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Mesoscopic Hamiltonian for Josephson traveling-wave parametric amplifiers

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We present a theory describing parametric amplification in a Josephson junction embedded transmission line. We will focus on the process of four-wave mixing under the assumption of an undepleted pump. However, the approach taken is quite general, such that a different parametric process or the process under different assumptions is easily derived. First the classical theory of the coupled-mode equations as presented by O'Brien *et al.* [O'Brien *et al.*, *Phys. Rev. Lett.* **113**, 157001 (2014)] is briefly reviewed. Then a derivation of the full quantum theory is given using mesoscopic quantization techniques in terms of discrete-mode operators. This results in a Hamiltonian that describes the process of parametric amplification. We show that the coupled-mode equations can be derived from this Hamiltonian in the classical limit and elaborate on the validity of the theory.

DOI: [10.1103/PhysRevA.99.063838](https://doi.org/10.1103/PhysRevA.99.063838)**I. INTRODUCTION**

Parametric amplification arises as a result of nonlinear optics. In the case where a nonlinear medium is traversed by a (weak) signal and a strong pump, a wave-mixing interaction causes the signal to be amplified. The main advantage of such amplifiers is their low added noise. In comparison, a conventional low-noise microwave amplifier has a noise temperature T_n of several Kelvins, which translates into $k_B T_n / \hbar \omega \approx 10$ photons of added noise for $T_n = 2$ K at a frequency of $\omega/2\pi = 4$ GHz [1–3]. This number can be reduced to 1/2 or even 0 in a parametric amplifier, depending on its configuration [4]. This makes parametric amplifiers ideal to amplify signals that are on a single-photon level.

In the past decade, many microwave parametric amplifiers have been developed to read out quantum bits in quantum information experiments (see, e.g., [5] for a review). In most of the designs [6–10], the amplifier is embedded in a resonator to increase the interaction time of pump and signal, and thus to increase the amplifier's gain. Due to such a design, these amplifiers, however, are inherently limited in their bandwidth, giving rise to scalability issues now that the number of quantum bits in a single experiment increases. For this reason, traveling-wave parametric amplifiers (TWPAs) have been developed [11–16]. As these are not based on resonance, they do not suffer from the intrinsic bandwidth limitation. However, to achieve a large gain, the amplifiers need to be long.

Currently, two sources of nonlinearity have been considered for TWPAs. First, one can base the amplifier design on the intrinsic nonlinear kinetic inductance of superconductors [11,14–16]. Second, one can embed Josephson

junctions in the transmission line, which have a nonlinear inductance [12,13]. Both versions of the TWPA have been described theoretically using classical coupled-mode equations [11,12,17,18]. However, a Hamiltonian description is necessary to describe the TWPA as a quantum device, which is needed for a recently proposed experiment testing the limits of quantum mechanics by entangling two TWPAs within a single-photon interferometer by adding them to each of the interferometer's arms [19]. To interpret the results of such an experiment, it is of utmost importance that one understands precisely how the amplifiers work and what the magnitude of the various coupling constants in the theory is. Moreover, a quantum theory allows one to calculate averages, standard deviations, and higher-order moments of measurement operators, and takes into account the effects of commutation relations, whereas a classical theory only allows averages to be calculated straightforwardly and the effect of noncommuting operators is neglected. Some authors consider such a Hamiltonian description impossible due to the difficulties of quantum mechanics in describing dispersion ([20] and references therein), an important characteristic in TWPAs. However, in the case of a TWPA based on Josephson junctions, such a description appears to be possible. The Josephson TWPA has already been described using a Hamiltonian based on continuous-mode operators [21]. This description was used to calculate average gain and squeezing effects. In this work, we use discrete-mode operators for our analysis and use the resulting Hamiltonian to calculate photon-number distributions, apart from gain effects.

