) = (\mu x + \lambda y, (x \mid y) + \lambda\mu),$$

that turns $H \oplus \mathbb{R}$ into a JB-algebra for the norm $\|(x, \lambda)\| := \|x\|_H + |\lambda|$, where $\|.\|_H$ is the norm on H. In this case, $(0, 1)$ is the algebraic unit. In Chapter 6, we characterise spin factors as the complete order unit spaces (V, C, u) with $\dim V \geq 3$ and C a strictly convex that admits an antihomogeneous order antimorphism on C°.

A *JBW-algebra* is the Jordan analogue of a von Neumann algebra; a JB-algebra that is a dual space. An element p of a JBW-algebra M is called a *projection* whenever $p^2 = p$. A non-zero projection that does not dominate another non-zero projection is called an *atom*. A JBW-algebra in which every non-zero projection dominates an atom is considered *atomic*.

We remark that the spin factor, as in Example 1.26, is self-dual as JB-algebra. In particular, a spin factor is a JBW-algebra. Another prime example of a JBW-algebra is given below.

Example 1.27. Let H be a Hilbert space and consider the space $B(H)$ of bounded linear operators on H. The self-adjoint operators $B(H)_{sa}$ now form a real vector space. Endowing this space with the Jordan product given by

$$T \circ S = \frac{1}{2}(TS + ST),$$

for $T, S \in B(H)_{sa}$, where TS and ST denote compositions, yields a JBW-algebra for the operator norm. Projections with respect to $\circ$ are exactly the orthogonal projections onto a closed subspace of H. Therefore, the atoms of $B(H)_{sa}$ are precisely the orthogonal projections of rank one, which shows that $B(H)_{sa}$ is atomic. We remark that in the cone $B(H)_{sa}^+$ of positive semi-definite operators the projections of rank one are precisely the extreme vectors.

The connection exhibited in $B(H)_{sa}$, as well as in Example 1.27, between the algebraic concept of atoms and extreme vectors of the cone, holds generally in the setting of atomic JBW-algebras. Furthermore, the atoms corresponding to disengaged extreme vectors, are precisely those that are orthogonal to the other atoms, which we will show in Lemma 5.6. Interplay between the algebraic structure of an atomic JBW-algebra and the order structure induced by its cone, will be used in our study on order isomorphisms between cones of JB-algebras in Chapter 5. There, a more detailed overview of the theory on JB-algebras, as developed by Alfsen and Schultz [AS01, AS03] and Hanche-Olsen and Størmer [H-OS84], will be given.

Chapter 2

Lattice structure preservers in $C(S)$

For a compact Hausdorff space S the space $C(S)$ of real valued continuous functions on S endowed with pointwise order is an example of a Riesz space. Indeed, any pair of functions $x, y \in C(S)$ has a supremum $x \vee y \in C(S)$ given by

$$x \vee y(s) = \max\{x(s), y(s)\}, \quad \text{for } s \in S.$$

The other lattice operations of $C(S)$ are also determined pointwise. A classic result of Kaplansky [Kap47] states that compact Hausdorff spaces S and T are homeomorphic if and only if $C(S)$ and $C(T)$ are lattice isomorphic, here a bijection $f\colon C(S) \to C(T)$ is considered a lattice isomorphism whenever $f(x \vee y) = f(x) \vee f(y)$ holds for all $x, y \in C(S)$. This reinforces the idea that the lattice structure of $C(S)$ has a strong connection with the underlying topological space. It is therefore unsurprising that Riesz homomorphisms on $C(S)$, as defined in 1.9, which preserve the lattice structure, have a simple description.

Theorem 2.1 (Theorem 4.25 in [AA02])**.** *Let S and T be compact Hausdorff spaces. A positive linear map $f\colon C(S) \to C(T)$ is a Riesz homomorphism if and only if there exist a map $\pi\colon T \to S$ and a weight $w \in C(T)^+$ such that we have*

$$f(x)(t) = w(t)x(\pi(t)), \qquad x \in C(S), t \in T. \tag{2.1}$$

Moreover, in this case, $w = f(\mathbb{1}_S)$ and the map π is uniquely determined and continuous on the set $\{w > 0\}$.

One naturally wonders whether such a result can be extended to a variety of subspaces of $C(S)$ and $C(T)$. However, many natural subspaces of $C(S)$ are themselves not a Riesz space. For example, in the case $S = [0, 1]$, the spaces $C^1[0, 1]$ and $P[0, 1]$ consisting of the continuously differentiable functions and polynomial functions on $[0, 1]$, respectively, are not Riesz spaces. However, $C^1[0, 1]$ and $P[0, 1]$ are examples of pre-Riesz spaces. We can therefore attempt to generalise Theorem 2.1 to an analogous statement concerning Riesz* homomorphism between pre-Riesz subspaces of $C(S)$ and $C(T)$.

An overview of pre-Riesz spaces is given in Section 1.2. We briefly recall some terminology for the reader's convenience. A partially ordered vector space X is a pre-Riesz space if it admits a vector lattice cover, here a pair (E, i) of a Riesz space E and a linear order embedding $i\colon X \to E$ is a *vector lattice cover* of X if $i[X]$ is order dense in E. If, in addition, $i[X]$ generates E as a Riesz space then the pair (E, i) is unique up to isomorphism for these properties and is called the *Riesz completion* of X. For short, we denote the Riesz completion of a pre-Riesz space X by X^ρ. Recall that a linear map $f\colon X \to Y$ between pre-Riesz spaces is a Riesz* homomorphism if and only if f extends to a Riesz homomorphism $f^\rho\colon X^\rho \to Y^\rho$.

The aim of this chapter, which is based on [vI18], is to generalise classic results concerning Riesz homomorphisms to the setting of Riesz* homomorphisms between pre-Riesz spaces. We start, as mentioned earlier, by generalising Theorem 2.1 to the setting of Riesz* homomorphisms between order dense subspace of $C(S)$ and $C(T)$. In doing so, we also highlight differences between Riesz* homomorphisms and other classes of operators, which appear in literature, that extend the notion of a Riesz homomorphism. After that we consider the case where S and T are locally compact spaces and our pre-Riesz spaces are order dense in $C_0(S)$ and $C_0(T)$, respectively. We find that any Riesz homomorphism is still of the form (2.1), but with more restriction on the maps w and π. The second vector lattice result we consider is the following.

Theorem 2.2 (Theorem 2.15 in [AB06])**.** *Let E and F be Riesz spaces and $f\colon E \to F$ a bijective Riesz homomorphism. Then f^{-1} is a Riesz homomorphism.*

An analogous statement for Riesz* homomorphisms fails to hold in general. We provide a counterexample in 2.19. However, on a pervasive pre-Riesz space it is true that the inverse of a bijective Riesz* homomorphism is again a Riesz* homomorphism. We then continue by studying the linear order isomorphisms of again order dense subspaces of $C_0(S)$ and $C_0(T)$, using our knowledge on Riesz* homomorphism obtained earlier. We then turn our attention to the last result we examine.

Theorem 2.3. *Let E and F be Riesz spaces. A linear map $f\colon E \to F$ is a Riesz homomorphism if and only if f is positive and disjointness preserving.*

We illustrate with an example, that it is not true in general, that a positive linear disjointness preserving map between pre-Riesz spaces is necessarily a Riesz* homomorphism. This example is based on a pre-Riesz space that does not contain a pair of non-trivial positive disjoint elements. It is reasonable to expect that on a pervasive pre-Riesz space any positive linear disjoint preserving operator is a pre-Riesz space. However, we are only able to prove this assertion under stronger conditions.

Weighted composition maps

Let S and T be compact Hausdorff spaces and $X \subseteq C(S)$ and $Y \subseteq C(T)$ linear subspaces. A map $f\colon X \to Y$ that satisfies (2.1) for some w and π is called a *weighted composition map*. We argue that if X and Y are order dense that any Riesz* homomorphism from X to Y is such a weighted composition map. We elaborate on this setting. Let $X \subseteq C(S)$ and $Y \subseteq C(T)$ be order dense. By Proposition 1.4, both X and Y are pre-Riesz spaces, as they are directed and Archimedean. Moreover, we can describe their Riesz completions as follows. Consider $L(X)$ to be the Riesz subspace of $C(S)$ generated by X, in other words, $x \in L(X)$ if and only if there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ such that

$$x = \bigvee_{i=1}^{n} x_i - \bigvee_{j=1}^{m} y_j.$$

By construction $L(X)$ is a Riesz subspace of $C(S)$. Remark that X is order dense in $L(X)$, since X is order dense in $C(S)$. Therefore, we can identify the Riesz completion (X^ρ, i) of X as the pair $(L(X), i)$, where $i \colon X \to L(X)$ is the canonical embedding. In conclusion, we can view the Riesz completion X^ρ of X as a majorizing Riesz subspace of $C(S)$. Analogously, we identify the Riesz completion Y^ρ as a majorizing Riesz subpsace of $C(T)$. We use this observation freely throughout.

Riesz* homomorphisms

Let $f \colon X \to Y$ be a Riesz* homomorphism and $f^\rho \colon X^\rho \to Y^\rho$ denote the Riesz homomorphism that extends f. An intuitive approach to show that f is a weighted composition operator is to extend f^ρ further to a Riesz homomorphism between $C(S)$ and $C(T)$ and apply the general theory, namely Theorem 2.1. However, generally not every Riesz* homomorphism on X is the restriction of a Riesz homomorphism on $C(S)$, which we illustrate with the following example.

Example 2.4. Consider the subspace X of $C[0,1]$ consisting of functions $x \in X$ that satisfy $x(0) = x(1)$. Straightforward verification yields that X is an order dense Riesz subspace of $C[0,1]$. We consider the *inside-out operator* $f \colon X \to X$ defined by $f(x)(t) = x(\pi(t))$ for all $x \in X$ and $t \in [0,1]$, where

$$\pi(t) = \frac{1}{2} - t + \mathbb{1}_{[t > \frac{1}{2}]}.$$

For all $x \in X$ the defining property $x(0) = x(1)$ guarantees that $f(x)$ is continuous. Moreover, we obtain from $f(x)(0) = x(\frac{1}{2}) = f(x)(1)$ that f maps into X. Since the lattice structure is determined pointwise, it is clear that f is a Riesz homomorphism. Suppose that $g \colon C[0,1] \to C[0,1]$ is a Riesz homomorphism that extends f. By Theorem 2.1, there exist $w \colon [0,1] \to \mathbb{R}_+$ and $\tau \colon [0,1] \to [0,1]$ such that $g(x)(t) = w(t)x(\tau(t))$ for all $x \in C[0,1]$ and $t \in T$. As X contains the constant functions we get $w(t) = 1$ for all $t \in T$. We get $\tau = \pi$, which contradicts that τ is continuous. We conclude that f does not extend to a Riesz homomorphism on $C[0,1]$.

As we will see in the following result, requiring our subspaces to separate the points of the underlying topological space, guarantees that the weight and composition map are automatically continuous. This leads to the following characterisation.

Theorem 2.5. *Let X and Y be order dense subspaces of $C(S)$ and $C(T)$, respectively.*

(i) For every Riesz homomorphism $f\colon X \to Y$ there exist $w\colon T \to \mathbb{R}_+$ and $\pi\colon T \to S$ such that*

$$f(x)(t) = w(t)x(\pi(t)), \qquad x \in X, t \in T. \tag{2.2}$$

Moreover, if in addition X separates the points of S, then w is continuous on T and π is continuous and uniquely determined on $\{w > 0\}$.

(ii) A linear map $f\colon X \to Y$ that satisfies (2.2) for some $w \in C(T)^+$ and $\pi\colon T \to S$ continuous on $\{w > 0\}$ is a Riesz homomorphism.*

Proof. Suppose X and Y are given as in the first statement of (i) and $f\colon X \to Y$ is a Riesz* homomorphism. Let $f^\rho\colon X^\rho \to Y^\rho$ be the Riesz homomorphism that extends f. We fix $t \in T$. Consider the Riesz homomorphism $f_t\colon X^\rho \to \mathbb{R}$ as the composition of f^ρ with the point evaluation at t, i.e., $f_t(x) = f^\rho(x)(t)$ for all $x \in X^\rho$. We apply the Lipecki-Luxemburg-Schep Theorem [AA02, Theorem 4.36] to f_t to obtain a Riesz homomorphism $\hat{f}_t\colon C(S) \to \mathbb{R}$ that extends f_t. The conditions of this theorem are satisfied as X^ρ is a majorizing Riesz subspace of $C(S)$ and $\mathbb{R}$ is Dedekind complete. Riesz homomorphims from $C(S)$ to $\mathbb{R}$ are characterized as scalar multiples of point evaluations, see for example [AA02, Lemma 4.23]. In other words, there exist $w(t) \in \mathbb{R}_+$ and $\pi(t) \in S$ such that $\hat{f}_t(x) = w(t)x(\pi(t))$ holds for all $x \in C(S)$. As $t \in T$ was chosen arbitrarily we obtain that f satisfies (2.2).

Suppose now that X separates the points of S. We redefine if necessary $w(t)$ to equal zero whenever $f(x)(t) = 0$ holds for all $x \in X$. Equation (2.2) remains satisfied. Let $x \in X$ be greater than the constant one function. Then $f(x)$ is a bounded function as element of $C(T)$, which implies by (2.2) that w is a bounded map. Fix $t \in T$. We argue that w is continuous at t and that π is continuous at t whenever $w(t)$ is non-zero.

Let (t_α) in T be a net that converges to t. We show that (t_α) has a subnet (t_β) such that $\lim_\beta w(t_\beta) = w(t)$ and that $\lim_\beta \pi(t_\beta) = \pi(t)$ whenever $w(t)$ is non-zero. For $x \in x$ we get by continuity of $f(x)$ and application of (2.2) that

$$w(t)f(\pi(t)) = f(x)(t) = \lim_\alpha f(x)(t_\alpha) = \lim_\alpha w(t_\alpha)x(\pi(t_\alpha)). \tag{2.3}$$

As we have shown that w is bounded, the fact that S is compact yields the existence of a subnet (t_β) of (t_α) such that $(w(t_\beta))$ converges to some $a \in \mathbb{R}$ and $(\pi(t_\beta))$ to $s \in S$. Therefore, for any $x \in X$ equation (2.3) yields $f(x)(t) = a \lim_\beta x(\pi(t_\beta))$. Moreover, by continuity of the functions $x \in X$ we obtain $f(x)(t) = ax(\lim_\beta \pi(t_\beta)) = ax(s)$ and, in particular, that $w(t)x(\pi(t)) = ax(s)$.

We remark that $a = 0$ whenever $w(t) = 0$ as X contains an element for which $x(s) \neq 0$ and, hence, we are done in that case as then $w(t) = a = \lim_\beta w(t_\beta)$. We consider the remaining case where $w(t) > 0$ holds. It is evident that $a > 0$. Consequently, for any $x \in X$ the equation $x(s) = cx(\pi(t))$ is satisfied, where $c = w(t)/a$ is non-zero and independent of f. As X separates the points of S we obtain the equalities $s = \pi(t)$ and $c = 1$. In other words, π is continuous at t. Plugging this into (2.3) yields

$w(t)x(\pi(t)) = \lim_\beta w(t_\beta)f(\pi(t))$ and, hence, applying that to a $x \in X$ with $x(\pi(t)) \neq 0$ yields that w is continuous at t. We conclude that w and π are continuous on T and $\{w > 0\}$, respectively. Additionally, π is uniquely determined on $\{w > 0\}$ due to separating property of X.

Lastly, suppose that $f\colon X \to Y$ satisfies (2.2) for suitable $w \in C(T)^+$ and $\pi\colon T \to S$ continuous on $\{w > 0\}$. The weighted compostion operator between the Riesz completions $X^\rho \subseteq C(S)$ and $Y^\rho \subseteq C(T)$ defined by w and π is a well-defined Riesz homomorphism that extends T, hence T is a Riesz* homomorphism. □

Henceforth, for notational convenience let $f_{w,\pi}\colon X \to Y$ denote the weighted composition operator between X and Y with weight map $w\colon T \to \mathbb{R}_+$ and composition map $\pi\colon T \to S$ that satisfies (2.2).

Riesz homomorphisms

A predecessor of the Riesz* homomorphism is the Riesz homomorphism, an alternative class of operators on a pre-Riesz space X that extend to Riesz homomorphisms on X^ρ. A linear operator $f\colon X \to Y$ between pre-Riesz spaces is called a *Riesz homomorphism* whenever $f(\{x, y\}^u)^l \subseteq \{f(x), f(y)\}^{ul}$ holds for all $x, y \in X$. Similarly to Riesz* homomorphisms these operators extend to Riesz homomorphisms. However, not all Riesz homomorphisms between the completions are obtained as such extensions. Another disadvantage of the class of Riesz homomorphisms is that it is not stable under composition, see [vH93]. We use our knowledge on the weighted composition structure of Riesz* homomorphisms to investigate similarities and differences between these two classes of operators.

Suppose $f\colon X \to Y$ is a positive linear operator. Positivity of f immediately yields that for any finite $A \subseteq X$ we have $f[A^{ul}] \subseteq f[A^u]^l$. We infer that any Riesz homomorphism is a Riesz* homomorphism. A converse statement does not generally hold on pre-Riesz spaces, which will be illustrated later by a counterexample in Example 2.8. However, we show that on a wide class of subspaces of $C(S)$, which is contained in the class of separating order dense subspaces, the notions of a Riesz homomorphism and a Riesz* homomorphism coincide.

A linear subspace X of $C(S)$ is called *pointwise order dense* if it satisfies

$$y(s) = \inf\{x(s)\colon x \in X, x \geq y\}$$

for all $y \in C(S)$ and $s \in S$. Straightforward verification yields that any pointwise order dense subspace X of $C(S)$ is separating and order dense. Moreover, it is routine to show that a norm dense subspace of $C(S)$ containing the constant functions is pointwise order dense.

Example 2.6. Consider the so-called *Namioka space* defined as $\mathcal{N} = \{x \in C([0, 1])\colon x(0) + x(1) = 2x(\frac{1}{2})\}$. $\mathcal{N}$ is a pervasive and order dense subspace of $C([0, 1])$, which is not pointwise order dense.

We consider an equivalent defining property of a Riesz homomorphism. A linear map $f\colon X \to Y$ is a Riesz homomorphism if and only if

$$\inf\{f(z)\colon z \in X, z \geq x, y\} = f(x) \vee f(y) \text{ in } Y^\rho, \qquad x, y \in X. \tag{2.4}$$

Here the infimum and supremum are taken within the Riesz space Y^ρ. We use this characterisation and Theorem 2.5 to prove that any Riesz* homomorphism on a pointwise order dense subspace of $C(S)$ is automatically a Riesz homomorphism.

Theorem 2.7. *Let X be a pointwise order dense subspace of $C(S)$, Y an order dense subspace of $C(T)$ and $f\colon X \to Y$ a linear operator. Then f is a Riesz* homomorphism if and only if f is a Riesz homomorphism.*

Proof. Let X and Y be as in the statement and suppose $f\colon X \to Y$ is a Riesz* homomorphism. Due to Theorem 2.5 there exist suitable w and π such that $f = f_{w,\pi}\colon X \to Y$. Suppose $x, y \in X$ are given. We argue that (2.4) is satisfied. As f is positive $f(x) \vee f(y)$ is a lower bound of $\{f(z)\colon z \in X, z \geq x, y\}$ in Y^ρ. Suppose w is another lower bound of $\{f(z)\colon z \in X, z \geq x, y\}$ in Y^ρ. We compute for all $t \in T$

$$\begin{aligned} w(t) &\leq \inf\{f(z)(t)\colon z \in X, z \geq x, y\} \\ &= \inf\{w(t)z(\pi(t))\colon z \in X, z \geq x, y\} \\ &= w(t)[\inf\{z(\pi(t))\colon z \in X, z \geq x, y\}] \\ &= w(t)(x \vee y)(\pi(t)) = (f(x) \vee f(y))(t). \end{aligned}$$

Here we used that X is pointwise order dense in $C(S)$ in the second last equality on $x \vee y \in C(S)$ and $\pi(t) \in S$. This shows that $w \leq f(x) \vee f(y)$ holds in the order induced by $C(T)$, which shows that f satisfies condition (2.4). Recall that the other implication holds for general pre-Riesz spaces as discussed earlier. □

We consider an example that shows that the above theorem fails to hold generally for separating order dense subspaces of $C(S)$.

Example 2.8. Let $\mathcal{N}$ be the Namioka space as considered in Example 2.6 and recall that $\mathcal{N}$ is indeed separating and order dense, however, not pointwise order dense in $C(S)$. Let $\varphi\colon \mathcal{N} \to \mathbb{R}$ be the functional that composes any $x \in \mathcal{N}$ with the point evaluation at $s = \frac{1}{2}$. Evidently φ is a Riesz* homomorphism by Theorem 2.5. However, letting $x, y \in \mathcal{N}$ be defined by $x(s) = s$ and $y(s) = 1 - s$, we obtain $\varphi(x) \vee \varphi(y) = \frac{1}{2} \vee \frac{1}{2} = \frac{1}{2}$, while for any $z \in \mathcal{N}$ with $z \geq x, y$ we get $z(0) \geq y(0) \geq 1$ and $z(1) \geq x(1) \geq 1$, hence $\varphi(z) = z(\frac{1}{2}) \geq 1$. Therefore, φ does not satisfy condition (2.4) and, hence, is not a Riesz homomorphism.

Complete Riesz Homomorphisms

First introduced and studied by Buskes and van Rooij in [BvR93] is the class of complete Riesz homomorphisms. Between Riesz spaces these complete Riesz homomorphisms are exactly the order continuous Riesz homomorphisms. Between pre-Riesz spaces the complete Riesz homomorphisms are exactly the operators that extend to order continuous Riesz homomorphisms between the completions [vH93, Theorem 5.12]. Our aim is to characterize the complete Riesz homomorphisms between order dense subspaces of $C(S)$ and, in doing so, characterize the order continuous Riesz homomorphisms between Riesz subspaces of $C(S)$. More specifically, our aim is to determine a necessary condition imposed on w and π that when imposed guarantees $f_{w,\pi}\colon X \to Y$ to be a complete Riesz homomorphism.

A linear map $f\colon X \to Y$ between partially ordered vector spaces is called a *complete Riesz homomorphism* whenever $Z \subseteq X$ with $\inf\ Z = 0$ implies $\inf\ f(Z) = 0$. As not all Riesz homomorphism are order continuous we easily construct an example of a Riesz* homomorphism that is not a complete Riesz homomorphism. Consider a weighted composition operator $f_{w,\pi}\colon C[0,1] \to C[0,1]$ where w is positive and non-vanishing and π is constant. There exists a sequence in $C[0,1]$ that descends to zero and is constantly one on the singleton $\pi\big[[0,1]\big]$. Therefore, f is indeed not a complete Riesz homomorphism. It holds generally, however, that for $w \geq 0$ non-vanishing and π an open map that $f_{w,\pi}$ is a complete Riesz homomorphism. It turns out that π being an open map is not a necessary condition, as will be shown in Theorem 2.10.

A function $\pi\colon T \to S$ is called *weak-open* if for all non-empty open $U \subseteq T$ the image $\pi[U]$ is dense somewhere, i.e., there exists a non-empty $V \subseteq S$ open such that $\pi[U] \cap V$ is dense in V, and π is called *nowhere constant* if for all non-empty $U \subseteq T$ open the image $\pi[U]$ is not a singleton. One easily verifies that the former implies the latter and π being open implies both properties.

We characterise subsets in $C(S)$ whose infimum exist and equal zero.

Lemma 2.9. *Let X be an order dense subspace of $C(S)$ and let $Z \subseteq X_+$ be given. Then* $\inf\ Z = 0$ *holds in X if and only if Z satisfies the following property*

$$\forall \epsilon > 0, U \subseteq S\backslash\{\emptyset\} \text{ open } \exists z \in Z, s \in U \text{ such that } z(s) \leq \epsilon. \tag{2.5}$$

Proof. Let $Z \subseteq X$ with $\inf\ Z = 0$. Suppose that the converse of (2.5) holds. Let $\epsilon > 0$ and $U \subseteq S$ be non-empty and open such that for all $z \in Z$ and $s \in U$ we have $z(s) > \epsilon$. An application of Urysohn's Lemma yields a non-zero positive $y \in C(S)$ whose support is contained in U. After rescaling if necessary y is a lower bound of Z in $C(S)$. As X is order dense in $C(S)$ there exists a $x \in X$ with $x \not\leq 0$ and $x \leq y$. This yields a contradiction with the assumption that $\inf\ Z = 0$.

Suppose $Z \subseteq X_+$ does not satisfy $\inf\ Z = 0$. Then there exists a lower bound $x \in X$ of Z such that $x \not\leq 0$. Remark that the positive part x^+ of x is a non-zero positive element of $C(S)$ and a lower bound of Z. By continuity there exists an $\epsilon > 0$ and $U \subseteq S$ non-empty and open such that $z(s) \geq x^+(s) > \epsilon$ holds for all $s \in U, z \in Z$. □

We are now ready to characterize complete Riesz homomorphisms on order dense subspaces of $C(S)$ and note that no additional conditions are imposed on the subspace Y of $C(T)$.

Theorem 2.10. *Let X be an order dense subspace of $C(S)$ and Y a subspace of $C(T)$. Let $w \in C(T)_+$ and $\pi\colon T \to S$ be such that $f_{w,\pi}\colon X \to C(T)$ maps into Y. Then $f_{w,\pi}\colon X \to Y$ is a complete Riesz homomorphism if and only if π is weak-open on $\{w > 0\}$.*

Proof. Let w and π be as in the statement and let us denote $f_{w,\pi}$ by f. Suppose π is weak open on $\{w > 0\}$. Let $Z \subseteq X$ with $\inf\ Z = 0$. Fix $\delta > 0$ and $U \subseteq T$ non-empty and open. Due to Lemma 2.9 it suffices to show existence of an $z \in Z$ and $t \in U$ such that $f(z)(t) \leq \delta$ holds. Suppose there exists a $t \in U \cap \{w = 0\}$. Then for all $z \in Z$ we have $f(z)(t) = 0 < \delta$ and, hence, we are done. Therefore, we assume that $U \subseteq \{w > 0\}$ holds. In particular, there exists a non-empty open $V \subseteq S$ with $\pi[U] \cap V$ dense in V, as π is weak-open on $\{w > 0\}$. Put $\epsilon := \delta(m+1)^{-1} > 0$, where $m := \sup\{w(t)\colon t \in T\}$. As $\inf\ W = 0$ holds Lemma 2.9 yields a $z \in Z$ and $s \in V$ such that $z(s) \leq \frac{\epsilon}{2}$. Therefore, by continuity there exists an $s_0 \in \pi[U] \cap V$ with $f(s_0) \leq \epsilon$. Let $t_0 \in \pi^{-1}(\{s_0\}) \cap U$. We compute

$$f(z)(t_0) = w(t_0)z(\pi(t_0)) \leq m \cdot z(\pi(y_0)) = m \cdot z(s_0) \leq M\epsilon \leq \delta,$$

and conclude that due to Lemma 2.9 f is a complete Riesz homomorphism.

Conversely, suppose that π is not weak open on $\{w > 0\}$. In other words, there exist $\delta > 0$ and non-empty and open $U \subseteq T$ with $U \subseteq \{w \geq \delta\}$ and $\pi[U]$ is nowhere dense in S. We recall that generally a complete Riesz homomorphism between pre-Riesz spaces extends to a order continuous Riesz homomorphism between the Riesz completions, which, therefore, is itself is a complete Riesz homomorphism. Hence, it suffices to show that $\hat{f} = f_{w,\pi}\colon E^\rho \to F^\rho$ is not a complete Riesz homomorphism. We define

$$Z\colon = \{z \in X^\rho\colon z \geq 0 \text{ and } z \geq 1 \text{ on } \pi[U]\}.$$

We argue by contradiction that $\inf\ Z = 0$. Suppose there exists a lower bound $z \in X^\rho_+$ of Z not smaller than zero. As X^ρ is a Riesz space we replace z by z^+ if necessary to obtain a non-zero positive lower bound z of Z. In particular, there exist $\epsilon > 0$ and $W \subseteq S$ non-empty and open such that $z \geq \epsilon$ on W. Recall that $\pi[U]$ is nowhere dense, so $\pi[U] \cap W$ is not dense in W. Therefore, the closure V of $\pi[U] \cap W$ in W is a closed strict subset of W. Let $W_0 \subseteq W$ be non-empty and open with $\overline{W}_0 \cap V = \emptyset$. By Urysohn's lemma there is a $z_0 \in C(S)^+$ with $z_0 = 1$ on $V \supseteq \pi[U]$ and $z_0 = 0$ on W_0. As X^ρ is order dense in $C(S)$ there exists an $x \in X^\rho$ with $x \geq z_0$ and $x(s) < \epsilon$ for some $s \in W_0$. By construction $x \geq z_0$ yields $x \in Z$ and from $W_0 \subseteq W$ we infer that $z \not\leq x$. This yields a contradiction with z being a lower bound of Z and we conclude that $\inf\ Z = 0$.

On the other hand, however, we argue that $f[Z]$ has a lower bound that is not negative. By construction of U any $y \in f[Z]$ satisfies $y \geq \delta$ on U. Since Y^ρ is an order dense Riesz subspace of $C(T)$, there exists a lower of $f[Z]$ in Y^ρ that is strictly positive on U. $\square$

Let us remark that Theorem 2.10 shows, in particular, that the order continuous Riesz homomorphisms between order dense Riesz subspaces of $C(S)$ and $C(T)$ are exactly the composition multiplication operators where the composition map is weak open on the set where the multiplication map is non-zero.

Concluding this section we remark that, in the special case that S and T are compact intervals of $\mathbb{R}$, the condition that π is weak-open on $\{w > 0\}$ in the above theorem can be relaxed to π being nowhere constant on $\{w > 0\}$. The proof follows immediately from the following lemma.

Lemma 2.11. *Let I and J be intervals in $\mathbb{R}$ and $\pi\colon I \to J$ a map. For all $U \subseteq I$ which are open in $\mathbb{R}$, π is weak-open on U if and only if π is nowhere constant on U.*

Proof. As mentioned earlier, it is evident that the former implies the latter. Let $U \subseteq I$ is open in $\mathbb{R}$. Without loss of generality we assume that U is non-empty. Suppose that π is nowhere constant on U. Then we can find distinct points $s, t \in J$ contained in $\pi[U]$ and say $s < t$. Consequently, there are $a, b \in U$ with $a < b$ such $\pi(a) = s$ and $\pi(b) = t$ or vice versa. We restrict π to the continuous map $\hat{\pi}\colon [a, b] \to J$. For any $r \in (s, t)$ we can find, by the Intermediate Value Theorem, a $c \in (a, b)$ with $\pi(c) = z$. Therefore, (s, t) is contained in $\pi[U]$ and we conclude that π is weak open on U. In the case that $\pi(a) = t$ and $\pi(b) = s$ we can interchange the roles of s and t and obtain similarly that (t, s) is contained in $\pi[U]$. □

Locally compact spaces

We have investigated Riesz* homomorphisms on separating order dense subspaces of the space of continuous functions on some compact Hausdorff space. With similar techniques we characterize the Riesz* homomorphisms between pre-Riesz spaces of continuous functions as weighted composition maps with some additional conditions on the weight and composition map.

In this section S and T are locally compact Hausdorff spaces. Consider the subspace $C_0(S)$ of $C(S)$ consisting of all functions $x \in C(S)$ that *vanish at infinity*, i.e., for all $\epsilon > 0$ the set $\{x \geq \epsilon\}$ is compact in S. Evidently, $C_0(S)$ is a Riesz space and it coincides with $C(S)$, whenever S is compact. We generalise the results in Theorem 2.5 to the setting of vanishing functions on locally compact spaces. We use the following result from [Fol84, 7.3]: the positive norm-bounded linear functionals on $C_0(S)$ are exactly those functionals that are given by integration against a finite Radon measure.

Theorem 2.12. *Let X and Y be separating order dense subspaces of $C_0(S)$ and $C_0(T)$, respectively. A linear map $f\colon X \to Y$ is a Riesz* homomorphism if and only if $T = T_{w,\pi}$ for some $w \in C_b(Y)^+$ and $\pi\colon Y \to X$ continuous on $\{w > 0\}$. Moreover, in this case π is proper on $\{w \geq \epsilon\}$ for each $\epsilon > 0$, i.e.,*

$$K \subseteq S \text{ is compact}, \epsilon > 0 \;\Rightarrow \pi^{-1}(K) \cap \{w \geq \epsilon\} \text{ is compact.} \tag{2.6}$$

Proof. Let $f\colon X \to Y$ be a Riesz* homomorphism and let $f^\rho\colon X^\rho \to Y^\rho$ be the Riesz homomorphism that extends f. Fix $t \in T$. We define a positive linear functional

$f_t\colon X^\rho \to \mathbb{R}$ by $f_t(x)\colon = f^\rho(x)(t)$ for $x \in X^\rho$. We extend f_t to a Riesz homomorphism $\hat{f}_t\colon C_0(S) \to \mathbb{R}$ using the Lipecki-Luxemburg-Schep extension theorem. By [Fol84, 7.3] the functional f_t is given by integration against a finite Radon measure, say μ_t. This result applies since positive operators between Banach lattices are norm continuous by [AB06, Theorem 4.3], and hence, f_t is norm-bounded.

We argue that μ_t is supported in at most a single point. Suppose that s and t are distinct points in the support of μ_t. Here the support of μ_t, denoted by $\text{supp}(\mu_t)$, consists of all $s \in S$ such that for all open $U \subseteq S$ that contain s we have $\mu_t(U) > 0$. By the Hausdorff property of S we obtain disjoint open sets $U, V \subseteq S$ with $s \in U$, $t \in V$. In particular, we get $\mu_t(U), \mu_t(V) > 0$. Furthermore, since the Radon measure μ_t is inner regular, we can assume without loss of generality, that U and V are contained in some compact set $K \subseteq S$. Applying Urysohn's lemma yields continuous $x, y\colon S \to [0,1]$ with $x(s) = 1$ and $x = 0$ on $S\backslash U$, and $y(t) = 1$ and $y = 0$ on $S\backslash V$. We remark that $x, y \in C_0(S)$ as both x and y are zero outside the compact set K. Moreover, by construction x and y are disjoint. As f is a Riesz homomorphism, we infer $f_t(x) \perp f_t(y)$. However, for $\epsilon \in (0,1]$, we have

$$f_t(x) = \int_S x \, d\mu_t \geq \int_{\{x \geq \epsilon\}} \epsilon \, d\mu_t = \epsilon \cdot \mu_t(\{x \geq \epsilon\}) > 0,$$

since the set $\{f \geq \epsilon\}$ contains s and, therefore, has strictly positive measure, as s is in the support of μ_t. Analogously, $f_t(y) > 0$. This contradicts our earlier conclusion that $f_t(x) \wedge f_t(y) = 0$. Therefore, the support of μ_t is either a singleton or the functional f_t is identically zero.

Suppose $t \in T$ is given such that $f_t \neq 0$. Let $\pi(t) \in S$ be the unique element in the support of μ_t and $w(t) := \mu_t(\{\pi(t)\}) > 0$. For $x \in C_0(S)$ we obtain the desired formula

$$f_t(x) = \int_S x \, d\mu_t = \int_{\{\pi(t)\}} x \, d\mu_t = w(t)x(\pi(t)).$$

We put $w(t) := 0$ whenever the corresponding functional f_t equals zero. We conclude that f is a weighted composition map and $f = f_{w,\pi}$.

In the proof of Theorem 2.5, where we considered the $C(S)$ case, we showed that w and π are automatically continuous on T and $\{w > 0\}$, respectively. Our arguments made there only used that any bounded net in S has a convergent subnet, as property of the topological space S. As any locally compact topological space has this property we infer that w and π are continuous on Y and $\{w > 0\}$, respectively. Straightforward verification yields that w inherits the positive and bounded property from f.

We argue that any weighted composition map $f_{w,\pi}\colon X \to Y$ satisfies (2.6). Let $K \subseteq S$ be compact and $\epsilon > 0$. Let $x \in C_0(S)$ be equal to one on K; such an x as K is compact. For $t \in \pi^{-1}(K) \cap \{w \geq \epsilon\}$ we have

$$f(x)(t) = w(t)x(\pi(t)) = w(t) \geq \epsilon.$$

Therefore, $\pi^{-1}(K) \cap \{w \geq \epsilon\}$ is compact since $f(x) \in C_0(T)$.

Let $f = f_{w,\pi}\colon X \to Y$ for suitable w and π. By the previous paragraph equation (2.6) is satisfied. Therefore, the weighted composition map $f^\rho = f_{w,\pi}\colon X^\rho \to Y^\rho$ is well-defined, as any $f^\rho(x)$ vanishes at infinity for $x \in X^\rho$ by (2.6). Moreover, f^ρ is clearly a Riesz homomorphism that extends f. We conclude that f is a Riesz* homomorphism. □

An application: Sobolev spaces

A result by Biegert [Bie10, Theorem 4.4] states that any Riesz homomorphism on the Sobolev space $W_0^{1,p}(\Omega)$ is a weighted composition map, where $\Omega \subseteq \mathbb{R}^d$ is open. A higher order Sobolev space $W_0^{m,p}(\Omega)$ with $m > 1$ is generally not a Riesz space. Our aim is to extend Biergert's characterization to Riesz* homormorphisms on the pre-Riesz space $W_0^{m,p}(\Omega)$. Our strategy is to use the classical Sobolev Embedding Theorem to embed the Sobolev space into $C_0(\Omega)$ and show that this embedding satisfies the conditions of Theorem 2.12.

We start by describing the setting and giving the necessary definitions. All definitions and terminology we introduce here are taken from Adams [Ada75]. Let $d \in \mathbb{N}$ be given and Ω a *domain* in $\mathbb{R}^d$, i.e., $\Omega \subseteq \mathbb{R}^d$ is open. For any $m \in \mathbb{N}$ and $1 \leq p < \infty$ we define the Sobolev space $W^{m,p}(\Omega)$ as the space consisting of L^p-functions x on Ω for which all distributional partial derivates $D^\alpha x$, with $1 \leq |\alpha| \leq m$, are in L^p. Equipped with the norm $\|.\|_{m,p}$ defined by

$$\|x\|_{m,p} = \Big(\sum_{0\leq|\alpha|\leq m} \|D^\alpha x\|_p^p\Big)^{\frac{1}{p}},$$

the space $W^{m,p}(\Omega)$ is a Banach space. For smooth functions the distributional and classical partial derivatives coincide, hence, we infer $C^\infty(\Omega) \subseteq W^{m,p}(\Omega)$.

Under some regularity conditions imposed on the domain Ω every equivalence class $u \in W^{m,p}(\Omega)$ contains a unique continuous function. In this case we define $W_0^{m,p}(\Omega)$ as the norm-closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For the reader's convenience we include these conditions here. They can be found on page 66 of [Ada75].

Definition 2.13. Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ open be given.

(i) Let $x \in \mathbb{R}^d$ and B_1 and B_2 open balls in $\mathbb{R}^d$ with $x \in B_1$ and $x \notin B_2$. The set $C_x\colon = B_1 \cap \{x + \lambda(y - x)\colon y \in B_2, \lambda > 0\}$ is a *finite cone* with vertex x.

(ii) Every domain Ω for which there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C is said to have the *cone property*.

The classical Sobolev Embedding Theorem [Ada75, Theorem 5.4 part III(C)] states that if Ω is a domain in $\mathbb{R}^d$ which has the cone property and $mp > d$ holds, then $W_0^{m,p}(\Omega) \subseteq C_0(\Omega)$ holds. In this case, we therefore get

$$C_0^\infty(\Omega) \subseteq W_0^{m,p}(\Omega) \subseteq C_0(\Omega).$$

In particular, $W_0^{m,p}(\Omega)$ is separating and order dense in $C_0(\Omega)$. Therefore, Theorem 2.12 yields the following.

Theorem 2.14. *Suppose Ω_1 and Ω_2 are domains in $\mathbb{R}^d$ having the cone property, $1 \leq p, q < \infty$ and $m, n \in \mathbb{N}$ be such that $pm > d$ and $qn > d$ hold. Any Riesz* homomorphism $f\colon W_0^{m,p}(\Omega_1) \to W_0^{n,q}(\Omega_2)$ is a weighted composition map.*

In his proof, Biegert does not use the order structure of the space $W^{1,p}(\Omega)$ nor the Sobolev Embedding Theorem. Due to the latter he does not need to impose the cone property on Ω or any condition on p and d. However, Theorem 2.14 can deal with Sobolev spaces up to arbitrary order. Remark that $W^{m,p}(\Omega)$ is a Riesz space exactly when $m = 1$ holds. In conclusion, under additional regularity conditions on the domain, we extend the result of Biegert to higher order Sobolev spaces by considering Riesz* homomorphisms.

Linear order isomorphisms

We have considered three types of homomorphisms defined on pre-Riesz spaces. In order from weak to strong, Riesz* homomorphisms, Riesz homomorphisms and complete Riesz homomorphisms. Analogous to Theorem 2.2, we consider whether the inverse of a bijective homomorphism of on of these types is again of the same type. Since the results that we obtain hold in general partially ordered vector spaces, we shall consider them in that setting.

Secondly, returning to the setting of order dense subspaces in spaces of continuous functions, we study linear order isomorphisms. Combining our observations with Theorem 2.5 we obtain a method of describing the automorphism group for any order dense subspace of $C(S)$. We then apply this method to the space of differentiable functions on a smooth manifold.

Inverses of Riesz homomorphisms*

We start with an elementary observation on linear order isomorphisms.

Proposition 2.15. *Suppose X and Y are partially ordered vector spaces and $f\colon X \to Y$ is a linear map. Then f is an order isomorphism if and only if f is bijective and both f and f^{-1} are complete Riesz homomorphisms.*

Proof. Suppose f is a linear order isomorphism. It suffices to show that f is a complete Riesz homomorphism, as f^{-1} is also a linear order isomorphism. Let $Z \subset X$ with $\inf\ Z = 0$. Suppose $y \in Y$ is a lower bound of $f[Z]$, then $f^{-1}(y)$ is a lower bound of Z. Therefore, $f^{-1}y \leq 0$ holds as the infimum of Z equals zero. Applying the positivity of f again yields $f(f^{-1}(x)) = x \leq 0$, which proves that $\inf\ f[Z] = 0$.

For the converse, it suffices to observe that complete Riesz homomorphisms are positive. Let $x \in X$ be positive. We put $Z\colon = \{0, x\}$. Evidently $\inf\ Z = 0$ holds and, hence, if $f\colon X\to Y$ is a complete Riesz homomorphism, this yields $\inf f[Z]=0$, which shows, in particular, $f(x)\geq 0$. □

In the following result we highlight the usefulness of the pervasive property.

Lemma 2.16. *Let X and Y be pre-Riesz spaces, $f\colon X\to Y$ a Riesz* homomorphism and $f^\rho\colon X^\rho\to Y^\rho$ the Riesz homomorphism that extends f. The following assertions hold:*

(i) If f is surjective, then f^ρ is surjective;

(ii) If X is pervasive and f is injective, then f^ρ is injective.

Proof. Suppose f is surjective. Let $y\in Y^\rho$ be given. Recall that Y^ρ is generated as Riesz space by Y. Let $x_1,...,x_n,y_1,...,y_m\in Y$ be such that $y=\bigvee_{i=1}^n x_i-\bigvee_{j=1}^m y_j$ in Y^ρ. Using that f is surjective we let $a_1,...,a_n,b_1,...,b_m\in X$ with $x_i=f(a_i)$ and $y_j=f(b_j)$, $i=1,...,n$ and $j=1,...,m$. We define $x:=\bigvee_{i=1}^n a_i-\bigvee_{j=1}^m b_i\in X^\rho$. The image of x under f^ρ is computed as follows

$$\begin{aligned} f^\rho(x) &= f^\rho(\bigvee_{i=1}^n a_i-\bigvee_{j=1}^m b_j)\\ &= \bigvee_{i=1}^n f^\rho(a_i)-\bigvee_{j=1}^m f^\rho(b_j)\\ &= \bigvee_{i=1}^n x_i-\bigvee_{j=1}^m y_j = y, \end{aligned}$$

and, hence, f is surjective.

Suppose X is pervasive and f is injective. Let $x\in X^\rho$ be non-zero and positive. As X is pervasive there exists a $y\in X_+$ with $0<y\leq x$. Since f^ρ is positive this yields $0\leq f(y)=f^\rho(y)\leq f^\rho(x)$. The injectivity of f yields $f^\rho(x)\neq 0$. From this we conclude that for any $x\in X^\rho$ with $f^\rho(x)=0$ that $x=0$, since $f^\rho(x_+)=f(x)_+=0$ and $f^\rho(x_-)=f(x)_-=0$ hold and both x_+ and x_- are positive. □

In particular, Lemma 2.16 shows that a bijective Riesz* homomorphism on a pervasive pre-Riesz space extends to a bijective Riesz homomorphism on the Riesz completion. This is fact is useful in studying properties of the inverse of bijective homomorphisms.

Theorem 2.17. *Suppose X and Y are pre-Riesz spaces, X is pervasive and that $f\colon X\to Y$ is a bijective Riesz* homomorphism. Then f^{-1} is a Riesz* homomorphism and, hence, f is an order isomorphism.*

Proof. Suppose f is a bijective Riesz* homomorphism. Lemma 2.16 yields that f extends to a bijective Riesz homomorphism $f^\rho\colon X^\rho\to Y^\rho$. The inverse $(f^\rho)^{-1}\colon Y^\rho\to X^\rho$ is a Riesz homomorphism by Theorem 2.2 that extends $f^{-1}\colon Y\to X$ and, hence, f^{-1} is a Riesz* homomorphism. □

In conjunction with Proposition 2.15 this yields the following list of equivalent statements.

Corollary 2.18. *Let X and Y be pre-Riesz spaces and X pervasive. For a linear bijection $f\colon X \to Y$ the following statements are equivalent:*

(i) f is a Riesz homomorphism.*

(ii) f is a Riesz homomorphism.

(iii) f is a complete Riesz homomorphism.

(iv) f^{-1} is a Riesz homomorphism.*

(v) f^{-1} is a Riesz homomorphism.

(vi) f^{-1} is a complete Riesz homomorphism.

(vii) f is an order isomorphism.

The main implication of Corollary 2.18 is that a bijective Riesz* homomorphism between pervasive pre-Riesz spaces is an order isomorphism, just as in the vector lattice case. We consider an example of a non-pervasive pre-Riesz space for which the statement in Theorem 2.17 fails to hold.

Example 2.19. Let X be the subspace of $C([0,1])$ consisting of all polynomials. Then X is a pre-Riesz space and its Riesz completion X^ρ is the Riesz subspace of $C([0,1])$ consisting of all piecewise polynomial functions. Since non-constant polynomials can only be zero in finitely many points, one easily verifies that X is not pervasive. We consider $f = f_{w,\pi}\colon X \to X$, where $w = \mathbb{1}$ and $\pi(s) = \frac{1}{2}s$ for $s \in [0,1]$. Theorem 2.5 yields that f is a Riesz* homomorphism. Moreover, since π is a weak-open map, f is even a complete Riesz homomorphism by Theorem 2.10. We argue that f is a bijective map. By definition it is evident that f is injective. Let $y \in X$ be of the form $y(s) = \alpha_n s^n + \ldots + \alpha_1 s + \alpha_0$, for $s \in [0,1]$ with $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$. The pre-image $f^{-1}(y)$ is given by $x(s) = \sum_{i=0}^n \beta_i s^i \in X$, where $\beta_i \colon = 2^i \alpha_i$ for $0 \leq i \leq n$. We conclude that T is a bijective Riesz* homomorphism.

Suppose there exist $v\colon [0,1] \to \mathbb{R}$ continuous and $\tau\colon [0,1] \to [0,1]$ continuous on $\{\theta > 0\}$ such that $f^{-1} = f_{v,\tau}$ on X. The equality $f(\mathbb{1}) = \mathbb{1}$, where $\mathbb{1}$ denote the constant one functions, immediately yields that v is identically zero. Let $x \in X$ satisfy $x(s) = s$ for all $s \in [0,1]$. We compute for all $s \in [0,1]$ that

$$s = f^{-1}(f(x))(s) = f(x)(\tau(s)) = x(\tfrac{1}{2}(\tau(s))) = \tfrac{1}{2}\tau(s).$$

However, the equality $\tau(s) = 2s$ can not be satisfied on all of $[0,1]$, therefore, f^{-1} is not a weighted composition operator. In particular, Theorem 2.5 yields that f^{-1} is not a Riesz* homomorphism.

Automorphism groups

Combining our characterisation of Riesz* homomorphisms on spaces of continuous functions, obtained in Theorem 2.12, and the general observation made in the previous section concerning linear order isomorphisms, we can describe the linear order isomorphisms between order dense subspaces $X \subseteq C_0(S)$ and $Y \subseteq C_0(T)$ that separate the points, where S and T are locally compact Hausdorff spaces.

Theorem 2.20. *A linear map $f\colon X \to Y$ is an order isomorphism if and only if $f = f_{w,\pi}$, where $w \in C(T)$ satisfies $0 < \delta\mathbb{1} \leq w \leq D\mathbb{1}$ for some $\delta, D > 0$ and $\pi\colon T \to S$ is a homeomorphism. In this case, w and π are uniquely determined by f.*

We remark that a homeomorphism is proper and, therefore, (2.6) is satisfied in this case.

Proof. Let $f\colon X \to Y$ be a linear order isomorphism. Due to Proposition 2.15, f is a complete Riesz homomorphism and, in particular, f is a Riesz* homomorphism. Therefore, Theorem 2.12 yields the existence of maps $w \in C_b(T)_+$ and $\pi\colon T \to S$ continuous on $\{w > 0\}$ such that $f = f_{w,\pi}$.

Suppose that $w(t) = 0$ holds for some $t \in T$. In that case, we get $y(t) = 0$ for all $y \in f(X)$. This yields an immediate contradiction with the fact that f is surjective and Y is majorizing. We conclude that w is indeed non-vanishing. We remark that, since $\{w > 0\} = T$ holds, π is continuous and uniquely determined everywhere.

Suppose π is not injective. Let $t_1, t_2 \in T$ be such that $t_1 \neq t_2$ and $\pi(t_1) = \pi(t_2)$. We obtain for all $x \in X$ that

$$f(x)(t_1) = w(t_1)x(\pi(t_1)) = w(t_2)\frac{w(t_1)}{w(t_2)}x(\pi(t_2)) = \frac{w(t_1)}{w(t_2)}f(x)(t_2).$$

Therefore, any $g \in f(X)$ satisfies $g(t_1) = \lambda g(t_2)$, where $\lambda = w(t_1)/w(t_2)$. Since Y separates the points of T this contradicts the surjectivity of f and we conclude that π is injective.

We argue that π is surjective. Recall that π is continuous. Hence, $\pi(T)$ is compact and, hence, $\pi(T)$ is closed in S. Therefore, supposing that π is not surjective yields a non-empty open $U \subseteq S \backslash \pi(T)$. Due to Urysohn's lemma we find a non-zero negative $y \in C(S)$ with $\operatorname{supp}(y) \subseteq U$. We infer the existence of an $x \in X$ that satisfies $x \geq y$ and $x \ngeq 0$, from the fact X is order dense in $C(S)$. We remark that $x \geq y$ yields $x(s) \geq 0$ for all $s \in \pi(T)$. Therefore, for all $t \in T$ we get $f(x)(t) = w(t)x(\pi(t)) \geq 0$ as w is positive. This contradicts that f^{-1} is positive as $f(x) \geq 0$ and $x \ngeq 0$.

Consider the weighted composition map $g\colon Y \to X$ with weight map η the reciprocal of $w \circ \pi^{-1}$ and composition map π^{-1}. This is well-defined as we have shown that w does not vanish and π is bijective. We verify that g is the inverse of f. For all $y \in Y$ and $t \in T$ we compute

$$f \circ g(y)(t) = w(t)g(y)(\pi(t)) = \frac{w(t)}{\eta(\pi(t))}y(\pi^{-1}(\pi(t))) = y(t).$$

Completely analogously one verifies that $g \circ f$ equals the identity on X. We conclude that $g = f^{-1}$. Theorem 2.12 applied to the linear order isomorphism g yields that η is bounded and π^{-1} is continuous. In particular, $0 < \delta\mathbb{1} \leq w \leq D\mathbb{1}$ holds for some $\delta, D > 0$ and π is a homeomorphism.

Conversely, suppose $f = f_{w,\pi}\colon X \to Y$ with w and π as in the assertion. The weighted composition map $g = g_{\eta,\pi^{-1}}\colon Y \to X$, where η is the reciprocal of $w \circ \pi^{-1}$, is a well-defined inverse of f by the assumptions on w and π. Therefore, f is bijective. Moreover, as both w and η are positive we get that f and $f^{-1} = g$ are positive. Indeed f is an order isomorphism. □

Remark 2.21. A class result [Kap47] by Kaplansky states that you can recover a locally compact Hausdorff space S from the lattice structure of $C(S)$, or in other words, that if there exists a lattice isomorphism from $C(S)$ to $C(T)$ then S and T must be homeomorphic. We remark that a lattice isomorphism here is not necessarily linear. One could ask whether a compact Hausdorff space S is also fully determined by the order structure of subspaces of $C(S)$. Theorem 2.20 yields a partial answer, namely we can determine S by the *linear* order structure of any order dense separating subspace of $C(S)$. This question is also studied in [LL13]. Their result [LL13, Theorem 1], for the case where S is compact, states that if $X \subseteq C(S)$ and $Y \subseteq C(T)$ contain the constant functions and precisely separates points from closed sets, then any linear order isomorphism $f\colon X \to Y$ is a weighted composition map, where the composition map is a homeomorphism from S to T. Here X *precisely separates points from closed sets* whenever for any closed $F \subseteq S$ and $s \notin F$ there exists an $x \in X$ with $x[S] \subseteq [0,1]$, $x[F] \subseteq \{0\}$ and $x(s) = 1$. We remark that these conditions in [LL13, Theorem 1] are more restrictive than ours in Theorem 2.20. Suppose $X \subseteq C(S)$ contains the constant functions and precisely separates the points from closed sets. The former conditions implies that X is majorizing in $C(S)$, and the latter yields both that X separates the points of S and that X is pervasive. Now Lemma 1.5 yields that X is order dense in $C(S)$.

Let (X, C) be a partially ordered vector space. The *automorphism group of* X, denoted by $\mathrm{Aut}(X)$, is the set consisting of all linear order isomorphisms from X onto itself equipped with the group action of composition. Due to Theorem 2.20 we can describe the automorphism group of $C_0(S)$.

Consider the group $C_b(S)_+$ consisting of all positive bounded continuous functions on S equipped with pointwise multiplication. We denote the interior of $C_b(S)_+$ relative to the maximum norm by $C_b(S)_+^\circ$. Note that $w \in C_b(S)_+^\circ$ exactly when $w \in C(S)$ satisfies $0 < \delta\mathbb{1} \leq w \leq D\mathbb{1}$ for some $\delta, D > 0$. We denote the group consisting of all homeomorphisms from S to itself equipped with the group action composition by $\mathrm{Hom}(S)$.

Theorem 2.22. $\mathrm{Aut}(C_0(S))$ *is isomorphic to* $C_b(S)_+^\circ \times$ *Hom*(S) *endowed with the group action* $(w, \pi) \bullet (\eta, \rho) = (\eta(w \circ \rho), \pi \circ \rho)$*. Here the group isomorphism is given by* $(w, \pi) \mapsto f_{w,\pi}$*.*

Proof. We verify that $(C_b(S)_+^\circ \times \text{Hom}(S), \bullet)$ is a group. The operation $\bullet$ is closed as composition with a homeomorphism $\rho\colon S \to S$ leaves $C_b(S)_+^\circ$ invariant. Straightforward verification yields that $\bullet$ is associative. For any pair (w, π) the inverse with respect to $\bullet$ is given by $((w \circ \pi^{-1})^{-1}, \pi^{-1})$. Lastly, the identity element is $(\mathbb{1}, \text{Id}_S)$.

Theorem 2.20 yields that the map $(w, \pi) \mapsto f_{w,\pi}$ is a bijection. The group action $\bullet$ is constructed to make $(w, \pi) \mapsto f_{w,\pi}$ a group homomorphism. □

Let $X \subseteq C_0(S)$ be a separating and order dense subspace. Due to Theorem 2.20 any $f \in \text{Aut}(X)$ is of the form $f = f_{w,\pi}$ with $(w, \pi) \in C_b(S)_+^\circ \times \text{Hom}(S)$. Therefore, $\text{Aut}(X)$ is a subgroup of $\text{Aut}(C_0(S))$. In particular, this shows that automorphisms on X extend to automorphisms on $C_0(S)$. The observation to use Theorem 2.20 to extend linear order isomorphisms, leads to a general statement in the setting of order unit spaces.

Theorem 2.23. *Let (V, C, u) and (W, K, u') be order unit spaces. A linear order isomorphism $f\colon V \to W$ extends to a linear order isomorphism $f^\rho\colon V^\rho \to W^\rho$ between their Riesz completions.*

Proof. By Proposition 1.23 it suffices to verify the statement for separating order dense subspaces $X \subseteq C(S)$ and $Y \subseteq C(T)$, with S and T compact Hausdorff spaces. Let $f\colon X \to Y$ be an order isomorphism. Due to Theorem 2.20 we have $f = f_{w,\pi}$ with $w \in C(T)_+^\circ$ and $\pi\colon T \to S$ a homeomorphism. Therefore, $\hat{f} = f_{w,\pi}\colon C(S) \to C(T)$ is a well-defined linear order isomorphism that extends f. The restriction of $\hat{f}$ to $X^\rho \subseteq C(S)$ now maps into Y^ρ, as a linear order isomorphism preserves infima, suprema and linear combinations and X^ρ is generated as a Riesz space by X. Similarly, the inverse $\hat{f}^{-1}$ restricted to Y^ρ maps into X^ρ. We conclude that $f^\rho = \hat{f}|_{X^\rho}$ is the desired linear order isomorphism. □

Generally, automorphisms of $C_0(S)$ do not restrict to separating order dense subspace. For example, an automorphism $f_{w,\pi}$ on $C[0,1]$ with either w or π not differentiable does not restrict to an automorphism on $C^k[0,1]$. It is possible, however, to fully describe the automorphism group of the differentiable functions up to arbitrary order on a locally compact space. We do so in the general context of smooth manifolds.

Smooth Manifolds

We recall several elementary definitions concerning smooth manifolds (see [Lee03]). Let (M, τ) be a second countable Hausdorff space. M is called a d-dimensional *topological manifold* if there exists an open cover $(U_i)_{i \in I}$ of M such that for all $i \in I$, U_i is homeomorphic to an open subset V_i of $\mathbb{R}^d$. In that case, the collection of triplets $\mathcal{A} = \{(U_i, h_i, V_i)\colon i \in I\}$ is called an *atlas* of M, where $h_i\colon U_i \to V_i$ are homeomorphisms. One such a triplet is then called a chart of the atlas $\mathcal{A}$. $M = (X, \mathcal{A})$ is an *m-smooth manifold* if in addition for all $i, j \in I$ the gluing map $(h_i \circ h_j^{-1})|_{h_j(U_i \cap U_j)}\colon h_j(U_i \cap U_j) \to h_i(U_i \cap U_j)$ is m-times differentable as a map on $\mathbb{R}^d$, or simply a smooth manifold whenever $m = \infty$.

Let M be an m-smooth d-dimensional manifold. A continuous map $x\colon M \to \mathbb{R}$ is called m-times *differentiable* if for all charts (U, h, V) of M the map $(x \circ h^{-1})\colon V \subseteq \mathbb{R}^d \to \mathbb{R}$ is m-times differentiable. Let $C_0^\infty(M)$ be the space consisting of continuous functions from M to $\mathbb{R}$ that vanish at infinity and are infinitely many times differentiable. A useful tool when dealing with the space $C_0^\infty(M)$ is the notion of a partition of unity. Suppose $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ is an open cover of M. A *partition of unity subordinate to* $\mathcal{U}$ is a collection of continuous functions $\varphi_\alpha\colon M \to [0, 1]$, $\alpha \in A$, such that $\operatorname{supp}(\varphi_\alpha) \subseteq U_\alpha$, $\{\operatorname{supp}(\varphi_\alpha)\colon \alpha \in A\}$ is a locally finite cover and $\sum_{\alpha\in A} \varphi_\alpha = 1$. Since the supports of the φ_α form a locally finite cover, $\sum_\alpha \varphi_\alpha$ has only finitely many non-zero terms in a neighborhood around every point and we encounter no convergence problems. Such a partition of unity is called *m-smooth* if every φ_α is a m-smooth function. An important result in the study of m-smooth manifolds is the existence of a m-smooth partition of unity subordinate to any given open cover (see [Lee03, Theorem 2.25, p.54]). A useful consequence of the existence of an m-smooth partition of unity is the existence of m-smooth *bump functions* on M. Let U and V be open subsets of M such that $\overline{V} \subseteq U$ holds. Letting $U_1 = U$ and $U_2 = M\backslash\overline{V}$ we get an open cover $\{U_1, U_2\}$ of M, hence there exists a subordinated m-smooth partition of unity $\{\varphi_1, \varphi_2\}$. Observe that φ_1 is an m-smooth map on M with values in $[0, 1]$, supported in U and constantly one on $\overline{V}$. A map φ_1 satisfying these properties is called an m-smooth bump function of V supported in U.

For the remainder of this section let M be an n-dimensional locally compact m-smooth manifold with $m \in \mathbb{N} \cup \{\infty\}$ and $k \le m$ an integer or $k = \infty$. We argue that our results concerning Riesz* homomorphisms and order isomorphisms apply to $C_0^k(M)$ in Proposition 2.25. Before proving this it is convenient to understand the pervasive property of spaces of continuous functions.

Lemma 2.24. *Let S be a compact Hausdorff space and $X \subseteq C_0(S)$ a pre-Riesz space. Then X is pervasive in $C_0(S)$ if and only if for every non-empty open $U \subseteq S$ there exists a positive non-zero $x \in X$ with supp$(x) \subseteq U$.*

Proof. Suppose X is pervasive in $C_0(S)$ and $U \subseteq S$ is non-empty and open. By Urysohn's lemma there exists a non-zero $y \in C_0(S)^+$ with $\operatorname{supp}(y) \subseteq U$. Due to the pervasive assumption on X, there exists a non-zero $x \in X$ with $0 \le x \le y$. In particular, we get $\operatorname{supp}(x) \subseteq \operatorname{supp}(y) \subseteq U$.

Conversely, suppose that latter condition is satisfied. Let $y \in C_0(S)_+$ be non-zero. We fix $0 < \epsilon < \|y\|_\infty$. Now the set $U := \{s \in S\colon y(s) > \epsilon\}$ is a non-empty open subset of S. Let $z \in X$ with $\operatorname{supp}(z) \subseteq U$. Then $x = \epsilon z/\|z\|_\infty \in X$ is non-zero and positive and is constructed to satisfy $x \le y$. We conclude that X is pervasive in $C_0(S)$. □

Proposition 2.25. *The space $C_0^k(M)$ is a separating, pervasive and order dense subspace of $C_0(M)$.*

Proof. It is sufficient to argue the case where $k = \infty$. The existence of smooth bump functions in $C_0^\infty(M)$ described above, immediately yields that $C_0^\infty(M)$ separates the points of M, and is pervasive due to Lemma 2.24. Due to Lemma 1.6 it suffices to show that $C_0^\infty(M)$ is majorizing in $C_0(M)$ to obtain order denseness.

Lemma 2.23 in [Lee03] states that there exists a countable locally finite cover $(U_n)_{n=1}^\infty$ of M consisting of precompact open sets. Let $W_1 = U_1$ and observe that (U_n) covers the compact set $\overline{W}_1$, hence there exist $n_1, ..., n_k \in \mathbb{N}$ such that $\overline{W}_1 \subseteq \bigcup_{j=1}^k U_{n_j} =: W_2$. Inductively, we obtain a cover (W_n) of M consisting of precompact open sets satisfying $\overline{W}_n \subseteq W_{n+1}$, for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ let $z_n \in C_0(M)$ be a bump function of W_n supported in W_{n+1} and let $z = \sum_n 2^{-n} z_n$. We conclude that $z \in C_0(M)$ is positive and vanishes nowhere.

Suppose $x \in C_0(M)$ is positive. We aim to construct a $y \in C_0^\infty(M)$ that dominates x. Remark that we may assume without loss of generality that f vanishes nowhere as we can replace x by $x \vee z$. For convenience sake we rescale x to have sup-norm equal to one. For $n \in \mathbb{N}$ we define the open set $V_n = \{p \in M : 2^{-(n+2)} < x(p) < 2^{-n}\}$ and $V_0 = \{p \in M : x(p) > 2^{-2}\}$. The collection $(V_n)_{n=0}^\infty$ is a locally finite countable open cover of $\{x > 0\}$, which equals M. Let $(\varphi_n : M \to \mathbb{R})_{n=0}^\infty$ be a smooth partition of unity subordinate to $(V_n)_{n=0}^\infty$ and define

$$y(p) := \sum_{j=0}^\infty 2^{-j} \varphi_j(p), \qquad p \in M.$$

For any point $p \in M$ only finitely many terms are non-zero in a neighborhood of p, hence y is well-defined and smooth. Let $\epsilon > 0$ be given and let $j_0 \in \mathbb{N}$ be such that $\epsilon > \sum_{j=j_0}^\infty 2^{-j}$, then we get

$$\{y \geq \epsilon\} \subseteq \bigcup_{n=0}^{j_0} V_n \subseteq \{x \geq 2^{-(j_0+2)}\}. \tag{2.7}$$

Indeed, whenever $p \in M \setminus \bigcup_{n=0}^{j_0} V_n$ we have $y(p) = \sum_{j=j_0}^\infty 2^{-j}\varphi_j(p) \leq \sum_{j=j_0}^\infty 2^{-j} < \epsilon$, showing the first inclusion while the second inclusion follows from the construction of the set V_n. Since x vanishes at infinity, the set on the right hand side of (2.7) is compact. Therefore, the closed set $\{y \geq \epsilon\}$ is compact, showing that y vanishes at infinity. We are left to show that $y \geq x$ holds. Let $p \in M$ and $n \in \mathbb{N}$ the largest index such that $p \in V_n$. Then we have $y(p) = \sum_{j=0}^n 2^{-j}\varphi_j(p) \geq 2^{-n} \sum_{j=0}^n \varphi_j(p) = 2^{-n}$. On the other hand, we have $x(p) < 2^{-n} \leq y(p)$ as $p \in V_n$ holds. □

We consider for a moment an open subset $S \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$. The following chain of inclusions is satisfied

$$C_0^k(S) \subseteq LC_0(S) \subseteq C_0^{k,\alpha}(S) \subseteq UC_0(S) \subseteq C_0(S).$$

Here $LC_0(S)$ denotes the Lipschitz continuous functions on S that vanish at infinity, $C_0^{k,\alpha}(S)$ is the subspace of $C_0^k(S)$ consisting of functions that are Hölder continuous with exponent $0 \leq \alpha \leq 1$, meaning that $|x(s) - x(t)| \leq C\|s-t\|^\alpha$ holds for all $s, t \in S$

and some constant $C > 0$, and $UC_0(S)$ denotes the space of all uniformly continuous functions on S that vanish at infinity. Proposition 2.25 yields that $LC_0(S), C_0^{k,\alpha}(S)$ and $UC_0(S)$ are separating, pervasive and order dense subspaces of $C_0(K)$.

Suppose M and N are m- and n-smooth manifolds of independent dimension and let $k \leq n, m$ be an integer or $k = \infty$. Combining Theorem 2.12 and Proposition 2.25 yields that any Riesz* homomorphism $f\colon C_0^k(M) \to C_0^k(N)$ is a weighted composition map $f = f_{w,\pi}$, where $w \in C_b(N)^+$ and $\pi\colon N \to M$ is continuous and proper on $\{w > 0\}$. As discussed in the previous section $\mathrm{Aut}(C_0^k(M))$ is a subgroup of $\mathrm{Aut}(C_0(M))$. We aim to give a full description of this subgroup. To this end we show that any bijective weighted composition operator on $C_0^k(M)$ has automatically k-smooth weight and composition maps. This is the content of Lemma 2.27. We consider an intermediate result concerning the existence of k-smooth maps on M that locally behave like coordinate projections in $\mathbb{R}^d$.

Lemma 2.26. *Let $p \in M$ and (U, h, V) a chart of M with $p \in U$. For any index $1 \leq n \leq d$ there exists a k-smooth function $x \in C_0(M)$ and a neighborhood U_0 of p contained in U such that $x = x_n \circ h$ on U_0, where $x_n(v_1, ..., v_d) = v_n$, for all $(v_1, .., v_d) \in V$.*

Proof. Suppose $p \in M$ is given and (U, h, V) is a chart in M containing p. Let U_0 be a neighborhood of p with $\overline{U}_0 \subseteq U$ and $\varphi\colon M \to \mathbb{R}$ a k-smooth bump function of U_0 supported in U. Define $y\colon M \to \mathbb{R}$ by $y(q) = x_n(h(q))$ for all $q \in U$ and $y(q) = 0$ elsewhere, where x_n is the n-th coordinate projection in $\mathbb{R}^d$ as in the statement. Since φ is supported in U the map x on M defined by $x = \varphi \cdot y$ is k-smooth. As φ is constantly equal to one on U_0 we conclude that $x = x_n \circ h$ on U_0. $\square$

Lemma 2.27. *If $f = f_{w,\pi}\colon C_0^k(M) \to C_0^k(N)$ is a linear order isomorphism, then w and π are k-smooth.*

Proof. Recall from Theorem 2.20 and Proposition 2.25 that there exist $\delta, D > 0$ such that $\delta\mathbb{1} \leq w \leq D\mathbb{1}$ and that π is a homeomorphism. Let $q \in N$ be given and C be a compact neighborhood of q in N. Then $\pi(C)$ is compact in M. Let K be a compact neighborhood of $\pi(C)$ and $x \in C_0^k(M)$ be a bump function of K. For all $p \in N$ we have $f(x)(p) = w(p)x(\pi(p))$. The functions $f(x)$ and w coincide on $\pi^{-1}(K)$, which contains C. As $f(x) \in C_0^k(N)$ is k-smooth, we infer that w is k-times differentiable at q.

Fix $q \in N$. Let (U, h, V) a chart of N with $q \in U$ and (U', h', V') be a chart of M with $\pi(q) \in U'$. Let $1 \leq n \leq d$. Due to Lemma 2.26 we obtain an $x \in C_0^k(M)$ and some neighborhood U_n of $\pi(q)$ contained in U' such that $x = x_n \circ h'$ holds on U_n, where x_n is the n-th coordinate projection on $\mathbb{R}^d$. Since the reciprocal of w is well-defined and k-times differentiable on N, we get $w^{-1} \cdot f(x) = x \circ \pi$ and, hence, $(x \circ \pi)$ is k-times differentiable on N. Therefore, the map $(x_n \circ h' \circ \pi)$ is k-times differentiable on $\pi^{-1}(U_n)$ which is a neighborhood of q, since π is bijective. In particular, $(f_n \circ h' \circ \pi \circ h^{-1})$ is k-times differentiable on $h(\pi^{-1}(U_n))$.

Let $W\colon = h(\pi^{-1}(U_1)) \cap ... \cap h(\pi^{-1}(U_d))$. W is a neighborhood of q. We observe that the map $(h' \circ \pi \circ h^{-1})$ is k-times differentiable on W when composed with any of the coordinate projection on $\mathbb{R}^d$. In conclusion, $(h' \circ \pi \circ h^{-1})$ is k-times differentiable at q and, hence, q is k-smooth. $\square$

We obtain the following description of the automorphism group of $C_0^k(M)$ as a corollary of Theorem 2.22 and Lemma 2.27.

Theorem 2.28. *Let M be an m-smooth manifold of arbitrary dimension and let $k \leq m$ be given, where $m, k \in \mathbb{N} \cup \{\infty\}$. The automorphism group of $C_0^k(M)$ can be described by*

$$\mathrm{Aut}(C_0^k(M)) \simeq (C_0^k(M)_+^\circ \times \mathrm{Diff}^k(M), \bullet),$$

where $\mathrm{Diff}^k(M)$ denotes the space of all k-diffeomorphisms on M and the group action is given by $(w, \pi) \bullet (\eta, \rho) = (\eta(w \circ \rho), \pi \circ \rho)$.

Positive disjointness preserving operators

In a Riesz space X two elements $x, y \in X$ are called *disjoint*, denoted by $x \perp y$, whenever $|x| \wedge |y| = 0$. A linear map $f\colon X \to Y$ between Riesz spaces is disjointness preserving whenever $x \perp y$ implies $f(x) \perp f(y)$. Recall Theorem 2.3 that states that a linear map $f\colon X \to Y$ between Riesz spaces is a Riesz homomorphism if and only if f is positive and disjointness preserving. We investigate whether an analogous result holds in the setting of Riesz* homomorphisms between pre-Riesz spaces.

We recall the concept of disjointness in general partially ordered vector spaces. A pair of elements $x, y \in X$ are defined to be *disjoint*, denoted by $x \perp y$, whenever

$$\{x + y, -x - y\}^u = \{x - y, -x + y\}^u.$$

Here the intuition is that the left- and right-hand side of this equality replace the moduli $|x + y|$ and $|x - y|$, which are equal for disjoint elements in a Riesz space. In [vGK18, Proposition 4.1.4] it is shown that two elements in a pre-Riesz space X are disjoint if and only if they are disjoint in a vector lattice cover of X in the usual sense. Therefore, if X is a pre-Riesz space as an order dense subspace of some $C(S)$, then elements $x, y \in X$ are disjoint if and only if for all $s \in S$ either $x(s)$ or $y(s)$ equals zero, as $C(S)$ is a vector lattice cover.

We start by considering a counterexample that shows that generally in pre-Riesz space not all positive disjointness preserving maps are Riesz* homomorphisms.

Example 2.29. Consider the pre-Riesz space $P[0,1]$ of all polynomials on the unit interval and the map $f\colon P[0,1] \to P[0,1]$ defined by $x \mapsto (s \mapsto \int_0^s x(t)dt)$. One easily verifies that f is positive. As $P[0,1]$ does not contain a non-trivial pair of disjoint elements, f is disjointness preserving. Suppose that f is a Riesz* homomorphism, then by Theorem 2.5 there exist $w\colon [0,1] \to \mathbb{R}_+$ and $\pi\colon [0,1] \to [0,1]$ such that for all $x \in P[0,1]$ and $s \in [0,1]$ we have $f(x)(s) = w(s)x(\pi(s))$. From the equality $f(t \mapsto 1) = (s \mapsto s)$ we obtain $w(s) = s$ for all $s \in [0,1]$ and, moreover, the equality $f(t \mapsto t) = (s \mapsto \frac{1}{2}s^2)$ then yields $\pi(s) = \frac{1}{2}s$ for all $s \in [0,1]$. Considering the polynomial $(t \mapsto t^2)$ on the one hand yields $f(t \mapsto t^2) = (s \mapsto w(s)(\pi(s))^2 = \frac{1}{2}s^3)$, while integration of the same polynomial yields $(s \mapsto \frac{1}{3}s^3)$. This contradiction yields that f is not a Riesz* homomorphism.

Example 2.29 is based on the fact that the pre-Riesz space $P[0,1]$ does not contain a non-trivial pair of disjoint elements. Therefore, it is natural to consider pervasive pre-Riesz spaces, since they contain pairs of positive disjoint elements below any pair of positive disjoint elements in the Riesz completion. We remark that $P[0,1]$ is indeed not pervasive.

A potential generalisation of the classical result in vector lattice theory then becomes that any linear map between pervasive pre-Riesz spaces is positive and disjointness preserving if and only if it is a Riesz* homomorphism. We contribute to this, currently open, problem with a less general statement. We consider pre-Riesz subspaces of $C(S)$ that have satisfy a property stronger than being prevasive. Recall that $X \subseteq C(S)$ is pervasive whenever for all non-empty and open $U \subseteq S$ there exists an $x \in X_+$ non-zero with $\operatorname{coz}(x) \subseteq U$, where $\operatorname{coz}(x) = \{s \in S \colon x(s) \neq 0\}$ denotes the co-zero set. We say that X is *pointwise pervasive* in $C(S)$ if for every $s \in S$ and neighborhood U of s there exists an $x \in X_+$ such that $s \in \operatorname{coz}(x) \subseteq U$.

Many ideas in the proof of the following theorem are inspired by [Jar90].

Theorem 2.30. *Suppose S and T are compact Hausdorff spaces and $X \subseteq C(S)$ and $Y \subseteq C(T)$ are pointwise pervasive order dense subspaces. Any linear positive disjointness preserving map $f \colon X \to Y$ is a weighted composition map and, in particular, a Riesz* homomorphism.*

Proof. Fix $t \in T$. We consider the functional $\varphi_t \colon X \to \mathbb{R}$ defined as the composition of f with the point evaluation at t:

$$\varphi_t(x) = f(x)(t), \qquad \forall x \in X.$$

We remark that φ_t is linear, positive, disjointness preserving and, in particular, order bounded. We aim to show that φ_t is given by integration against a point measure to obtain the desired $w(t)$ and $\pi(t)$. Even though we can not immediately construct a measure, we introduce the notion of a support for our functional φ_t. The *support* of φ_t, denoted by $\operatorname{supp}(\varphi_t)$, is the set of all $s \in S$ such that for all neighborhoods U of s there is an $x \in X$ with $\operatorname{coz}(x) \subseteq U$ and $\varphi_t(x) \neq 0$.

Suppose that $t \in T$ is such that $\operatorname{supp}(\varphi_t)$ contains at least two distinct points. Using that S is Hausdorff we obtain disjoint open set $U_1, U_2 \subseteq S$ and, hence, by the pervasive property of X this yields $x_1, x_2 \in X$ with $\operatorname{coz}(x_i) \subseteq U_i$ and $\varphi_t(x_i) \neq 0$ for $i = 1, 2$. However, this yields that $x_1 \perp x_2$ and, hence, contradicts φ_t being disjointness preserving.

We argue that $\operatorname{supp}(\varphi_t) = \emptyset$ if and only if $\varphi_t = 0$. Suppose the former. For all $s \in S$ there exists an open $U_s \subseteq S$ with $s \in U_s$ such that $x \in X$ with $\operatorname{coz}(x) \subseteq U_s$ implies $\varphi_t(x) = 0$. Fix $\epsilon > 0$. As X is pointwise pervasive there exist $x_s \in X$ with $s \in \operatorname{coz}(x_s) \subseteq U_s$. After rescaling if necessary we assume without loss of generality that $f_s(s) > \epsilon$. We define $V_s := \{t \in S\colon f_s(t) \geq \epsilon\}$ for all $s \in S$. We remark that the V_s are non-empty open neighborhoods of the $s \in S$, respectively. By compactness of S there exists a finite set $\{s_1, ..., s_n\} \subseteq S$ such that $S \subseteq V_{s_1} \cup ... \cup V_{s_n}$. Therefore, $x := x_1 + ... + x_n \in X$ satisfies $x \geq \epsilon \mathbb{1}_S$. Now for any $y \in X$, letting $\lambda = \|y\|_\infty/\epsilon$, we get

$$0 = -\lambda \sum_{i=1}^{n} \varphi_t(x_i) = -\lambda\varphi_t(x) \leq \varphi_t(y) \leq \lambda\varphi_t(x) = \lambda \sum_{i=1}^{n} \varphi_q(x_i) = 0.$$

This show that $\varphi_t = 0$. Conversely, the support of the zero functional is empty. We conclude for each $t \in T$ that $\mathrm{supp}(\varphi_t)$ consists of exactly one point, or is empty in which case $\varphi_t = 0$.

Let $T_N := \{t \in T\colon \varphi_t = 0\}$ and $T_C := T\backslash T_N$. The map $\pi\colon T_C \to S$ that satisfies $\mathrm{supp}(\varphi_t) = \{\pi(t)\}$ is well-defined. We argue that π is continuous. Supposing the converse, yields a net $(t_\alpha)_{\alpha\in A}$ in T_C converging to $t_0 \in T_C$ such that $s_\alpha := \pi(t_\alpha)$ converges to $s_1 \neq s_0 := \pi(t_0)$. By the Hausdorff property of S there exists disjoint open neighborhoods U_0 and U_1 of s_0 and s_1, respectively. Let $x_0 \in X$ be such that $\mathrm{coz}(x_0) \subseteq U_0$ and $\varphi_{t_0}(x_0) \neq 0$. There is an $\alpha_0 \in A$ such that $s_{\alpha_0} \in U_1$, as $s_\alpha \to s_1$, and simultaneously that $f(x_0)(s_{\alpha_0}) \neq 0$, as $t_\alpha \to t_0$ and $f(x_0) \in C(T)$. However, now we can find $x_1 \in X$ with $\mathrm{coz}(x_1) \subseteq U_1$ and $f(x_1)(t_{\alpha_0}) \neq 0$, which by construction contradicts f from being disjointness preserving.

For any $t \in T$ we extend φ_t to a positive linear functional $\tilde{\varphi}_t\colon C(S) \to \mathbb{R}$, using the extension theorem of Kantorovich. The Riesz-Markov-Kakutani representation theorem now yields a Borel measure μ_t such that

$$\tilde{\varphi}_t(x) = \int_S x \, d\mu_t, \qquad x \in C(S).$$

We aim to show for $t \in T_C$ that μ_t is the Dirac measure at $\pi(t)$. Let $t \in T_C$ be given. It suffices to argue that the support of the measure μ_t, which we denote by $\mathrm{supp}(\mu_t)$, equals the singleton $\{\pi(t)\}$. By construction it is clear that $\pi(t) \in \mathrm{supp}(\mu_t)$. In order to verify the other inclusion we let $s \in \mathrm{supp}(\mu_t)\backslash\{\pi(t)\}$. Let $U \subseteq S$ open with $s \in U$ and $\pi(t) \neq \overline{U}$. Using the pointwise pervasive property of X yields an $x \in X$ such that $s \in \mathrm{coz}(x) \subseteq U$. As $\{s > \frac{1}{2}\}$ is an open neighborhood of s it has strictly positive measure and, hence

$$\int_S x \, d\mu_t \geq \frac{1}{2}\mu_t(\{x > \frac{1}{2}\}) > 0.$$

In particular, this yields $\varphi_t(x) = \int_S x \, d\mu_t > 0$, which contradicts $\pi(t)$ being the unique supporting point of φ_t. We conclude that for all $t \in T_C$ and $x \in X$ we have

$$\int_S x \, d\mu_t = x(\pi(t))\tilde{\varphi}_t(\mathbb{1}_S).$$

For $t \in T_C$ letting $w(t)\colon = \tilde{\varphi}_t(\mathbb{1}_S) > 0$, we obtain for all $x \in X$

$$\varphi_t(x) = w(t)x(\pi(t)). \tag{2.8}$$

Now we extend π to a map from T to S and put $w(t) := 0$ for all $t \in T_N$. We remark that (2.8) is now satisfies for all $x \in X$ and $t \in T$. As X is majorizing in $C(S)$, X contains a strictly positive function $x \in X$. The fact that $f(x)$ is continuous on T, and $x \circ \pi$ does not vanish, ensures that w is continuous on T.

The last part of the assertion, which states that f is a Riesz* homomorphism, follows from Theorem 2.5(ii). □

Chapter 3

Linearity of order isomorphisms

A fundamental problem in the study of partially ordered vector spaces is understanding the structure of order isomorphisms, i.e., an order preserving bijection whose inverse is also order preserving. Of particular interest is understanding in which partially ordered vector spaces all order isomorphisms are linear after translation. Research on this question originates from Special Relativity where causal order is considered on the Minkowski spacetime. During the 1950s and 1960s various results appeared dealing with finite dimensional spaces. Alexandrov and Ovčinnikova [AO53] and Zeeman [Zee64] have shown that the order isomorphisms from the causal cone onto itself are linear. Alexandrov [Ale67] extended his result to order isomorphisms on finite dimensional spaces ordered by a cone of which every extreme ray is engaged, where an extreme ray is considered engaged whenever it is contained in the linear span of the other extreme rays. Rothaus [Rot66] proved a similar result where the domain of the order isomorphism could also be the interior of the cone. In the 1970s Noll and Schäffer made numerous contributions to this area in a series of papers, [NS78] [NS77] [Sch77] [Sch78]. Like Alexandrov they considered the case where the cone is the sum of engaged extreme rays viewed in the general setting of infinite dimensional spaces.