We will first put the concept of parametric amplification on solid ground by introducing the necessary terminology in Sec. II. Then, in Sec. III, a review is given of O'Brien *et al.* [18], where the coupled-mode equations were derived, which can be used for predicting the classical response of a TWPA in the case where the nonlinearity in the transmission line is weak. In Sec. IV, we proceed to derive the Hamiltonian of the Josephson junction embedded transmission line in the limit of a weak nonlinearity, which we apply to the specific case of a nondegenerate parametric amplifier with undepleted

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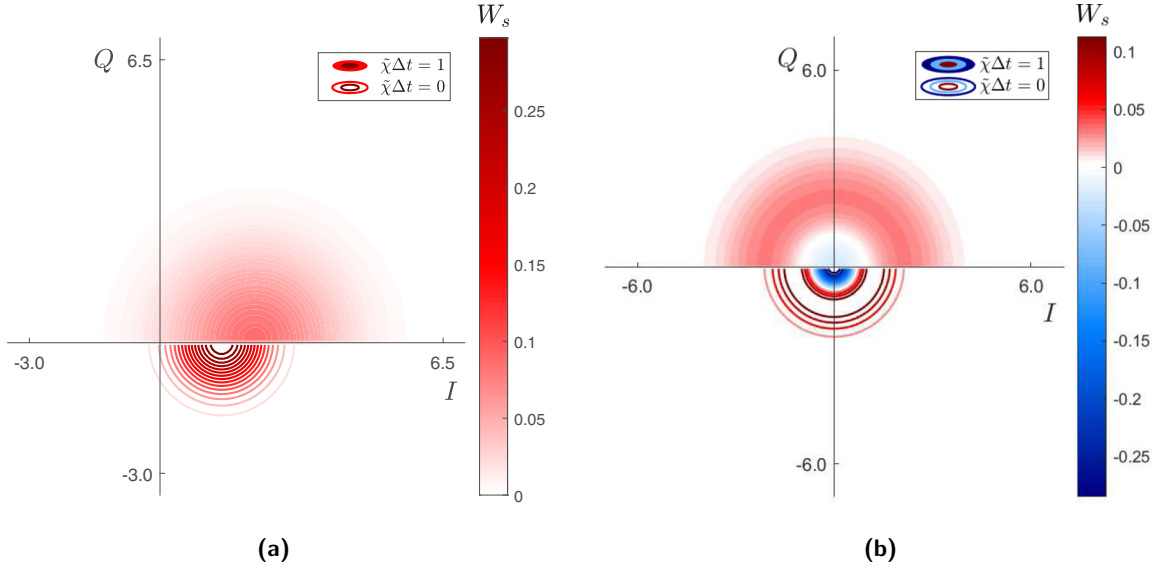


FIG. 1. Effect of nondegenerate (or phase-preserving) amplification with an undepleted, degenerate pump on (a) a coherent state ($\alpha = 1$) and (b) a single-photon-number state in the I , Q -quadrature plane. The lower-half plane depicts (half of the) Wigner function of the unamplified state using contours, whereas the upper-half plane shows the Wigner function of the state after amplification with filled contours. The increased width of the latter indicates the increase of noise in the amplified state. The Wigner functions are calculated using QUTIP [22].

degenerate pump in Sec. V. In this section, we will also briefly discuss other implementations of the Hamiltonian. After revisiting the terminology in Sec. VI, we derive the classical coupled-mode equations once more in Sec. VII, but now from the quantum Hamiltonian. Thus we show that the classical and quantum theories converge in the classical limit. The paper is concluded by a discussion of the validity of the theories in terms of the strength of the nonlinearity, i.e., to which value of the nonlinearity it can be considered weak, in Sec. VIII.

II. TERMINOLOGY

For parametric amplifiers, a specific terminology is used that can be confusing at times. Here an overview of the terminology is presented and it is discussed under which circumstances the terms play a role. These circumstances are fully determined by the Hamiltonian that describes the process.