In many natural settings, such as in operator algebras, their result however is not applicable. Molnár in [Mol01] considered order isomorphisms on $B(H)_{sa}^+$, the cone consisting of self-adjoint positive semi-definite operators on a Hilbert space H, and showed among other results that they are linear. While $B(H)_{sa}^+$ contains many engaged extreme rays, namely those rays spanned by projections of rank 1, $B(H)_{sa}^+$ does not satisfy the conditions of Noll and Schäffer's result. Molnár's proof uses the spectral theorem to reduce to Rothaus's finite dimensional case and is, therefore, not suitable for extension to a larger class of partially ordered vector spaces. In this chapter, which is based on [LvGvI], we provide a generalisation of Noll and Schäffer's result in the form of a condition on infinite dimensional cones that guarantees that all order isomorphisms are affine, which is sufficiently mild to include Molnár's result.

Before outlining our approach and results, we interpose with a short discussion concerning the domains of the order isomorphisms we study. Šemrl in [Sem17] characterised order isomorphisms on order intervals of $B(H)_{sa}$. From his results we see that all order isomorphisms on an interval are affine only when the interval is unbounded from above. Our results concern domains, which contain all upper bounds of their elements, called upper sets. In particular, the whole space, the cone, the interior of the cone and translations thereof are upper sets.

Our approach is to use the fact that order isomorphisms preserve infima and suprema. This allows us to weaken the necessary condition imposed on the cone by Noll and Schäffer to the cone being merely equal to the inf-sup hull of the span of its engaged extreme rays. From the results stated in [NS77] it is not clear, however, how they can be extended with this method. Restricting an order isomorphism to the span of the engaged extreme rays is generally not possible. So we carefully rework the ideas of their proofs to obtain a more general alternative to their result. We change their assertion that all order isomorphism are affine provided that the cone is the sum of its engaged extreme rays, to order isomorphims being affine on the sum of the engaged extreme rays of the cone. From here we can extend the set on which the order isomorphism is affine to all elements in the domain that we can reach with taking infima and suprema of positive sums of engaged extreme vectors. This approach also leads us to some interesting corollaries.

In finite dimensions the existence of a disengaged extreme ray always yields a non-linear order isomorphism. This for example follows from the work by Artstein-Avidan and Slomka [A-AS11], where they show in Theorem 1.7 that order isomorphisms have a certain *diagonal* form as follows. Let K be a closed generating cone in $\mathbb{R}^n$ and $f : X \to X$ and order isomorphism, where X is K, K° or $\mathbb{R}^n$, then there exist linearly independent extreme vector $v_1, \ldots, v_n \in K$ and bijective increasing maps $f_1, \ldots, f_n \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(\sum_{i=1}^{n} \lambda_i v_i) = \sum_{i=1}^{n} f_i(\lambda_i) f(v_i).$$

In this case, we have $f_j(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$ whenever the corresponding v_j is engaged. We provide an infinite dimensional analogue to this description of order isomorphisms.

Another example of a non-linear order isomorphism is taking the piontwise third power in a space of continuous functions. More precisely, for a compact Hausdorff space S the map $f \colon C(S) \to C(S)$ defined by $f(x)(s) = x(s)^3$ for all $x \in C(S)$ and $s \in S$ is a non-linear order isomorphism. Schäffer [Sch77] shows that a homogeneous order isomorphisms on $C(S)_+^\circ$, and on suitable subcones, are always linear. In [Sch78] he further strengthens this result by showing that the same result holds for the interior of a cone in an order unit space. We are able to apply our infinite dimensional analogue of the diagonal form of Artstein-Avidan and Slomka to derive a condition, alternative to the result of Schäffer, under which homogeneous order isomorphism are necessarily linear.

Noll and Schäffer's approach

Throughout this section (X, C) and (Y, K) will be Archimedean partially ordered vector spaces. We briefly recall the definitions of linear and affine maps, as introduced in Section 1.1. Let $U \subseteq X$ be a subset. A map $f\colon U \to Y$ is called *linear* of *affine* if it is the restriction of a linear map $F\colon \mathrm{span}(U) \to Y$ or a an affine map $F\colon \mathrm{aff}(U) \to Y$, respectively.

Initially we only consider order isomorphisms $f\colon [a, \infty) \to [b, \infty)$, where $a \in X$ and $b \in Y$. However, in the main result, Theorem 3.15, we remove this constraint and obtain a result that holds for more general domains.

A key idea to analyse order isomorphisms is to consider extreme half-lines. This idea has been exploited to study order isomorphism on finite dimensional partially ordered vector space [A-AS11], as well as in infinite dimensions in [NS77]. In finite dimensions, a closed cone equals the positive linear span of its extreme rays. In infinite dimensions, however, the extreme rays are not necessarily so plentiful. Namely, there exist non-trivial cones that have none or only very few extreme rays.

We briefly recall some terminology on extreme rays, for a thourough overview see Section 1.3. A vector $s \in C\backslash\{0\}$ is called an *extreme vector* (of the cone C) if $0 \leq x \leq s$ implies that $x = \lambda s$ for some $\lambda \geq 0$. For an element $x \in C\backslash\{0\}$ we define the *ray through* x as $R_x = \{\lambda x : \lambda \geq 0\}$. If $e \in C$ is an extremal vector, R_e is said to be an *extreme ray*. Given an extreme ray R and $z \in X$, then we call $z + R$ an *extreme half-line with apex* z. In the sequel we encounter the use of negative extreme vectors and hence we adopt the following terminology. A vector $x \in X$ is an *extreme vector* in X if $x \in (R \cup -R)\backslash\{0\}$ for an extreme ray R of the cone, in the case that $x \in R\backslash\{0\}$ we say that x is an extreme vector of C.

The following order theoretic characterisation of extreme half-lines is due to Noll and Schäffer, see [NS77, Proposition 1]. For completeness we provide a proof.

Proposition 3.1. *Let $x \in X$ be given. A subset $H \subseteq [x, \infty)$ is an extreme half-line with apex x if and only if H is maximal among subsets $G \subseteq [x, \infty)$ with $x \in G$ that satisfy:*

(P1) G is directed.

(P2) For any $y \in G$ the order interval $[x, y]$ is totally ordered.

(P3) G contains at least two distinct points.

Proof. Suppose $H \subseteq X$ is maximal among subsets $G \subseteq [x, \infty)$ that satisfy properties (P1)–(P3). We first argue that H is contained in a half-line. Let $y, w \in H$ be given, so $x \leq y, w$. Due to (P1) there exists a $z \in H$ such that $y, w \leq z$. Since $\leq$ is preserved under addition, (P2) guarantees that the order interval $[0, z - x]$ is totally ordered. Clearly, it contains $y - x$, $w - x$, and $\lambda(z - x)$ for all $0 \leq \lambda \leq 1$. Therefore, by Lemma 1.14 there exist $\alpha, \beta \geq 0$ such that $y - x = \alpha(z - x)$ and $w - x = \beta(z - x)$. This shows that y and w are on the half-line through z with apex x. We conclude that any pair of points in H lie on a half-line with apex x, and hence H is contained in a half-line with apex x. Let R be a ray in C such that $H \subseteq x+R$.

By (P3) there exists an $r \in C\setminus\{0\}$ such that $x+r \in H$, so $x+R = \{x+\lambda r\colon \lambda \geq 0\}$. Clearly $x+R$ satisfies properties (P1) and (P3). We now show that $x+R$ also satisfies (P2). Consider $y = x+\lambda r$ with $\lambda > 0$. Then $[x,y] = [x, x+\lambda r]$ equals the interval $[x,r]$ up to dilation. We know that $[x, x+r]$ is totally ordered, as $x+r \in H$ and H satisfies property (P2). Hence $[x,y]$ is also totally ordered. It now follows from the maximality assumption on H that $H = x+R$.

To see that $x+R$ is an extreme half-line, we note that $[0,r]$ is totally ordered, as $[x, x+r]$ is totally ordered. It follows from Lemma 1.14 that r is an extreme vector.

Conversely, suppose $H = x+R$ is an extreme half-line. Clearly H satisfies properties (P1)–(P3). Suppose $G \supseteq H$ also satisfies (P1)–(P3) and $y \in G$. Since G is directed, there exists a $z \in G$ with $z \geq y, x+r$. Moreover, $[x,z]$ is totally ordered by (P2). Hence, $[0, z-x]$ is totally ordered and $y-x, r \in [0, z-x]$. If $y-x \leq r$, then there is a $\mu \geq 0$ such that $y-x = \mu r$, as r is extreme, so that $y = x+\mu r \in H$. Otherwise, we have $r \leq y-x$ and for each $0 \leq \lambda \leq 1$ we have $\lambda(y-x) \in [0, z-x]$, so $r \leq \lambda(y-x)$ or $\lambda(y-x) \leq r$. By Lemma 1.14 it follows that there is a $\sigma \geq 0$ such that $r = \sigma(y-x)$. Then $\sigma \neq 0$ and $y = x+\sigma^{-1}r \in H$. □

We note that property (P3) is only a necessary condition if C does not have any extreme rays and can be dropped otherwise.

As a direct corollary we obtain the following result.

Corollary 3.2. *If $f\colon [a,\infty) \to [b,\infty)$ is an order isomorphism, then f maps an extreme half-line with apex $x \in [a,\infty)$ onto an extreme half-line with apex $f(x) \in [b,\infty)$.*

Proof. Suppose that R is an extreme ray of C. Then $f(x+R) \subseteq f(x)+K$ and satisfies properties (P1)–(P3), as f is an order isomorphism. So by Proposition 3.1 we find that $f(x+R) = f(x)+S$, where S is an extreme ray of K. □

Our next step is to show that order isomorphisms $f\colon [a,\infty) \to [b,\infty)$ possess an additive property on extreme half-lines, which is a combination of Proposition 3 and Lemma 1 in [NS77]. We provide an alternative proof for the reader's convenience.

Proposition 3.3. *Let R and S be distinct extreme rays of C and $f\colon [a,\infty) \to [b,\infty)$ be an order isomorphism. For all $x \in [a,\infty)$, $r \in R$ and $s \in S$ we have*

$$f(x+s+r) - f(x+s) = f(x+r) - f(x). \tag{3.1}$$

Proof. The equality in the statement holds trivially if either r or s equals zero. Assume $r \neq 0$ and $s \neq 0$. Then $R_j = x+js+R$ for $j \in \{0,1,2\}$ are three distinct parallel extreme half-lines. Due to Corollary 3.2, their images $f(R_j)$ are extreme half-lines in Y and they are distinct as f is injective. For each $\lambda \geq 0$, the set $x+S+\lambda r$ is an extreme half-line that intersects R_j for each $j \in \{0,1,2\}$, so, by Corollary 3.2, $f(x+S+\lambda r)$ is an extreme half-line and

$$f(x+S+\lambda r) \text{ intersects } f(R_j) \text{ for each } j \in \{0,1,2\} \text{ and } \lambda \geq 0. \tag{3.2}$$

We obtain that $f(x+S+\lambda r)$ is not parallel to any of the $f(R_j)$, as R and S are distinct and f is injective.

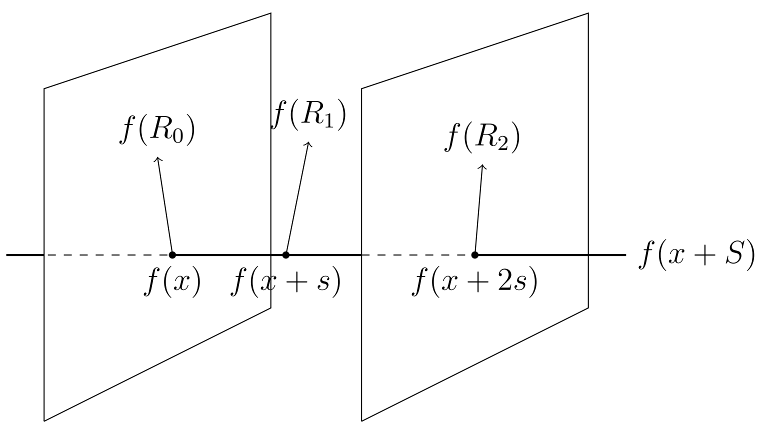
We aim to show that $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel. We do so in two steps. As a first step we show that if two of them are parallel, then all three of them are parallel. Indeed, assume that $f(R_j)$ and $f(R_k)$ are parallel, with $j,k \in \{0,1,2\}$, $j \neq k$. Since $f(R_j)$ and $f(R_k)$ are distinct parallel half-lines, it follows from (3.2) that the half-line $f(x+S+\lambda r)$ is in their affine span for every $\lambda \geq 0$. Then the half-line $f(R_i)$ with $i \in \{0,1,2\} \setminus \{j,k\}$ is in that affine span, too, as it intersects $f(x+S+\lambda r)$ for two distinct values of λ. Thus, $f(x+S)$, $f(R_i)$, and $f(R_j)$ are three extreme half-lines in the affine plane spanned by $f(R_j)$ and $f(R_k)$. By Lemma 1.13, it follows that at least two of the half-lines $f(x+S)$, $f(R_i)$, and $f(R_j)$ must be parallel, which yields that $f(R_i)$ and $f(R_j)$ must be parallel. Thus, $f(R_i)$, $f(R_j)$, and $f(R_k)$ are parallel.

As a second step we argue by contradiction that at least two of the half-lines $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel. For $i \in \{0,1,2\}$, take $w_i \in Y$ such that

$$f(R_i) = \{f(x+is) + \lambda w_i \colon \lambda \geq 0\}.$$

Suppose that no two of the three extreme half-lines $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel. After translation they correspond to three distinct extremal rays, so that Lemma 1.13 yields that w_0, w_1, and w_2 are linearly independent. Define

$$\begin{aligned} W_0 &= f(x) + \operatorname{span}\{w_0, w_2\}, \\ W_2 &= f(x+2s) + \operatorname{span}\{w_0, w_2\}, \\ \ell_1 &= \{f(x+s) + \lambda w_1 \colon \lambda \in \mathbb{R}\}. \end{aligned}$$



We observe that W_0 and W_2 are parallel and distinct planes. Moreover, $f(R_0) \subseteq W_0$, $f(R_2) \subseteq W_2$ and $f(R_1) \subseteq \ell_1$. The affine span $\operatorname{aff}(W_0, W_2)$ of W_0 and W_2 is three dimensional and contains ℓ_1. Indeed, for every $z \in f(R_1)$ there is $\lambda \geq 0$ with $z = f(x+s+\lambda r)$, and by (3.2), $\operatorname{aff}(W_0, W_2)$ contains the half-line $f(x+S+\lambda r)$. This shows that $f(R_1) \subseteq \operatorname{aff}(W_0, W_2)$, and hence $\ell_1 \subseteq \operatorname{aff}(W_0, W_2)$. Since w_1 is linearly independent of w_0 and w_2, we conclude that ℓ_1 intersects W_0 and W_2.

We proceed by showing that the half-line $f(R_1)$ intersects W_0 or W_2. Loosely speaking, the point $f(x+s)$ on ℓ_1 lies between W_0 and W_2 and, therefore, the points where ℓ_1 intersects W_0 and W_2 cannot be both at the same side of $f(x+s)$. To make this idea precise, let $v \in Y$ be such that

$$f(x+S) = \{f(x) + \lambda v \colon \lambda \geq 0\}.$$

Observe that $v \in K$, as $f(x+S) \subseteq [f(x), \infty)$. Then

$$\operatorname{aff}(W_0, W_2) = \{f(x+s) + \lambda w_0 + \mu w_2 + \sigma v \colon \lambda, \mu, \sigma \in \mathbb{R}\}.$$

As $f(x+s) + w_1 \in f(R_1) \subseteq \operatorname{aff}(W_0, W_2)$, there are $\lambda, \mu, \sigma \in \mathbb{R}$ such that $w_1 = \lambda w_0 + \mu w_2 + \sigma v$. By linear independence of w_0, w_1 and w_2, we have $\sigma \neq 0$. Consider the case $\sigma < 0$. Then $f(R_1)$ intersects W_0, so there is a $t > 0$ such that $f(x+s+tr) \in W_0$. As $f(x+R) = f(R_0) \subseteq W_0$, it follows that the half-line $f(x+S+tr)$ contains two distinct points of W_0, so that $f(x+S+tr) \subseteq W_0$. Therefore $f(x+2s+tr) \in W_0 \cap f(R_2) \subseteq W_0 \cap W_2$, which is a contradiction. Otherwise, in case $\sigma > 0$, then $f(R_1)$ intersects W_2, and we similarly arrive at a contradiction. Hence at least two of the half-lines $f(R_0)$, $f(R_1)$, and $f(R_2)$ are parallel, so by the first step all three of them are parallel.

Now we complete the proof. As $f(R_0)$ and $f(R_1)$ are parallel, we have that the vectors $f(x+r) - f(x)$ and $f(x+s+r) - f(x+s)$ have the same direction. By interchanging the roles of R and S we obtain that the vectors $f(x+s) - f(x)$ and $f(x+s+r) - f(x+r)$ have the same direction. Thus, $f(x)$, $f(x+r)$, $f(x+s+r)$, and $f(x+s)$ are the consecutive corners of a parallellogram, which concludes the proof. □

It is interesting to note that the proof of Proposition 3.3 does not work if the domain of the order isomorphism is bounded. In fact, there exist examples of order isomorphisms on bounded order intervals for which equation (3.1) does not hold, see for example [Sem17] where order isomorphisms on order intervals in $B(H)_{sa}$ are studied.

The followin observation is a simple consequence of the previous proposition.

Corollary 3.4. *Suppose r and s are extreme vectors in X such that $r \neq \lambda s$ for all $\lambda \in \mathbb{R}$ and $f \colon [a, \infty) \to [b, \infty)$ is an order isomorphism. If $x \in [a, \infty)$ is such that $x+r+s, x+r, x+s \in [a, \infty)$ then*

$$f(x+r+s) - f(x+r) = f(x+s) - f(x).$$

Proof. We only discuss the proof for the case $r < 0$ and $s < 0$, and leave the other two remaining cases to the reader, as they are proved in a similar way. By writing $y = x + r + s$, we get

$$\begin{aligned} f(x+r+s) - f(x+s) &= f(y) - f(y-r) \\ &= f(y-s) - f(y-r-s) \\ &= f(x+r) - f(x), \end{aligned}$$

by Proposition 3.3. □

Using this corollary we show the following lemma.

Lemma 3.5. *Let $f\colon [a,\infty)\to[b,\infty)$ be an order isomorphism. Suppose $s_1,\dots,s_n,r$ be extreme vectors in X such that $r\neq\lambda s_i$ for all $\lambda\in\mathbb{R}$ and $i=1,\dots,n$. If $x, x+r+s_1+\dots s_n, x+s_1+\dots s_n, x+r\in[a,\infty)$, then*

$$f\left(x+r+\sum_{i=1}^n s_i\right)-f\left(x+\sum_{i=1}^n s_i\right)=f(x+r)-f(x).$$

Proof. By relabelling we may assume that $s_1,\dots,s_k\geq 0$ and $s_{k+1},\dots,s_n<0$ for some $k\in\{1,\dots,n\}$. Then $x+r+\sum_{i=1}^k s_i\in[a,\infty)$ and $x+\sum_{i=1}^k s_i\in[a,\infty)$ for $k=1,\dots,n$. By Corollary 3.4 we have

$$f\left((x+\sum_{i=1}^{n-1}s_i)+s_n+r\right)-f\left((x+\sum_{i=1}^{n-1}s_i)+s_n\right)=f\left(x+\sum_{i=1}^{n-1}s_i+r\right)-f\left(x+\sum_{i=1}^{n-1}s_i\right).$$

Repeating this argument yields the desired conclusion. □

Lemma 3.6. *Let $f\colon [a,\infty)\to[b,\infty)$ be an order isomorphism. Suppose $x\in[a,\infty)$ and $s_1,\dots,s_n$ are extreme vectors in X such that $s_i\neq\lambda s_j$ for all $\lambda\in\mathbb{R}$ and $i\neq j$, $x+s_1+\dots+s_n\in[a,\infty)$, and $x+s_i\in[a,\infty)$ for all $i=1,\dots,n$, then*

$$f\left(x+\sum_{i=1}^n s_i\right)-f(x)=\sum_{i=1}^n\left(f(x+s_i)-f(x)\right).$$

Proof. By relabelling we may assume that $s_1,\dots,s_k\geq 0$ and $s_{k+1},\dots,s_n<0$ for some $k\in\{1,\dots,n\}$. Then $x+\sum_{i=1}^k s_i\in[a,\infty)$ for all $k=1,\dots,n$. Using a telesoping sum and Lemma 3.5 we obtain

$$\begin{aligned} f\left(x+\sum_{i=1}^n s_i\right)-f(x) &= f\left(x+\sum_{i=1}^n s_i\right)-f\left(x+\sum_{i=1}^{n-1} s_i\right)+\dots+f(x+s_1)-f(x)\\ &= \sum_{m=1}^n (f(x+s_m)-f(x)). \end{aligned}$$

□

Let $\mathcal{R}$ denote the collection of extreme rays in C. For $\mathcal{S}\subseteq\mathcal{R}$ define

$$[a,\infty)_{\mathcal{S}}=\{a+r_1+\dots+r_n\in[a,\infty)\colon \text{there exist } S_i\in\mathcal{S} \text{ with } r_i\in S_i, \text{ for } i=1,\dots,n\}. \tag{3.3}$$

Lemma 3.7. *Let $f\colon [a,\infty)\to[b,\infty)$ be an order isomorphism. Suppose that $x,y\in[a,\infty)_{\mathcal{R}}$ and $y-x=s_1+\dots+s_n$ with s_i an extreme vector in X for $i=1,\dots,n$. If r is an extreme vector in X with $r\neq\lambda s_i$ for all $\lambda\in\mathbb{R}$ and $i=1,\dots,n$, and $x+r, y+r\in[a,\infty)$, then*

$$f(x+r)-f(x)=f(y+r)-f(y).$$

Proof. Note that we get

$$\begin{aligned} f(y+r)-f(y) &= f(x+(y-x)+r)-f(x+(y-x)) \\ &= f(x+s_1+\ldots+s_n+r)-f(x+s_1+\ldots+s_n) \\ &= f(x+r)-f(x), \end{aligned}$$

by Lemma 3.5. □

In the setting of Lemma 3.7, if $r = \lambda s_i$ for some λ and i, and $r \in \operatorname{span}\{s\colon s \in S \text{ and } S \in \mathcal{R}\setminus\{R\}\}$ where $R = \{\lambda r\colon \lambda \geq 0\}$, then one could replace s_i by a linear combination of extreme vectors not contained in $R \cup -R$ and thus obtain $y - x = s_1' + \cdots + s_m'$ with $r \neq \lambda s_j'$ for all λ and j. Then the conclusion of Lemma 3.7 still holds. This motivates the following definition due to [NS77] .

Definition 3.8. Let $\mathcal{S}$ be a collection of rays in a cone C in a vector space X. A ray $R \in \mathcal{S}$ is called *engaged (in $\mathcal{S}$)* whenever

$$R \subseteq \operatorname{span}(\mathcal{S}\setminus\{R\}) = \operatorname{span}\{s\colon\ s \in S \text{ and } S \in \mathcal{S}\backslash\{R\}\}$$

holds, and R is called *disengaged (in $\mathcal{S}$)* otherwise.

It can be shown that an extreme ray of a finite dimensional cone is disengaged (in the set of extreme rays) if and only if the cone equals the Cartesian product of the ray and another subcone. Cones that do not allow such a decomposition are considered in [Ale67].

Recall that $\mathcal{R}$ denotes the collection of all extreme rays of C. We denote the collection of all engaged extreme rays in $\mathcal{R}$ by $\mathcal{R}_E$ and the collection of all disengaged extreme rays in $\mathcal{R}$ by $\mathcal{R}_D$. We remark that being an engaged ray is relative to the collection it is viewed in. Nevertheless, we have that the elements of $\mathcal{R}_E$ are again engaged in $\mathcal{R}_E$. For simplicity we say that an extreme vector $r \in R \cup -R$ is *engaged* if $R \in \mathcal{R}_E$.

Lemma 3.9. *If r is an extreme vector in X, then the following assertions hold:*

(i) *$f(x+\lambda r)-f(x)$ is a scalar multiple of $f(x+r)-f(x)$ for $x \in [a,\infty)$ and $\lambda \in \mathbb{R}$ such that $x+r, x+\lambda r \in [a,\infty)$;*

(ii) *If r is engaged and $x, y, x+r, y+r \in [a,\infty)$ and $y - x \in \operatorname{span}\mathcal{R}$, then*

$$f(x+r)-f(x) = f(y+r)-f(y).$$

Proof. Assertion (i) follows from Corollary 3.2. Remark that if r is engaged then there exist extreme vectors $s_1, \ldots, s_n$ with $y - x = s_1 + \ldots + s_n$ such that $r \neq \lambda s_i$ for all $\lambda \in \mathbb{R}$ and $i = 1, \ldots, n$. So (ii) follows from Lemma 3.7. □

The following result is an extension of [NS78, Corollary A1]. Recall that $\mathcal{R}_E$ denotes the collection of engaged extreme rays in $\mathcal{R}$. Following our definition in (3.3) we get

$$[a,\infty)_{\mathcal{R}_E} = \{a+r_1+\ldots+r_n \in [a,\infty)\colon r_i \in C \text{ an engaged extreme vector for } i = 1,\ldots,n\}.$$

Theorem 3.10. *If $f\colon [a,\infty) \to [b,\infty)$ is an order isomorphism, then f is affine on $[a,\infty)_{\mathcal{R}_E}$.*

Proof. Let R be an engaged extreme ray of C and fix $r \in R\backslash\{0\}$. Let $\lambda \in \mathbb{R}$ and $x \in [a,\infty)_{\mathcal{R}_E}$ be such that $x + \lambda r \geq a$. Then $x, x + r, x + \lambda r \in [a,\infty)$. So by Lemma 3.9(i) there exists a unique $g_{r,x}(\lambda) \in \mathbb{R}$ with

$$f(x + \lambda r) - f(x) = g_{r,x}(\lambda)(f(x + r) - f(x)).$$

It follows from Lemma 3.9(ii) that $g_{r,x}(\lambda)$ does not depend on x. Thus there is a unique function $g_r\colon \mathbb{R} \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$ and $x \in [a,\infty)_{\mathcal{R}_E}$ with $x + \lambda r \geq a$ we have

$$f(x + \lambda r) - f(x) = g_r(\lambda)(f(x + r) - f(x)). \tag{3.4}$$

Clearly, $g_r(1) = 1$ and g_r is order preserving. Moreover, for $\lambda, \mu \in \mathbb{R}$ there exists $x \in [a,\infty)_{\mathcal{R}_E}$ such that $x + \lambda r \geq a$, $x + \mu r \geq a$, and $x + \lambda r + \mu r \geq a$. Then

$$\begin{aligned} g_r(\lambda + \mu)(f(x + r) - f(x)) &= f(x + (\lambda + \mu)r) - f(x) \\ &= f(x + \lambda r + \mu r) - f(x + \lambda r) + f(x + \lambda r) - f(x) \\ &= g_r(\mu)(f(x + \lambda r + r) - f(x + \lambda r)) + g_r(\lambda)(f(x + r) - f(x)). \end{aligned}$$

Since r is engaged, Lemma 3.9(ii) gives

$$f(x + \lambda r + r) - f(x + \lambda r) = f(x + r) - f(x).$$

Note that $f(x + r) - f(x) \neq 0$, as $r \neq 0$ and f is injective. So, it follows that

$$g_r(\lambda + \mu) = g_r(\lambda) + g_r(\mu).$$

As g_r is monotone increasing, additive, and $g_r(1) = 1$, a result by Darboux (see [**?**, Theorem 1 in Section 2.1]) yields that $g_r(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$.

To show that f is convex on $[a,\infty)_{\mathcal{R}_E}$, let $x, y \in [a,\infty)_{\mathcal{R}_E}$ and $0 \leq t \leq 1$. Then $x = a + \sum_{i=1}^n \lambda_i r_i$ and $y = a + \sum_{i=1}^n \mu_i r_i$ where each $r_i \in C\backslash\{0\}$ is an engaged extreme vector and $r_i \neq \lambda r_j$ for all $\lambda \in \mathbb{R}$ and $i \neq j$. Moreover, $\lambda_i, \mu_i \geq 0$ and $\lambda_i + \mu_i \neq 0$ for all i. Put $s_i = (t\lambda_i + (1-t)\mu_i)r_i$. As $a + s_i \in [a,\infty)$ for all i, we can apply Lemma 3.6 to get

$$\begin{aligned}
f(tx+(1-t)y)-f(a) &= f(a+\sum_{i=1}^{n} s_i)-f(a)\\
&= \sum_{i=1}^{n}\big(f(a+s_i)-f(a)\big)\\
&= \sum_{i=1}^{n}\big(f(a+(t\lambda_i+(1-t)\mu_i)r_i)-f(a)\big)\\
&= \sum_{i=1}^{n}\big(t\lambda_i+(1-t)\mu_i\big)\big(f(a+r_i)-f(a)\big)\\
&= t\sum_{i=1}^{n}\lambda_i(f(a+r_i)-f(a))+(1-t)\sum_{i=1}^{n}\mu_i(f(a+r_i)-f(a))\\
&= t\big(f(a+\sum_{i=1}^{n}\lambda_i r_i)-f(a)\big)+(1-t)\big(f(a+\sum_{i=1}^{n} r_i)-f(a)\big)\\
&= tf(x)+(1-t)f(y)-f(a),
\end{aligned}$$

where we have used (3.4) and the fact that each r_i is engaged in the forth and sixth equality, and Lemma 3.6 in the seventh one. This completes the proof. □

Remark 3.11. It is interesting to note that in the proof of Theorem 3.10 we have only used the assumption that r is an engaged extreme vector to show that the map $g_r\colon \mathbb{R}\to\mathbb{R}$ satisfying (3.4) is additive, from which it follows that g_r is in fact the identity map. However, if r is a (possibly disengaged) extreme vector, then (3.4) also holds for each x that can be written as $a+\sum_{i=1}^{n} s_i$ with $s_i\in C$ extreme and $s_i\neq\lambda r$ for all $\lambda\in\mathbb{R}$ and i. In Section 5 we will exploit this observation. Moreover, we remark it is necessary to work with the positive linear span of engaged positive extreme vectors, $[a,\infty)_{\mathcal{R}_E}$. Indeed, to apply Lemma 3.6 we need for each i that $a+s_i$ is in the domain of f.

Generalisation using inf-sup hull

Let us now show how we can use Theorem 3.10 to prove that order isomorphisms are affine in a variety of new settings.

Suppose $V\subseteq X$ is given and $\sup V$ exists. If f is an order isomorphisms, then $f(\sup V)=\sup f(V)$. Likewise an order isomorphism preserves infima. With these basic observations in mind we make the following definition.

Definition 3.12. For $V\subseteq X$ the *inf-sup hull of* V is the set

$$\{x\in X\colon \text{ there exist } v_{\alpha,\beta}\in V \text{ for } \alpha\in A \text{ and } \beta\in B \text{ such that } x=\inf_{\alpha\in A}(\sup_{\beta\in B} v_{\alpha,\beta})\}.$$

Note that if $V \subseteq X$ and x and y are in the inf-sup hull of V, then $x = \inf_{\alpha\in A}(\sup_{\beta\in B} x_{\alpha,\beta})$ and $y = \inf_{\sigma\in S}(\sup_{\tau\in T} y_{\sigma,\tau})$, with all $x_{\alpha,\beta}$ and $y_{\sigma,\tau}$ in V, and hence for all $\lambda, \mu \geq 0$ we have that

$$\begin{aligned}\lambda x + \mu y &= \inf_{\alpha\in A}(\sup_{\beta\in B} \lambda x_{\alpha,\beta}) + \inf_{\sigma\in S}(\sup_{\tau\in T} \mu y_{\sigma,\tau}) = \inf_{\alpha\in A}(\sup_{\beta\in B} \lambda x_{\alpha,\beta} + \inf_{\sigma\in S}(\sup_{\tau\in T} \mu y_{\sigma,\tau}))\\ &= \inf_{\alpha\in A}(\inf_{\sigma\in S}(\sup_{\beta\in B} \lambda x_{\alpha,\beta} + \sup_{\tau\in T} \mu y_{\sigma,\tau})) = \inf_{\alpha\in A}(\inf_{\sigma\in S}(\sup_{\beta\in B}(\sup_{\tau\in T} \lambda x_{\alpha,\beta} + \mu y_{\sigma,\tau})))\\ &= \inf_{(\alpha,\sigma)\in A\times S}(\sup_{(\beta,\tau)\in B\times T} \lambda x_{\alpha,\beta} + \mu y_{\sigma,\tau}), \end{aligned} \tag{3.5}$$

which shows that $\lambda x + \mu y$ is also in the inf-sup hull. In particular we see that the inf-sup hull of a convex subset of X is again a convex set.

Lemma 3.13. *Let $f\colon [a,\infty) \to [b,\infty)$ be an order isomorphism and let $D \subseteq [a,\infty)$ be convex. If f is affine on D, then f is affine on the inf-sup hull of D.*

Proof. Suppose $V \subseteq [a,\infty)$ and $v \in [a,\infty)$ are such that $v = \sup(V)$. Then $f(v)$ is an upper bound of $f(V)$ in $[b,\infty)$. Moreover, if $w \in [b,\infty)$ is another upper bound of $f(V)$, then $f^{-1}(w)$ is an upper bound of V, since f^{-1} is order preserving. As $v = \sup(V)$ we deduce that $v \leq f^{-1}(w)$, so that $f(v) \leq w$. This implies that $f(v) = \sup(f(V))$ in $[b,\infty)$. In the same way it can be shown that if $W \subseteq [a,\infty)$ and $w \in [a,\infty)$ are such that $w = \inf(W)$, then $f(w) = \inf(f(W))$ in $[b,\infty)$.

To complete the proof it suffices to show that f is convex-linear on the inf-sup hull of D. Indeed, the inf-sup hull of D is a convex set by (3.5). Suppose that x and y are in the inf-sup hull of D and $0 \leq t \leq 1$. Write $x = \inf_\alpha \sup_\beta x_{\alpha,\beta}$ and $y = \inf_\sigma \sup_\tau y_{\sigma,\tau}$, with $x_{\alpha,\beta}, y_{\sigma,\tau} \in D$ for all α, β, σ and τ.

By repeatedly using the fact that f preserves infima and suprema and Theorem 3.10 we get

$$\begin{aligned}f(tx + (1-t)y) &= \inf_{\alpha\in A}(\sup_{\beta\in B}(\inf_{\sigma\in S}(\sup_{\tau\in T} f(tx_{\alpha,\beta} + (1-t)y_{\sigma,\tau}))))\\ &= \inf_{\alpha\in A}(\sup_{\beta\in B}(\inf_{\sigma\in S}(\sup_{\tau\in T} tf(x_{\alpha,\beta}) + (1-t)f(y_{\sigma,\tau}))))\\ &= tf(\inf_{\alpha\in A}(\sup_{\beta\in B} x_{\alpha,\beta})) + (1-t)f(\inf_{\sigma\in S}(\sup_{\tau\in T} y_{\sigma,\tau})) = tf(x) + (1-t)f(y).\end{aligned}$$

□

Combination of Theorem 3.10 and Lemma 3.13 yields the next conclusion.

Proposition 3.14. *Every order isomorphism $f\colon [a,\infty) \to [b,\infty)$ is affine on the inf-sup hull of $[a,\infty)_{\mathcal{R}_E}$.*

Let us now generalise the previous proposition to order isomorphisms on more general domains. A set $U \subseteq X$ is called an *upper set* if $x \in U$ and $y \geq x$ imply $y \in U$. We remark that X, C and C° and translations thereof are all upper sets in (X, C).

Theorem 3.15. *Let (X, C) and (Y, K) be Archimedean partially ordered vector spaces, $U \subseteq X$ and $V \subseteq Y$ upper sets and $f \colon U \to V$ be an order isomorphism. If C is generating and equals the inf-sup hull of $\{r_1 + \cdots + r_n \colon r_i \in \mathcal{R}_E$ for $i = 1, \ldots, n\}$, then f is affine.*

Proof. Let $a \in U$ be given. As C is the inf-sup hull of $\mathcal{R}_E$, by mere translation we get that the interval $[a, \infty)$ equals the inf-sup hull of $[a, \infty)_{\mathcal{R}_E}$. So it follows from Proposition 3.14 that f is affine on $[a, \infty)$. As the cone C is generating, this implies that there exists a unique affine map $g : X \to Y$ such that g restricted to $[a, \infty)$ coincides with f.

In the same way find that for any $b \in U$ the map f is affine on $[b, \infty)$. Using that C is directed, we know there exists $c \in U$ such that $c \geq a, b$. We remark that the intersection $[a, \infty) \cap [b, \infty)$ contains the interval $[c, \infty)$. Therefore, f and g also coincide on $[b, \infty)$. Since $b \in U$ was chosen arbitrarily and the fact that U equals the union of all intervals $[x, \infty)$ for $x \in U$, we conclude that g coincides with f on U, which completes the proof. □

Theorem 3.15 is a generalisation of [NS77, Theorem A] by Noll and Schäffer. It would be interesting to have a complete characterisation of the (infinite dimensional) directed Archimedean partially ordered vector spaces (X, C) for which every order isomorphism $f \colon C \to C$ is linear. To our knowledge, Theorem 3.15 is the most general result at present. It can, however, not be applied in a variety of settings such as the space $C([0, 1]) \oplus \mathbb{R}$ with cone $\{(f, \alpha) \colon \|f\|_\infty \leq \alpha\}$. In this space the cone has exactly two disengaged extreme rays: $\{\lambda(\mathbb{1}, 1) \colon \lambda \geq 0\}$ and $\{\lambda(-\mathbb{1}, 1) \colon \lambda \geq 0\}$, where $\mathbb{1}(x) = 1$ for all $x \in [0, 1]$, but it has no engaged extreme rays. We believe, however, that each order isomorphism on the cone is linear in this space.

We end this section with a simple observation concerning direct sums. Let (X_1, C_1) and (X_2, C_2) be directed Archimedean partially ordered vector spaces. Then the direct sum $X_1 \oplus X_2$ is a directed Archimedean partially ordered vector space with cone $C_1 \times C_2$. Moreover $(r, s) \in C_1 \times C_2$ is an (engaged) extreme vector if and only if r is an (engaged) extreme vector and $s = 0$, or, s is an (engaged) extreme vector and $r = 0$. It is straightforward to infer that if (X_1, C_1) and (X_2, C_2) satisfy the conditions on (X, C) in Theorem 3.15, then so does $(X_1 \oplus X_2, C_1 \times C_2)$.

Self-adjoint operators on a Hilbert space

Let H be a Hilbert space and $B(H)_{\text{sa}}$ be the space of bounded self-adjoint operators on H, ordered by the cone $B(H)_{\text{sa}}^+$ of positive semi-definite operators. In this section we show that $B(H)_{\text{sa}}$ satisfies the conditions of Theorem 3.15.

It is easy to show that the extreme rays of $B(H)_{\text{sa}}^+$ are the rays spanned by rank-one projections. We will denote the collection of all extreme rays of $B(H)_{\text{sa}}^+$ by $\mathcal{R}$. Furthermore, for a closed subspace V of H we denote the orthogonal projection onto by V by P_V, and for $x \in H$ we write $P_x = P_{\text{span}(\{x\})}$.

We remark that what we show here is a special case of Lemma 5.7, where we prove that the cone in an atomic JBW-algebra is the sup-hull of the positive linear span of its engaged atoms. However, here we supply a more direct approach that offers insight in the order structure of $B(H)_{sa}$.

Theorem 3.16. *If H is a Hilbert space, with $\dim H \geq 2$, and $U, W \subseteq B(H)_{\mathrm{sa}}$ are upper sets, then every order isomorphism $f\colon U \to W$ is affine.*

Proof. We verify that $B(H)_{\mathrm{sa}}$ satisfies the conditions of Theorem 3.15. Evidently, $B(H)_{\mathrm{sa}}$ is directed and Archimedean. We first show that all extreme rays of $B(H)_{\mathrm{sa}}^{+}$ are engaged. So, suppose $P \in \mathcal{R}$. Then there exists an $x \in H$ such that $P = P_x$. As $\dim H \geq 2$ we can find non-zero $y, z \in H$ such that y and z are orthogonal and x, y, z lie in a two-dimensional subspace V. Then $P_V = P_y + P_z$, so that

$$P_x = P_V - (I - P_x)P_V = P_y + P_z - P_{\{x\}^{\perp}}P_V = P_y + P_z - P_w,$$

where $w \in \{x\}^{\perp} \cap (V \backslash \{0\})$. We conclude that P_x can be written as a linear combination of rank-one projections different from P_x and, hence, the ray spanned by P_x is engaged in $\mathcal{R}$.

Note that the positive linear span of the extreme rays equals the set of positive finite rank operators, which will be denoted F. To verify the condition in Theorem 3.15 it suffices to show that the inf-sup hull of F equals $B(H)_{\mathrm{sa}}^{+}$, as the inf-sup hull is closed under positive sums by (3.5).