Basically, parametric amplifiers work by the principle of wave mixing. This mixing process occurs due to a nonlinear response of the device to a transmitting electromagnetic field and causes energy transfer between the different transmitting modes. Suppose that the nonlinearity occurs as a result of a nonlinear polarization of the material,

$$\mathbf{P} = (\chi_e^{(1)} + \chi_e^{(2)}\mathbf{E} + \chi_e^{(3)}\mathbf{E}^2 + \dots)\mathbf{E}. \quad (1)$$

Then the Hamiltonian contains a term

$$H_{EP} \propto \mathbf{E} \cdot \mathbf{P} = \chi_e^{(1)}\mathbf{E}^2 + \chi_e^{(2)}\mathbf{E}^3 + \chi_e^{(3)}\mathbf{E}^4 + \dots \quad (2)$$

If the material has a strong $\chi_e^{(2)}$ contribution, the \mathbf{E}^3 term in the Hamiltonian leads to a three-wave-mixing (3WM) process and, consequently, to a mixing term in the Hamiltonian of the form

$$\hat{H}_{3WM} = \hbar\chi\hat{a}_p\hat{a}_s^\dagger\hat{a}_i^\dagger e^{i(-\Delta\Omega t + \Delta\phi)} + \text{H.c.} \quad (3)$$

This Hamiltonian enables a photon in the pump mode (p) to be scattered into a photon in the signal mode (s) that is to be amplified and some rest energy, which is generally referred to as the idler mode (i). As the Hamiltonian conserves energy, $\omega_i = \omega_p - \omega_s$. Here, $\Delta\Omega$ is a phase-mismatching term resulting from dispersion and modulation in the device, to be discussed in Sec. III [Eq. (16)] and Sec. VI [Eqs. (79) and (81)]. $\Delta\phi = \phi_p - \phi_s - \phi_i$ is the phase difference between the pump, signal, and idler that enter the device.

Contrarily, if the material has a dominant $\chi_e^{(3)}$ contribution, the Hamiltonian contains a term

$$\hat{H}_{4WM} = \hbar\chi\hat{a}_p\hat{a}_p\hat{a}_s^\dagger\hat{a}_i^\dagger e^{i(-\Delta\Omega t + \Delta\phi)} + \text{H.c.}, \quad (4)$$

and a four-wave-mixing (4WM) process takes place, where $\Delta\phi = \phi_p + \phi_p - \phi_s - \phi_i$. In this case, two pump photons are scattered into a signal and an idler photon and $\omega_i = \omega_p + \omega_p - \omega_s$. In the case where $\hat{a}_p \neq \hat{a}_p$ the pump is said to be nondegenerate, whereas it is degenerate if $\hat{a}_p = \hat{a}_p$.

Generally, the pump(s) in Eqs. (3) and (4) are treated as classical modes, which are undepleted. This implies that the corresponding operators are replaced by a constant amplitude and can be absorbed in the coupling constant. This results in a contribution to the Hamiltonian that is identical for 3WM and 4WM,

$$\hat{H}_{3/4WM} = \hbar\tilde{\chi}\hat{a}_s^\dagger\hat{a}_i^\dagger e^{i(-\Delta\Omega t + \Delta\phi)} + \text{H.c.}, \quad (5)$$

in which $\tilde{\chi} = \chi|A_p|$ for 3WM and $\tilde{\chi} = \chi|A_p||A_{p'}|$ for 4WM, respectively.

Apart from a distinction in 3WM and 4WM devices, parametric amplifiers can be phase preserving and phase sensitive. Phase-preserving amplification occurs if the signal and idler are in two distinct modes [$\hat{a}_s \neq \hat{a}_i$ as in Eqs. (3) and (4)]. For this reason, such amplifiers are also referred to as nondegenerate. The amplification is independent of $\Delta\phi$

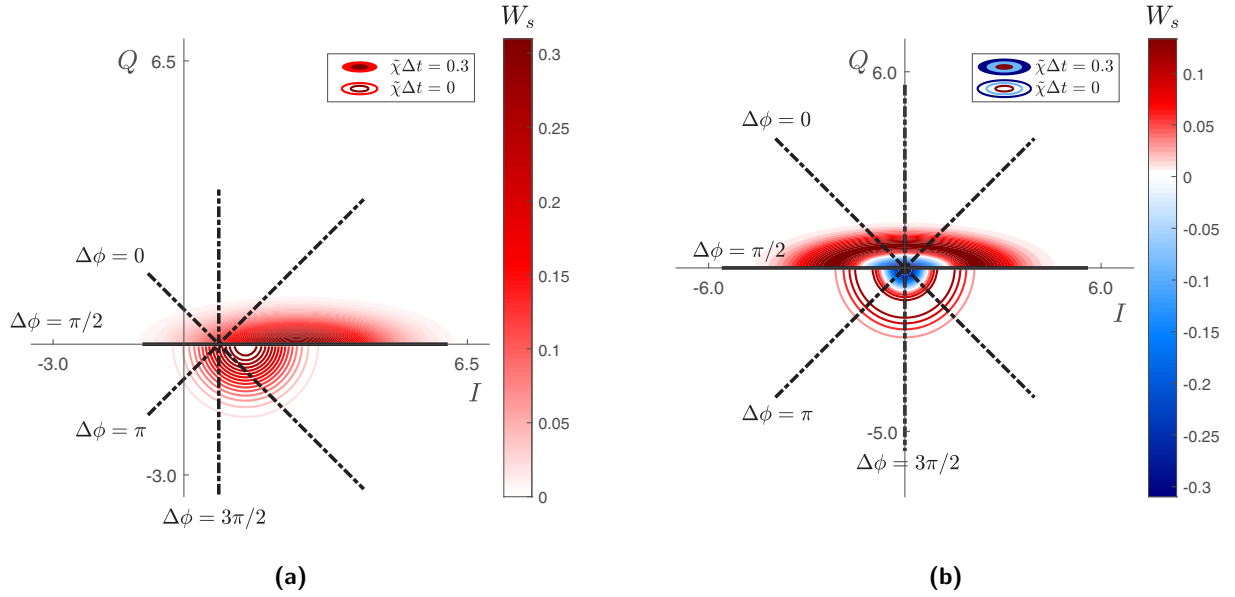


FIG. 2. Effect of degenerate (or phase-sensitive) amplification with an undepleted pump on (a) a coherent state ($\alpha = 1$) and (b) a single-photon-number state in the I , Q -quadrature plane. In the lower-half plane, (half of the) the Wigner functions of the unamplified states are depicted using contours. The upper-half plane depicts the Wigner functions of the amplified states for $\Delta\phi = \pi/2$ with filled contours, whereas the long symmetry axis of the Wigner functions for some different $\Delta\phi$ s are indicated by dashed lines. The width of one of the quadratures I and Q is amplified, whereas the other is deamplified, such that the total added noise can be 0. If the input state is a coherent state, the power gain varies with $\Delta\phi$. The Wigner functions are calculated using QUTIP [22].

and a minimum of half a photon of noise per unit bandwidth is added to the signal [4]. The process is illustrated in Fig. 1.

If the signal and idler are in nondistinct modes [$\hat{a}_s = \hat{a}_i$ in Eqs. (3) and (4)], however, the amplifier is said to be degenerate and works in a phase-sensitive mode. The latter term results from a critical dependence of the amplification process on $\Delta\phi$, which causes one quadrature of the signal to be amplified, whereas the other is deamplified; see Fig. 2. This implies that for input signal states with an explicit phase, such as coherent states, the amplifier's power gain depends on the phase difference between the signal and pump. The gain is maximized for $\Delta\phi = \pi/2$, and for $\Delta\phi = 3\pi/2$ the gain is less than unity, attenuating the signal. For input states that do not have such an explicit phase, e.g., number states and thermal states, the amplifier power gain is phase independent. In this process, amplification is possible without adding noise to the signal [4].