We start by showing that the identity I belongs to the inf-sup hull of F. Note that $I \geq P_x$ for all $x \in H$. Suppose that $B \in B(H)_{\mathrm{sa}}$ is an upper bound of P_x for all $x \in H$. Then we have for any $x \in H$ that

$$\langle Bx, x\rangle \geq \langle P_x x, x\rangle = \langle Ix, x\rangle. \tag{3.6}$$

Therefore, $B \geq I$ holds and we conclude that $I = \sup\{P_x\colon x \in H\}$. Note that it follows from (3.6) that for each $Q_0 \in F$ with $Q_0 \leq I$ we have that

$$I = \sup\{Q \in F\colon Q_0 \leq Q \leq I\},$$

as for all $x \in H$ there exists a $Q \in F$ with $Q_0 \leq Q \leq I$ and $Q \geq P_x$.

Now suppose that $A \in B(H)_{\mathrm{sa}}^{+}$ is invertible. Let $T_A\colon B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$ be given by $T_A(Q) = A^{\frac{1}{2}}QA^{\frac{1}{2}}$. Then T_A is a linear order isomorphism, so that

$$A = T_A(I) = T_A(\sup\{Q \in F\colon Q_0 \leq Q \leq I\}) = \sup\{T_A(Q) : Q \in F,\ Q_0 \leq Q \leq I\}.$$

As T_A is a bijection from F onto itself, we get that $A = \sup\{Q \in F : T_A(Q_0) \leq Q \leq A\}$.

Finally, suppose $A \in B(H)_{\mathrm{sa}}^{+}$. Remark that $A + I$ is invertible. For $P \in F$, with $P \leq I$ we let $Q_0 = T_{(A+I)^{-1}}(P)$. Then $A + I = \sup\{Q \in F\colon T_{A+I}(Q_0) \leq Q \leq A + I\}$, from which it follows that $A + I - P = \sup\{Q - P\colon Q \in F,\ P \leq Q \leq A + I\}$. Thus,

$$A = \inf\{A+I-P\colon P \in F, P \leq I\} = \inf\{\sup\{Q-P\colon Q \in F,\ P \leq Q \leq A+I\}\colon P \in F,\ P \leq I\}.$$

This shows that $B(H)_{\mathrm{sa}}^{+}$ is the inf-sup hull of the positive linear span of its extreme rays, and hence Theorem 3.15 yields the desired result. □

We remark that Theorem 3.16 was first proved, using different arguments, by Molnár [Mol01] and does not follow from [NS77, Theorem A].

Order isomorphisms in related problems

In this section we proceed the discussion of Section 3.2 and relate to results by Artstein-Avidan and Slomka and Schäffer in settings somewhat different than in Theorem 3.15. We obtain three results. First, we present a "diagonalization formula" for order isomorphisms between cones, see (3.7) below. Second, we apply the results of Section 3.2 to positively homogeneous order isomorphisms between cones and obtain that they must be linear if one of the cones equals the inf-sup hull of the positive span of its extreme rays. Third, we consider separable complete order unit spaces where in one of them the inf-sup hull of the positive linear span of the engaged extreme rays is big enough to intersect the interior of the cone. In that case we derive from Theorem 3.15 that every order isomorphism between upper sets must be affine.

We begin with the following infinite dimensional analogue of a result by Artstein-Avidan and Slomka [A-AS11, Theorem 1.7].

Proposition 3.17. *Let (X, C) and (Y, K) be Archimedean partially ordered vector spaces and suppose that $f\colon C \to K$ is an order isomorphism. Let $(v_\alpha)_{\alpha\in A}$ be a collection of linearly independent extreme vectors in C. Then there exist corresponding monotone increasing bijections $g_\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$, for $\alpha \in A$, such that for all $\lambda_1, \ldots, \lambda_n \geq 0$ and $\alpha_1, \ldots, \alpha_n \in A$ we have*

$$f\left(\sum_{i=1}^{n} \lambda_i v_{\alpha_i}\right) = \sum_{i=1}^{n} g_{\alpha_i}(\lambda_i) f(v_{\alpha_i}). \tag{3.7}$$

Proof. Note that $f(0) = 0$. Let $r \in C$ be an extreme vector. According to Corollary 3.2, f maps the extreme ray through r bijectively onto the extreme ray through $f(r)$. Hence there exists a nonnegative scalar $g_r(\lambda)$ such that $f(\lambda r) = g_r(\lambda) f(r)$, for all $\lambda \geq 0$. Moreover, the function $g_r\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing bijection. Equation (3.7) now follows from Lemma 3.6. □

In [A-AS11, Theorem 1.7], also the finite dimensional cases $f\colon X \to X$ and $f\colon C^\circ \to C^\circ$ are considered. In the situation of Proposition 3.17, if f is an order isomorphism from X to Y and $f(0) = 0$, then one can easily verify that the maps g_r are actually defined on $\mathbb{R}$ and that (3.7) holds for all $\lambda \in \mathbb{R}$. The infinite dimensional version of the case where $f\colon C^\circ \to K^\circ$ is not so strong. Indeed, if (X, C) and (Y, K) are infinite dimensional order unit spaces, then one can adapt the proof of Proposition 3.17 to show that for each order isomorphism $f\colon C^\circ \to K^\circ$ and each collection $(v_\alpha)_{\alpha\in A}$ of linearly independent extreme vectors of C, there are linearly independent extreme vectors $(w_\alpha)_{\alpha\in A}$ of K and monotone increasing bijections $g_\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha \in A$, such that for all $\lambda_1, \ldots, \lambda_n \geq 0$ and $\alpha_1, \ldots, \alpha_n \in A$ we have (3.7) where $f(v_{\alpha_i})$ is replaced by w_{a_i}, provided that $\sum_{i=1}^{n} \lambda_i v_{\alpha_i} \in C^\circ$. However, in general infinite dimensional order unit spaces most elements of the interior of the cone cannot be written as a positive linear combination of finitely many positive extreme vectors and, thus, the use of this result is limited.

Let us next consider positively homogeneous order isomorphisms. If $U \subseteq X$ and $V \subseteq Y$ are such that $\lambda u \in U$ and $\lambda v \in V$ for every $u \in U$, $v \in V$, and $\lambda > 0$, then a map $f\colon U \to V$ is called *positively homogeneous* if $f(\lambda u) = \lambda f(u)$ for every $u \in U$ and $\lambda > 0$. If U and V are generating Archimedean cones, then this condition implies that $f(0) = 0$, which yields the more common definition that includes $\lambda = 0$. The definition given here also applies to maps on interiors of cones.

In [Sch78, Theorem B], Schäffer provides the next result.

Theorem 3.18 (Schäffer)**.** *Let (X, C, u) and (Y, K, v) be order unit spaces. Then every positively homogeneous order isomorphism $f\colon C^\circ \to K^\circ$ is linear.*

The results of Section 3.2 yield the following alternative statement, in which the requirement of an order unit is replaced by a condition involving extreme rays.

Theorem 3.19. *Let (X, C) and (Y, K) be Archimedean partially ordered vector spaces such that (X, C) is directed and C equals the inf-sup hull of $[0, \infty)_\mathcal{R}$. Then every positively homogeneous order isomorphism $f\colon C \to K$ is linear.*

Proof. We first show that f is additive on $[0, \infty)_\mathcal{R}$. Let $s_1, \ldots, s_n$ be extreme vectors in C. It suffices to show that $f\left(\sum_{i=1}^n s_i\right) = \sum_{i=1}^n f(s_i)$. In order to apply Lemma 3.6, we combine terms of s_i that lie on the same ray. Indeed, for $j = 1, \ldots, m$, let $I_j \subseteq \{1, \ldots, n\}$ be disjoint with $\bigcup_{j=1}^m I_j = \{1, \ldots, n\}$ such that for every $i, k \in \{1, \ldots, n\}$ we have $s_i = \lambda s_k$ for some $\lambda \geq 0$ if and only if there exists $j \in \{1, \ldots, m\}$ with $i, k \in I_j$. Denote $r_j = \sum_{i \in I_j} s_i$ and for every $i \in I_j$ let λ_i be such that $s_i = \lambda_i r_j$. Then $\sum_{i \in I_j} \lambda_i = 1$ for $j = 1, \ldots, m$. With the aid of Lemma 3.6 and the positive homogeneity of f we obtain

$$\begin{aligned} f\left(\sum_{i=1}^n s_i\right) &= f\left(\sum_{j=1}^m r_j\right) = \sum_{j=1}^m f(r_j) = \sum_{j=1}^m \sum_{i \in I_j} \lambda_i f(r_j) \\ &= \sum_{j=1}^m \sum_{i \in I_j} f(\lambda_i r_j) = \sum_{i=1}^n f(s_i). \end{aligned}$$

As f is positively homogeneous, it follows that f is linear on $[0, \infty)_\mathcal{R}$. Due to Lemma 3.13 we obtain that f is linear on the inf-sup hull of $[0, \infty)_\mathcal{R}$, which equals C. □

If in Theorem 3.19 f is an order isomorphism from X to Y and f is homogeneous instead of only positively homogeneous, then it can be shown along similar lines that f is affine.

It is useful to compare Theorem 3.18 and Theorem 3.19 and identify the differences. Let (X, C, u) and (Y, K, v) be order unit spaces. Suppose that $f\colon C \to K$ is a positively homogeneous order isomorphism. Then straightforward verification yields $f(C^\circ) = K^\circ$. Hence it follows by Theorem 3.18 that f is linear on C°. As C is the inf hull of the convex set C°, it follows from Lemma 3.13 that f is linear on C. Thus, any homogeneous order isomorphism between cones of order unit spaces is linear. Theorem 3.19 provides a condition, alternative to having an order unit, that yields the same conclusion. For example, the space $\ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$ with coordinate-wise order satisfies the conditions of Theorem 3.19 but fails to have an order unit. Hence Schäffer's Theorem 3.18 does not imply our Theorem 3.19.

Our third interest in this section is an intermediate result by Schäffer, which has a milder homogeneity condition than Theorem 3.18. In [Sch78, Corollary A1] Schäffer shows for order unit spaces (X, C, u) and (Y, K, v), where either $(X, \|.\|_u)$ or $(Y, \|.\|_v)$ is separable and complete, that any order isomorphism $f\colon C^\circ \to K^\circ$ is linear, provided there exists a $w \in C^\circ$ such that $f(\lambda w) = \lambda f(w)$ for all $\lambda \geq 0$. Compared to [Sch78, Theorem B], the positively homogeneous condition of f is weakened to only being positively homogeneous on a ray through the interior of the cone, at the cost of one of the order unit spaces being separable and complete. In conjunction with Theorem 3.15 this yields the following.

Theorem 3.20. *Let (X, C, u) and (Y, K, v) be order unit spaces, and $U \subseteq X$ and $V \subseteq Y$ be upper sets. Suppose that the inf-sup hull of $[0, \infty)_{\mathcal{R}_E}$ has a non-empty intersection with C°, and that either $(X, \|.\|_u)$ or $(Y, \|.\|_v)$ is separable and complete. Then every order isomorphism $f\colon U \to V$ is affine.*

Proof. Firstly, we consider the case $U = C^\circ$ and $V = K^\circ$. Let C_E denote the inf-sup hull of the positive linear span of the engaged extreme rays of C. By assumption there exists $x \in C_E \cap C^\circ$. We recall that an order unit space is directed and Archimedean. Hence, Proposition 3.14 says that f is affine on $C_E \cap C^\circ$. As f is an order isomorphism mapping C° onto K°, it is straightforward to infer that f is in fact linear on $C_E \cap C^\circ$. In particular, $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$. Now [Sch78, Corollary A1] yields that f is linear on C°.

Next we consider the case $U = C$ and $V = K$. Just as in the previous paragraph, there exists an $x \in C^\circ$ such that $f(\lambda x) = \lambda x$ for all $\lambda \geq 0$. We infer that $f(C^\circ) = K^\circ$. Indeed, let $y \in K$. As $x \in C^\circ$ there exists $\lambda \geq 0$ such that $\lambda x \geq f^{-1}(y)$. This yields that $\lambda f(x) = f(\lambda x) \geq y$. Therefore, $f(x)$ is an order unit in (Y, K) and hence $f(x) \in K^\circ$. Now let $y \in C^\circ$. Then there exists $m > 0$ such that $mx \leq y$. We get $mf(x) = f(mx) \leq f(y)$. In particular, $f(y)$ is an order unit and we conclude that $f(y) \in K^\circ$. Hence $f(C^\circ) \subseteq K^\circ$. We remark that for all $\lambda \geq 0$ we have $f^{-1}(\lambda f(x)) = \lambda x = \lambda f^{-1}(f(x))$, in other words f^{-1} is positively homogeneous along the ray through $f(x)$. Therefore, the previous steps applied to f^{-1} instead of f yield the converse inclusion $K^\circ \subseteq f(C^\circ)$. By the first part of the proof we obtain that f is linear on C°. Since C is the inf hull of the convex set C°, it follows from Lemma 3.13 that f is linear on C.

Suppose $a \in X$ and $b \in Y$ are such that $U = [a, \infty)$ and $V = [b, \infty)$. The order isomorphism $\hat{f}$ defined by $\hat{f}(c) = f(c+a)-b$ maps C to K. By the previously considered case $\hat{f}$ is linear, and hence f is affine.

The general case where $U \subseteq X$ and $V \subseteq Y$ are upper sets follows by arguments similar to those made in the proof of Theorem 3.15. Indeed, for every $a \in U$, f is an order isomorphism from $[a, \infty)$ to $[f(a), \infty)$, so that f is affine on $[a, \infty)$ by the previous case. Then $f|_{[a,\infty)}$ extends to a unique affine map $F\colon X \to Y$, which is independent of $a \in U$, as (X, C) is directed. $\square$

To conclude the paper we provide an example to which Theorem 3.20 applies, but not Theorem 3.15. Consider the order unit space (X, C, u) consisting of the real vector space $X = C([0,1] \cup [2,3]) \oplus \mathbb{R}$, the Archimedean cone

$$C = \{(f, \lambda)\colon \|f\|_\infty \leq \lambda\}$$

and the order unit $u = (0, 1) \in C$. Then $(X, \|.\|_u)$ is complete and separable. The unit ball

$$B = \{f \in C([0,1] \cup [2,3])\colon \|f\|_\infty \leq 1\}$$

has four extreme points: $\pm\mathbb{1}_{[0,1]}$ and $\pm\mathbb{1}_{[2,3]}$, where $\mathbb{1}_{[0,1]}$ and $\mathbb{1}_{[2,3]}$ denote the indicator functions of $[0,1]$ and $[2,3]$, respectively. Therefore, C has four extreme rays, namely the rays through $(\pm\mathbb{1}_{[0,1]}, 1)$ and $(\pm\mathbb{1}_{[2,3]}, 1)$. As

$$(\mathbb{1}_{[0,1]}, 1) + (-\mathbb{1}_{[0,1]}, 1) = 2u = (\mathbb{1}_{[2,3]}, 1) + (-\mathbb{1}_{[2,3]}, 1),$$

all four extreme rays are engaged, and u which lies in C° is contained in the positive linear span of the engaged extreme rays. We conclude that the order unit space (X, C, u) satisfies the conditions of Theorem 3.20. However, the inf-sup hull of the sum of the engaged extreme rays consist only of elements of the form $(\lambda\mathbb{1}_{[0,1]} + \mu\mathbb{1}_{[2,3]}, \nu)$, with $\lambda, \mu \geq 0$ and $|\lambda|, |\mu| \leq \nu$, and hence (X, C) does not satisfy the conditions of Theorem 3.15.

Chapter 4

Monotone dynamical systems

The theory of monotone dynamical systems concerns the behaviour of dynamical systems on subsets of real vector spaces that preserve a partial ordering induced by a cone in the vector space. Pioneering studies by M. Hirsch [Hir82, Hir85, Hir88] and numerous subsequent works, see [DH91, HS06, LN12, PT92, Smi95] and the references therein, showed that under suitable additional conditions the generic behaviour of such dynamical systems cannot be very complex. Common additional conditions include smoothness conditions on the system and various strong forms of monotonicity.

We will consider discrete time dynamical systems (f, Ω), where Ω is an open connected subset of a finite dimensional real vector space V and $f : \Omega \to \Omega$ is a homeomorphism which is monotone with respect to a generating closed cone $C \subseteq V$. For such discrete dynamical systems the complexity of the generic behaviour is not well understood. Recently, however, M. Hirsch [Hir17] showed that if the cone C is polyhedral, as in Example 1.1, then the system cannot display chaotic behaviour in the following sense. He showed that if such a monotone dynamical system (f, Ω) has a dense set of periodic points in Ω, then f is periodic, that is to say, there exists an integer $p \geq 1$ such that $f^p(x) = x$ for all $x \in \Omega$. Furthermore, he conjectured that this result holds for generating closed cones in finite dimensional vector spaces. We confirm this conjecture. These results can be found in [LvGvI18]. As in [Hir17] we will also use the following theorem, which is a direct consequence of Montgomery [Mon37].

Theorem 4.1 (Montgomery)**.** *If $f : \Omega \to \Omega$ is a homeomorphism of an open connected subset Ω of a finite dimensional real vector space V and each $x \in \Omega$ is a periodic point of f, then f is periodic.*

Connection with order isomorphisms

Throughout let (V, C) be a finite dimensional partially ordered vector space with a generating cone C. The *dual cone* of C is given by $C^* = \{\varphi \in V^* : \varphi(x) \geq 0 \text{ for all } x \in C\}$. Let us fix $u \in C^\circ$. Recall that the *state space* of C is defined by

$$S_C = \{\varphi \in C^* : \varphi(u) = 1\},$$

which is a compact convex set that spans V^*. We write ∂S_C to denote the boundary of S_C with respect to its affine span.

The partial order induced by C will simply be denoted by $\leq$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and write $x \ll y$ if $y - x \in C^\circ$. Note that if $x \ll z$, then the order interval $[x, z]$ is a compact, convex set and its interior is non-empty and is given by $[x, z]^\circ = \{y \in V : x \ll y \ll z\}$.

We also use the following basic dynamical systems terminology. We denote a discrete time dynamical system by a pair (f, Ω), so $f : \Omega \to \Omega$. We say that $x \in \Omega$ is a *periodic point* in (f, Ω) if there exists an integer $p \geq 1$ such that $f^p(x) = x$. The least such $p \geq 1$ is called the *period* of x. A periodic point with period 1 is called a *fixed point*. The set of all periodic points of (f, Ω) is denoted $\mathrm{Per}(f)$, and the set of all fixed points of f is denoted $\mathrm{Fix}(f)$. A periodic point $x \in \Omega$ of f with period p is said to be *stable* if there exists a neighbourhood $U \subseteq \Omega$ of x such that $f^p(U) \subseteq U$.

Under the same conditions as in the conjecture by Hirsh, our monotone dynamical system locally has the behaviour of an order isomorphism.

Lemma 4.2. *If (f, Ω) is a monotone dynamical system, where $f : \Omega \to \Omega$ is a homeomorphism, $\Omega \subseteq V$ is open and connected, and Per(f) is dense in Ω, then for any $x, y \in \Omega$ with $x \ll y$ the restriction $f|_{[x,y]} : [x, y] \to [f(x), f(y)]$ is an order isomorphism.*

Proof. We first consider the case where $x, y \in \Omega$ with $x \ll y$ are both periodic points. Let p and q be the periods of x and y, respectively, and r the least common multiple of p and q. Since f is monotone we have $f([x, y]) \subseteq [f(x), f(y)]$. We argue that the restriction of f to $[x, y]$ maps onto $[f(x), f(y)]$.

Suppose $w \in [f(x), f(y)]^\circ$. Let (w_n) be a sequence in $[f(x), f(y)]^\circ \cap P(f)$ that converges to w. For $n \in \mathbb{N}$ we let s_n be the period of w_n, r_n the least common multiple of r and s_n and define $v_n := f^{r_n - 1}(w_n)$. We get for for all $n \in \mathbb{N}$ that

$$v_n = f^{r_n - 1}(w_n) \in [f^{r_n - 1}(f(x)), f^{r_n - 1}(f(y))] = [x, y].$$

As $[x, y]$ is compact there exists a convergent subsequence (v_{n_k}) of (v_n), with limit say $v \in [x, y]$. By continuity of f we have

$$f(v) = \lim_{k \to \infty} f(v_{n_k}) = \lim_{k \to \infty} f^{r_{n_k}}(w_{n_k}) = \lim_{k \to \infty} w_{n_k} = w.$$

We conclude $w \in f([x, y])$ and, hence, $[f(x), f(y)]^\circ \subseteq f([x, y])$. Since the latter set is compact by continuity of f and the interval $[f(x), f(y)]$ is closed, we conclude $[f(x), f(y)] = f([x, y])$. It remains to argue that for $w, z \in [f(x), f(y)]$ with $w \leq z$ we have $f^{-1}(w) \leq f^{-1}(z)$.

Consider the case $w, z \in [f(x), f(y)]^\circ$ with $w \ll z$. Then the interval $[f(x), w]$ has non-empty interior. Let (w_n) be a sequence in $[f(x), w] \cap \mathrm{Per}(f)$, with periods say p_n, that converges to w. Analogously, let (z_n) be a sequence in $[z, f(y)] \cap \mathrm{Per}(f)$, with periods say q_n, that converges to z. Letting r_n be the least common multiple of p_n and q_n, for each $n \in \mathbb{N}$, yields

$$f^{-1}(w_n) = f^{r_n - 1}(w_n) \leq f^{r_n - 1}(z_n) = f^{-1}(z_n).$$

We conclude from the continuity of f^{-1} that $f^{-1}(w) \leq f^{-1}(z)$.

Remark that for $w, z \in [f(x), f(y)]^\circ$ with $w \leq z$ and sufficiently small $\epsilon > 0$ we have $f(x) \ll w - \epsilon u \ll z + \epsilon u \ll y$. It follows by the previous paragraph that $f^{-1}(w - \epsilon u) \leq f^{-1}(z + \epsilon u)$. Letting ϵ decrease to zero, the continuity of f^{-1} guarantees $f^{-1}(w) \leq f^{-1}(z)$.

Finally, consider the general case $w, z \in [f(x), f(y)]$ with $w \leq z$. Let $x', y' \in \mathrm{Per}(f)$ with $x' \ll x$ and $y \ll y'$. Now we have $w, z \in [f(x'), f(y')]^\circ$ as f is a homeomorphism and we conclude $f^{-1}(w) \leq f^{-1}(z)$, by repeating all previous argument from the restriction of f to $[x', y']$.

We have proven the assertion in the case that x and y are periodic points. Suppose now that $x, y \in \Omega$ with $x \ll y$ are given. Let $x', y' \in \Omega \cap \mathrm{Per}(f)$ be such that $[x, y] \subseteq [x', y']$. Then $f|_{[x',y']} \colon [x', y'] \to [f(x'), f(y')]$ is an order isomorphism. In particular, f maps $[x, y]$ onto $[f(x), f(y)]$ and the assertion follows. □

In light of Lemma 4.2, understanding the structure of order isomorphisms between intervals is important. Remark that Theorem 3.15 is not applicable here as the intervals are bounded from above. In [Sem17] order isomorphisms between intervals in $B(H)_{sa}$ are studied and fully described. Already in the case $\dim H = 2$, nonlinear order isomorphisms exist. Nevertheless, a description of order isomorphisms between intervals would yield much information on the dynamics of (f, Ω).

Dense periodic points

We continue with our analysis of monotone dynamical systems with dense periodic points.

Lemma 4.3. *If (f, Ω) is a monotone dynamical system, where $f \colon \Omega \to \Omega$ is a homeomorphism and $\mathrm{Per}(f)$ is dense in Ω, then*

(i) for each $x \in \Omega$ there exist $y, z \in \mathrm{Per}(f)$ such that $y \ll x \ll z$.

(ii) each periodic point of f is stable.

Proof. The first assertion follows directly from the fact that $(x + C^\circ) \cap \Omega$ and $(x - C^\circ) \cap \Omega$ are non-empty open sets, and $\mathrm{Per}(f)$ is dense in Ω.

To prove the second assertion let x be a periodic point of f with period p. Let $y, z \in \mathrm{Per}(f)$ such that $y \ll x \ll z$. Suppose that y has period q, and z has period r. Let s be the least common multiple of q and r. Then

$$U = \bigcap_{k=0}^{s-1} [f^{kp}(y), f^{kp}(z)]_C^\circ$$

is a neighbourhood of x such that $f^p[U] \subseteq U$. □

The next proposition is a consequence of [Hir17, Proposition 6]. As the proof in [Hir17] uses advanced results from algebraic topology, we include a more elementary proof for the reader's convenience.

Proposition 4.4. *Let (f,Ω) be a monotone dynamical system, where $f\colon \Omega \to \Omega$ is a homeomorphism and suppose that* $\mathrm{Per}(f)$ *is dense in* Ω. *If $x \in \Omega$, then* $\mathrm{Per}(f)$ *is dense in $(x+\partial C)\cap\Omega$ and $(x-\partial C)\cap\Omega$.*

Proof. We will only give the proof for $(x+\partial C)\cap\Omega$ and leave the other case, which can be proved in an analogous way, to the reader. First note that by considering the monotone dynamical system $(g,\Omega - x)$, where $g(v) = f(v+x)-x$, we may as well assume that $x = 0$. Now let $v \in \partial C \cap \Omega$ and $S \subseteq \Omega$ be a neighbourhood of v. Then, given $u \in C^\circ$, there exists an $\epsilon > 0$ such that $[v-\epsilon u, v+\epsilon u]_C^\circ \subseteq S$. Now $(v + C^\circ) \cap [v-\epsilon u, v+\epsilon u]_C^\circ$ is an non-emtpy open subset of Ω, and hence contains a periodic point of f, say z. Likewise, $(v - C^\circ)\cap[v-\epsilon u, v+\epsilon u]_C^\circ$ contains a periodic point of f, say y. So, $v - \epsilon u \ll y \ll v \ll z \ll v+\epsilon u$, and $[y,z]_C \subseteq [v-\epsilon u, v+\epsilon u]_C^\circ$.

Let r be the least common multiple of the periods of y and z, and write $g = f^r$. So, $y,z \in \mathrm{Fix}(g)$. Now let $\mathcal{M}$ be the collection of $M \subseteq \mathrm{Fix}(g)$ such that M is totally $\leq$-ordered with $\min M = y$ and $\max M = z$, and order $\mathcal{M}$ by inclusion. Then each chain (M_α) in $(\mathcal{M},\subseteq)$ has an upper bound, namely $\cup_\alpha M_\alpha$. Indeed, if $a,b \in \cup_\alpha M_\alpha$, then there exists an α such that $a,b\in M_\alpha$, and hence either $a \leq b$ or $b\leq a$, as M_α is totally $\leq$-ordered. Thus, by Zorn's Lemma $(\mathcal{M},\subseteq)$ has a maximal element, say M.

We claim that M is a connected subset of Ω. To show this we argue by contradiction. So, suppose that there exist $U,W \subseteq M$ non-empty and relatively open such that $U\cap W = \emptyset$ and $M = U \cup W$. We may as well assume that $y \in U$. Note that both U and W are totally $\leq$-ordered. Thus, $\{x_w\}$, $w \in (W,\preceq)$, forms a net, where $w_1 \preceq w_2$ if $w_2 \leq w_1$ and $x_w = w$ for all $w \in W$. Now for each $\varphi \in S_C$ we have that $\{\varphi(x_w)\}$ is a decreasing net that is bounded below by $\varphi(y)$, and hence it converges. As S_C spans V^*, we conclude that $\{x_w\}$ converges to say $w^* \in [y,z]_C$. As C is closed, we get that $w^* \leq w$ for all $w \in W$.

Now let $U_0 = \{u \in U\colon u \leq w^*\}$. Then $y \in U_0$ and U_0 is totally $\leq$-ordered. Thus, $\{x_u\}$, $u \in (U_0,\preceq)$ forms a net, where $u_1 \preceq u_2$ if $u_1 \leq u_2$ and $x_u = u$ for all $u \in U_0$. As before, $\{x_u\}$, $u\in U_0$, converges to say $u^* \in [y,z]_C$, and $u \leq u^*$ for all $u \in U_0$.

Note that as $\mathrm{Fix}(g)$ is closed, w^* and u^* are fixed points of g. To derive a contradiction that proves the claim we distinguish two cases: $u^* \neq w^*$ and $u^* = w^*$. If $u^* \neq w^*$, then $u^* < w^*$ and u^* and w^* are stable fixed points of g by Lemma 4.3(ii). It now follows from [DH91, Proposition 1] that there exists $\eta \in \mathrm{Fix}(g)$ with $u^* < \eta < w^*$. Note that $U \cup W = M$ implies that $\eta \notin M$ and $M \cup \{\eta\} \in \mathcal{M}$, as each $w \in W$ satisfies $\eta < w^* \leq w$, each $u \in U_0$ satisfies $u \leq u^* < \eta$, and for each $u \in U\setminus U_0$ there exists a $w\in W$ with $\eta < w^* \leq w \leq u$. This, however, contradicts the maximality of M. On the other hand, if $u^* = w^*$, then writing $\xi = u^* = w^*$ we see that $\xi \notin M$. Indeed, if $\xi \in M$, then either $\xi \in U$ or $\xi \in W$, which is impossible as $\xi \in \overline{U}\cap\overline{W}$, U and W are relatively open, and $U \cap W = \emptyset$. As in the previous case, one can check that $M \cup \{\xi\}$

belongs to $\mathcal{M}$, which again contradicts the maximality of M. This shows that M is connected.

To complete the proof we consider the continuous function $h\colon V \to \mathbb{R}$ given by

$$h(x) = \inf_{\varphi \in S_C} \varphi(x) \quad \text{for } x \in V.$$

Recall that $0 \leq v \ll z$ and $y \ll v$. So, for each $\varphi \in S_C$ we have that $0 \leq \varphi(v) < \varphi(z)$, which implies that $h(z) > 0$, since S_C is compact. Moreover, there exists $\varphi^* \in S_C$ such that $\varphi^*(v) = 0$, so that $\varphi^*(y) < \varphi^*(v) = 0$. This implies that $h(y) < 0$. Now, as h is continuous and M is connected, $h[M]$ is a connected subset of $\mathbb{R}$, and hence there exists $\zeta \in M$ such that $h(\zeta) = 0$. It follows that $\varphi(\zeta) \geq 0$ for all $\varphi \in S_C$, and there exists $\varphi' \in S_C$ such that $\varphi'(\zeta) = 0$, since S_C is compact. Thus, $\zeta \in \partial C \cap [y, z]_C \subseteq \partial C \cap S$ and $f^r(\zeta) = g(\zeta) = \zeta$, which completes the proof. □

The proof of the main result relies on a couple of geometric lemmas concerning convexity of finite dimensional generating closed cones. For basic convexity notions we follow the terminology of [Roc97]. Given $x \in \partial C$ we write

$$\nu(x) = \{\varphi \in \partial S_C \colon \varphi(x) = 0\}.$$

Note that $\nu(x) = \nu(\lambda x)$ for all $\lambda > 0$, and $\nu(x)$ is non-empty for each $x \in \partial C$, as each $x \in \partial C$ has a supporting functional. We will consider ∂S_C and ∂C as topological spaces with the induced norm topology from V^* and V, respectively.

Lemma 4.5. *If $U \subseteq \partial S_C$ is open, then $\{x \in \partial C \colon \nu(x) \subseteq U\}$ is open.*

Proof. Suppose by way of contradiction that there exists $z \in \partial C$ with $\nu(z) \subseteq U$ and a sequence $\{z_n\}$ in ∂C converging to z with $\nu(z_n) \not\subseteq U$ for all $n \geq 1$. Then for each $n \geq 1$ there exists a $\varphi_n \in \nu(z_n)$ with $\varphi_n \notin U$. As ∂S_C is compact, we may assume after taking a subsequence that $\{\varphi_n\}$ converges to $\varphi \in \partial S_C$. Now note that

$$0 \leq \varphi(z) \leq |\varphi(z) - \varphi(z_n)| + |\varphi(z_n) - \varphi_n(z_n)| \leq \|\varphi\|\|z - z_n\| + \|\varphi - \varphi_n\|\|z_n\|$$

for all $n \geq 1$. Since the right-hand side converges to 0 as $n \to \infty$, we conclude that $\varphi(z) = 0$, and hence $\varphi \in \nu(z) \subseteq U$. This is impossible, since $\{\varphi_n\}$ converges to $\varphi \in U$, $\varphi_n \notin U$ for all $n \geq 1$, and U is open. □

Remark 4.6. Note that given $U \subseteq \partial S_C$ open, the set $\{x \in \partial C \colon \nu(x) \subseteq U\}$ may be empty. If, however, $\varphi \in \partial S_C$ is an exposed point of S_C, then by definition there exists $y \in \partial C$ such that $\varphi(y) = 0$ and $\psi(y) > 0$ for all $\psi \neq \varphi$ in ∂S_C. In that case $\nu(\lambda y) = \{\varphi\}$ for all $\lambda > 0$. Thus, for any neighbourhood U of an exposed point φ in ∂S_C we know that $\{x \in \partial C \colon \nu(x) \subseteq U\} \cap W$ is non-empty and open, for all non-empty neighbourhoods W of 0 in V.

By Straszewicz's Theorem [Roc97, Theorem 18.6] the exposed points of S_C are dense in the extreme points of S_C. As S_C is the convex hull of its extreme points, S_C is also the convex hull of its exposed points. Let $\psi_1, \dots, \psi_d$ be linearly independent exposed points of S_C, where $d = \dim V^* = \dim V$, and let

$$K = \left\{ \sum_{i=1}^{d} \lambda_i \psi_i \colon \lambda_1, \dots, \lambda_d \geq 0 \right\}, \tag{4.1}$$

which is a generating closed cone in V^*. The dual cone of K is $K^* = \{x \in V \colon \psi_i(x) \geq 0 \text{ for all } i = 1, \dots, d\}$, which is also closed and generating. Furthermore let

$$\psi = \frac{1}{d} \sum_{i=1}^{d} \psi_i. \tag{4.2}$$

Then ψ is a strictly positive functional for K^*, that is to say, $\psi(x) > 0$ for all $x \in K^* \setminus \{0\}$.

Lemma 4.7. *For $i = 1, \dots, d$ there exist neighbourhoods U_i of ψ_i in ∂S_C such that if $\varphi_i \in U_i$ for $i = 1, \dots d$, then*

(i) $\varphi_1, \dots, \varphi_d$ are linearly independent.

(ii) ψ is a strictly positive functional for the generating closed cone

$$K' = \{x \in V \colon \varphi_i(x) \geq 0 \text{ for all } i = 1, \dots, d\}.$$

Proof. The first assertion follows directly from the fact that the set of invertible linear maps $L \colon \mathbb{R}^d \to V^*$ is open by considering the invertible linear map $A \colon x \mapsto \sum_{i=1}^d x_i \psi_i$ for $x \in \mathbb{R}^d$.

The second assertion follows from the fact that the map $L \mapsto L^{-1}$ is continuous on the set of invertible linear maps from $\mathbb{R}^d$ onto V^*. Indeed, consider A as above. Now if $\varphi_1, \dots, \varphi_d$ are linearly independent, then the linear map $B \colon x \mapsto \sum_{i=1}^d x_i \varphi_i$ is invertible. Thus,

$$\|B^{-1}\psi - (1/d, \dots, 1/d)\| = \|B^{-1}\psi - A^{-1}\psi\| \leq \|B^{-1} - A^{-1}\| \|\psi\|,$$

implies that $(B^{-1}\psi)_i > 0$ for all $i = 1, \dots d$, if B^{-1} is sufficiently close to A^{-1}. As $\psi = B(B^{-1}\psi) = \sum_{i=1}^d (B^{-1}\psi)_i \varphi_i$, we conclude that $\psi(x) = \sum_{i=1}^d (B^{-1}\psi)_i \varphi_i(x) > 0$ for all $x \in K' \setminus \{0\}$, since $\varphi_j(x) > 0$ for some j and $(B^{-1}\psi)_i > 0$ for all i. $\square$

Proof of periodicity: Hirsch's conjecture

Theorem 4.8. *If (f, Ω) is a monotone dynamical system, where $f \colon \Omega \to \Omega$ is a homeomorphism on an open connected subset Ω of a finite dimensional real vector space V and $\operatorname{Per}(f)$ is dense in Ω, then f is periodic.*

Proof. By Montgomery's Theorem 4.1 it suffices to show that each $x \in \Omega$ is periodic. So let $x \in \Omega$. By considering the dynamical system $(g, \Omega - x)$, where $g(v) = f(v+x) - x$ for $v \in \Omega - x$, we may as well assume that $x = 0$. From Proposition 4.4 we know that the periodic points of f are dense in $\partial C \cap \Omega$ and $-\partial C \cap \Omega$.

Now as above choose $\psi_1, \ldots, \psi_d$ linearly independent exposed points of S_C, and let $\psi = \frac{1}{d} \sum_{i=1}^d \psi_i$. By Lemma 4.7 there exist open neighbourhoods U_i of ψ_i in ∂S_C for $i = 1, \ldots, d$ such that if we take $\varphi_i \in U_i$ for $i = 1, \ldots, d$, then $\varphi_1, \ldots, \varphi_d$ are linearly independent, and ψ is a strictly positive functional for the generating closed cone $K' = \{x \in V \colon \varphi_i(x) \geq 0 \text{ for all } i = 1, \ldots, d\}$.

Lemma 4.5 implies that for each i we have that $W_i = \{x \in \partial C \colon \nu(x) \subseteq U_i\}$ is open in ∂C. It follows that $W_i \cap \Omega$ and $-W_i \cap \Omega$ are non-empty and open in $\partial C \cap \Omega$ and $-\partial C \cap \Omega$, respectively, as ψ_i is an exposed point of ∂S_C, see Remark 4.6. As the periodic points of f are dense in $-\partial C \cap \Omega$, there exists a periodic point $x_i \in -W_i \cap \Omega$ of f with period, say p_i, for $i = 1, \ldots, d$. Let $\rho_i \in U_i$ be such that $\rho_i \in \nu(x_i)$. So, $\rho_1, \ldots, \rho_d$ are linearly independent and $\rho_i(x_i) = 0$ for $i = 1, \ldots, d$.

Now consider the set $S_1 = \{y \in \Omega \colon x_i \leq y \text{ for all } i = 1, \ldots, d\}$. Then $0 \in S_1$ and for each $y \in S_1$ we have that $\rho_i(y) \geq \rho_i(x_i) = 0$ for all $i = 1, \ldots, d$. So y belongs to the generating closed cone $K_1' = \{v \in V \colon \rho_i(v) \geq 0 \text{ for all } i = 1, \ldots, d\}$. Moreover, ψ is a strictly positive functional for K_1'.

Likewise, there exist periodic points $z_1, \ldots, z_d$ with $z_i \in W_i \cap \Omega$ for $i = 1, \ldots, d$. Let q_i be the period of z_i, and take $\sigma_i \in \nu(z_i) \subseteq U_i$ for all i. Then $\sigma_1, \ldots, \sigma_d$ are linearly independent and $\sigma_i(z_i) = 0$ for all i. Now consider $S_2 = \{y \in \Omega \colon y \leq z_i \text{ for all } i = 1, \ldots, d\}$. So, $0 \in S_2$ and for each $y \in S_2$ we have that $\sigma_i(y) \leq \sigma_i(z_i) = 0$ for all i, and hence y belongs to $-K_2'$, where $K_2' = \{v \in V \colon \sigma_i(v) \geq 0 \text{ for all } i = 1, \ldots, d\}$ is a generating closed cone. Again ψ is a strictly positive linear functional for K_2'.

Thus, $S_1 \cap S_2 \subseteq K_1' \cap (-K_2')$ and $0 \in S_1 \cap S_2$. In fact, we have that $S_1 \cap S_2 = \{0\}$. Indeed, if $y \in S_1 \cap S_2$, then $\psi(y) \geq 0$, as $y \in K_1'$. But also $\psi(y) \leq 0$, as $y \in -K_2'$. Thus $\psi(y) = 0$, which implies that $y = 0$, since ψ is a strictly positive linear functional for K_1'.

To complete the proof, let r be the least common multiple of $p_1, \ldots, p_d, q_1, \ldots, q_d$, and then $f^r(0) \in S_1 \cap S_2$, as f is monotone. Thus, $f^r(0) = 0$ and we are done. □

Chapter 5

Order isomorphisms in JB-algebras

An important class of partially ordered vector spaces are the self-adjoint parts of C^*-algebras, or more generally, JB-algebras. The self-adjoint part of a C^*-algebra equipped with the Jordan product

$$x \circ y := \tfrac{1}{2}(xy + yx)$$

tuns it into a JB-algebra. It is interesting that order isomorphisms in JB-algebras, which a priori only preserve the partial order, often also preserve the underlying Jordan algebraic structure of the space. It follows from a theorem of Kadison's in [Kad52] that a linear order isomorphism between C^*-algebras mapping the unit to the unit is in fact is a C^*-isomorphism, or a Jordan isomorphism between the self-adjoint parts of the C^*-algebras. Molnár studied order isomorphisms on the cone of positive semi-definite operators on a complex Hilbert space in [Mol01] and proved that they must be linear. The linearity of the order isomorphism, in turn, then yields a Jordan homomorphism on the self-adjoint operators. An interesting problem to solve is to classify the JB-algebras for which all order isomorphism on the cones are automatically linear. The analogue for JB-algebras of the theorem of Kadison's proved by Isidro and Rodrígues-Palacios in [IR-P95] would then also yield a Jordan isomorphism if the unit is mapped to the unit.

The theory developed in Chapter 3, concerning the linearity of order isomorphisms in general partially ordered vector space, has an application in the setting of JB-algebras. Results presented here are based on [vIR]. Before considering cones in JB-algebras, we restrict our study to order isomorphisms between cones in JBW-algebras, the Jordan analogue of a von Neumann algebra. In the cone of a JBW-algebra the extreme rays correspond precisely with the rays through minimal projections, or atoms. A JBW-algebra is considered atomic if every positive element dominates an atom. From the spectral theorem [AS03, Theorem 2.20] for JBW-algebras it follows readily that the any element in the cone of an atomic JBW-algebra is the supremum of positive linear combinations of orthogonal atoms. Furthermore, in such an atomic JBW-algebra we have a convenient algebraic description of engaged atoms; an atom is disengaged if and only if the atom is central. In Proposition 5.9 we show that an atomic JBW-algebra has an algebraic decomposition in a part containing the engaged atoms and a part containing the disengaged atoms, and we proceed to show that an order isomorphism between cones in atomic JBW-algebras is always linear on the engaged part of the algebra according to this decomposition in Theorem 5.10. The disengaged part of an atomic JBW-algebra has a simple structure; it is an algebraic direct sum of one dimensional factors. Combining these observation we describe all order isomorphisms between cones of atomic JBW-algebras, in Theorem 5.12.