In both 3WM and 4WM devices, the amplification process is most efficient if the phase mismatch $\Delta\Omega = 0$, as is illustrated in Fig. 3. A nonzero $\Delta\Omega$ arises from dispersion and modulation effects, which are therefore beneficial to be canceled.

III. THE NONDEGENERATE PARAMETRIC AMPLIFIER WITH UNDEPLETED DEGENERATE PUMP—CLASSICAL THEORY

The classical theory for Josephson junction embedded transmission lines is worked out in detail in Refs. [17,18]. In Ref. [17], such a line, as schematically depicted in Fig. 4, is

considered and, as a result, a nonlinear wave equation

$$C_g \frac{\partial^2 \Phi}{\partial t^2} - \frac{a^2}{L_{1,0}} \frac{\partial^2 \Phi}{\partial z^2} - C_1 a^2 \frac{\partial^4 \Phi}{\partial z^2 \partial t^2} = -\frac{a^4}{2I_c^2 L_{1,0}^3} \frac{\partial^2 \Phi}{\partial z^2} \left(\frac{\partial \Phi}{\partial z} \right)^2 \quad (6)$$

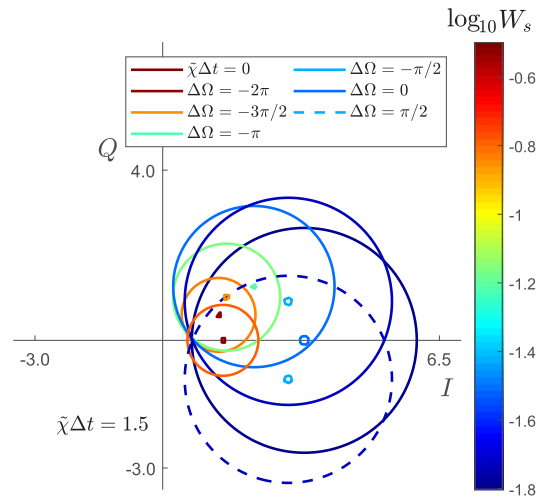


FIG. 3. Effect of phase mismatch on the phase-preserving amplification of a coherent state for an undepleted pump. Depicted are the full width at half maximum and the maximum of the Wigner function of the initial state $\tilde{\chi}\Delta t = 0$ and the final states $\tilde{\chi}\Delta t = 1.5$ with various amounts of phase mismatch $\Delta\Omega$ ($\Delta t = 1$) in the I , Q -quadrature plane. The legend refers to the maximum of the Wigner functions. Increasing the phase mismatch reduces the power gain of the amplifier. The Wigner functions are calculated using QUTIP [22].

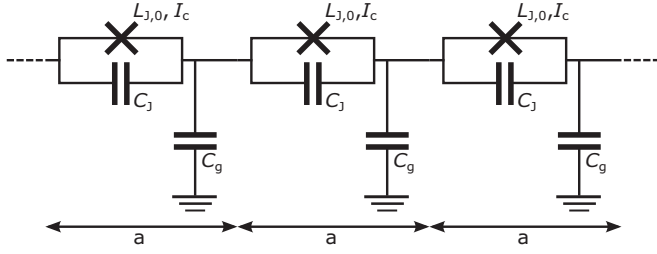


FIG. 4. Schematic overview of a Josephson junction embedded transmission line. The junctions are modeled as a parallel LC circuit with a nonlinear inductor L_J .

is derived that describes the evolution of the flux $\Phi = \Phi(z, t)$ through the line. Here, C_g is the capacitance to ground, a is the length of a unit cell of the transmission line, $L_{J,0}$ is the Josephson inductance of the junctions at 0 flux, C_J is the capacitance of the Josephson junction, and I_c is its critical current. $L_{J,0}$ and I_c are related by $L_{J,0} = \varphi_0/I_c$, with $\varphi_0 = \hbar/2e$ the reduced magnetic flux quantum; see Sec. IV D. In deriving this equation, it is assumed that $a \ll \lambda$, the wavelength of the propagating modes, and that the nonlinearity provided by the Josephson junction is weak, such that only the first-order nonlinear term [right-hand side of Eq. (6)] resulting from the presence of the Josephson junction needs to be taken into account.