Proceeding with cones in general JB-algebras, an essential observation is that a JB-algebra can be embedded into the atomic part of its bidual, and that this bidual is a JBW-algebra. This embedding has convenient properties with respect to the order structure; it is an order embedding and, due to a deep result [H-OS84, Theorem 4.4.10], the whole atomic part of the bidual can be reached by subsequently adjoining the limits of monotone increasing and decreasing nets of elements in the original JB-algebra in finitely many steps. Due to these properties, if an order isomorphism between cones of JB-algebras extends to a homeomorphism between the cone of the atomic parts of the biduals with respect to a suitable topology, then this extension is necessarily an order isomorphism. Existence of disengaged atoms in the bidual is related to the the existence of norm closed ideals of codimension one in the JB-algebra. This leads to the result, Theorem 5.19, that an order isomorphism between cones of JB-algebras, that do not contain any ideals of codimension one, is linear if and only if it extends to a homeomorphism between the atomic parts of the biduals. We conclude with an example that illustrates the necessity of the absense of norm closed ideals of codimension one.

Overview of the theory on JB-algebras

As the theory on JB-algebras, developed by Alfsen and Schultz [AS01, AS03] and Hanche-Olsen and Størmer [H-OS84], is rather extensive, we give an overview of the results and terminology that will be used in the sequel for the sake of convenience. Consequently, nothing presented in this section is new.

A *Jordan algebra* $(A, \circ)$ is a commutative, not necessarily associative algebra such that

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \quad \text{for all } x, y \in A.$$

A *JB-algebra* A is a real Jordan algebra with a norm $\|.\|$ such that $(A, \|.\|)$ is complete and,

$$\begin{aligned} \|x \circ y\| &\leq \|x\| \, \|y\| \, , \\ \|x^2\| &= \|x\|^2 \, , \\ \|x^2\| &\leq \|x^2 + y^2\| \end{aligned}$$

for all $x, y \in A$. As mentioned in the introduction, an important example of a JB-algebra is the set of self-adjoint elements of a C^*-algebra, equipped with the Jordan product $x \circ y := (xy + yx)/2$.

The elements $x, y \in A$ are said to *operator commute* if $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in A$. An element $x \in A$ is said to be *central* if it operator commutes with all elements of A.

For JB-algebras A with algebraic unit e, the *spectrum* of $x \in A$, $\sigma(x)$, is defined to be the set of $\lambda \in \mathbb{R}$ such that $x - \lambda e$ is not invertible in $\mathrm{JB}(x, e)$, the JB-algebra generated by x and e in A, see [H-OS84, Section 3.2.3]. The algebra $JB(x, e)$ is associative. Furthermore, there is a continuous functional calculus: $\mathrm{JB}(x, e) \cong C(\sigma(x))$, see [AS03, Corollary 1.19]. The cone of elements in A with non-negative spectrum is denoted by A_+, and equals the set of squares by the functional calculus, and its interior A_+° consists of all elements with strictly positive spectrum. This cone turns A into an order unit space with order unit e, that is,

$$\|x\| = \inf\{\lambda > 0 : -\lambda e \leq x \leq \lambda e\}.$$

Assumption. *Every JB-algebra under consideration is unital with unit e.*

The *Jordan triple product* $\{\cdot, \cdot, \cdot\}$ is defined as

$$\{x, y, z\} := (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y,$$

for $x, y, z \in A$. The linear map $U_x \colon A \to A$ defined by $U_x y := \{x, y, x\}$ will play an important role and is called the *quadratic representation* of x. In case x is invertible, it follows that U_x is an automorphism of the cone A_+ and its inverse is $U_{x^{-1}}$ by [AS03, Lemma 1.23] and [AS03, Theorem 1.25]. A *state* φ of A is a positive linear functional on A such that $\varphi(e) = 1$. The set of states on A is called the *state space* of A and is w^*-compact by the Banach-Alaoglu theorem and therefore must have sufficiently many extreme points by the Krein-Milman theorem. These extreme points are referred to as *pure states* on A (cf. [AS03, A 17]).

A *JBW-algebra* M is the Jordan analogue of a von Neumann algebra: it is a JB-algebra with unit e which is monotone complete and has a separating set of normal states, or equivalently, a JB-algebra that is a dual space. Here a partially ordered vector space is monotone complete if any monotone increasing net that is bounded from above has a supremum. A state φ of M is said to be *normal* if for any bounded increasing net $(x_i)_{i \in I}$ with supremum x we have $\varphi(x_i) \to \varphi(x)$. The linear space of normal states on M is called the *normal state space* of M. The topology on M defined by the duality of M and the normal state space of M is called the σ-weak topology. That is, we say a net $(x_i)_{i \in I}$ converges σ-weakly to x if $\varphi(x_i) \to \varphi(x)$ for all normal states φ on M. The Jordan multiplication on a JBW-algebra is separately σ-weakly continuous in each variable and jointly σ-weakly continuous on bounded sets by [AS03, Proposition 2.4] and [AS03, Proposition 2.5]. Furthermore, for any x the corresponding quadratic representation U_x is σ-weakly continuous by [AS03, Proposition 2.4]. If A is a JB-algebra, then one can extend the product to its bidual A^{**} turning A^{**} into a JBW-algebra, see [AS03, Corollary 2.50]. In JBW-algebras the spectral theorem [AS03, Theorem 2.20] holds, which implies in particular that the linear span of projections is norm dense, see [H-OS84, Proposition 4.2.3].

If p is a projection, then the orthogonal complement $e - p$ will be denoted by $p^\perp$ and a projection q is *orthogonal* to p precisely when $q \leq p^\perp$, see [AS03, Proposition 2.18].

The collection of projections forms a complete orthomodular lattice by [AS03, Proposition 2.25], which means in particular that every set of projections has a supremum. We remark that this supremum is the least upper bound in the lattice of projections, and in general this is not a supremum in M.

Any central projection c decomposes the JBW-algebra M as a direct sum of JBW-subalgebras $M = U_c M \oplus U_{c^\perp} M$, see [AS03, Proposition 2.41]. A minimal non-zero projection is called an *atom* and a JBW-algebra in which every non-zero projection dominates an atom is called *atomic*. Furthermore, by [AS03, Lemma 5.58] we have that the normal state space of an atomic JBW-algebra is the closed convex hull of the set of pure states of M, where a normal state φ is considered *pure* whenever there exists an atom $p \in M$ such that $\varphi(p) = 1$.

A standard application of Zorn's lemma shows that in an atomic JBW-algebra M every non-zero projection q dominates a maximal set of pairwise disjoint atoms $\mathcal{P}$. If we denote the finite subsets of such a maximal set by $\mathcal{F}$, it follows that $\mathcal{F}$ is directed by set inclusion and we obtain an increasing net $(p_F)_{F\in\mathcal{F}}$ where $p_F := \sum_{p\in F} p$ for all $F \in \mathcal{F}$. This net has a least upper bound in M since the normal states determine the order on M by [AS03, Corollary 2.17] and in fact

$$\sup\{p_F \colon F \in \mathcal{F}\} = q.$$

By [AS03, Propositin 2.25] and [AS03, Proposition 2.5] this net converges σ-weakly to a projection, say r. Suppose that there is an atom $s \leq q - r$. Then s and r are orthogonal and so s is orthogonal to all atoms p where $\{p\} \in \mathcal{F}$, contradicting the maximality of $\mathcal{P}$. Hence $r = q$. This is a standard argument in the theory of JBW-algebras even without the presence of atoms. Nevertheless, the following lemma is useful in our study of atomic JBW-algebras, and is therefore recorded for future reference.

Lemma 5.1. *Let M be an atomic JBW-algebra and let $q \in M$ be a non-zero projection. Then there exists a maximal set $\mathcal{P}$ of pairwise disjoint atoms dominated by q, and the increasing net $(p_F)_{F\in\mathcal{F}}$ indexed by the finite subsets $\mathcal{F}$ of $\mathcal{P}$ such that $p_F := \sum_{p\in F} p$ for all $F \in \mathcal{F}$ converges σ-weakly to its least upper bound q.*

The linear isomorphisms between JB-algebras can be completely described. An important result for describing the linear order isomorphisms we will use is [IR-P95, Theorem 1.4], which we state here for the convenience of the reader. A *symmetry* is an element s satisfying $s^2 = e$. Note that s is a symmetry if and only if $p := (s + e)/2$ is a projection, and $s = p - p^\perp$.

Theorem 5.2 (Isidro, Rodríguez-Palacios)**.** *Let A and B be JB-algebras. The bijective linear isometries from A onto B are the mappings of the form $x \mapsto sJx$, where s is a central symmetry in B and $J \colon A \to B$ a Jordan isomorphism.*

This theorem uses the fact that a bijective unital linear isometry between JB-algebras is a Jordan isomorphism, see [MWY78, Theorem 4]. We cite Corollary 2.2 and Proposition 2.3 from [LRW], that use this property of linear isometries, to obtain a description of the linear order isomorphisms between JB-algebras.

Corollary 5.3. *Let A and B be order unit spaces, and $T\colon A \to B$ be a unital linear bijection. Then T is an isometry if and only if T is an order isomorphism. Moreover, if A and B are JB-algebras, then these statements are equivalent to T being a Jordan isomorphism.*

Proof. Suppose T is an isometry, and let $x \in A_+$, $\|x\| \leq 1$. Then $\|e - x\| \leq 1$, and so $\|e - Tx\| \leq 1$, showing that Tx is positive. So T is a positive map, and by the same argument T^{-1} is a positive map. (This argument is taken from the first part of [MWY78, Theorem 4].)

Conversely, if T is an order isomorphism, then $-\lambda e \leq x \leq \lambda e$ if and only if $-\lambda e \leq Tx \leq \lambda e$, and so T is an isometry.

Now suppose that A and B are JB-algebras. If T is an isometry, then T is a Jordan isomorphism by [MWY78, Theorem 4]. Conversely, if T is a Jordan isomorphism, then T preserves the spectrum, and then also the norm since $\|x\| = \max|\sigma(x)|$. □

Proposition 5.4. *Let A and B be JB-algebras. A map $T\colon A \to B$ is a linear order isomorphism if and only if T is of the form $T = U_y J$, where $y \in B_+^\circ$ and $J\colon A \to B$ is a Jordan isomorphism. Moreover, this decomposition is unique and $y = (Te)^{1/2}$.*

Proof. If T is of the above form, then T is an order isomorphism as a composition of two order isomorphisms. Conversely, if T is an order isomorphism, then $T = U_{(Te)^{1/2}} U_{(Te)^{-1/2}} T$, and by the above corollary $U_{(Te)^{-1/2}} T$ is a Jordan isomorphism.

For the uniqueness, if $T = U_y J$, then $Te = U_y Je = U_y e = y^2$ which forces $y = (Te)^{1/2}$. This implies that $J = U_{(Te)^{-1/2}} T$, so J is also unique. □

Atomic JBW-algebras

In this section we give a complete description of order isomorphisms between cones in atomic JBW-algebras. Furthermore, we characterise under which conditions on the atomic JBW-algebra all order isomorphisms are linear.

The class of atomic JBW-algebras provides a natural setting for Theorem 3.15. Indeed, we proceed by describing the relation between the order theoretical notions stated in Theorem 3.15 with the atomic structure of the JBW-algebra. More precisely, in an atomic JBW-algebra the extreme vectors of the cone correspond to multiples of atoms, the disengaged atoms are precisely the central atoms, and the cone is the sup-hull of the positive linear span of the atoms.

Lemma 5.5. *The atoms of a JBW-algebra M correspond precisely to the normalised extreme vectors of the cone M_+.*

Proof. Let M be a JBW-algebra. If $x \in M_+$ is a normalized extremal vector, then x lies in the boundary of M_+, so $0 \in \sigma(x)$. Suppose that there are two distinct non-zero $s, t \in \sigma(x)$. Then an application of Urysohn's lemma yields a non-zero positive function $f \in C(\sigma(x))$ such that $x \pm f \in M_+$ by the continuous functional calculus. This contradicts the extremality of $x \in M_+$, so $\sigma(x) = \{0, 1\}$ since $\|x\| = 1$. Hence

x is a projection. Again, by the extremality of x, this projection must be minimal, or equivalently, it is an atom.

Conversely, if M is a JBW-algebra and $p \in M$ is an atom. Then by [AS01, Lemma 3.29] we have $\mathbb{R}p = U_pM$ and [AS01, Proposition 2.32] in turn implies $\text{face}(p) = \mathbb{R}_+p$ from which we conclude that p is an extremal vector. □

Note that the first part of the proof of Lemma 5.5 is also valid in general JB-algebras, and hence any normalized extreme vector in the cone of a JB-algebra is a minimal projection. It follows from Corollary 3.2 that an order isomorphism between cones in JBW-algebras must map the rays corresponding to atoms bijectively onto each other.

Engaged and disengaged part

An atomic JBW-algebra M can be decomposed as a direct sum $M = M_D \oplus M_E$, where M_D and M_E are atomic JBW-algebras that contain all disengaged and engaged atoms of M, respectively. In this case, the cone M_+ is the direct product $M_+ = (M_E)_+ \times (M_D)_+$, and $(M_E)_+$ equals the sup hull of the positive linear span of the engaged atoms of M, which is of interest to us in light of Theorem 3.15. To carry out the construction of this decomposition we characterise the disengaged atoms in an atomic JBW-algebra.

Lemma 5.6. *Let M be an atomic JBW-algebra and $p \in M$ be an atom. The following are equivalent:*

(i) p is disengaged;

(ii) p is orthogonal to all other atoms;

(iii) p is central.

Proof. Let p be a disengaged atom and let q be an atom distinct from p. By [AS03, Lemma 3.53] the sum $p + q$ can be written as an orthogonal sum of atoms $p + q = \sum_{i=1}^n \lambda_i q_i$. Suppose $p = q_j$ for some $j \in \{1, \ldots, n\}$. If $\lambda_j = 1$ holds, then $q = \sum_{i \neq j} \lambda_i q_i$. Since p equals q_j, it is orthogonal to all other q_i Hence $p \circ q = 0$ and thus p and q are orthogonal. If instead $\lambda_j \neq 1$ holds, then p can be written as the non-trivial linear combination

$$p = \frac{1}{1 - \lambda_j} \left(-q + \sum_{i \neq j} \lambda_i q_i \right)$$

of atoms different from p which contradicts the assumption that p is disengaged. The last case to consider is that p does not equal any q_i, in which case $p = -q + \sum_i \lambda_i q_i$ is a non-trivial linear combination contradicting that p is disengaged yet again. We conclude that (i) implies (ii).

Suppose p is orthogonal to all other atoms. In particular, p operator commutes with all other atoms. Let $q \in M$ be a projection. Let $(q_F)_{F \in \mathcal{F}}$ be a net directed by the finite subsets of a maximal set of pairwise orthogonal atoms dominated by q as in Lemma 5.1.

As multiplication is separately σ-weakly continuous and p operator commutes with the finite sums q_F, the relation $p \circ (q \circ x) = q \circ (p \circ x)$ holds for all $x \in M$. Hence, p is a central projection by [H-OS84, Lemma 4.2.5] showing that (ii) implies (iii).

Lastly, suppose that p is central. Then $U_p M = \mathbb{R}p$ by [AS03, Lemma 3.29] and we get $M = \mathbb{R}p \oplus U_{p^\perp} M$. It follows that p is disengaged, showing that (iii) implies (i). □

Lemma 5.7. *The cone M_+ of an atomic JBW-algebra M is the sup-hull of the positive linear span of its atoms.*

Proof. Let $x \in M_+$. By the spectral theorem [AS01, Theorem 2.20] there exists an increasing net $(x_i)_{i \in I}$ in M_+ consisting of positive linear combinations of orthogonal projections

$$x_i := \sum_{k=1}^{n_i} \lambda_{i,k} p_{i,k}$$

for all $i \in I$ that converges in norm to its supremum x. For each $i \in I$ and $1 \leq k \leq n_i$ there also exist increasing nets $(p_{F_{i,k}})_{F_{i,k} \in \mathcal{F}_{i,k}}$ of finite sums of orthogonal atoms that converges σ-weakly to its supremum $p_{i,k}$ as in Lemma 5.1. But now the set

$$\left\{ \sum_{k=1}^{n_i} \lambda_{i,k} p_{F_{i,k}} \colon i \in I,\ F_{i,k} \in \mathcal{F}_{i,k} \right\}$$

is in the positive linear span of the atoms in M and has supremum x. Indeed, this set has upper bound x and if $y \in M_+$ is an upper bound for each of these positive linear combinations of atoms, then $x_i \leq y$ for all $i \in I$, as M_+ is σ-weakly closed. Hence $x \leq y$ and x is therefore in the sup hull of the positive linear span of the atoms in M. □

Our next goal is to construct a central projection that dominates all disengaged atoms and its orthogonal complement dominates all engaged atoms. To that end, we define

$$\mathcal{D}_M := \{p \in M \colon p \text{ is a disengaged atom}\}$$

and let p_D be the supremum of $\mathcal{D}_\mathcal{M}$ in the lattice of projections of M. The projection p_D has the desired properties as shown in the following proposition.

Proposition 5.8. *Let M be an atomic JBW-algebra. Then p_D is a central projection and any atom of M is either dominated by p_D or is orthogonal to p_D.*

Proof. Let p be an engaged atom. Then p is orthogonal to all disengaged atoms by Lemma 5.6 and therefore $q \leq p^\perp$ for all $q \in \mathcal{D}_M$. Thus $p_D \leq p^\perp$, or equivalently p is orthogonal to p_D. On the other hand, if p is an atom orthogonal to p_D, then it cannot be disengaged. It follows that every atom of M is either dominated by p_D or is orthogonal to p_D.

Let $p \in M$ be a projection and $(p_F)_{F \in \mathcal{F}}$ be an increasing net that converges σ-weakly to p consisting of finite sums induced by a maximal set of pairwise orthogonal atoms dominated by p as in Lemma 5.1. Since any atom in M is either dominated by p_D or orthogonal to p_D, it follows from [AS03, Proposition 2.18, Proposition 2.26] that p_D operator commutes with all atoms in M. Hence p_D operator commutes with p_F for all $F \in \mathcal{F}$ and as multiplication is separately σ-weakly continuous, p_D operator commutes with p. We conclude that p_D operator commutes with all elements in M by [H-OS84, Lemma 4.2.5], hence p_D is a central projection. □

The central projection p_D and its orthogonal complement $p_E := p_D^\perp$ now decomposes the atomic JBW-algebra M as a direct sum of JBW-algebras $M = U_{p_D}M \oplus U_{p_E}M$. We refer to $M_D := U_{p_D}M$ as the *disengaged* part of M and $M_E := U_{p_E}M$ as the *engaged* part of M. We proceed to show that the disengaged part of M is a sum of copies of $\mathbb{R}$, and the cone in the engaged part of this decomposition is the sup hull of the positive span of the engaged atoms.

Proposition 5.9. *Let M be an atomic JBW-algebra. Then there exist JBW-algebras M_D and M_E such that $M = M_D \oplus M_E$ which satisfy the following properties:*

(i) $M_D = \bigoplus_{p\in\mathcal{D}_M} \mathbb{R}p$;

(ii) $(M_E)_+$ *equals the sup-hull of the positive linear span of the engaged atoms.*

Proof. Decompose M into its disengaged and engaged part $M = M_D \oplus M_E$. By Lemma 5.6 all disengaged atoms are central projections in M_D, so we can write

$$M_D = \bigoplus_{p\in\mathcal{D}_M} U_pM = \bigoplus_{p\in\mathcal{D}_M} \mathbb{R}p,$$

as U_pM is one-dimensional by [AS03, Lemma 3.29]. Since M_E is an atomic JBW-algebra with unit p_E by [AS03, Proposition 2.9], the second statement follows from Lemma 5.7 as the atoms in M_E are precisely the engaged atoms of M by Proposition 5.8. □

Describing the order isomorphisms

Using Proposition 5.9 we can now characterize the atomic JBW-algebras M and N for which every order isomorphism $f\colon M_+ \to N_+$ is linear.

Theorem 5.10. *Let M and N be atomic JBW-algebras with order isomorphic cones. Then any order isomorphism $f\colon M_+ \to N_+$ is linear on $(M_E)_+$, and $f[(M_E)_+] = (N_E)_+$, where M_E and N_E are the engaged parts of M and N, respectively. In particular, any order isomorphism $f\colon M_+ \to N_+$ is linear if and only if M does not contain any central atoms.*

Proof. Proposition 5.9(ii) in conjunction with Theorem 3.15 yields that f is linear on $(M_E)_+$. Consequently, the rays corresponding to the engaged atoms of M must be mapped bijectively to the rays corresponding to the engaged atoms of N. In particular, the order isomorphism f maps $(M_E)_+$ into $(N_E)_+$ since these cones are the sup hull of the positive linear span of the engaged atoms by Proposition 5.9. Applying a similar argument to f^{-1} yields $f[(M_E)_+] = (N_E)_+$.

For the second part of the statement, suppose that M does not contain any central atoms. Then $M = M_E$ by Proposition 5.9 and so $M_+ = (M_E)_+$. In particular, all order isomorphisms $f\colon M_+ \to N_+$ must be linear. Conversely, suppose that M does contain central atoms and that all order isomorphisms $f\colon M_+ \to N_+$ are linear. Let $f\colon M_+ \to N_+$ be such a linear order isomorphism. Using the notation of Proposition 5.9(i), define the map $g\colon (M_D)_+ \to (M_D)_+$ by

$$g((x_p p)_{p\in\mathcal{D}_M}) := (x_p^2 p)_{p\in\mathcal{D}_M}.$$

Note that g is a non-linear order isomorphism and, therefore, by Proposition 5.9 the map $g \oplus \mathrm{Id}$ defined on $(M_D)_+ \times (M_E)_+ = M_+$ is a non-linear order isomorphism. It follows that $f \circ (g \oplus \mathrm{Id})\colon M_+ \to N_+$ is not linear either, which contradicts the assumption. □

A statement similar to Theorem 5.10 is also valid when order isomorphisms between upper sets are considered instead of cones, where the conclusion is that the order isomorphism is affine instead of linear as $f(0) = 0$ is no longer automatic.

Theorem 5.11. *Let M and N be atomic JBW-algebras such that $M_+ = (M_E)_+$, then every order isomorphism $f\colon \Omega \to \Theta$ between upper sets $\Omega \subseteq M$ and $\Theta \subseteq N$ is affine.*

Proof. Suppose that $M_+ = (M_E)_+$ and let $f\colon \Omega \to \Theta$ be an order isomorphism. For any $x \in \Omega$ we have that f restricts to an order isomorphism from $x + M_+$ onto $f(x) + N_+$. Define the map $\hat{f}\colon M_+ \to N_+$ by

$$\hat{f}(y) := f(x+y) - f(x).$$

It follows that $\hat{f}$ is an order isomorphism as it is the composition of two translations and the restriction of f. By Theorem 5.10, $\hat{f}$ must be linear and therefore the restriction of f to $x + M_+$ must be affine. Hence there exists an affine map $g\colon M \to N$ that coincides with f on $x + M_+$. We proceed to show that f coincides with g on all of Ω. To that end, let $y \in \Omega$. Analogously, there is an affine map $h\colon M \to N$ that coincides with f on $y + M_+$. Since M is an order unit space, there exists a $z \in M$ such that $x, y \leq z$ and we have $z \in \Omega$ as Ω is an upper set. It follows that $z + M_+ \subseteq (x + M_+) \cap (y + M_+)$, so g and h coincide on $z + M_+$. Note that we can write y as the affine combination

$$y = z - (z-y) = -(z + (z-y)) + 2z$$

of the elements $z + (z-y), z \in z + M_+$. We find that

$$\begin{aligned} f(y) = h(y) = h(-(z+(z-y))+2z) &= -h(z+(z-y)) + 2h(z) = -g(z+(z-y)) + 2g(z) \\ &= g(-(z+(z-y))+2z) = g(y), \end{aligned}$$

and we conclude that f coincides with g on Ω. □

We can now completely describe the order isomorphisms between cones of atomic JBW-algebras.

Theorem 5.12. *Let M and N be atomic JBW-algebras and let $f\colon M_+ \to N_+$ be an order isomorphism. We denote $\mathcal{D}_M$ and $\mathcal{D}_N$ for the collections of disengaged atoms and M_E and N_E for the engaged parts of M and N, respectively. Then there exist $y \in N_+^\circ$, order isomorphisms $f_p\colon \mathbb{R}_+ \to \mathbb{R}_+$ for all $p \in \mathcal{D}_M$, a bijection $\sigma\colon \mathcal{D}_M \to \mathcal{D}_N$, and a Jordan isomorphism $J\colon M_E \to N_E$, such that for all $x = x_D + x_E \in M_+$ with $x_D = (x_p p)_{p\in\mathcal{D}_M}$ we have*

$$f(x) = (f_p(x_p)\sigma(p))_{p\in\mathcal{D}_M} + U_y J x_E.$$

Proof. Let $f\colon M_+ \to N_+$ be an order isomorphism. By Proposition 5.9 we can decompose M_+ and N_+ as $M_+ = (M_D)_+ \times (M_E)_+$ and $N_+ = (N_D)_+ \times (N_E)_+$. By Theorem 5.10 we have $f[(M_E)_+] = (N_E)_+$, and therefore also $f[(M_D)_+] = (N_D)_+$. Furthermore, the rays corresponding to the disengaged atoms of M are mapped bijectively to the rays corresponding to the disengaged atoms of N. In particular, there exists a bijection $\sigma\colon \mathcal{D}_M \to \mathcal{D}_N$ and for each $p \in \mathcal{D}_M$ there is an order isomorphism $f_p\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(\lambda p) = f_p(\lambda)\sigma(p)$.

For $x \in M_+$ we have $x = x_D + x_E = \sup\{x_D, x_E\}$ and we find that

$$f(x_D + x_E) = f(\sup\{x_D, x_E\}) = \sup\{f(x_D), f(x_E)\} = f(x_D) + f(x_E),$$

where the last equality is due to $f(x_D) \in (N_D)_+$ and $f(x_E) \in (N_E)_+$. This shows that f decomposes as the sum of order isomorphisms $f_D\colon (M_D)_+ \to (N_D)_+$ and $f_E\colon (M_E)_+ \to (N_E)_+$, by defining $f_D(x_D) = f((x_D, 0))$ and $f_E(x_E) = f((0, x_E))$. Every $x_D \in (M_D)_+$ is of the form $x_D = (x_p p)_{p\in\mathcal{D}_M}$ and satisfies $x_D = \sup\{x_p p\colon p \in \mathcal{D}_M\}$, hence

$$\begin{aligned} f_D(x_D) = f((x_p p)_{p\in\mathcal{D}_M}) &= f(\sup\{x_p p\colon p \in \mathcal{D}_M\}) = \sup\{f(x_p p)\colon p \in \mathcal{D}_M\} \\ &= \sup\{f_p(x_p)\sigma(p)\colon p \in \mathcal{D}_M\} \\ &= (f_p(x_p)\sigma(p))_{p\in\mathcal{D}_M}. \end{aligned}$$

Moreover, since f_E is a linear order isomorphism, it follows that $f(x_E) = U_y J x_E$ for an element $y \in N_+^\circ$ and a Jordan isomorphism $J\colon M_E \to N_E$ by Proposition 5.4. □

An interesting and immediate consequence of Theorem 5.12 is the following corollary.

Corollary 5.13. *Let M and N be atomic JBW-algebras. Then the cones M_+ and N_+ are order isomorphic if and only if M and N are Jordan isomorphic.*

Proof. Suppose that M_+ and N_+ are order isomorphic, and let $f\colon M_+ \to N_+$ be an order isomorphism. By Theorem 5.12 there is a bijection $\sigma\colon \mathcal{D}_M \to \mathcal{D}_N$ and a Jordan isomorphism $J\colon M_E \to N_E$. Then $G\colon M \to N$ defined for $x = (x_D, x_E) \in M$ with $x_D = (x_p p)_{p\in\mathcal{D}_M}$ by

$$G((x_D, x_E)) := ((x_p\sigma(p))_{p\in\mathcal{D}_M}, Jx_E),$$

is a Jordan isomorphism. The converse implication is immediate. □

JB-algebras

The results in the previous section completely describe the order isomorphisms between cones of atomic JBW-algebras, and our goal is to investigate how these results can be used to study order isomorphisms between cones in general JB-algebras. A key observation is that any JB-algebra can be embedded isometrically, as a JB-subalgebra, into an atomic JBW-algebra, namely the atomic part of the bidual. We start by determining under which conditions an order isomorphism between cones of JB-algebras can be extended to an order isomorphism between the cones of the corresponding atomic JBW-algebras obtained via this embedding. It turns out that it is sufficient to extend to a σ-weak homeomorphism for the preduals of the atomic JBW-algebras, guaranteeing that the extension is an order isomorphism. Furthermore, by relating the ideal structure of a JB-algebra to central atoms of its bidual, we obtain an analogue of Theorem 5.10 for cones of JB-algebras.

The atomic representation of a JB-algebra

The canonical embedding of a JB-algebra A into its bidual $\hat{}\colon A \hookrightarrow A^{**}$ is not only an isometry, but also extends the product of A to A^{**} by [AS03, Corollary 2.50]. Furthermore, let z be the central projection in A^{**} as in [AS03, Lemma 3.42] such that

$$A^{**} = U_z A^{**} \oplus U_{z^\perp} A^{**}$$

where $U_z A^{**}$ is atomic and $U_{z^\perp} A^{**}$ is purely non-atomic. In the sequel we will denote the atomic part $U_z A^{**}$ of A^{**} by A^{**}_a. The quadratic representation $U_z\colon A^{**} \to A^{**}_a$ corresponding to the central projection z defines a surjective Jordan homomorphism by [AS03, Proposition 2.41]. Hence we obtain a Jordan homomorphism $U_z \circ \hat{}\colon A \to A^{**}_a$. It is a standard result for C^*-algebras that the composition of the canonical embedding $\hat{}$ and the multiplication by z is an isometric algebra embedding, see for example the preliminaries in [Ake71], and the proof for JB-algebras is the same; see [FR86, Proposition 1] for a proof for JB*-triples, which are a generalization of JB-algebras. Hence we can view A as a JB-subalgebra of A^{**}_a, and we shall do so freely throughout.

As A^{**}_a is a JBW-algebra, it is a dual space, and it follows from [RW, Corollary 2.11] that it is the dual of

$$A' := \overline{\mathrm{Span}}\{\varphi\colon \varphi \text{ is a pure state on } A\} \qquad \text{(norm closure in } A^*).$$

In particular, this yields $A' = U_z^* A^*$. Indeed, if φ is a pure state on A, then it is a normal pure state on A^{**}, so there is an atom $p \in A^{**}$ such that $\varphi(p) = 1$. It follows that $0 \leq \varphi(z^\perp) \leq \varphi(p^\perp) = 0$, so $\varphi(z^\perp) = 0$. Thus for any $x \in A^{**} = U_z A^{**} \oplus U_{z^\perp} A^{**}$ we have $-\|x\| z^\perp \leq U_{z^\perp} x \leq \|x\| z^\perp$ since $z^\perp$ is an order unit in $U_{z^\perp} A^{**}$, and so

$$\varphi(x) = \varphi(U_z x) + \varphi(U_{z^\perp} x) = U_z^* \varphi(x).$$

Hence $A' \subseteq U_z^*A^*$ as $U_z^*A^*$ is norm closed. Conversely, if φ is a state on A, then $U_z^*\varphi$ is σ-weakly continuous on A^{**}. Suppose that $U_z^*\varphi \neq 0$, then $\varphi(z)^{-1}U_z^*\varphi$ is a normal state on A^{**}. Since this state annihilates $U_{z^\perp}A^{**}$, it defines a normal state on the atomic part of A^{**} and by [AS03, Lemma 5.61] it follows that $U_z^*\varphi \in A'$. As the state space of A generates A^*, this proves the inclusion $U_z^*A^* \subseteq A'$.

Since the cone in A_a^{**} is monotone complete, our next objective is to study how the cone of A lies inside the cone of A_a^{**} with respect to bounded monotone increasing and decreasing nets, respectively. To this end, we introduce the following notation. For a subset $B \subseteq (A_a^{**})_+$ we denote by B^m the set where the suprema of all bounded monotone increasing nets in B are adjoined. Similarly, we denote by B_m the subset of $(A_a^{**})_+$ where all the infima of bounded monotone decreasing nets in B are adjoined. If we obtain $(A_a^{**})_+$ from B by adjoining suprema and infima inductively in any order, but in finitely many steps, we say that B is *finitely monotone dense* in $(A_a^{**})_+$. A consequence of a result by Pedersen [Ped72, Theorem 2] is that the cone of the self adjoint part $\mathcal{A}_{sa}$ of a C^*-algebra is finitely monotone dense in $(\mathcal{A}_{sa}^{**})_+$. In [H-OS84, Theorem 4.4.10] the analogue of Pedersen's theorem is given for JB-algebras, where it is shown that the cone of a JB-algebra A is finitely monotone dense in the cone of its bidual. The next proposition verifies that the cone of a JB-algebra is finitely monotone dense in the cone of the atomic part of its bidual.

Proposition 5.14. *Let A be a JB-algebra. Then A_+ is finitely monotone dense in $(A_a^{**})_+$.*

Proof. Let A be a JB-algebra, canonically embedded into its bidual, and let $A_+ \subseteq \Omega \subseteq A_+^{**}$. Furthermore, let U_z be the Jordan homomorphism mapping A into A_a^{**}. If $(x_i)_{i\in I} \subseteq \Omega$ is a bounded monotone increasing net with supremum x in A_+^{**}, then the net $(U_zx_i)_{i\in I}$ is a bounded increasing net in $(A_a^{**})_+$ with supremum y in $(A_a^{**})_+$, as A_a^{**} is a JBW-algebra and U_z is order preserving. Since U_z is the projection onto A_a^{**}, it follows that $U_zy = y$. For any normal state φ on A^{**} we have

$$\begin{aligned}\varphi(y - U_zx) &= \varphi(y - U_zx_i) + \varphi(U_zx_i - U_zx) = \varphi(U_zy - U_z^2x_i) + \varphi(U_zx_i - U_zx)\\ &= U_z^*\varphi(y - U_zx_i) + U_z^*\varphi(x_i - x) \to 0\end{aligned}$$

since $U_zx_i \to y$ for $\sigma(A_a^{**}, A')$, $x_i \to x$ for $\sigma(A^{**}, A^*)$ and $U_z^*\varphi \in A'$. Hence $y = U_zx$ as the normal states separate the points of A^{**}. We have shown that

$$U_zA_+ \subseteq U_z(\Omega^m) \subseteq (U_z\Omega)^m \subseteq (A_a^{**})_+$$

and the fact that the analogous inclusions hold for Ω_m follows verbatim. Therefore, we conclude that the assertion holds as A_+ is finitely monotone dense in A_+^{**} by [H-OS84, Theorem 4.4.10]. □

The Kaplansky density theorem for JB-algebras [AS03, Proposition 2.69] in conjunction with [AS03, Proposition 2.68] states that the unit ball of a JB-algebra A, which is canonically embedded into its bidual, is σ-weakly dense in the unit ball of A^{**}. The unit ball of A corresponds to the order interval $[-e,e]$ as it is an order unit space, so by applying the affine map $x \mapsto \frac{1}{2}(x+e)$, we find that consequently the unit interval $[0,e]$ of A is σ-weakly dense in the unit interval $[0,e]$ of A^{**}. The analogue for the atomic representation also holds.

Lemma 5.15. *The unit interval* $[0,e]$ *of a JB-algebra* A *is* $\sigma(A_a^{**},A')$*-dense in the unit interval* $[0,e]$ *of* A_a^{**}.

Proof. Let x be in the unit interval of A_a^{**}. Then x lies in the unit interval of A^{**} and $U_z x = x$. By the Kaplansky density theorem for JB-algebras [AS03, Proposition 2.69] in conjunction with [AS03, Proposition 2.68] there is a net $(x_i)_{i\in I}$ in the unit interval of A that converges σ-weakly to x. But then the net $(U_z x_i)_{i\in I}$ lies in the unit interval of A and converges σ-weakly, and therefore also for the $\sigma(A_a^{**},A')$-topology, to $U_z x = x$. □

Extending the order isomorphism

Consider again the situation where A and B are JB-algebras and $f\colon A_+ \to B_+$ is an order isomorphism. Our aim now is to extend f to an order isomorphism from $(A_a^{**})_+$ onto $(B_a^{**})_+$. Since A_+ and B_+ are finitely monotone dense in $(A_a^{**})_+$ and $(B_a^{**})_+$, respectively, by Proposition 5.14, it turns out that it is sufficient to extend f to a homeomorphism with respect to the $\sigma(A_a^{**},A')$-topology and the $\sigma(B_a^{**},B')$-topology.

Proposition 5.16. *Let* A *and* B *be JB-algebras and suppose* $f\colon A_+ \to B_+$ *is an order isomorphism that extends to a homeomorphism* $\hat{f}\colon (A_a^{**})_+ \to (B_a^{**})_+$ *with respect to the* $\sigma(A_a^{**},A')$*-topology and the* $\sigma(B_a^{**},B')$*-topology. Then the extension* $\hat{f}$ *is an order isomorphism.*

Proof. Let $f\colon A_+ \to B_+$ be an order isomorphism and let $\hat{f}\colon (A_a^{**})_+ \to (B_a^{**})_+$ be a homeomorphism with respect to the $\sigma(A_a^{**},A')$-topology and the $\sigma(B_a^{**},B')$-topology that extends f. Suppose that $A_+ \subseteq \Omega \subseteq (A_a^{**})_+$ and $B_+ \subseteq \Theta \subseteq (B_a^{**})_+$ are subsets for which $\hat{f}$ restricts to an order isomorphism from Ω onto Θ. We argue that $\hat{f}$ also restricts to an order isomorphism from Ω^m onto Θ^m, and from Ω_m onto Θ_m. The assertion then follows as A_+ and B_+ are finitely monotone dense in $(A_a^{**})_+$ and $(B_a^{**})_+$ respectively, by Proposition 5.14.

We derive some useful properties of $\hat{f}$. For all $x \in \Omega^m$ we have

$$\hat{f}(x) = \sup\left\{\hat{f}(y)\colon y\in\Omega,\ y\leq x\right\}. \tag{5.1}$$

To see this, let $x \in \Omega^m$. We first argue that $\hat{f}(x)$ is an upper bound of $\hat{f}(y)$ for all $y \in \Omega$ with $y \leq x$. To that end, suppose $y \in \Omega$ with $y \leq x$. Remark that $x - y \in (A_a^{**})_+$. After rescaling we can apply Lemma 5.15 to obtain a net $(y_i)_{i\in I}$ in A_+ that converges to $x-y$. By the continuity of $\hat{f}$, it follows that $\hat{f}(y_i+y)$ converges to $\hat{f}(x)$. By our assumption that $\hat{f}$ is order preserving on Ω we have $\hat{f}(y) \leq \hat{f}(y_i+y)$ for all $i \in I$ and therefore $\hat{f}(y) \leq \hat{f}(x)$ follows as $(B_a^{**})_+$ is closed. Suppose now that $z \in (B_a^{**})_+$ is such that $\hat{f}(y) \leq z$ for all $y \in \Omega$ with $y \leq x$. As $x \in \Omega^m$, there is a monotone increasing net

$(x_i)_{i\in I}$ in Ω with supremum x. Then $(x_i)_{i\in I}$ converges to x by monotone completeness and so $\hat{f}(x_i)$ converges to $\hat{f}(x)$. Hence $\hat{f}(x) \leq z$, again as $(B_a^{**})_+$ is closed, showing (5.1) holds.

Secondly, for all $x \in \Omega^m$ we have

$$y \in \Omega \text{ and } \hat{f}(y) \leq \hat{f}(x) \text{ imply } y \leq x. \tag{5.2}$$

Indeed, let $y \in \Omega$ with $\hat{f}(y) \leq \hat{f}(x)$ and $(z_i)_{i\in I}$ a net in B_+ that converges to $\hat{f}(x)-\hat{f}(y)$. As $\hat{f}^{-1}$ is continuous we infer $\hat{f}^{-1}(z_i + \hat{f}(y))$ converges to x. For all $i \in I$ we have $y = \hat{f}^{-1}(\hat{f}(y)) \leq \hat{f}^{-1}(z_i + \hat{f}(y))$ since $\hat{f}^{-1}$ is order preserving on Θ. Then $y \leq x$ follows from the fact that $(A_a^{**})_+$ is closed. This shows (5.2). Now for all $x, y \in \Omega^m$ we have

$$\begin{aligned} x \leq y &\iff \{z \in \Omega \colon z \leq x\} \subseteq \{z \in \Omega \colon z \leq y\} \\ &\iff \left\{\hat{f}(z) \colon z \in \Omega,\ z \leq x\right\} \subseteq \left\{\hat{f}(z) \colon z \in \Omega,\ z \leq y\right\} \\ &\implies \sup\left\{\hat{f}(z) \colon z \in \Omega,\ z \leq x\right\} \leq \sup\left\{\hat{f}(z) \colon z \in \Omega,\ z \leq y\right\} \\ &\implies \hat{f}(x) \leq \hat{f}(y), \end{aligned}$$

where the last implication is due to (5.1). Conversely, by (5.2) we have for all $x, y \in \Omega^m$ that

$$\begin{aligned} \hat{f}(x) \leq \hat{f}(y) &\implies \{z \in \Omega \colon z \leq x\} \subseteq \{z \in \Omega \colon z \leq y\} \\ &\implies x \leq y. \end{aligned}$$

This shows that $\hat{f}$ is an order embedding of Ω^m into $(B_a^{**})_+$, and it remains to be shown that $\hat{f}$ maps Ω^m onto Θ^m. For $x \in \Omega^m$ and a monotone increasing net $(x_i)_{i\in I}$ in Ω with supremum x we have that $\hat{f}(x)$ is the supremum of the monotone increasing net $(\hat{f}(x_i))_{i\in I}$ which is contained in Θ, showing that $\hat{f}$ maps Ω^m into Θ^m. Similarly, $\hat{f}^{-1}$ maps Θ^m into Ω^m and we conclude that $\hat{f}$ restricts to an order isomorphism from Ω^m to Θ^m. Analogously, $\hat{f}$ restricts to an order isomorphism from Ω_m to Θ_m by reversing all inequalities and replacing the suprema by infima. □

To the best of our knowledge it is presently unknown whether every order isomorphism between cones of JB-algebras always extends to a homeomorphism between the cones of the atomic part of their bidual. If this open question is answered in the positive, then the results in the next section characterise the JB-algebras for which all order isomorphisms between their cones are linear.

Automatic linearity of order isomorphisms

Provided that an order isomorphism between cones of JB-algebras extends to an order isomorphism between the cones of the corresponding atomic parts of their biduals, its linearity depends on the absence of central atoms in these biduals by Theorem 5.10.

Therefore, it is crucial to understand for which JB-algebras this absence is guaranteed. Since the disengaged part of an atomic JBW-algebra is an associative direct summand by Proposition 5.9, one leads to believe that the existence of central atoms in the bidual corresponds to having a non-zero associative direct summands in the original JB-algebra. This, however, is not the case, as is illustrated by the following example. The idea of this example is partly due to M. Wortel.

Example 5.17. Consider the JB-algebra $C([0,1];\mathrm{Sym}_2(\mathbb{R}))$ consisting of symmetric 2×2 matrices with continuous functions on $[0,1]$ as entries. Note that the dual of this JB-algebra is $M([0,1];\mathrm{Sym}_2(\mathbb{R}))$ consisting of symmetric 2×2 matrices with regular Borel measures on $[0,1]$ as entries with the dual pairing

$$\left\langle \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix} \right\rangle = \int_0^1 x_1(t)\ d\mu_1(t) + \int_0^1 x_2(t)\ d\mu_2(t) + 2\int_0^1 x_3(t)\ d\mu_3(t)$$

Define the JB-subalgebra A by

$$A := \left\{ \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} \in C([0,1];\mathrm{Sym}_2(\mathbb{R})) : x_3(t) = 0 \text{ for all } 0 \le t \le \tfrac{1}{2} \right\}. \tag{5.3}$$

Note that A does not have any non-trivial direct summands as $[0,1]$ is connected and $\mathrm{Sym}_2(\mathbb{R})$ is a factor. In particular, A does not contain an associative direct summand. However, the atomic part of the bidual equals

$$A_a^{**} = \left\{ \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} \in \ell^\infty([0,1];\mathrm{Sym}_2(\mathbb{R})) : x_3(t) = 0 \text{ for all } 0 \le t \le \tfrac{1}{2} \right\}$$

and the elements of the form

$$\begin{bmatrix} \delta_t & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 0 & \delta_t \end{bmatrix}$$

where δ_t denotes the point mass function at t for $0 \le t \le \frac{1}{2}$ are central atoms in A_a^{**}.

This example shows that an alternative condition on the JB-algebra is needed.

Lemma 5.18. *Let A be a JB-algebra. Then A^{**} contains a central atom if and only if A contains a norm closed ideal of codimension one.*

Proof. Suppose $p \in A^{**}$ is a central atom. Then $U_p\colon A^{**} \to \mathbb{R}p$ is a σ-weakly continuous Jordan homomorphism by [AS03, Proposition 2.4] and [AS03, Proposition 2.41]. Hence the corresponding multiplicative functional φ_p defined by $U_p x = \varphi_p(x)p$ for all $x \in A^{**}$ is an element of A^*. We conclude that $\ker\varphi$ is a norm closed ideal in A of codimension one.

Conversely, if I is a norm closed ideal in A of codimension one, then $A/I \cong \mathbb{R}$ and the corresponding quotient map $\pi\colon A \to \mathbb{R}$ extends uniquely to a normal homomorphism $\tilde{\pi}\colon A^{**} \to \mathbb{R}$ by [AS03, Theorem 2.65]. Since $\ker\tilde{\pi}$ is a σ-weakly closed ideal of A^{**}, it follows from [AS03, Proposition 2.39] that there is a central projection $p \in A^{**}$ such that $\ker\tilde{\pi} = U_p A^{**}$. But as this implies $U_{p^\perp} A^{**} \cong \mathbb{R}$, the central projection $p^\perp$ must be an atom by [AS03, Lemma 3.29]. □

Theorem 5.19. *Let A and B be JB-algebras such that A does not contain any norm closed ideals of codimension one. Let $f\colon A_+ \to B_+$ be an order isomorphism. Then f is linear if and only if it extends to a homeomorphism $\hat{f}\colon (A_a^{**})_+ \to (B_a^{**})_+$ with respect to the $\sigma(A_a^{**}, A')$-topology and the $\sigma(B_a^{**}, B')$-topology.*

Proof. If f is linear then there is an element $y \in B_+^\circ$ and a Jordan isomorphism $J\colon A \to B$ such that $f = U_y J$ by Proposition 5.4. Since the adjoint of $U_y J$ is an order isomorphism between the duals of B and A, it must map B' bijectively onto A'. If we denote this restriction by $(U_y J)'$, then its adjoint $(U_y J)'^*$ in turn, is a bounded linear bijection from A_a^{**} onto B_a^{**}, which must be a homeomorphism with respect to the $\sigma(A_a^{**}, A')$-topology and the $\sigma(B_a^{**}, B')$-topology. As the points of a JB-algebra are separated by the pure states, we conclude that $(U_y J)'^*$ is an extension of f.

Conversely, suppose that f extends to a homeomrphism $\hat{f}\colon (A_a^{**})_+ \to (B_a^{**})_+$ with respect to the $\sigma(A_a^{**}, A')$-topology and the $\sigma(B_a^{**}, B')$-topology. Then $\hat{f}$ is an order isomorphism by Proposition 5.4 and as A_a^{**} does not contain any central atoms by Lemma 5.18, it must be linear by Theorem 5.10. □

The condition in Theorem 5.19 of the JB-algebra not having any norm closed ideals of codimension one is necessary. Indeed, if we consider the JB-algebra A defined in (5.3), then we can define a non-linear order isomorphism on A_+ that does extend to a $\sigma(A_a^{**}, A')$-homeomorphism on the atomic part of its bidual. Indeed, let $\lambda\colon [0,1] \to \mathbb{R}_+$ be a non-constant strictly positive continuous map such that $\lambda(t) = 1$ on $(\frac{1}{2}, 1]$. Define the map $f\colon A_+ \to A_+$ by $f(x)(t) := x(t)^{\lambda(t)}$. Since taking a coordinate-wise strictly positive power is an order isomorphism on $\mathbb{R}_+^2$, the map f defines an order isomorphism. However, f is not homogeneous and therefore not linear, and f extends to a $\sigma(A_a^{**}, A')$-homeomorphism $\hat{f}\colon (A_a^{**})_+ \to (A_a^{**})_+$ by the same formula that defines f.

Chapter 6

Order theoretic characterisation of spin factors

The famous Koecher-Vinberg theorem ([Koe57] and [Vin60]) characterises the Euclidean Jordan algebras among the finite dimensional order unit spaces as the ones that have a symmetric cone. As JB-algebras are merely Banach spaces instead of Hilbert spaces, no such characterisation exists in infinite dimensions. It is, however, interesting to ask if one could characterise the JB-algebras among the complete order unit spaces in order theoretic terms. One such characterization was obtained by Kai [Kai08] who characterized the symmetric cones among the homogeneous cones. More recently, Walsh [Wal13] gave an alternative characterization of Euclidean Jordan algebras. He showed that the Euclidean Jordan algebras correspond to the finite dimensional order unit spaces (V, C, u) for which there exists a gauge-reversing bijection $g\colon C^\circ \to C^\circ$, or in other words an antihomogeneous order antimorphism. In this chapter, which is based on [LRvI17], we make the first steps towards extending this order theoretic charcterisation to classes of infinite dimensional JB-algebras.

A special class of JB-algebras are spin factors. A *spin factor* M is a real vector space with $\dim M \geq 3$ such that $M = H \oplus \mathbb{R}e$ (vector space direct sum) with $(H, (\cdot \mid \cdot))$ a Hilbert space and $\mathbb{R}e$ the linear span of e, where M is given the Jordan product

$$(a + \alpha e) \circ (b + \beta e) = \beta a + \alpha b + ((a \mid b) + \alpha\beta)e \tag{6.1}$$

and norm $\|a+\lambda e\| := \|a\|_2 + |\lambda|$, with $\|\cdot\|_2$ the norm of H. We characterize spin factors among order unit spaces that are complete with respect to the order unit norm and which have a strictly convex cone. Recall that a cone C is *strictly convex* if for each linearly independent $x, y \in \partial C$, the segment $\{(1-\lambda)x + \lambda y \colon 0 < \lambda < 1\}$ is contained in C°.

Theorem 6.1. *If (V, C, u) is a complete order unit space with a strictly convex cone and $\dim V \geq 3$, then there exists a bijective antihomogeneous order antimorphism $g\colon C^\circ \to C^\circ$ if and only if (V, C, u) is a spin factor.*

As our general approach is similar to Walsh's [Wal13], we briefly discuss the main similarities and differences. To prove that the cone is homogeneous Walsh uses in

[Wal13, Lemma 3.5] the fact that a bijective antihomogeneous order antimorphism is a locally Lipschitz map, and hence almost everywhere Fréchet differentiable by Rademacher's Theorem. There is, however, no infinite dimensional version of Rademacher's Theorem. To overcome this difficulty, we show that a bijective antihomogeneous order antimorphism is Gateaux differentiable at each point in a strictly convex cone, and work with the Gateaux derivative, see Proposition 6.5. Like Walsh we will also use ideas from metric geometry such as Hilbert's and Thompson's metrics. In particular, Walsh applies his characterization of the Hilbert's metric horofunctions [Wal08], which, at present, is not known for infinite dimensional spaces. Instead we shall show that if there exists a bijective antihomogeneous order antimorphism on a strictly convex cone, then the cone is smooth, see Theorem 6.14. This will allow us to avoid the use of horofunctions completely, but implicitly some of Walsh's horofunction method is still present in the proof of Proposition 6.18.

Order antimorphisms and symmetries

For $x, y \in V$ linearly independent we write $V(x,y) := \mathrm{span}(x,y)$, $C(x,y) := V(x,y) \cap C$, and $C^\circ(x,y) := V(x,y) \cap C^\circ$. Note that as C is Archimedean, $C(x,y)$ is a closed 2-dimensional cone in $V(x,y)$, if $x \in C^\circ$.

Useful tools in the analysis of antihomogeneous order antimorphism are Hilbert's and Thompson's metrics on C°. We briefly recall their definitions. A more detailed description and elementary properties of these metrics is given in Section 1.5.

For $x \in C$ and $y \in C^\circ$ let

$$M(x/y) := \inf\{\beta > 0 \colon x \leq_C \beta y\}.$$

Now *Hilbert's metric* on C° is defined by

$$d_H(x,y) := \log M(x/y) + \log M(y/x),$$

and *Thompson's metric* on C° is given by

$$d_T(x,y) := \max\{\log M(x/y), \log M(y/x)\}$$

for $x, y \in C^\circ$. Hilbert's metric is not a metric on C°, however, instead a metric on pairs of rays of C°. Thompson's metric is a metric on C° whose topology coincides with the order unit norm topology on C°.

The following basic lemma is well known, see e.g., [NS77], and implies that each antihomogeneous order antimorphism is an isometry under d_H and d_T. For the reader's convenience we include the simple proof.

Lemma 6.2. *Let (V, C, u) be an order unit space. Then $g \colon C^\circ \to C^\circ$ is an antihomogeneous order antimorphism if and only if $M(x/y) = M(g(y)/g(x))$ for all $x, y \in C^\circ$. In particular, a bijective antihomogeneous order antimorphism $g \colon C^\circ \to C^\circ$ is an isometry under d_H and d_T, and the inverse $g^{-1} \colon C^\circ \to C^\circ$ is an antihomogeneous order antimorphism.*

Proof. Clearly, if $g\colon C^\circ \to C^\circ$ is an antihomogeneous order antimorphism and $x \leq_C \beta y$, then $g(\beta y) \leq_C g(x)$, so that $g(y) \leq_C \beta g(x)$. This implies that $M(g(y)/g(x)) \leq M(x/y)$. On the other hand, $g(y) \leq_C \beta g(x)$ implies $g(\beta y) \leq_C g(x)$, so that $x \leq_C \beta y$ from which we conclude that $M(x/y) \leq M(g(y)/g(x))$. This shows that $M(x/y) = M(g(y)/g(x))$ for all $x, y \in C^\circ$.

Now suppose that $M(x/y) = M(g(y)/g(x))$ for all $x, y \in C^\circ$. If $x \leq_C y$, then $M(g(y)/g(x)) = M(x/y) \leq 1$, so that $g(y) \leq_C g(x)$. Likewise $g(y) \leq_C g(x)$ implies $M(x/y) = M(g(y)/g(x)) \leq 1$, so that $x \leq_C y$, which shows that g is an order antimorphism. To see that g is antihomogeneous note that if $x \in C^\circ$ and $\lambda > 0$, then $y := \lambda x$ satisfies $M(g(y)/g(x)) = M(x/y) = 1/\lambda$ and $M(g(x)/g(y)) = M(y/x) = \lambda$. This implies that $\lambda g(y) \leq_C g(x) \leq_C \lambda g(y)$ from which we conclude that $g(\lambda x) = g(y) = \frac{1}{\lambda} g(x)$. □

Every JB-algebra A has a bijective antihomogeneous order antimorphism namely, the map $\iota\colon A_+^\circ \to A_+^\circ$ given by $\iota(a) = a^{-1}$. As shown in [LRW, Section 2.4], we have that $M(\iota(a)/\iota(b)) = M(b/a)$ for all $a, b \in A_+^\circ$, and hence ι is a bijective antihomogeneous order antimorphism by Lemma 6.2.

If (V, C, u) is an order unit space with a strictly convex cone, then there exists a strictly positive state on V as the following lemma shows.

Lemma 6.3. *If (V, C, u) is an order unit space with a strictly convex cone, then there exists a strictly positive state $\rho \in S(V)$.*

Proof. Let $r \in \partial C \setminus \{0\}$. Then $C(r, u)$ is a 2-dimensional closed cone in V. By [LN12, A.5.1] there exists an $s \in \partial C \setminus \{0\}$ such that $C(r, u) = \{\alpha r + \beta s\colon \alpha, \beta \geq 0\}$. Let ϕ and ψ be linear functionals on $V(r, u)$ such that $\phi(r) = 0 = \psi(s)$, $\phi(s), \psi(r) > 0$, and $\phi(u) = 1 = \psi(u)$. By the Hahn-Banach theorem we can extend ϕ and ψ to linear functionals on V such that $\|\phi\| = \phi(u) = 1$ and $\|\psi\| = \psi(u) = 1$. It follows from [AS01, 1.16 Lemma] that $\phi, \psi \in S(V)$.

Now let $\rho := \frac{1}{2}(\phi + \psi) \in S(V)$. Note that $\phi(x) = 0$ for $x \in C$ if and only if $x = \lambda r$ for some $\lambda \geq 0$, as C is strictly convex. Likewise, $\psi(x) = 0$ for $x \in C$ if and only if $x = \lambda s$ for some $\lambda \geq 0$. This implies that $\rho(x) > 0$ for all $x \in C \setminus \{0\}$. □

Next we shall show that antihomogeneous order antimorphisms on strictly convex cones map 2-dimensional subcones to 2-dimensional subcones. To prove this we use unique geodesics for Hilbert's metric.

If (V, C, u) is an order unit space with a strictly positive functional $\rho \in S(V)$, then d_H is a metric on

$$\Sigma_\rho := \{x \in C^\circ\colon \rho(x) = 1\}.$$

Straight line segments are geodesic in the Hilbert's metric space (Σ_ρ, d_H). Moreover, if the cone is strictly convex, then it is well known, see for example [Bus55, Section 18], that each geodesic in the Hilbert's metric space (Σ_ρ, d_H) is a straight line segment.

Lemma 6.4. *Let (V, C, u) be an order unit space with a strictly convex cone, and $g\colon C^\circ \to C^\circ$ be a bijective antihomogeneous order antimorphism. If $x, y \in C^\circ$ are* *linearly independent, then $g(x)$ and $g(y)$ are linearly independent and g maps $C^\circ(x,y)$ onto $C^\circ(g(x),g(y))$.*

Proof. Let $\rho \in S(V)$ be a strictly positive state, which we know exists by Lemma 6.3. Now define $f\colon \Sigma_\rho \to \Sigma_\rho$ by

$$f(x) := \frac{g(x)}{\rho(g(x))} \quad \text{for all } x \in \Sigma_\rho.$$

Then f is an isometry on (Σ_ρ, d_H) by Lemma 6.2. If $x,y \in C^\circ$ are linearly independent, then the straight line ℓ through $x/\rho(x)$ and $y/\rho(y)$ intersected with Σ_ρ is a geodesic line in (Σ_ρ, d_H). Thus, $f(\ell \cap \Sigma_\rho)$ is also a geodesic line, and hence a straight line segment, as C is strictly convex. In fact, its image is the intersection of the straight line through $g(x)/\rho(g(x))$ and $g(y)/\rho(g(y))$ and Σ_ρ. It follows that $g(x)/\rho(g(x))$ and $g(y)/\rho(g(y))$ are linearly independent and that g maps $C^\circ(x,y)$ onto $C^\circ(g(x),g(y))$, as g is antihomogeneous. □

We note that the proof of Lemma 6.4 goes through if one only assumes that (Σ_ρ, d_H) is uniquely geodesic.

Using this lemma we can now prove the following proposition.

Proposition 6.5. *Let (V,C,u) be an order unit space with a strictly convex cone. If $g\colon C^\circ \to C^\circ$ is a bijective antihomogeneous order antimorphism, then the following assertions hold.*

(1) For each linearly independent $x,y \in C^\circ$ the restriction g_{xy} of g to $C^\circ(x,y)$ is a Fréchet differentiable map, and its Fréchet derivative $Dg_{xy}(z)$ at $z \in C^\circ(x,y)$ is an invertible linear map from $V(x,y)$ onto $V(g(x),g(y))$.

(2) For each $x \in C^\circ$ and $z \in V$ we have that

$$\Delta_x^z g(x) := \lim_{t\to 0} \frac{g(x+tz)-g(x)}{t}$$

exists, and $-\Delta_x^z g(x) \in C$ for all $z \in C$.

(3) For each $x \in C^\circ$ we have $\Delta_x^{\lambda x} g(x) = -\lambda g(x)$ for all $\lambda \in \mathbb{R}$.

Proof. Let $x,y \in C^\circ$ be linearly independent and $g\colon C^\circ \to C^\circ$ be an antihomogeneous order antimorphism. By Lemma 6.4 the restriction g_{xy} of g maps $C^\circ(x,y)$ onto $C^\circ(g(x),g(y))$. The 2-dimensional closed cones $C(x,y)$ and $C(g(x),g(y))$ are order isomorphic to

$$\mathbb{R}^2_+ := \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1,x_2 \geq 0\},$$

in other words, there exist linear maps $A\colon V(x,y) \to \mathbb{R}^2$ and $B\colon V(g(x),g(y)) \to \mathbb{R}^2$ such that $A(C(x,y)) = \mathbb{R}^2_+$ and $B(C(g(x),g(y))) = \mathbb{R}^2_+$. Thus, the map $h\colon (\mathbb{R}^2_+)^\circ \to$

$(\mathbb{R}^2_+)^\circ$ given by $h(z) = B(g_{xy}(A^{-1}(z)))$ is a bijective antihomogeneous order antimorphism on $(\mathbb{R}^2_+)^\circ$, and hence h is a d_T-isometry on $(\mathbb{R}^2_+)^\circ$. We know from [LRW, Theorem 3.2] that h is of the form:

$$h((z_1, z_2)) = (a_1/z_{\sigma(1)}, a_2/z_{\sigma(2)}) \quad \text{for } (z_1, z_2) \in (\mathbb{R}^2_+)^\circ,$$

where σ is a permutation on $\{1, 2\}$ and $a_1, a_2 > 0$ are fixed. Clearly the map h is Fréchet differentiable on $(\mathbb{R}^2_+)^\circ$, and hence g_{xy} is Fréchet differentiable on $C^\circ(x, y)$. Moreover, the Fréchet derivative $Dh(z)$ is an invertible linear map on $\mathbb{R}^2$ at each $z \in (\mathbb{R}^2_+)^\circ$, so that $Dg_{xy}(z)$ an invertible linear map from $V(x, y)$ onto $V(g(x), g(y))$ for all $z \in C^\circ(x, y)$.

To prove the second statement note that if z is linearly independent of x, then there exists a $y \in C^\circ$ such that $z \in V(x, y)$. From (1) we get that $\Delta^z_x g(x) = Dg_{xy}(x)(z)$, as g_{xy} is Fréchet differentiable on $C^\circ(x, y)$. Also, if $z = \lambda x$ for some $\lambda \neq 0$, then

$$\Delta^{\lambda x}_x g(x) = \lim_{t\to 0} \frac{g(x + t\lambda x) - g(x)}{t} = \lim_{t\to 0} \frac{-\lambda t}{t(1 + \lambda t)} g(x) = -\lambda g(x),$$

and $\Delta^0_x g(x) = 0$. Furthermore, if $z \in C$, then

$$\Delta^z_x g(x) = \lim_{t\to 0} \frac{g(x + tz) - g(x)}{t} \in -C,$$

as g is an order antimorphism. This completes the proofs of (2) and (3). □

Given a bijective antihomogeneous order antimorphism $g \colon C^\circ \to C^\circ$ on a strictly convex cone C in an order unit space, and $x \in C^\circ$ we define $G_x = G_{g,x} \colon V \to V$ by

$$G_x(z) := -\Delta^z_x g(x) \quad \text{for all } z \in V.$$

Lemma 6.6. *If $x \in C^\circ$ and $G_x(x) = x$, then $g(x) = x$.*

Proof. Simply note that $x = G_x(x) = -\Delta^x_x g(x) = g(x)$ by Proposition 6.5(3). □

The map G_x has the following property.

Proposition 6.7. *The map $G_x \colon V \to V$ is a bijective homogeneous order isomorphism with inverse $G_{g^{-1},g(x)} \colon V \to V$.*

Proof. Let $z \in V(x, y)$, $x, y \in C^\circ$ linearly independent, and $\lambda \neq 0$. Then

$$G_x(\lambda z) = -\lim_{t\to 0} \frac{g(x + t\lambda z) - g(x)}{t} = -\lambda \lim_{t\to 0} \frac{g(x + t\lambda z) - g(x)}{\lambda t} = \lambda G_x(z).$$

Also if $w \leq_C z$, then

$$G_x(w) = -\lim_{t\to 0} \frac{g(x + tw) - g(x)}{t} \leq_C -\lim_{t\to 0} \frac{g(x + tz) - g(x)}{t} = G_x(z),$$

as $x + tw \leq_C x + tz$ for all $t > 0$ and g is an order antimorphism.

To show that G_x is a surjective map on V let $h := g_{xy} \circ g^{-1}_{g(x)g(y)}$. So, $h\colon C^\circ(g(x), g(y)) \to C^\circ(g(x), g(y))$ and $h(z) = z$ for all $z \in C^\circ(g(x), g(y))$. For each $w \in V(g(x), g(y))$ we have by the chain rule that

$$w = Dh(g_{xy}(x))(w) = Dg_{xy}(x)Dg^{-1}_{g(x)g(y)}(g_{xy}(x))w = G_x(G_{g^{-1},g(x)}(w)).$$

Interchanging the roles of g and g^{-1} we also have that $G_{g^{-1},g(x)}(G_x(v)) = v$ for all $v \in V(x, y)$, and hence $G_{g^{-1},g(x)}$ is the inverse of G_x on V. □

Combining Proposition 6.7 and [NS77, Theorem B] we conclude that $G_x \in \mathrm{Aut}(C) := \{T \in \mathrm{GL}(V)\colon T(C) = C\}$ and G_x is continuous with respect to $\|\cdot\|_u$ on V, as $\|G_x\|_u = \|G_x(u)\|_u$.

Now for $x \in C^\circ$ define the *symmetry at x* by

$$S_x := G_x^{-1} \circ g. \tag{6.2}$$

So, $S_x\colon C^\circ \to C^\circ$ is a bijective antihomogeneous order antimorphism, with inverse $S_x^{-1} = g^{-1} \circ G_x$. We derive some further properties of the symmetries. Let us begin by making the following useful observation.

Lemma 6.8. *Let $x \in C^\circ$ and $y \in V$ be linearly independent of x. Then for each $w \in V(x, y)$ we have that $D(S_x)_{xy}(x)(w) = -w$.*

Proof. Note that

$$\begin{aligned} D(S_x)_{xy}(x)(w) &= \lim_{t\to 0} \frac{S_x(x+tw) - S_x(x)}{t} \\ &= \lim_{t\to 0} \frac{G_x^{-1}(g(x+tw)) - G_x^{-1}(g(x))}{t} \\ &= G_x^{-1}\left(\lim_{t\to 0} \frac{g(x+tw) - g(x)}{t}\right) \\ &= G_x^{-1}(-G_x(w)) \\ &= -w, \end{aligned}$$

as $G_x^{-1} = G_{g^{-1},g(x)}$ is a bounded linear map on $(V, \|\cdot\|_u)$ by Proposition 6.7. □

Theorem 6.9. *For each $x \in C^\circ$ we have that*

(1) $S_x(x) = x$.

(2) $S_x \circ S_x = \mathrm{Id}$ on C°.

Proof. To prove (1) note that for $x \in C^\circ$ we have by Propositions 6.5(3) and 6.7 that

$$S_x(x) = G_x^{-1}(g(x)) = G_{g^{-1},g(x)}(g(x)) = g^{-1}(g(x)) = x.$$

To show (2) let $x, y \in C^\circ$ be linearly independent. For simplicity we write $T := (S_x)_{S_x(x)S_x(y)}$ and $S := (S_x)_{xy}$, so $(S_x^2)_{xy} = T \circ S$ and S, T are Fréchet differentiable on $C^\circ(x, y)$ and $C^\circ(S_x(x), S_x(y))$ respectively. Then using the chain rule and Lemma 6.8 we find that

$$\Delta_x^y S_x^2(x) = \lim_{t \to 0} \frac{T(S(x+ty)) - T(S(x))}{t} = DT(S(x))(DS(x))(y) = -DS(x)(y) = y.$$

Note that S_x^2 is a homogeneous order isomorphism on C°, and hence by [NS77, Theorem B] we know that it is linear. So, it follows from the previous equality that $S_x^2 = \text{Id}$ on C°. □

To proceed it is useful to recall a few facts about unique geodesics for Thompson's metric from Section 1.5. A type I geodesic γ through x is the image of the geodesic path,

$$\gamma(t) := e^t r + e^{-t} s \quad \text{for } t \in \mathbb{R},$$

with $r, s \in \partial C$ and $r + s = x$. A geodesic $\gamma[\mathbb{R}]$ is of type I exactly whenever $M(u/v) = M(v/u)$ for all $u, v \in \gamma$. We remark that, in a strictly convex cone C, all type I geodesic lines are unique by [LR15, Theorem 4.3].

Lemma 6.10. *Let (V, C, u) be an order unit space with a strictly convex cone. If $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ is a geodesic path with $\gamma(0) = x$, and $\gamma[\mathbb{R}]$ is a type I geodesic line, then $S_x(\gamma(t)) = \gamma(-t)$ for all $t \in \mathbb{R}$.*

Proof. If $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ is a geodesic path with $\gamma(0) = x$, and $\gamma[\mathbb{R}]$ is a type I geodesic line, then there exist $r, s \in \partial C$ with $r + s = x$ and $\gamma(t) = e^t r + e^{-t} s$ for all $t \in \mathbb{R}$ by [LR15, Lemma 3.7]. As C is strictly convex, we know from [LR15, Theorem 4.3] that $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ is a unique geodesic path. This implies that $\hat{\gamma}(t) := S_x(\gamma(t))$, $t \in \mathbb{R}$, is also a unique geodesic path in (C°, d_T), as S_x is an isometry under d_T. Moreover, as $M(S_x(y)/S_x(z)) = M(z/y)$ for all $y, z \in C^\circ$, we know that

$$M(S_x(\gamma(t_1))/S_x(\gamma(t_2))) = M(\gamma(t_2)/\gamma(t_1)) = M(\gamma(t_1)/\gamma(t_2)) = M(S_x(\gamma(t_2))/S_x(\gamma(t_1))),$$

so that $\hat{\gamma}[\mathbb{R}]$ is a type I geodesic line though x.

It now follows again from [LR15, Lemma 3.7] that there exists $u, v \in \partial C$ such that $u + v = x$ and $\hat{\gamma}(t) = e^t u + e^{-t} v$ for all $t \in \mathbb{R}$. Recall from Proposition 6.5 that the restriction $(S_x)_{rx}$ of S_x to $C^\circ(r, x)$ is Fréchet differentiable, and hence

$$\hat{\gamma}'(0) = D(S_x)_{rx}(\gamma(0))(\gamma'(0)) = D(S_x)_{rx}(x)(r - s) = -r + s$$

by Lemma 6.8. But also $\hat{\gamma}'(0) = u - v$. Combining this with the equalities $r + s = x = u + v$, we find that $u = s$ and $v = r$. Thus, $S_x(\gamma(t)) = \hat{\gamma}(t) = e^t s + e^{-t} r = \gamma(-t)$ for all $t \in \mathbb{R}$. □

Proposition 6.11. *Let (V, C, u) be an order unit space with a strictly convex cone. For each $x \in C^\circ$ we have that S_x has x as a unique fixed point.*

Proof. Suppose by way of contradiction that $y \in C^\circ$ is a fixed point of S_x and $y \neq x$. Then y is linearly independent of x, as S_x is antihomogeneous and $S_x(x) = x$. Define $\mu := M(x/y)^{1/2}M(y/x)^{-1/2}$ and $z := \mu y \in C^\circ$. Then $M(x/z) = M(z/x)$ and hence there exists a type I geodesic path $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ through x and z, with $\gamma(0) = x$. From Lemma 6.10 it follows that $S_x\big[\gamma[\mathbb{R}]\big] = \gamma[\mathbb{R}]$, As z is the unique point of intersection of $\gamma[\mathbb{R}]$ with the invariant ray $R_y := \{\lambda y \colon \lambda > 0\}$, we conclude that $S_x(z) = z$. This, however, contradicts Lemma 6.10, as $z \neq x$. □

Remark 6.12. The metric space (C°, d_T) is a natural example of a Banach-Finsler manifold, see [Nus00]. So, the results in this section show that if there exists a bijective antihomogeneous order antimorphism on C° in a complete order unit space with strictly convex cone, then (C°, d_T) is a *globally symmetric* Banach-Finsler manifold, in the sense that for each $x \in C^\circ$ there exists an isometry $\sigma_x\colon C^\circ \to C^\circ$ such that $\sigma_x^2 = \mathrm{Id}$ and x is an isolated fixed point of σ_x. Indeed, we can take $\sigma_x = S_x$. It is interesting to understand which complete order unit spaces (C°, d_T) are globally symmetric Banach-Finsler manifolds. It might well be true that these are precisely the JB-algebras.

Smoothness of the cone

Throughout this section we will assume that $\dim V \geq 3$.

We will show that if (V, C, u) is a complete order unit space with a strictly convex cone and there exists an antihomogeneous order antimorphism $g\colon C^\circ \to C^\circ$, then C is a *smooth* cone, that is to say, for each $w \in \partial C$ with $w \neq 0$ there exists a unique $\phi \in S(V)$ such that $\phi(w) = 0$. Before we prove this we make the following elementary observation.

Lemma 6.13. *If (V, C, u) is an order unit space and $w \in \partial C$ with $w \neq 0$, then for each $x \in C^\circ$ and $y := (1 - s)w + sx$, with $0 < s \leq 1$, we have that*

$$M(x/y) = \frac{\phi(x)}{\phi(y)} = \frac{1}{s}$$

for each $\phi \in S(V)$ with $\phi(w) = 0$.

Proof. By [LN12, Section 2.1] we know that

$$M(x/y) = \frac{\|w - x\|_u}{\|w - y\|_u} = \frac{1}{s}.$$

But also $1/s = \phi(x)/\phi(y)$ for all states $\phi \in S(V)$ with $\phi(w) = 0$. □

Theorem 6.14. *If (V, C, u) is an order unit space with a strictly convex cone and there exists a bijective antihomogeneous order antimorphism $g\colon C^\circ \to C^\circ$, then C is a smooth cone.*

Proof. Let $\rho \in S(V)$ be a strictly positive state, which exists by Lemma 6.3. Suppose by way of contradiction that there exist $w \in \partial C$ with $\rho(w) = 1$ and states $\phi \neq \psi$ such that $\phi(w) = 0 = \psi(w)$. As $\phi \neq \psi$, there exists $x \in V$ such that $\phi(x) \neq \psi(x)$. Note that if $\alpha x + \beta w + \gamma u = 0$ for some $\alpha, \beta, \gamma \in \mathbb{R}$, then $\alpha\phi(x) + \gamma = \alpha\psi(x) + \gamma = 0$, which yields $\alpha = 0$ and $\gamma = 0$. This shows that x, w and u are linearly independent.

Let $W := \operatorname{span}(x, w, u)$ and $K := W \cap C$. As $\dim V \geq 3$ and $u \in C^\circ$, K is a 3-dimensional, strictly convex, closed cone in W containing u in its interior. Let $S(W)$ be the state space of the order unit space (W, K, u). Note that the restrictions of ϕ, ψ, ρ to W, denoted $\bar{\phi}$, $\bar{\psi}$, and $\bar{\rho}$ respectively, are in $S(W)$. Moreover $\bar{\rho}(w) > 0$ for all $w \in K \setminus \{0\}$, and hence

$$\Omega := \{w \in K : \bar{\rho}(w) = 1\}$$

is a 2-dimensional, strictly convex, compact set, with w in its (relative) boundary. We also know that $S(W)$ is a compact, convex subset of W^*.

Let $F := \{\zeta \in S(W) : \zeta(w) = 0\}$, which is a closed face of $S(W)$. As F contains $\bar{\phi}$ and $\bar{\psi}$ which are not equal, F is a straight line segment, say $[\tau, \nu]$ with $\tau \neq \nu$. Let $x, y \in \partial\Omega$ be such that u is between the straight line segments $[w, x]$ and $[w, y]$, as in Figure 6.1.

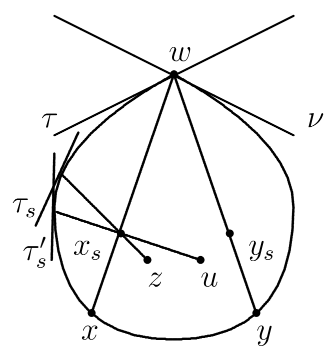


Figure 6.1: Point of non-smoothness

Now let $z \in \Omega \cap C^\circ$ also be between the segments $[w, x]$ and $[w, y]$ such that $\operatorname{span}(z, w, u) = W$. For $0 < s < 1$, let $x_s := (1 - s)w + sx$ and $y_s := (1 - s)w + sy$. By Lemma 6.13 there exists $\tau_s, \tau_s' \in S(W)$ such that

$$M(z/x_s) = \frac{\tau_s(z)}{\tau_s(x_s)} \quad \text{and} \quad M(u/x_s) = \frac{\tau_s'(u)}{\tau_s'(x_s)}$$

for $0 < s < 1$.

Then

$$\tau_s'(z) = \frac{\tau_s'(z)}{\tau_s'(x_s)} \frac{\tau_s'(x_s)}{\tau_s'(u)} \leq M(z/x_s) M(u/x_s)^{-1} \leq \frac{\tau_s(z)}{\tau_s(x_s)} \frac{\tau_s(x_s)}{\tau_s(u)} \leq \tau_s(z)$$

for all $0 < s < 1$. As $\tau_s(z) \to \tau(z)$ and $\tau_s'(z) \to \tau(z)$ as $s \to 0$, we conclude that

$$\lim_{s \to 0} M(z/x_s) M(u/x_s)^{-1} = \tau(z).$$

In the same way it can be shown that

$$\lim_{s\to 0} M(z/y_s)M(u/y_s)^{-1} = \nu(z).$$

We will now show that $\tau(z) = \nu(z)$, which implies that $\tau = \nu$, as $\tau(w) = \nu(w) = 0$, $\tau(u) = \nu(u) = 1$ and $\mathrm{span}(z, w, u) = W$. This gives the desired contradiction. To prove the equality we use the symmetry $S_u\colon C^\circ \to C^\circ$ at u. Let $f\colon \Sigma_\rho \to \Sigma_\rho$ be given by

$$f(v) = \frac{S_u(u)}{\rho(S_u(v))} \quad \text{for all } v \in \Sigma_\rho = \{w \in C^\circ\colon \rho(w) = 1\}.$$

Thus, f is an isometry on (Σ_ρ, d_H). As C is strictly convex, the segments (x, w) and (y, w) are unique geodesic lines in (Σ_ρ, d_H). So, $f((x, w))$ and $f((y, w))$ are unique geodesic lines, and hence there exist $x', y', \zeta_1, \zeta_2 \in \partial\Sigma_\rho$ so that $f((x, w)) = (x', \zeta_1)$ with $\lim_{s\to 0} f(x_s) = \zeta_1$, and $f((y, w)) = (y', \zeta_2)$ with $\lim_{s\to 0} f(y_s) = \zeta_2$.

We claim that $\zeta_1 = \zeta_2$. Suppose by way of contradiction that $\zeta_1 \neq \zeta_2$. Then using [KN02, Theorem 5.2] we know that there exists a constant $C_0 < \infty$ such that

$$\limsup_{s\to 0}\, d_H(f(x_s), u) + d_H(f(y_s), u) - d_H(f(x_s), f(y_s)) \leq C_0, \tag{6.3}$$

as Σ_ρ is strictly convex.

However, we know (see [LN12, Section 2.1]) that

$$d_H(x_s, y_s) = \log \frac{\|y_s - w'_s\|}{\|x_s - w'_s\|}\frac{\|x_s - v'_s\|}{\|y_s - v'_s\|}$$

for all $0 < s < 1$, where $w'_s, v'_s \in \partial\Omega$. Let w_s, v_s be on the lines ℓ_1 and ℓ_2 as in Figure 6.2, where ℓ_1 and ℓ_2 are fixed. For $s > 0$ sufficiently small

$$\frac{\|y_s - w'_s\|}{\|x_s - w'_s\|}\frac{\|x_s - v'_s\|}{\|y_s - v'_s\|} \leq \frac{\|y_s - w_s\|}{\|x_s - w_s\|}\frac{\|x_s - v_s\|}{\|y_s - v_s\|}.$$

By projective invariance of the cross-ratio we know there exists $C_1 < \infty$ such that

$$\frac{\|y_s - w_s\|}{\|x_s - w_s\|}\frac{\|x_s - v_s\|}{\|y_s - v_s\|} = C_1 \quad \text{for all } s > 0 \text{ sufficiently small.}$$

Thus, $\limsup_{s\to 0} d_H(x_s, y_s) \leq \log C_1$.

As f is an isometry under d_H with $f(u) = u$, we deduce that

$$d_H(f(x_s), u) + d_H(f(y_s), u) - d_H(f(x_s), f(y_s)) = d_H(x_s, u) + d_H(y_s, u) - d_H(x_s, y_s) \to \infty,$$

as $s \to 0$. This contradicts (6.3), and hence $\zeta_1 = \zeta_2$.

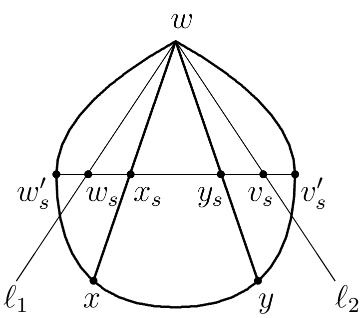


Figure 6.2: cross-ratios

Now note that

$$\begin{aligned}\tau(z) &= \lim_{s\to 0} M(z/x_s)M(u/x_s)^{-1}\\ &= \lim_{s\to 0} M(S_u(x_s)/S_u(z))M(S_u(x_s)/u)^{-1}\\ &= \lim_{s\to 0} M(f(x_s)/S_u(z))M(f(x_s)/u)^{-1}\\ &= M(\zeta_1/S_u(z))M(\zeta_1/u)^{-1}.\end{aligned}$$

Likewise $\nu(z) = M(\zeta_2/S_u(z))M(\zeta_2/u)^{-1}$, which shows that $\tau(z) = \nu(z)$, as $\zeta_1 = \zeta_2$. This completes the proof. □

Lemma 6.15. *Let (V, C, u) be an order unit space with a smooth cone, $w \in \partial C \setminus \{0\}$, and $\phi \in S(V)$ be such that $\phi(w) = 0$. Suppose that $z \in C$ with $\phi(z) > 0$, and for $0 < s \leq 1$ let $y_s := (1-s)w + su$ and $z_s := (1-s)z + su$ in C°. If $\phi_s \in S(V)$ is such that $M(z_s/y_s) = \phi_s(z_s)/\phi_s(y_s)$ for $0 < s \leq 1$, then $\phi_s(w) \to 0$, as $s \to 0$, and (ϕ_s) w*-converges to ϕ.*

Proof. Note that $M(z_s/y_s) = \phi_s(z_s)/\phi_s(y_s) \geq \phi(z_s)/\phi(y_s) = \frac{1-s}{s}\phi(z) + 1 \to \infty$, as $s \to 0$. As $|\phi_s(z_s)| \leq \|z_s\|_u \leq (1-s)\|z\|_u + s\|u\|_u \leq \|z\|_u + 1$, we deduce that $\phi_s(y_s) \to 0$ as $s \to 0$. So,

$$|\phi_s(w)| \leq |\phi_s(w) - \phi_s(y_s)| + |\phi_s(y_s)| \leq \|w - y_s\|_u + |\phi_s(y_s)| \to 0 \quad \text{as } s \to 0.$$

Now consider any subnet $(\phi_{s'})$ of (ϕ_s) in $S(V)$. It has a w*-convergent subnet with limit say ψ, as $S(V)$ is w*-compact. By the first part of the lemma we know that $\psi(w) = 0$, and hence $\psi = \phi$, since C is smooth. This shows that (ϕ_s) w*-converges to ϕ. □

Proposition 6.16. *Let (V, C, u) be an order unit space with a smooth cone, $w \in \partial C \setminus \{0\}$, and $\phi \in S(V)$ be such that $\phi(w) = 0$. Suppose that $z \in C$ with $\phi(z) > 0$ and for $0 < s \leq 1$ let $y_s := (1-s)w + su$ and $z_s := (1-s)z + su$ in C°. Then*

$$\lim_{s\to 0} M(z_s/y_s)M(u/y_s)^{-1} = \phi(z).$$

Proof. For $0 < s \leq 1$ let $\phi_s \in S(V)$ be such that $M(z_s/y_s) = \phi_s(z_s)/\phi_s(y_s)$. So, (ϕ_s) w*-converges to ϕ by Lemma 6.15. Note that

$$\begin{aligned} M(z_s/y_s)M(u/y_s)^{-1} &\leq \frac{\phi_s(z_s)}{\phi_s(y_s)}\left(\frac{\phi(u)}{\phi(y_s)}\right)^{-1} \\ &= \frac{\phi_s(z_s)}{\hat{\phi}(u)}\frac{\phi(y_s)}{\phi_s(y_s)} \\ &= \phi_s(z_s)\frac{\phi((1-s)w+su)}{\phi_s((1-s)w+su)} \\ &\leq \phi_s(z_s) \end{aligned}$$

as $\phi(w) = 0$ and $\phi_s(w) \geq 0$ for all $0 < s \leq 1$. The right-hand side of the inequality converges to $\phi(z)$ as $s \to 0$, since (ϕ_s) w*-converges to ϕ.

On the other hand, if we let $\psi_s \in S(V)$ be such that $M(u/y_s) = \psi_s(u)/\psi_s(y_s)$, then (ψ_s) w*-converges to ϕ by taking $z = u$ in Lemma 6.15. Moreover,

$$\begin{aligned} M(z_s/y_s)M(u/y_s)^{-1} &\geq \frac{\phi(z_s)}{\phi(y_s)}\left(\frac{\psi_s(u)}{\psi_s(y_s)}\right)^{-1} \\ &= \frac{\phi(z_s)}{\psi_s(u)}\frac{\psi_s(y_s)}{\phi(y_s)} \\ &\geq \phi(z_s), \end{aligned}$$

as $\psi_s(w) \geq 0$. The right-hand side converges to $\phi(z)$ as $s \to 0$, which completes the proof. □

Construction of Jordan structure

Define

$$\mathcal{P} := \{p \in \partial C \colon M(p/u) = \|p\|_u = 1\}.$$

Lemma 6.17. *If (V, C, u) is an order unit space, then for each $p \in \mathcal{P}$ there exists a unique $p' \in \mathcal{P}$ with $p + p' = u$.*

Proof. Note that $p \leq_C M(p/u)u = u$, so that $w := u - p \in (\partial C \setminus \{0\}) \cap V(p, u)$. So,

$$M(w/u) := \inf\{\beta > 0 \colon u - p \leq_C \beta u\} = \inf\{\beta > 0 \colon 0 \leq_C (\beta - 1)u + p\} = 1,$$

as otherwise $p - \delta u \in C$ for some $\delta > 0$. This would imply that $p = \delta u + (p - \delta u) \in C^\circ$, as $\delta u \in C^\circ$, which is impossible. Thus, if we let $p' := w$, then clearly p' is unique, $p' \in \mathcal{P}$ and $p + p' = u$. □

Note that $V = \operatorname{span}(\mathcal{P})$. Indeed, if $v \in V$ is linearly independent of u, then $V(u, v)$ is a 2-dimensional subspace with a 2-dimensional closed cone $C(u, v)$. By [LN12, A.5.1] there exists $r, s \in \partial C$ such that $C(u, v) = \{\lambda r + \mu s \colon \lambda, \mu \geq 0\}$ and $\operatorname{span}(r, s) = V(u, v)$.

So, if we let $p := M(r/u)^{-1}r$ and $q := M(s/u)^{-1}s$, then $p, q \in \mathcal{P}$ and $v \in \operatorname{span}(p, q)$. On the other hand, if $v = \lambda u$ with $\lambda \in \mathbb{R}$, then $v = \lambda(p + p')$ for some $p \in \mathcal{P}$ by Lemma 6.17.

Now let (V, C, u) be an order unit space with a strictly convex cone and $\dim V \geq 3$. Suppose there exists a bijective antihomogeneous order antimorphism $g\colon C^\circ \to C^\circ$. Then C is a smooth cone by Theorem 6.14. Denote by $\phi_p \in S(V)$ the unique supporting functional at $p \in \mathcal{P}$, so $\phi_p(p) = 0$ and $\phi_p(p') = \phi_p(u) = 1$. For $p \in \mathcal{P}$ define the linear form $B(p, \cdot)$ on V by

$$B(p, v) := \phi_{p'}(v) \quad \text{for all } v \in V.$$

Proposition 6.18. *If $p, q \in \mathcal{P}$, then $B(p, q) = B(q, p)$.*

Proof. Let $p, q \in \mathcal{P}$ and for $0 < s \leq 1$ define

$$p_s := (1 - s)p + su, \qquad p'_s := (1 - s)p' + su,$$
$$q_s := (1 - s)q + su, \qquad q'_s := (1 - s)q' + su.$$

We wish to show that $S_u(p_s) = \frac{1}{s}p'_s$ and $S_u(q_s) = \frac{1}{s}q'_s$. By interchanging the roles of p_s and q_s it suffices to prove the first equality.

Note that if $\beta > 0$ is such that $u \leq_C \beta p_s$, then $(1 - \beta s)u \leq_C \beta(1 - s)p$, so that $\beta s \geq 1$, as $p \in \partial C$ and $u \in C^\circ$. Thus, $M(u/p_s) = 1/s$. The same argument shows that $M(u/p'_s) = 1/s$. Furthermore, it is easy to check that $M(p_s/u) = 1 = M(p'_s/u)$, and hence $d_T(u, p_s) = -\log s = d_T(u, p'_s)$ for all $0 < s \leq 1$.

Let $\delta_s := M(u/p_s)^{1/2}M(p_s/u)^{-1/2} = 1/\sqrt{s}$ and put $x_s := \delta_s p_s$ and $y_s := \delta_s p'_s$. Then $M(x_s/u) = M(u/x_s) = 1/\sqrt{s} = M(y_s/u) = M(u/y_s)$. Thus, x_s and y_s are on the unique type I geodesic line γ through u in $C^\circ(p, p')$. Let $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ be the geodesic path with $\gamma = \gamma[\mathbb{R}]$ and $\gamma(0) = u$. As S_u is a d_T-isometry and $S_u(u) = u$, we find that $d_T(u, x_s) = d_T(u, S_u(x_s)) = -\log \sqrt{s} = d_T(u, y_s)$. Using Lemma 6.10 and the fact that $x_s \neq y_s$, we conclude that $S_u(x_s) = y_s$. Thus, $S_u(\delta_s p_s) = \delta_s p'_s$, which shows that $S_u(p_s) = \frac{1}{s}p'_s$.

Now let $p, q \in \mathcal{P}$ and suppose that $q \neq p'$. Then by Proposition 6.16 we have that

$$\begin{aligned}
B(p, q) &= \phi_{p'}(q) \\
&= \lim_{s \to 0} M(q_s/p'_s)M(u/p'_s)^{-1} \\
&= \lim_{s \to o} M(q_s/S_u(p_s))M(u/S_u(p_s))^{-1} \\
&= \lim_{s \to 0} M(p_s/S_u(q_s))M(p_s/u)^{-1} \\
&= \lim_{s \to 0} M(p_s/S_u(q_s)),
\end{aligned}$$

where we used the identity $S_u(p_s) = \frac{1}{s}p'_s$ and the fact that $S_u^2 = \mathrm{Id}$ (Theorem 6.9) in the third equality.

Likewise,

$$B(q, p) = \lim_{s \to 0} M(q_s/S_u(p_s)).$$

Now using the fact that $M(p_s/S_u(q_s)) = M(q_s/S_u(p_s))$ for all $0 < s \leq 1$, we deduce that $B(p,q) = B(q,p)$ if $q \neq p'$. On the other hand, if $q = p'$, then $B(p,q) = 0$ and $B(q,p) = 0$. □

We now extend B linearly to V by letting

$$B\left(\sum_{i=1}^{n} \alpha_i p_i, v\right) := \sum_{i=1}^{n} \alpha_i B(p_i, v) \quad \text{for all } v \in V.$$

To see that B is a well-defined bilinear form suppose that $w = \sum_i \alpha_i p_i = \sum_j \beta_j q_j$ for some $\alpha_i, \beta_j \in \mathbb{R}$ and $p_i, q_j \in \mathcal{P}$. Write $v = \sum_k \gamma_k r_k$ with $r_k \in \mathcal{P}$. Then by Proposition 6.18 we get that

$$\begin{aligned} \sum_i \alpha_i B(p_i, v) &= \sum_{i,k} \alpha_i \gamma_k B(p_i, r_k) \\ &= \sum_{i,k} \gamma_k \alpha_i B(r_k, p_i) \\ &= \sum_k \gamma_k B(r_k, w). \end{aligned}$$

Likewise $\sum_j \beta_j B(q_j, v) = \sum_k \gamma_k B(r_k, w)$, which shows that B is a well defined symmetric bilinear form on $V \times V$.

Let $H := \operatorname{span}\{p - p' \colon p \in \mathcal{P}\}$ and $\mathbb{R}u := \operatorname{span}(u)$.

Lemma 6.19. *We have that $V = H \oplus \mathbb{R}u$ (vector space direct sum), and H is a closed subspace of $(V, \|\cdot\|_u)$.*

Proof. Note that for each $v \in V$ there exists $p \in \mathcal{P}$ and $\alpha, \beta \in \mathbb{R}$ such that $v = \alpha p + \beta p'$. So,

$$v = \frac{1}{2}(\alpha - \beta)(p - p') + \frac{1}{2}(\alpha + \beta)u, \tag{6.4}$$

by Lemma 6.17. This shows that $V = H + \mathbb{R}u$. Now let $\psi_u \colon V \to \mathbb{R}$ be given by $\psi_u(v) := B(v, u)$ for all $v \in V$. Note that if $v = p - p'$, then

$$\psi_u(v) = B(p, u) - B(p', u) = \phi_{p'}(u) - \phi_p(u) = 1 - 1 = 0,$$

and hence $H \subseteq \ker(\psi_u)$. Moreover, $B(u,u) = B(p,u) + B(p',u) = 2$. Also for $v = \alpha s + \beta u$ with $s = p - p' \in H$ we have that $\psi_u(v) = 2\beta = 0$ if and only if $\beta = 0$. Thus, $H = \ker(\psi_u)$, which shows that $V = H \oplus \mathbb{R}u$.

To see that H is closed it suffices to show that ψ_u is bounded with respect to $\|\cdot\|_u$. Let $v = \alpha p + \beta p' \in V$. Then

$$\|v\|_u = \inf\{\lambda > 0 \colon -\lambda u \leq_C \alpha p + \beta p' \leq_C \lambda u\} = \max\{|\alpha|, |\beta|\}. \tag{6.5}$$

It follows that

$$|\psi_u(v)| \leq |\alpha|\psi_u(p) + |\beta|\psi_u(p') = |\alpha| + |\beta| \leq 2\|v\|_u,$$

and hence ψ_u is bounded. □

Define a bilinear form $(x \mid y)$ on H by

$$(x \mid y) := \frac{1}{2}B(x, y) \quad \text{for all } x, y \in H.$$

Proof of Theorem 6.1. We will first show that $(H, (\cdot \mid \cdot))$ is a Hilbert space. Note that if $x \in H$, then there exists $p \in \mathcal{P}$ and $\alpha \in \mathbb{R}$ such that $x = \alpha(p - p')$ by (6.4). Clearly

$$\|x\|_2^2 = (x \mid x) = \frac{1}{2}\left(\alpha^2 B(p, p - p') - \alpha^2 B(p', p - p')\right) = \frac{\alpha^2}{2}(1 + 1) = \alpha^2 = \|x\|_u^2, \quad (6.6)$$

by (6.5). It follows that $(x \mid x) \geq 0$ for all $x \in H$, $(x \mid x) = 0$ if and only if $x = 0$, and $(H, (\cdot \mid \cdot))$ is a Hilbert space, as H is closed in $(V, \|\cdot\|_u)$.

We already know from Lemma 6.19 that $V = H \oplus \mathbb{R}u$, where $(H, (\cdot \mid \cdot))$ is a Hilbert space. Note that if $x = \alpha(p - p') \in H$, then $\|x + \beta u\|_u = \max\{|\alpha + \beta|, |\alpha - \beta|\} = |\alpha| + |\beta| = \|x\|_u + |\beta|$ by (6.5). So, we deduce from equality (6.6) that

$$\|x + \beta u\|_u = \|x\|_2 + |\beta| \quad \text{for } x \in H \text{ and } \beta \in \mathbb{R}.$$

It remains to show that $\{a^2 \colon a \in V\} = C$, where the Jordan product is given by (6.1). Note that if $a = x + \sigma u$ where $x = \delta(p - p') \in H$ and $\sigma, \delta \in \mathbb{R}$, then

$$\begin{aligned}
a^2 &= 2\sigma x + ((x \mid x) + \sigma^2)u \\
&= 2\sigma\delta(p - p') + \left(\frac{\delta^2}{2}B(p - p', p - p') + \sigma^2\right)u \\
&= 2\sigma\delta(p - p') + (\delta^2 + \sigma^2)(p + p') \\
&= (\sigma + \delta)^2 p + (\sigma - \delta)^2 p' \in C.
\end{aligned}$$

Conversely, if $v \in C$, then $v = \lambda p + \mu p'$ for some $\lambda, \mu \geq 0$ and $p, p' \in \mathcal{P}$. Let

$$w := \sqrt{\lambda}p + \sqrt{\mu}p' = \frac{1}{2}\left((\sqrt{\lambda} - \sqrt{\mu})(p - p') + (\sqrt{\lambda} + \sqrt{\mu})(p + p')\right).$$

So,

$$w^2 = \frac{1}{4}\left(2(\sqrt{\lambda} - \sqrt{\mu})(\sqrt{\lambda} + \sqrt{\mu})(p - p') + ((\sqrt{\lambda} - \sqrt{\mu})^2 + (\sqrt{\lambda} + \sqrt{\mu})^2)(p + p')\right) = \lambda p + \mu p' = v,$$

which shows that $v \in \{a^2 \colon a \in V\}$. $\square$

Chapter 7

Symmetric cones and order antimorphisms

In a finite dimensional vector space the interior of a closed cone is considered *symmetric* if it is homogeneous and self-dual. A variety of characterisations are given for this special class of cones, see Section 1.6 for more details. The famous Koecher-Vinberg theorem ([Koe57] and [Vin60]) shows that a symmetric cone arises precisely as the interior of the cone of squares for a formally real Jordan algebra. An infinite dimensional analogue of this result for JB-algebras does not exist, since the notion of a symmetric cone is not well-defined in a Banach space, which in general cannot be realised as a Hilbert space. An alternative characterisation of symmetric cones in finite dimensions is given, due to Walsh [Wal13], in terms of the existence of an antihomogeneous order antimorphism on the interior of the cone. In Chapter 6, we considered strictly convex cones in arbitrary dimensions where an antihomogeneous order antimorphism exists on the interior, and we obtained precisely the spin factors. With the techniques developed there, we further investigate the relation between the existence of an antihomogeneous order antimorphism on the interior of the cone and the symmetric property of that cone.

Let $(H, (\cdot \mid \cdot))$ be a Hilbert space and $C \subseteq H$ be a cone. Then C° is considered a *symmetric* cone if C° is *homogeneous*, meaning for $x, y \in C^\circ$ there exists a linear order isomorphism $S\colon C^\circ \to C^\circ$ such that $S(x) = y$, and *self-dual* with respect to the inner product $(\cdot \mid \cdot)$, meaning

$$C^\circ = \{x \in H\colon (x \mid y) > 0 \text{ for all } y \in C\backslash\{0\}\}.$$

In the sequel, when we have an order unit space (V, C, u) we say that C° is symmetric whenever it is homogeneous and there exists an inner product on V that turns it into a Hilbert space and with respect to which C° is self-dual. Our aim is to characterise symmetric cones in complete order unit spaces in an order theoretic way without a priori imposing Hilbert space structure. We remark that the interior of the cone of a spin factor is symmetric. Therefore, characterising properties for symmetric cones should be weaker than those imposed in Theorem 6.1 for spin factors. It turns out that one should replace the condition on C of being strictly convex with being the sum of its extreme rays. In other words, instead of every element of the cone being the sum of two positive extreme vectors, we merely require that they are a finite positive linear combination of postive extreme vectors.

Our general strategy to obtain this characterisation is to reduce problems to finite dimensional subcones. A key observation is that, the results of [NS77] concerning order isomorphisms and the way they interact with line segments that are parallel to extreme rays, which we outlined in Section 3.1, for the most part also apply to order antimorphisms. This will yield that any subcone of C° that is spanned by finitely many extreme rays will be mapped by the antimorphism $g\colon C^\circ \to C^\circ$ onto a subcone that is again spanned by finitely many extreme rays. Before we make the ideas more rigorous, we summarise the results obtained in [Wal13], that are relevant for our purpose.

Theorem 7.1 (Walsh)**.** *Suppose K is a finite dimensional closed cone with non-empty interior and $g\colon K^\circ \to K^\circ$ is an antihomogeneous order antimorphism. Then K is a symmetric cone that is self-dual for an inner product $(\cdot \mid \cdot)$ which satisfies*

$$(y \mid x) = M(x/g(y)), \tag{7.1}$$

for all $y \in K^\circ$ and $x \in K$ extreme.

We recall that by definition for $x \in K$ and $y \in K^\circ$ we have

$$M(x/y) = \inf\{\beta > 0 \colon x \leq \beta y\}.$$

In Proposition 3.1 and Proposition 3.3 and their subsequent corollaries, we have shown that for an order isomorphism $f\colon C \to K$ and extreme vector $r \in C$, the element $f(x + r) - f(x)$ is an extreme vector of K. Furthermore, if the difference of $x, y \in C$ is a linear combination of extreme vectors, then $f(x + r) - f(x)$ is a scalar multiple of $f(y + r) - f(y)$. In our setting, where C is assumed to be the sum of its extreme rays, we obtain that $f(x + r) - f(x)$ and $f(y + r) - f(y)$ lies on the same extreme ray for all $x, y \in C$ and $r \in C$ extreme. We draw a similar conclusion for order antimorphisms. Recall that for a cone C the set $\mathcal{E}(C)$ denotes the collection of its extreme rays.

Proposition 7.2. *Let (X, C) and (Y, K) be Archimedean partially ordered vector spaces, where C is the sum of its extreme rays and $U \subseteq X$ and $V \subseteq Y$ are upper sets. For any order anti- or isomorphism $f\colon U \to V$, there exists a bijection $\varphi\colon \mathcal{E}(C) \to \mathcal{E}(K)$ such that any line segment in U parallel to some $R \in \mathcal{E}(C)$ is mapped by f onto a line segment in V parallel to $\varphi(R) \in \mathcal{E}(K)$. In symbols, for every $x \in U$ and $R \in \mathcal{E}(C)$*

$$f((x \pm R) \cap U) = (f(x) \pm \varphi(R)) \cap V. \tag{7.2}$$

Proof. Consider the case where f is an order isomorphism. Equation (7.2) follows from Theorem 3.10 for $R \in \mathcal{E}(C)$ that are engaged and from Proposition 3.17 for $R \in \mathcal{E}(C)$ that are disengaged.

Suppose that f is an order antimorphism. It follows from Proposition 3.1, that a subset L of U is of the form $(x \pm R) \cap U$ for some extreme ray $R \in \mathcal{E}(C)$ if and only if L is maximal among those subset of U that are directed and whose subintervals are totally ordered. These properties are not only preserved by order isomorphisms, but also by order antimorphisms. The only difference being that an order antimorphism reverses the order within an extreme half-line. In symbols, for $x \in U$, $r \in X$ an extreme vector and $\lambda \in \mathbb{R}$ with $x + r, x + \lambda r \in U$ we have that

$$f(x + \lambda r) - f(x) = c(f(x + r) - f(x)), \tag{7.3}$$

for some $c \in \mathbb{R}$. Let R and S be different extreme rays of C. Suppose $x \in U$, $r \in -R$ and $s \in -S$ are given such that $x, x + r, x + s, x + r + s \in U$. We construct $R_j := (x + js - R) \cap U$, for $j \in \{0, 1, 2\}$. Their images $f(R_j)$ are disctinct half-lines with apexes $f(x + js)$, for $j \in \{0, 1, 2\}$ respectively, and they are unbounded in the direction of a positive extreme vector. The half-lines $f(R_0)$, f(R_1) and $f(R_2)$ satisfy the same conditions as their namesakes in the proof of Proposition 3.3. So we deduce that $f(x), f(x+s), f(x+r), f(x+r+s)$ are the consecutive corners of a parallellogram, and hence

$$f(x + r + s) - f(x + s) = f(x + r) - f(x). \tag{7.4}$$

We considered negative extreme vectors $r \in -R$ and $s \in -S$ to guarantee that the lines R_j were flipped by the antimorphism f to a positive direction. Suppose now that $r \in R$, then we can apply the above arguments to $x + r \in U$, $-r \in -R$ and $s \in -S$ and obtain (7.4). Similarly the sign of s is irrelevant for the conclusion.

By repeated application of (7.4), we obtain that for $x \in U$ and $s_1, \ldots, s_n, r \in X$ extreme vectors such that $r \neq \lambda s_i$ for all $\lambda \in \mathbb{R}$ and $i = 1, \ldots, n$ with $x, x + r, x + \sum_{i=1}^n s_i, x + r + \sum_{i=1}^n s_i \in U$ that

$$f(x + r + \sum_{i=1}^{n} s_i) - f(x + \sum_{i=1}^{n} s_i) = f(x + r) - f(x). \tag{7.5}$$

Let $x \in U$, $R \in \mathcal{E}(C)$ and $r \in (R \cup -R)\backslash\{0\}$ be such that $x + r \in U$. Now $s := f(x + r) - f(x)$ is an extreme vector of K. Let $y \in U$ with $y + r \in U$. By our assumption we can write $y - x = \sum_{i=1}^n r_i$, where $r_i \in X$ is an extreme vector, for $i = 1, \ldots, n$. Remark that one of these r_i might be a multiple of r. By relabelling we assume that $r_1, \ldots, r_k < 0$ and $r_{k+1}, \ldots, r_n > 0$ and, moreover, that if $r = \lambda r_i$ for some $\lambda \in \mathbb{R}$ then in the new labelling r_i becomes r_1 if $\lambda < 0$ and r_{k+1} if $\lambda > 0$. We assume the latter to be the case, since the other cases follow in fewer steps. Combining (7.3) and (7.5) yields

$$\begin{aligned} f(y+r)-f(y) &= f(x+r+\sum_{i=1}^{n} r_i) - f(x+\sum_{i=1}^{n} r_i) \\ &= f(x+r+\sum_{i=k+1}^{n} r_i) - f(x+\sum_{i=k+1}^{n} r_i) \\ &= c\left(f(x+r+\sum_{i=k+2}^{n} r_i) - f(x+\sum_{i=k+2}^{n} r_i)\right) \\ &= c\left(f(x+r)-f(x)\right), \end{aligned}$$

for some non-zero $c \in \mathbb{R}$. This shows that $f(y+r) \in f(y)+S$, where S denotes the extreme ray in K spanned by s. Defining $\varphi(R) := S$ yields a map $\varphi\colon \mathcal{E}(C) \to \mathcal{E}(K)$ that satisfies (7.2). That φ is bijective follows from the fact that f is bijective. □

The standing hypotheses in the sequel are as follows.
Let (V, C, u) be a complete order unit space. The cone C equals *the sum of its extreme rays*, that is, any element of C can be written as a positive linear combination of extreme vectors of C. Furthermore, let $g\colon C^\circ \to C^\circ$ be an antihomogeneous order antimorphism. Lastly, we denote by $\mathcal{E}(C)$ the collection of extreme rays of C and $\varphi\colon \mathcal{E}(C) \to \mathcal{E}(C)$ for the bijection corresponding to g, that satisfies (7.2), as obtain in Proposition 7.2.

We argue in steps that under these conditions C° is a symmetric cone.

Homogeneous cone

It is convenient to introduce some notation. For a finite subset $F \subseteq \mathcal{E}(C)$ we let $C(F) = \operatorname{span}(F) \cap C$ and $C^\circ(F) = \operatorname{span}(F) \cap C^\circ$. We remark that as the finite dimensional subspace $\operatorname{span}(F)$ is closed, that the relative interior of $C(F)$ equals $C^\circ(F)$ if $\operatorname{span} F \cap C^\circ$ is non-empty. Henceforth, *any* finite subset $F \subseteq \mathcal{E}(C)$ is assumed to yields a non-empty $C^\circ(F)$.

In the following result our approach is similar to that of [Wal18, Lemma 3.9].

Lemma 7.3. *For any finite subset $F \subseteq \mathcal{E}(C)$ with* $\operatorname{span} F \cap C^\circ$ *non-empty, we have* $g[C^\circ(F)] = C^\circ(\varphi[F])$.

Proof. Suppose $F \subseteq \mathcal{E}(C)$ is finite. Fix $x \in C^\circ(F)$. We define $W = g(x) + \operatorname{span}(\varphi[F])$, an affine subspace of V. Let $y \in C^\circ(F)$ and write $y - x = \sum_{i=1}^{n} r_i$ with $r_i \in R_i \cup -R_i$ and $R_i \in F$, for $i = 1, \ldots, n$. We reorder the indices if necessary so that all r_i are positive for $i \leq m$ and negative for $i > m$ for some m. Now we define $x_0 = x$ and $x_k = x + \sum_{i=1}^{k} r_i$, for $k = 1, \ldots, n$. We remark that our reordering guarantees that all $x_k \in C^\circ(F)$. Also, by construction $x_n = y$. Note that $g(x_0) \in W$. By Proposition 7.2, we now iteratively obtain that all subsequent $g(x_i)$, and in particular $g(x_n) = g(y)$, are contained in W. As y was chosen arbitrarily, we conclude $g[C^\circ(F)] \subseteq W$. We remark that g^{-1} satisfies (7.2) for the bijection $\varphi^{-1}\colon \mathcal{E}(C) \to \mathcal{E}(C)$. Therefore, we obtain the reverse inclusion $W \cap C^\circ \subseteq g[C^\circ(F)]$.

It remains to argue that W is a linear subspace of V. Let $x \in C^\circ(F)$. Then $x \in C^\circ$ and hence x is an order unit. For all $z \in C^\circ$ there exists an $n \in \mathbb{N}$ such that $nx > g^{-1}(z)$, so for all $m \geq n$ we have $0 \leq g(mx) < z$. In particular, $\|g(nx) - 0\|_u \to 0$ as $n \to \infty$. W is a finite dimensional affine subspace of V and, hence, is closed. We conclude $0 \in W$ and that W is a linear subspace. □

In the sequel we will denote the restriction $g|_{C^\circ(F)}\colon C^\circ(F) \to C^\circ(\varphi[F])$ simply by g_F. Similar as in Chapter 6, our first step is to construct a point symmetry for each $x \in C^\circ$.

Proposition 7.4. *Let (V, C, u) be an order unit space, such that C equals the sum of its extreme rays and $g\colon C^\circ \to C^\circ$ an antihomogeneous order antimorphism. For $x \in C^\circ$ and $y \in V$ the following limit exists*

$$\Delta_x^y g(x) := \lim_{t \to 0} \frac{g(x + ty) - g(x)}{t}.$$

Moreover, for any finite $F \subseteq \mathcal{E}(C)$ with span $F \cap C^\circ$ *non-emtpy the restriction $g_F\colon C^\circ(F) \to C^\circ(\varphi[F])$ is Fréchet differentiable.*

Proof. Let $F \subseteq \mathcal{E}(C)$ with span $F \cap C^\circ$ non-empty. As the restriction g_F of g is an antihomogeneous order antimorphism from $C^\circ(F)$ onto $C^\circ(\varphi[F])$, [Wal13, Corollary 1.2] yields that $C^\circ(F)$ and $C^\circ(\varphi[F])$ are linearly isomorphic. Let $h\colon C^\circ(F) \to C^\circ(\varphi[F])$ be a linear order isomorphism and $f = h^{-1} \circ g_F\colon C^\circ(F) \to C^\circ(F)$. Then $C^\circ(F)$ is a symmetric cone by Theorem 7.1, since f is an antihomogeneous order antimorphism. Therefore, span$F$ is a Euclidean Jordan algebra with $C^\circ(F)$ as the interior of its cone of squares, by the Koecher-Vinberg theorem. In [LRW, Theorem 3.2] the isometries for Thompson's metric on the interior of the cone of a JB-algebra are characterised. This yields in our case that f is the composition of a linear bijection and the inversion map with respect to the Jordan product. In general, the inversion map on the interior of a cone in a JB-algebra is smooth, whose derivative at x is given by $-Q_x^{-1}$. We conclude that f, and hence g_F, is Fréchet differentiable. Let $Dg_F(x)\colon$ span $F \to$ span F denote the Fréchet derivative of g_F at x.

Now let $x \in C^\circ$ and $y \in V$ be given. Let $F \subseteq \mathcal{E}(C)$ be finite with $x, y \in$ span F. Then $\Delta_x^y g(x)$ exists and is given by $Dg_F(x)(y)$.

□

Results in Chapter 6, from Proposition 6.5 up to and including Theorem 6.9, now directly follow in our case with the following modification. For any pair $x, y \in C^\circ$ we consider the restriction g_F of g to a subcone $C^\circ(F)$, for some $F \subseteq \mathcal{E}(C)$ finite with $x, y \in$ span(F), instead of the restriction g_{xy} of g to the 2-dimensional subcone $C^\circ(x, y)$. We summarise these results here for convenience.

Let $x \in C^\circ$. The map $G_{g,x}\colon V \to V$ defined by $G_{g,x}(y) = -\Delta_x^y g(x)$ is a linear order isomorphism whose inverse is given by $G_{g^{-1},g(x)}$. The *symmetry at* x defined by

$$S_x = G_{g,x}^{-1} \circ g, \tag{7.6}$$

is an antihomogeneous order antimorphism, whose Gateaux derivative satisfies $\mathcal{D}S_x = -\mathrm{Id}$ and satisfies both $S_x(x) = x$ and $S_x \circ S_x = \mathrm{Id}$ on C°.

Even though unnecessary for showing that C is homogeneous, we argue that the point symmetries S_x have x as their unique fixed point in our setting, giving rise to a globally symmetric Banach-Finsler manifold similar as in Remark 6.12. We need to employ arguments different to those used in the strictly convex cone case. First we make the following observation on closed balls of Hilbert's metric.

Lemma 7.5. *Let (V, C, u) be a complete order unit space. If $y \in C^\circ$ and $r > 0$, then $B_r(y) \cup \{0\}$ is a $\|.\|_u$-closed subcone of C, where $B_r(x)$ is the closed ball for Hilbert's metric centered at y with radius r.*

Proof. Suppose $y \in C^\circ$ and $r > 0$ are given. Let $B := B_r(y) \cup \{0\}$. In [LN12, Lemma 2.6.1] it shown that a closed ball for Hilbert's metric in the interior of a finite dimensional closed cone is projectively convex. From the arguments, however, it follows that any closed ball for Hilbert's metric is convex. Since Hilbert's metric is constant on rays it follows that B is a cone. We verify that B is closed for $\|.\|_u$. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in B that converges to $x \in C$. We consider several cases. If $x \in C^\circ$ holds, then eventually all $x_n \in B$ and as $(a, b) \mapsto M(a/b)$ is a continuous map from $V \times C^\circ$ to $\mathbb{R}$ by [LLNW18, Lemma 2.2], and $x \in B$ follows. In the case $x \in \partial C\backslash\{0\}$, we obtain a contradiction as eventually all $x_n \in B$ and $d_H(x_n, y)$ tends to infinity. The last case to consider is $x = 0$, which follows from $0 \in B$. □

Now we can show that the map S_x, for some $x \in C^\circ$, has x as a *unique* fixed piont, by using that S_x is an isometry of Hilbert's metric.

Lemma 7.6. *For each $x \in C^\circ$ we have that S_x has x as a unique fixed point.*

Proof. Suppose $y \in C^\circ$ is a fixed point of S_x and $y \neq x$. Due to S_x being antihomogeneous it follows that y is not a scalar multiple of x. Consider the closed d_H-ball $B_r(y)$ centered at y with radius $r = d_H(x, y)$. Then $B := B_r(y) \cup \{0\}$ is a $\|.\|_u$-norm closed subcone of C by Lemma 7.5 with $y \in B^\circ$ and $x \in \partial B$. By the Hahn-Banach theorem, let $\psi\colon V \to \mathbb{R}$ be a $\|.\|_u$-norm continuous functional that supports B at x. Then $\psi(x) = 0$, $\psi(y) > 0$ and $\psi(v) \geq 0$ for all $v \in B$. Consider the d_H-geodesic defined by $\gamma(t) = tx + (1-t)y$, for $t \in [0, 1]$, which is fully contained in $B_r(y)$. We remark that $S_x[B_r(y)] \subseteq B_r(y)$ holds, as y is a fixed point of the d_H- isometry S_x. Therefore, the d_H-geodesic $t \mapsto \hat{\gamma}(t) := S_x(\gamma(t))$ is also contained in $B_r(y)$.

Consider the composition $\psi \circ S_x : C^\circ \to \mathbb{R}$ and remark that it is Gateaux differentiable. We compute the Gateaux derivative of $\psi \circ S_x$ at x in the direction of $y - x$. As $(\psi \circ S_x)(x) = \psi(S_x(x)) = \psi(x) = 0$, we get

$$\lim_{t>0} \frac{(\psi \circ S_x)(x + t(y - x)) - (\psi \circ S_x)(x)}{t} = \lim_{t>0} \frac{\psi(\hat{\gamma}(t))}{t} \geq 0.$$

However, as ψ is linear we can compute the same derivative as follows

$$\psi(\mathcal{D}S_x(x)(y-x)) = \psi(x-y) = -\psi(y) < 0.$$

Here we used that $\mathcal{D}S_x = -\mathrm{Id}$. This yields the desired contradiction and we conclude that indeed x is the unique fixed point of S_x. □

We continue our analysis of the point symmetries S_x induced by g and how their existence guarantees that C° is a homogeneous cone. First off, we study the interaction between such symmetries and unique geodesics with respect to Thompson's metric. A point symmetry S_x maps a unique d_T-geodesic through x onto itself and reverses its orientation, as we show below. A similar statement is made in Lemma 6.10 with the difference being that there the cone is strictly convex, hence uniqueness of the geodesic is automatic. Here we have to restrict the scope to unique geodesics. For an overview of geodesics and their properties, see Section 1.5.

Lemma 7.7. *Let $x \in C^\circ$. For a unique d_T-geodesic line $\gamma\colon \mathbb{R} \to C^\circ$ with $\gamma(0) = x$ we have $S_x(\gamma(t)) = \gamma(-t)$, for all $t \in \mathbb{R}$.*

Proof. Let $x \in C^\circ$ and $\gamma\colon \mathbb{R} \to C^\circ$ a unique geodesic line with $\gamma(0) = x$. We remark that $\hat{\gamma}\colon \mathbb{R} \to C^\circ$ defined by $\hat{\gamma}(t) = S_x(\gamma(t))$, for $t \in \mathbb{R}$, is a unique geodesic as S_x is an isometry under d_T, and satisfies $\hat{\gamma}(0) = x$. Suppose that γ is of type II. Then $\gamma(t) = e^t x$ for all $t \in \mathbb{R}$. In this case, the antihomogeneity of S_x immediately yields

$$\hat{\gamma}(t) = S_x(e^t x) = e^{-t} S_x(x) = e^{-t} x = \gamma(-t).$$

In particular, this yields that as S_x is an involution that γ and $\hat{\gamma}$ are necessarily of the same type, as each unique geodesic is either of type I or of type II.

Suppose now that γ is a unique geodesic of type I. Then there two pairs $r, s \in \partial C$ and $u, v \in \partial C$ with $r + s = x$ and $u + v = x$ such that $\gamma(t) = e^t r + e^{-t} s$, and $\hat{\gamma}(t) = e^t u + e^{-t} v$, for all $t \in \mathbb{R}$. Recall that S_x is Gateaux differentiable and satisfies $\mathcal{D}S_x = -\mathrm{Id}$, and hence

$$\hat{\gamma}'(0) = \mathcal{D}S_x(\gamma(0))(\gamma'(0)) = \mathcal{D}S_x(x)(r-s) = -r + s.$$

Computing the same derivative directly yields $\hat{\gamma}'(0) = u - v$. In combination with $r + s = x = u + v$ this yields $r = v$ and $s = u$. Therefore, we conclude

$$S_x(\gamma(t)) = \hat{\gamma}(t) = e^t u + e^{-t} v = e^{-t} r + e^t s = \gamma(-t).$$

□

We verify that our cone has sufficiently many unique geodesics for Thompson's metric. Even though this is a direct consequence of [LR15, Theorem 4.3], we provide a proof for the reader's convenience.

Lemma 7.8. *Let $x \in C^\circ$ and $r, s \in \partial C$ with $x = r + s$. The type I geodesic $\gamma\colon \mathbb{R} \to C^\circ$ through x defined by $\gamma(t) = e^t r + e^{-t} s$, for all $t \in \mathbb{R}$, is unique whenever r or s is an extreme vector of C.*

Proof. Let $x \in C^\circ$ and $r, s \in \partial C$ be given with $x = r + s$. We remark that $r \neq s$. Let γ denote the type I geodesic given by $\gamma(t) = e^t r + e^{-t} s$, for all $t \in \mathbb{R}$. Without loss of generality we assume that r is an extreme vector.

By [LR15, Theorem 4.3] the geodesic γ is unique if no $y \in V\backslash\{0\}$ and $\epsilon > 0$ exist such that $r + \lambda y$ and $s + \lambda y$ are elements of $\partial C(r, s, y)$ for all $|\lambda| < \epsilon$, where $C(r, s, y) = C \cap \operatorname{span}(r, s, y)$. Suppose the converse holds. Let $\lambda \in \mathbb{R}$ with $|\lambda| < \epsilon$. Then both $r - \lambda y$ and $r + \lambda y$ are in C and r lies on the straight line segment connecting them. Therefore, as the extreme ray spanned by r is a face of the cone, we infer that $r + \lambda y$ is a scalar multiple of r. In particular, y is a multiple of r. Now $s - \mu r \in \partial C(r, s)$ for some $\mu > 0$, which yields a contradiction. □

Combining the existence of unique type I geodesics for Thompson's metric in the direction of extreme rays, as given by Lemma 7.8, with the fact that a point symmety S_x mirrors a unique geodesic through x, as shown in Lemma 7.7, we obtain information on the automorphism group $\operatorname{Aut}(C^\circ)$.

Lemma 7.9. *For any $x \in C^\circ$ and $r \in V$ an extreme vector of C with $x + r \in C^\circ$, there exists an $S \in Aut(C^\circ)$ such that $S(x) = x + r$.*

Proof. Let $\lambda \in \mathbb{R}$ and $s \in \partial C$ such that $x = r' + s$ with $r = \lambda r'$ and $r' \in C$. We remark that $x + \lambda r' = x + r \in C^\circ$, so $\lambda > -1$. Indeed, if $\lambda \leq -1$ then $x + r \leq x - r' = s$ holds and $x + r \notin C^\circ$ follows. Consider the geodesic line $\gamma\colon \mathbb{R} \to C^\circ$ for Thompson's metric defined by $\gamma(t) = e^t r' + e^{-t} s$, for $t \in \mathbb{R}$. By Lemma 7.8, γ is unique. Let $\alpha = \sqrt{1 + \lambda} > 0$. Now $\gamma_\alpha(t) := \alpha\gamma(t)$ is a unique geodesic through αx such that

$$\begin{aligned}\gamma_\alpha(\ln\alpha) &= \alpha e^{\ln\alpha} r' + \alpha e^{-\ln\alpha} s \\ &= \alpha^2 r' + s \\ &= (r' + s) + \lambda r' = x + r.\end{aligned}$$

Let $y = \gamma_\alpha(\frac{1}{2}\ln\alpha)$. Then $t \mapsto \gamma_\alpha(t + \frac{1}{2}\ln\alpha)$ is a unique d_T-geodesic line and by Lemma 7.7 we get

$$S_y(\alpha x) = S_y(\gamma_\alpha(-\frac{1}{2}\ln\alpha + \frac{1}{2}\ln\alpha) = \gamma_\alpha(\ln\alpha) = x + r.$$

Now consider the type II geodesic $\mu(t) = e^t x$, which is necessarily unique. Let $z = \mu(\frac{1}{2}\ln\alpha)$. Then

$$S_z(x) = S_z(\mu(-\frac{1}{2}\ln\alpha + \frac{1}{2}\ln\alpha) = \mu\ln\alpha = \alpha x.$$

Consider the composition $S = S_y \circ S_z$. We get $S(x) = S_y(\alpha x) = x + r$. Moreover, S is a composition of two antihomogeneous order antimorphisms and is, therefore, a homogeneous order isomorphism. By [NS77, Theorem B], S is linear and we conclude $S \in \operatorname{Aut}(C^\circ)$. □

We are in position to show that under the standing hypotheses C° is a homogeneous cone.

Theorem 7.10. *Let (V, C, u) be an order unit space, such that C equals the sum of its extreme rays and $g\colon C^\circ \to C^\circ$ be an antihomogeneous order antimorphism. Then C° is homogeneous.*

Proof. Let $x, y \in C^\circ$ be given. By assumption we can write $y - x = \sum_{i=1}^n \sigma_i r_i$ with all $r_i \in C$ extreme vectors and $\sigma_i \in \{-1, 1\}$. We reorder the indices if necessary such that the σ_i form a sequence of exclusively positive signs followed by negative signs. Let $x_0 = x$ and for $k \in \{1, \ldots n\}$ we let $x_k = x + \sum_{i=1}^k \sigma_i r_i$. For each $k \in \{1, \ldots, n\}$, we denote by S_k the automorphism of C°, obtained by Lemma 7.9, that maps x_{k-1} to x_k. The automorphism defined as the composition $S := S_n \circ \ldots \circ S_1$ satisfies $S(x) = x_n = y$. □

Self-dual cone in a Hilbert space

We show that under the standing hypotheses, our vector space V can be endowed with an inner product that makes V a Hilbert space, and that C° is self-dual for this inner product.

Construction of an inner product

Our strategy in constructing a bilinear form on $V \oplus V$, is to build it up from inner products induced by the finite dimensional subcones of C° that are symmetric. From Lemma 7.3 we know that for any finite $F \subseteq C^\circ$ with span $F \cap C^\circ$ non-empty we have $g[C^\circ(F)] = C^\circ(\varphi[F])$. In the situation $F = \varphi[F]$, the restriction g_F is an antihomogeneous order antimorphism from the interior of a finite dimensional closed cone to itself and, due to Theorem 7.1(Walsh), $C^\circ(F)$ is a symmetric cone. A priori, it is not apparent that there exists a finite subset $F \subseteq \mathcal{E}(C)$ with $\varphi[F] = F$.

We remark, however, that the point symmetry $S_u\colon C^\circ \to C^\circ$ given by (7.6) is an antihomogeneous order antimorphism that, in addition, is an involution. Henceforth, we assume without loss of generality that g is an involution. This means that the corresponding φ is also an involution. For a finite subset $F \subseteq \mathcal{E}(C)$ with span $F \cap C^\circ$ non-empty we now define

$$F^* := F \cup \varphi[F],$$

and remark that $\varphi[F^*] = F^*$. Now $g_{F^*}\colon C^\circ(F^*) \to C^\circ(F^*)$ is an antihomogeneous order antimorphism. By Theorem 7.1 there exists an inner product $(\cdot \mid \cdot)_{F^*}$ on $\operatorname{span}(F^*) \oplus \operatorname{span}(F^*)$, for which $C^\circ(F^*)$ is self-dual and that satisfies (7.1).

It is convenient to introduce some notation. Recall that the M-function is defined by

$$M(x/y) = \inf\{\alpha \geq 0\colon x \leq \alpha y\},$$

for $x \in C$ and $y \in C^\circ$. If multiple cones are under consideration, we denote the same functions by $M(\cdot/\cdot\,; C)$ to emphasize the dependence on the cone C. An obvious fact is that for any finite $F \subseteq \mathcal{E}(C)$ with span $F \cap C^\circ$ non-empty we have $M(x/y\,; C(F)) = M(x/y\,; C)$, for all $x \in C(F)$ and $y \in C^\circ(F)$.

Lemma 7.11. *Let F and G be finite subsets of $\mathcal{E}(C)$. Then we have*

$$(x \mid y)_{F^*} = (x \mid y)_{G^*},$$

for all $x, y \in \operatorname{span}(F^*) \cap \operatorname{span}(G^*)$.

Proof. It is sufficient to consider the case $F \subseteq G$, since for any $F, G \subseteq \mathcal{E}(C)$ finite, the union $F \cup G$ is also finite and contains both F and G. Let $x \in C(F^*)$ and $y \in C^\circ(F^*)$ be given. By (7.1) we get for all $y \in C^\circ(F^*)$ and $x \in R$, for some $R \in F^*$, that

$$(x \mid y)_{F^*} = M(x/g(y);\ C^\circ(F^*)) = M(x/g(y);\ C^\circ(G^*)) = (x \mid y)_{G^*}.$$

As $(\cdot \mid \cdot)_{F^*}$ and $(\cdot \mid \cdot)_{G^*}$ are bilinear, and both $C^\circ(F^*)$ and the union of rays in F^* generate $\operatorname{span}(F^*)$, this yields the assertion. □

We are now in position to construct an inner product on $V \oplus V$ for whch C° is a domain of positivity.

Theorem 7.12. *Let (V, C, u) be an order unit space, such that C equals the sum of its extreme rays and $g\colon C^\circ \to C^\circ$ be an antihomogeneous order antimorphism. Then there exists an inner product $(\cdot \mid \cdot)$ on V such that*

$$C^\circ = \big\{v \in V : (v \mid x) > 0 \text{ for all } x \in C \backslash \{0\}\big\}. \tag{7.7}$$

Proof. As before, we assume without loss of generality that g is an involution, by replacing it with S_u if necessary. For any pair $(x, y) \in V \oplus V$ there exists a finite $F \subseteq \mathcal{E}(C)$ with $x, y \in \operatorname{span}(F)$. The quantity

$$B(x, y) = (x \mid y)_{F^*},$$

is well-defined according to Lemma 7.11. The properties of $B\colon V \oplus V \to \mathbb{R}$ of being bilinear, symmetric and positive-definite follow, as they are verified on a finite set of vectors. Indeed, any such finite set of vectors is contained in a subcone $C^\circ(F^*)$, whose corresponding inner product $(\cdot \mid \cdot)_{F^*}$ has these listed properties by Theorem 7.1 and determines B. Thus $(\cdot \mid \cdot) := B(\cdot, \cdot)$ defines an inner product on $V \oplus V$.

We prove (7.7). Let $v \in C^\circ$. For $x \in C\backslash\{0\}$ there exists a finite $F \subseteq \mathcal{E}(C)$ with $v, x \in \operatorname{span}(F)$. Then $(v \mid x) = (v \mid x)_{F^*} > 0$, as $C^\circ(F^*)$ is self-dual for $(\cdot \mid \cdot)_{F^*}$ due to Theorem 7.1. For the reverse inclusion, suppose $v \in V$ is contained in the right-hand side of (7.7). Let $F \subseteq \mathcal{E}(C)$ be finite with $v \in \operatorname{span}(F)$. Then for all $x \in C(F^*)\backslash\{0\}$ we have $(v \mid x)_{F^*} = (v \mid x) > 0$. Hence, the self-duality of $C^\circ(F^*)$ for $(\cdot \mid \cdot)_{F^*}$ yields that $v \in C^\circ(F^*) \subseteq C^\circ$. □

Finite rank and completeness

We argue that the inner product constructed in Proposition 7.12 induces a complete norm, by showing that the induced inner product norm is equivalent to the order unit norm. For this purpose, we introduce a concept of rank in our space. For $x \in V$ we define its *rank*, denoted by $\rho(x)$, as the smallest number of extreme vectors needed to linearly span x. Remarkbly, there exists a global bound on the rank of elements in V, as a consequence of C° being a homogeneous cone.

Lemma 7.13. *For $x \in V$ we have $\rho(x) \leq 2\rho(u)$.*

Proof. Put $m := \rho(u)$. Let $r_1, \ldots, r_m \in C$ be extreme vectors such that $u = \sum_{i=1}^m \lambda_i r_i$, for some $\lambda_i \in \mathbb{R}$, for $i = 1, \ldots, m$. Let $y \in C^\circ$. By Theorem 7.10 there exists a linear order isomorphism $S \colon C \to C$ such that $S(u) = y$. We get $y = S(u) = \sum_{i=1}^m \lambda_i S(r_i)$. By Corollary 3.2, the vectors $S(r_i)$ are extreme, for $i = 1, \ldots, m$. Hence $\rho(y) \leq m$. As C° generates V, we obtain for all $x \in V$ that $\rho(x) \leq 2m = 2\rho(u)$. □

We briefly recall the spectral theory for finite dimensional formally real Jordan algebras, as also outlined in Section 1.6. The spectral theory allows us to express the norm induced by the inner product in terms of the eigenvalues of an element. Let $(\mathcal{A}, \circ)$ be a finite dimensional formally real Jordan algebra with unit e. A $c \in \mathcal{A}$ is said to be an *idempotent* if $c^2 = c$. An idempotent $c \in \mathcal{A}$ is considered *primitive* if c is non-zero and cannot be written as the sum of two non-zero idempotents. A set $\{c_1, \ldots, c_k\} \subseteq \mathcal{A}$ of primitive idempotents is called a *Jordan frame* if $c_i \circ c_j = 0$, for all $i \neq j$, and $\sum_{i=1}^k c_i = e$. The Spectral Theorem [FK94, Theorem III.1.2] says that for each $a \in \mathcal{A}$ there exists a Jordan frame $\{c_1, \ldots, c_k\}$ and unique real numbers $\lambda_1 \leq \ldots \leq \lambda_k$ such that $a = \sum_{i=1}^k \lambda_i c_i$. In fact, $\sigma(a) = \{\lambda_1, \ldots, \lambda_k\}$. Here the number $k \in \mathbb{N}$ is indepedent of a and satisfies $k \leq \dim \mathcal{A}$.

Proposition 7.14. *Suppose (V, C, u) be a complete order unit space, such that C equals the sum of its extreme rays and $g \colon C^\circ \to C^\circ$ an antihomogeneous order antimorphism. Then V can be endowed with an inner product $(\cdot \mid \cdot)$ such that (7.7) holds and $(V, (\cdot \mid \cdot))$ is a Hilbert space.*

Proof. Let $(\cdot \mid \cdot)$ be the inner product as obtained in Theorem 7.12. Then (7.7) is satisfied. Let $\|.\|_2$ denote the norm induced by $(\cdot \mid \cdot)$, i.e., for $x \in V$ we have $\|x\|_2 = \sqrt{(x \mid x)}$. As (V, C, u) is assumed to be complete, it suffices to argue that $\|.\|_2$ and $\|.\|_u$ are equivalent. Let $n := 6\rho(u)$ and $m := \|u\|_2^2$. We define

$$\tau(x, y) = \frac{n}{m}(x \mid y), \quad x, y \in V.$$

Remark that τ is a positive definite symmetric bilinear form on $V \oplus V$.

Let $x \in V$ be given. By Lemma 7.13 there exists a $F \subseteq \mathcal{E}(C)$ such that $x, u \in \operatorname{span}(F)$ and $\dim \operatorname{span}(F) \leq 3\rho(u)$. Let $\mathcal{A} := \operatorname{span}(F^*)$ and remark that $\dim \mathcal{A} \leq 6\rho(u) = n$. By construction $\tau(u, u) = n$. Since $C^\circ(F^*)$ is a symmetric cone for $(\cdot \mid \cdot)$, by [Koe62, Theorem VI.15], there exists a bilinear product $\circ \colon \mathcal{A} \oplus \mathcal{A} \to \mathcal{A}$ such that

$(\mathcal{A},\circ)$ is a formally real Jordan algebra, such that $\mathcal{A}_+^\circ = C^\circ(F^*)$. Moreover, due to [Koe62, Theorem III.13] the unit of $(\mathcal{A},\circ)$ is u and $\tau(a,b) = \mathrm{Tr}L(a\circ b)$, for all $a,b\in\mathcal{A}$. Here $L(a\circ b)$ denotes left multiplication by $a\circ b$. By the Spectral Theorem there exists a Jordan frame $\{c_1,\ldots,c_k\}$ and unique real numbers $\lambda_1\leq\ldots\leq\lambda_k$ such that $x=\sum_{i=1}^k\lambda_i c_i$. As the c_i are pairwise orthogonal idempotents we get $x^2=\sum_{i=1}^k\lambda_i^2 c_i$. Now we compute

$$\|x\|_2 = \frac{m}{n}\sqrt{\tau(x,x)} = \frac{m}{n}\sqrt{\mathrm{Tr}L(x^2)} = \frac{m}{n}\sqrt{\sum_{i=1}^k \lambda_i^2\ \mathrm{Tr}L(c_i)}. \tag{7.8}$$

The possible eigenvalues of $L(c)$ for an idempotent $c\in\mathcal{A}$ are $0,\frac{1}{2}$ and 1 by [FK94, Proposition III.1.3] and, hence, for $i=1,\ldots,k$ we have $1\leq\mathrm{Tr}L(c_i)\leq n$. Next we want to describe $\|x\|_u$. Remark that as $u\in C(F^*)=\mathcal{A}_+$, that computing the order unit norm of x in (V,C,u) yields the same as in $(\mathcal{A},\mathcal{A}_+,u)$. So from $u=\sum_{i=1}^k c_i$ we obtain $\|x\|_u=\max\{|\lambda_1|,\ldots,|\lambda_k|\}$. Let $j\in\{1,\ldots,k\}$ be such that $|\lambda_j|=\|x\|_u$. We then get $\|x\|_u\leq\sqrt{\sum_{i=1}^k\lambda_i^2}\leq k\|x\|_u$. Combining this with $k\leq\dim\mathcal{A}\leq n$ and (7.8) we get

$$\frac{m}{n}\|x\|_u\leq\|x\|_2\leq m\sqrt{n}\|x\|_u.$$

Since n and m are defined independent on the choice of x we conclude that $\|.\|_2$ and $\|.\|_u$ are equivalent. □

JH-algebras

A real Jordan algebra $\mathcal{H}$ that is a Hilbert space with an inner product $(\cdot\mid\cdot)$ which is *associative*, that is,

$$(a\circ b\mid c)=(b\mid a\circ c)\quad a,b,c\in\mathcal{H},$$

is called a *JH-algebra*. It is shown in [Chu17, Theorem 3.1] that, as an infinite dimensional generalisation of the Koecher-Vinberg theorem, a symmetric cone in a Hilbert space arises precisely as the interior of the cone of squares of unital JH-algebra. We have shown that a complete order unit space (V,C,u), such that C equals the sum of its extreme rays and C° admits an antihomogeneous order antimorphism, can be endowed with an inner product turning V into a Hilbert space in which C° is a symmetric cone. Therefore, by the result of Chu, under those assumptions V is a unital JH-algebra with C as its cone of squares. The converse of this last statement also holds. In order to prove this, it is convenient to consider the following characterisation of unital JH-algebras, which including its proof is due to Roelands and Wortel through personal communication.

Lemma 7.15. *A unital JH-algebra is a finite direct sum of formally real Jordan algebras and spin factors.*

Proof. Let $\mathcal{H}$ be a unital JH-algebra. By [Chu17, Lemma 2.6] the order unit norm is equivalent with the norm from the inner product. Hence, $\mathcal{H}$ is reflexive and has a predual. In particular, $\mathcal{H}$ is a JBW-algebra. Let $z \in \mathcal{H}$ be the central projection such that $z\mathcal{H}$ is the nonatomic part of $\mathcal{H}$. Suppose $z \neq 0$. We split z into a sum of two non-trivial orthogonal projections. One of them we split again into the sum of two non-trivial orthogonal projections, and so on. This process does not terminate in finite steps as $z\mathcal{H}$ is purely non-atomic. We obtain an infinite sequence $(p_n)_{n=1}^{\infty}$ of pairwise orthogonal projections. The JB-subalgebra generated by these projections is associative, and is isometrically isomorphic to some $C_0(S)$, for a locally compact Hausdorff space S. Then the map

$$(\lambda_n) \mapsto \sum_{n=1}^{\infty} \lambda_n p_n,$$

is an isometric embedding of c_0 into $\mathcal{H}$, contradicting the reflexivity of $\mathcal{H}$. Hence, $z = 0$ and $\mathcal{H}$ is an atomic JBW-algebra. Therefore, by [AS03, Proposition 3.45], $\mathcal{H}$ is a direct summand of type I JBW-factors. Since by the above arguments $\mathcal{H}$ cannot contain an infinite collection of orthogonal projections, this is a finite direct sum and each factor is of finite type. A finite type I JBW-factor is a spin factor or the self-adjoint matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or the 3×3-matrices over $\mathbb{O}$. □

Using these results on JH-algebras we can now fully characterise the symmetric cones in infinite dimensions.

Theorem 7.16. *Let (V, C, u) be a complete order unit space. Then C equals the sum of its extreme rays and there exists an antihomogeneous order antimorphism $g \colon C^\circ \to C^\circ$ if and only if C° is a symmetric cone.*

Proof. Suppose that C equals the sum of its extreme rays and $g \colon C^\circ \to C^\circ$ is an antihomogeneous order antimorphism. Then by Theorem 7.12 there exists an inner product $(\cdot \mid \cdot)$ on V, which by Proposition 7.14 turns V into a Hilbert space, for which C° is a self-dual cone by (7.7). Moreover, C° is a homogeneous cone due to Theorem 7.10. We conclude that C° is in fact a symmetric cone.

Conversely, suppose that $(V, (\cdot \mid \cdot))$ is a Hilbert space such that C° is a symmetric cone with respect to $(\cdot \mid \cdot)$. Then by [Chu17, Theorem 3.1], V can be endowed with a Jordan product turning into a unital JH-algebra such that C° is the interior of the cones of squares. Now by our characterisation of unital JH-algebras in Lemma 7.15 we know that V is a finite direct summand of formally real Jordan algebras and spin factors. The cone in any formally real Jordan algebra or spin factor equals the sum of its extreme rays and, hence, as C is a finite direct sum of such cones it also equals the sum of its extreme rays. Lastly, the inversion map $\iota \colon C^\circ \to C^\circ$ defined by $x \mapsto x^{-1}$ is an antihomogeneous order antimorphism. □

Automatic antihomogeneity

In this last section we argue that the antihomogeneous condition imposed on g in Theorem 7.16 is superfluous. Our strategy is to decompose our cone C into an engaged part and a disengaged part. Using Proposition 7.2 we can show that an antimorphism $g\colon C^\circ \to C^\circ$ respects this decomposition. Restricted to the engaged part g will automatically become antihomogeneous, by results from [Wal18], and by Theorem 7.16 this part is symmetric. The disengaged part of C must be finite dimensional, and as all its extreme rays are linearly independent it is isometrically isomorphic to a standard Euclidean cone, which is symmetric.

Henceforth let (V, C, u) be a complete order unit space, such that C equals the sum of its extreme rays and $g\colon C^\circ \to C^\circ$ an order antimorphism. As before let $\varphi\colon \mathcal{E}(C) \to \mathcal{E}(C)$ be the bijection, as obtained in Proposition 7.2, corresponding to g. Furthermore, we introduce the notations $\mathcal{R}_D$ and $\mathcal{R}_E$ for the collection of disengaged extreme rays and engaged extreme rays of C, respectively. Let $V_D := \operatorname{span} \mathcal{R}_D$ and $V_E := \operatorname{span} \mathcal{R}_E$.

Lemma 7.17. *Under the standing hypotheses* $(V, C) = (V_D, C(\mathcal{R}_D)) \oplus (V_E, C(\mathcal{R}_E))$, *as a direct sum of partially ordered vector spaces.*

Proof. Let $x \in C$. By assumption there exist extreme vectors $x_1, \ldots, x_n \in C$ such that $x = \sum_{i=1}^n \lambda_i x_i$. Let I be the subset of $\{1, \ldots, n\}$ consisting of those indices for which the corresponding x_i is a disengaged vector and let J be the complement of this set. Then by construction $x_D = \sum_{i\in I} x_i \in C(\mathcal{R}_D)$ and $x_E = \sum_{j\in J} x_j \in C(\mathcal{R}_E)$.

Suppose now that $x \in V_D \cap V_E$ and $x \neq 0$. We write

$$x = \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j,$$

with $x_i \in V$ disengaged extreme vectors and $y_j \in V$ engaged extreme vectors. It follows from

$$x_1 = \sum_{j=1}^{m} y_j - \sum_{i=2}^{n} x_i$$

that x_1 is engaged, which yields a contradiction. We conclude $V_D \cap V_E \subseteq \{0\}$. □

Due to Lemma 7.17 we can write $u = (u_D, u_E)$ with $u_D \in C(\mathcal{R}_D)$ and $u_E \in C(\mathcal{R}_E)$. Then u_D and u_E are order units in $C(\mathcal{R}_D)$ and $C(\mathcal{R}_E)$, respectively. Moreover, we get

$$C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E).$$

In what follows we argue that g factors of this direct sum.

Lemma 7.18. *Under the standing hypotheses,* $\varphi[\mathcal{R}_D] = \mathcal{R}_D$ *and* $\varphi[\mathcal{R}_E] = \mathcal{R}_E$.

Proof. Let $R \in \mathcal{R}_E$ be given. Then there exists a finite $F \subseteq \mathcal{R}_E \backslash \{R\}$ with $R \in \operatorname{span} F$ and $\operatorname{span} F \cap C^\circ \neq \emptyset$. By Lemma 7.2 we get

$$\varphi(R) \in g[C^\circ(F)] = C^\circ(\varphi[F]).$$

As φ is injective, $\varphi(R) \notin \varphi[F]$ so $\varphi(R)$ is engaged. We conclude $\varphi[\mathcal{R}_E] \subseteq \mathcal{R}_E$. Since g^{-1} is also an order antimorphism we get by Lemma 7.2 the reverse inclusion $\mathcal{R}_E \subseteq \varphi[\mathcal{R}_E]$. Due to φ being bijective we also obtain $\varphi[\mathcal{R}_D] = \mathcal{R}_D$. □

Lemma 7.19. *There exist order antimorphisms $g_D\colon C^\circ(\mathcal{R}_D) \to C^\circ(\mathcal{R}_D)$ and $g_E\colon C^\circ(\mathcal{R}_E) \to C^\circ(\mathcal{R}_E)$ such that for all $(x_D, x_E) \in C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$ we have*

$$g((x_D, x_E)) = (g_D(x_D), g_E(x_E)).$$

Proof. Let $x = (x_D, x_E)$ and $y = (y_D, y_E)$ be given in $C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$. Suppose $x_D = y_D$ holds. Then $x = y + \sum_{i=1}^n r_i$ for some $r_i \in R_i$ with $R_i \in \mathcal{R}_E$, for $i = 1, \dots, n$. Relabelling the indices such that $x_1, \dots, x_j < 0$ and $x_{j+1}, \dots, x_n > 0$ for some $j \in \{1, \dots, n\}$, guarantees that $y + \sum_{i=1}^k r_i \in C^\circ$ for all $k = 1, \dots, n$. Due to Lemma 7.2 we obtain

$$g(x) - g(y) = g\left(y + \sum_{i=1}^n r_i\right) - g(y) \in \operatorname{span} \varphi[\mathcal{R}_E].$$

As $\varphi[\mathcal{R}_E] = \mathcal{R}_E$ by Lemma 7.18, we remark that $g(x) - g(y) \in V_E$. In other words, $g(x)$ and $g(y)$ coincide in their first argument with respect to the decomposition obtained in Lemma 7.17. Similarly, if $g(x)$ and $g(y)$ coincide in the first argument, then by applying the same arguments to g^{-1} also x and y coincide in the first argument. This shows that $g_D(x_D) = g(x_D, x_E)$ is well-defined independent of x_E. Analogously, as Lemma 7.18 yields $\varphi[\mathcal{R}_D] = \mathcal{R}_D$, we obtain that $g_E(x_E) = g(x_D, x_E)$ is well-defined independent of x_E. That both g_D and g_E are order antimorphisms now follows from the fact that g is an order antiomorphism. □

On the engaged part of our cone the order antimorphism automatically becomes antihomogeneous, as a consequence of the finite dimensional result by Walsh, which states that an order antimorphism between the interiors of two closed cones is antihomogeneous whenever one of the cones does not contain a disengaged extreme vector.

Lemma 7.20. *The order antimorphism g_E as in Lemma 7.19 is antihomogeneous.*

Proof. Let $x \in C^\circ(\mathcal{R}_E)$ and $\lambda \in \mathbb{R}_+^\circ$ be given. Let $F \subseteq \mathcal{R}_E$ be finite such that $x \in C^\circ(F)$. Then g_E restricts to an order antimorphism $\tilde{g}_E\colon C^\circ(F) \to C^\circ(\varphi[F])$ by Lemma 7.3. We remark that $C^\circ(F)$ does not contain a disengaged extreme ray. Now [Wal18, Theorem 1.1] yields that $\tilde{g}_E$ is antihomogeneous and

$$g_E(\lambda x) = \tilde{g}_E(\lambda x) = \lambda^{-1} g_E(x).$$

□

In contrast to the engaged part, the order antimorphism g_D need not be antihomogeneous. This, however, is not of importance as the disengaged part of the cone $C^\circ(\mathcal{R}_D)$ is linearly isomorphic to a standard finite dimensional cone and, therefore, is symmetric.

Lemma 7.21. *Let (V, C, u) be an order unit space. If all extreme rays of C are disengaged and C equals the sum of its extreme rays, then C is linearly isomorphic to a standard Euclidean cone.*

Proof. Let $\mathcal{R}$ denote the collection of extreme rays of C. By assumption we can write the order unit u as a linear combination of finitely many extreme vectors, say $u = \sum_{i=1}^{n} r_i$ with $r_i \in R_i \in \mathcal{R}$ for all i. Suppose there exists an $R \in \mathcal{R}$ with $R \neq R_i$ for $i = 1, \dots, n$. Let $r \in R$ with $r \leq u$. Now write $u - r = \sum_{j=1}^{m} s_j$ with all s_j extreme vectors. Then we compute

$$r = u - (u - r) = \sum_{i=1}^{n} r_i - \sum_{j=1}^{m} s_j.$$

In particular, this contradicts that r is a disengaged extreme vector by assumption. We have shown that $\mathcal{R}$ does not contain additional rays besides $R_1, \dots, R_n$. Thus $\mathcal{R}$ is finite. Since V is the linear span of the extreme rays, V is finite dimensional. As all extreme rays are disengaged, any collection of reprensentatives form an algebraic basis for V. The basis transformation that maps this basis onto the standard coordinate basis is the desired linear order isomorphism from (V, C) onto $(\mathbb{R}^n, \mathbb{R}^n_+)$. □

We obtain a slight improvement for our characterisation of symmetric cones in infitine dimensions, Theorem 7.16, by dropping the antihomogeneous condition on g.

Theorem 7.22. *Let (V, C, u) be a complete order unit space. Then C equals the sum of its extreme rays and there exists an order antimorphism $g\colon C^\circ \to C^\circ$ if and only if C° is a symmetric cone.*

Proof. Let (V, C, u) be a complete order unit space, C the sum of its extreme rays and $g\colon C^\circ \to C^\circ$ be an order antimorphism. By Lemma 7.17 we get $C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$. Let g_D and g_E be the order antimorphisms as obtained in Lemma 7.19. Since $g_E : C^\circ(\mathcal{R}_E) \to C^\circ(\varphi[\mathcal{R}_E])$ is antihomogeneous by Lemma 7.20 the cones $C^\circ(\mathcal{R}_E)$ and $C^\circ(\varphi[\mathcal{R}_E])$ are linearly isomorphic. In particular, all extreme rays in $C^\circ(\varphi[\mathcal{R}_E])$ are engaged. Thus we get $C^\circ(\varphi[\mathcal{R}_E]) \subseteq C^\circ(\mathcal{R}_E)$ and hence $\varphi[\mathcal{R}_E] \subseteq \mathcal{R}_E$. As all arguments also apply to g^{-1} we get $\varphi[\mathcal{R}_E] = \mathcal{R}_E$. We conclude that $g_E\colon C^\circ(\mathcal{R}_E) \to C^\circ(\mathcal{R}_E)$ is an antihomogeneous order antimorphism. By Theorem 7.16, $C^\circ(\mathcal{R}_E)$ is a symmetric cone. Due to Lemma 7.21 the cone $C^\circ(\mathcal{R}_D)$ is linearly isomorphic to a standard Euclidean cone and, in particular, is a symmetric cone. Hence, the product $C^\circ = C^\circ(\mathcal{R}_D) \times C^\circ(\mathcal{R}_E)$ is also symmetric. The converse statement follows from Theorem 7.16. □

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Summary

In the study of partially ordered vector spaces a central problem is to understand the structure of order isomorphisms. Of particular interest is to classify the cones in such partially ordered vector spaces between which every order isomorphism is linear. Research on this question dates back to 1953 motivated by Relativity theory, wherein the causal cone is modelled as the three dimensional Lorentz cone. In subsequent years, the automatic linearity of order isomorphisms has been studied frequently for more general classes of cones. Noteworthy contributions to this area are both a result of Molnár, who shows that any order isomorphism between cones consisting of positive semi-definite bounded linear operators on a Hilbert space is linear using operator algebra techniques, and the result by Noll and Schäffer that states that any order isomorphism is linear provided that either cone is the sum of its engaged extreme rays. Noll and Schäffer describe an order theoretic condition that is sufficient to guarantee automatic linearity of order isomorphisms, unfortunately however, their condition is too restrictive to include Molnár's result. We extend their methods to hold in a significantly more general setting of partially ordered vector spaces, and consequently generalise the existing results concerning automatic linearity of order isomorphisms in an order theoretic framework.

A deep connection between Jordan algebras structure and symmetric geometry of cones in finite dimensions was discovered independently by Koecher and Vinberg. In order to formulate their result more precisely, we briefly introduce various concepts. The interior of a closed finite dimensional cone is considered a symmetric cone if it is both homogeneous, in the sense that its automorphism group acts transitively on it, and is self-dual with respect to an inner-product. Motivated by Quantum mechanics, a space of Hermitian matrices can be endowed with algebra structure by means of the Jordan product: $A \circ B := (AB + BA)/2$. An algebra which is commutative and satisfies the Jordan identity, a property weaker than associativity, is called a Jordan algebra. Furthermore, a Jordan algebra is considered formally real if the sum of squares of elements can only be zero if all the elements themselves are zero. The famous Koecher-Vinberg theorem asserts that the interior of the cone consisting of squares of a formally real Jordan algebra is symmetric, and that all finite dimensional symmetric cones arise in this way from a formally real Jordan algebra. With the aid of this result, one can also endow a symmetric cone with a Riemannian metric, making it a prime example of a Riemannian symmetric space. Results outlining these deep connections between symmetric cones in Euclidean spaces, formally real Jordan algebras and Riemannian

symmetric spaces are limited to finite dimensions. A central topic of this thesis is to develop pioneering steps towards similar theories in infinite dimensions. The notion of a formally real Jordan algebra has been generalised in the infinite dimensional setting by Alfsen, Schulz and Størmer, to a Jordan Banach algebra, or JB-algebra for short. In general, a JB-algebra cannot be realised as an inner-product space, and hence there is no natural notion of self-duality, nor can one endow the interior of the cone of squares with a Riemannian metric. Instead, we explore an order theoretic way to characterise the cone of a JB-algebra. In recent work, Walsh has characterised the finite dimensional symmetric cones as those that admit an antihomogeneous order antimorphism. We contribute to generalising this approach to the infinite dimensional setting of JB-algebras, in twofold. First, we characterise the spin factors, a special class of JB-algebras, among the complete order unit spaces as those that have a strictly convex cone and admit an order antimorphism on the interior of that cone. Secondly, we obtain an order theoretic characterisation of symmetric cones in infinite dimensional order unit space. We do so with the aid of the metric geometric techniques that we developed for the analysis of spin factors, and by adapting the results of Noll and Schäffer to instead concern order antimorphisms. More precisely, we show that the cone of a complete order unit space is the sum of its extreme rays and admits an order antimorphism on its interior if and only if the order unit space can be endowed with an inner-product, making it a Hilbert space, with respect to which the interior of the cone is symmetric. Due to recent work by Chu, this latter condition is equivalent for the cone to be the cone of squares for some Jordan Hilbert algebra.

The self-adjoint part of a C^*-algebra endowed with the canonical Jordan product is a JB-algebra. This further supports the idea that a JB-algebra is the natural infinite dimensional analogue of a formally real Jordan algebra. Hanche-Olsen and Størmer have lifted much of the theory on C^*-algebras and von Neumann algebras to the setting of JB-algebras and, their von Neumann analogues, JBW-algebras. A classic result by Kadison states that any linear order isomorphism between C^*-algebras, which carries the unit of one algebra onto the unit of the other algebras, is a C^*-isomorphism and, in particular, a Jordan isomorphism between the self-adjoint parts. Based on this result, it can be shown that if unital JB-algebras are linearly order isomorphic that then they are also Jordan isomorphic. This motivates us to understand the structure of order isomorphisms between cones in JB-algebras, and moreover, to find conditions under which they are necessarily linear. Due to the inherent connection between geometric properties of the cone in a JB-algebra and its algebraic properties, our results concerning the automatic linearity of order isomorphisms developed in the general setting of partially ordered vector spaces are applicable here. We start by fully describing the order isomorphisms between cones of atomic JBW-algebras and, in particular, characterise for which of these spaces every order isomorphism between cones is linear. By a canonical construction involving the bidual we can view any unital JB-algebra as a subalgebra of an atomic JBW-algebra. With the aid of deep results by Hanche-Olsen and Størmer, related to this embedding, we are able to use our description of order isomorphism between cones of atomic JBW-algebras to study order isomorphisms between cones of unital JB-algebras. This leads to the description of a rich class of JB-algebras for which an order isomorphism between cones in such spaces is linear provided it satisfies a mild continuity property.

Nederlandse samenvatting

In het onderzoek naar partieel geordende vector ruimtes is het een centraal probleem om de structuur van orde isomorfismen te begrijpen. Van bijzonder belang is het classificeren van kegels in zulke partieel geordende vector ruimtes waartussen elk orde isomorfisme lineair is. Onderzoek naar deze vraag dateert van 1953 gemotiveerd door Relativiteitstheorie, waarin de lichtkegel gemodelleerd wordt als de driedimensionale Lorentz kegel. In de daaropvolgende jaren werd de automatische lineariteit van orde isomorfisme onderzocht voor klassen van kegels in toenemende algemeenheid. Twee opmerkelijke bijdragen aan dit gebied zijn een resultaat van Molnár, die zegt dat elk orde isomorfisme tussen kegels bestaande uit positief semi-definiete begrensde lineaire operatoren op een Hilbert ruimte noodzakelijk lineair is gebruik makend van operator algebra technieken, en een resultaat van Noll en Schäffer, dat luidt dat elk orde isomorfisme lineair is onder de voorwaarde dat de kegel de som is van zijn betrokken extreme stralen. Noll en Schäffer beschrijven een orde theoretische voorwaarde die voldoende is om de automatische lineariteit van order isomorfismen te garanderen, echter, helaas is hun voorwaarde te restrictief om Molnár's resultaat te bevatten. We breiden hun methoden uit naar een significant algemenere klasse van partieel geordende vector ruimtes, en zodoende generaliseren we de huidige resultaten over automatische lineariteit van order isomorfismen vanuit een orde theoretisch kader.

Een diepe connectie tussen Jordan algebra structuur en symmetrische geometrie van kegels is onafhankelijk van elkaar ontdekt door Koecher en Vinberg. Om dit resultaat preciezer te beschrijven, introduceren wij kort verschillende concepten. Het inwendige van een gesloten eindigdimensionale kegel wordt beschouwd als een symmetrische kegel als deze zowel homogeen is, in de zin dat zijn automorfismen groep transitief op hem werkt, als zelf-duaal is ten opzichte van een inwendig product. Gemotiveerd vanuit de Quantum mechanica, kan een ruimte van Hermitische matrices voorzien worden van algebra structuur door middel van het Jordan product: $A \circ B := (AB + BA)/2$. Een algebra die commutatief is en aan de Jordan identiteit voldoet, een eigenschap zwakker dan associativiteit, wordt een Jordan algebra genoemd. Verder wordt een Jordan algebra formeel reëel genoemd als de som van kwadraten van elementen alleen nul kan zijn als de elementen zelf nul zijn. De welbekende Koecher-Vinberg stelling beweert dat het inwendige van de kegel van kwadraten in een formeel reële Jordan algebra een symmetrische kegel is, en bovendien dat alle eindigdimensionale symmetrische kegels op deze manier verkregen kunnen worden. Met behulp van dit resultaat kan men een symmetrische kegel voorzien van een Riemannse metriek, waarbij het een belangrijk voorbeeld wordt van een Riemanns symmetrische ruimte. Bestaande resultaten die deze diepe connecties tussen symmetrische kegels in Euclidische ruimten, formeel reële Jordan algebra's en Riemanns symmetische ruimten beschrijven zijn beperkt tot eindige dimensies. Een centraal onderwerp van deze dissertatie is om baanbrekende stappen te zetten naar soortgelijke theorieëen in oneindige dimensies. De notie van een formeel reële Jordan algebra is gegeneraliseerd naar de oneindig dimensionale setting door Alfsen, Schulz en Størmer als zogenaamde Jordan Banach algebra, of afgekort JB-algebra. In het algemeen kan een JB-algebra niet gerealiseerd worden al inproductruimte, en vandaar is er geen natuurlijk concept van een zelf-duale kegel, noch kan men het inwendige van de kegel van kwadraten voorzien van een Riemannse metriek. In plaats daarvan, onderzoeken wij een orde theoretische wijze om de kegel van een JB-algebra te karakteriseren. In recent werk heeft Walsh de eindigdimensionale symmetrische kegels gekarakteriseerd als zijnde de kegels die een antihomogeen orde antimorfisme toelaten. Wij dragen bij om deze aanpak te generaliseren naar oneindig dimensionale JB-algebra's op twee manieren. Allereerst, karakteriseren wij spin factoren, een speciale klasse van JB-algebra's, onder de volledige orde eenheid ruimten als degene die een strikt convexe kegel hebben die een orde antimorfisme toelaten op het inwendige. Ten tweede, geven wij een orde theoretische karakterisatie van symmetrische kegels in oneindig dimensionale orde eenheid ruimten. Dit doen wij met behulp van metrisch geometrische technieken de we ontwikkeld hebben in onze analyse van spin factoren en door de resultaten van Noll en Schäffer aan te passen om toepasbaar te zijn op orde antimorfismen. Wat we precies laten zien is dat de kegel van een volledige orde eenheid ruimte de som is van zijn betrokken extreme stralen en een orde antimorfisme toelaat op zijn inwendige dan en slechts dan als de orde eenheid ruimte voorzien kan worden van een inproduct, wat het een Hilbert ruimte maakt, en ten opzichte waarvan het inwendige van de kegel symmetrisch is. Dankzij recent werk van Chu is deze tweede uitspraak equivalent met de bewering dat de kegel optreedt als kegel van kwadraten van een Jordan Hilbert algebra.

Het zelf-geadjungeerde deel van een C^*-algebra voorzien van het kanonieke Jordan product is een JB-algebra. Dit versterkt het idee dat een JB-algebra het natuurlijke oneindig dimensionale analogon is van een formeel reële Jordan algebra. Hanche-Olsen en Størmer hebben de theorie van C^*-algebra's en von Neumann algebra's opgetild naar het kader van JB-algebra's en, hun von Neumann analoga, JBW-algebra's. Een klassiek resultaat van Kadison luidt dat elk lineair orde isomorfisme tussen C^*-algebra's die de eenheid van de ene algebra overhevelt naar de eenheid van de andere algebra noodzakelijk een C^*-isomorfisme is en, in het bijzonder, een Jordan isomorfisme is tussen de zelf-geadjungeerde delen van de C^*-algebra's. Gebaseerd op deze bevinding, kan men laten zien dat als unitaire JB-algebra's lineair orde isomorf zijn dat ze dan ook Jordan isomorf zijn. Dit motiveert ons om de structuur van orde isomorfismen tussen kegels in JB-algebra's te begrijpen, en bovendien, om voorwaarden te vinden waaronder ze noodzakelijk lineair zijn. Vanwege inherente verbanden tussen geometrische eigenschappen van de kegel in een JB-algebra en zijn algebraïsche eigenschappen, zijn onze resultaten omtrent de automatische lineariteit van orde isomorfismen, ontwikkeld in het algemene kader van partieel geordende vector ruimten, hier van toepassing. Wij beschrijven volledig de orde isomorfisme tussen kegels van atomaire JBW-algebra's en, in het bijzonder, karakteriseren voor welke van deze ruimten allen order isomorfismen lineair zijn. Door middel van een kanonieke constructie, waarbij de biduaal betrokken is, kunnen we elke unitaire JB-algebra zien als een deelalgebra van een atomaire JBW-algebra. Gebruikmakende van diepe resultaten van Hanche-Olsen en Størmer, gerelateerd aan deze inbedding, zijn wij in staat om onze beschrijving van orde isomorfismen tussen kegels in atomaire JBW-algebra's te gebruiken om orde isomorfismen tussen kegels in unitaire JB-algebra's te bestuderen. Dit leidt tot de beschrijving van een rijke klasse van JB-algebra's waarvoor elk orde isomorfisme tussen kegels in zulke ruimten die voldoet aan een milde continuïteitseigenschap lineair is.

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Curriculum vitae

Hent van Imhoff was born in The Hague, on 16th September 1990. In 2008 he obtained his secondary school diploma at Christelijk Gymnasium Sorghvliet. Initially intrigued by the Mechanical Engeering studies at TU Delft, he made a switch to Mathematics at Leiden University after a year. This turned out to be a great decision and a great passion for mathematics was formed. In 2012 he obtained his Bachelor degree after writing a thesis on order convergence in partially ordered vector spaces. Being introduced to ordered vector spaces his field of interest solidified in functional analysis with an emphasis on order structures. This led to the thesis *Riesz* homomorphisms on pre-Riesz spaces consisting of continuous functions*, which awarded him the Master degree in 2015. Following this degree he started his 4 year PhD studies at Leiden University under local supervision of dr. Onno van Gaans and external supervision by dr. Bas Lemmens at University of Kent. During this period, he attended numerous functional analysis seminars in Leiden, the 50th Anniversary meeting of the Northern British Functional Analysis Seminar in Edinburgh and the workshop *Order structures, Jordan algebras and geometry* at the Lorentz centre in Leiden. He gave contributed talks at the international conference *Positivity*, edition VIII in Chengdu, China, in 2015, and edition IX in Edmonton, Canada, in 2017. He also visited Technische Universität Dresden as an invited speaker and made visits to the University of Kent for collaboration with Bas Lemmens and Mark Roelands. The research conducted in this 4 year period culminated in this thesis.