Starting from this equation, Ref. [18] derives the coupled-mode equations. This is a set of coupled nonlinear differential equations that describe the evolution of the considered modes of the flux through the parametric amplifier. For the nondegenerate 4WM parametric amplifier with degenerate pump, it is assumed that only three modes of the field play a role. These are generally referred to as the pump, the signal, and the idler. The pump is the mode that delivers the energy for the amplification of the small-amplitude signal. As a result of energy conservation, an idler mode is created which also has a small amplitude.

Reference [18] suggests a trial solution for Eq. (6) in the form of

$$\Phi = \sum_{n=p,s,i} \text{Re}\{A_n(z)e^{i(k_n z - \omega_n t)}\} = \frac{1}{2} \sum_n A_n(z)e^{i(k_n z - \omega_n t)} + \text{c.c.}, \quad (7)$$

which describes a superposition of waves that may have varying amplitudes A_n while propagating through the line.

Furthermore, the slowly varying amplitude approximation is invoked, i.e., it is assumed that $|d^2 A_n/dz^2| \ll |k_n dA_n/dz|$, and that the change in amplitude within a wavelength of transmission line is small, $|dA_n/dz| \ll |k_n A_n|$, such that the first-order derivatives on the right-hand side of Eq. (6) can be neglected. Furthermore, terms of the order of $|A_s|^2$ and $|A_i|^2$ are neglected, as these are assumed to be small. Then, the amplitudes of the various modes are described by the following differential equations, upon substituting the trial solution into Eq. (6):

$$\frac{\partial A_p}{\partial z} = i\Xi_p |A_p|^2 A_p + 2iX_p A_p^* A_s A_i e^{i\Delta k z}, \quad (8)$$

$$\frac{\partial A_{s(i)}}{\partial z} = i\Xi_{s(i)} |A_p|^2 A_{s(i)} + iX_{s(i)} A_p^2 A_{i(s)}^* e^{i\Delta k z}, \quad (9)$$

where $\Delta k = 2k_p - k_s - k_i$ and¹

$$k_n = \frac{\omega_n \sqrt{L_{J,0} C_g}}{a \sqrt{1 - L_{J,0} C_J \omega_n^2}}. \quad (10)$$

The coupling constants Ξ_n and X_n follow to be

$$\Xi_n = \frac{a^4 k_p^2 k_n^3 (2 - \delta_{pn})}{16 C_g I_c^2 L_{J,0}^3 \omega_n^2}, \quad (11)$$

$$X_n = \frac{a^4 k_p^2 k_s k_i (k_n - \varepsilon_n \Delta k)}{16 C_g I_c^2 L_{J,0}^3 \omega_n^2}, \quad (12)$$

with $\varepsilon_p = 1$ and $\varepsilon_{s,i} = -1$. As can be noted, the Ξ_n s modulate the wave number of the modes if the pump amplitude is large. Ξ_p is therefore referred to as the self modulation of the pump, while $\Xi_{s,i}$ are the cross modulation terms between the pump and the signal or idler.

Under the undepleted pump approximation and assuming $A_{s,i} \ll A_p$, we can drop the interaction term in Eq. (8) and treat $|A_p|^2$ as a constant. As a result, the equation can be solved analytically as

$$A_p = |A_{p,0}| e^{i(\Xi_p |A_{p,0}|^2 z + \phi_p)}. \quad (13)$$

Since we describe 4WM, which is phase preserving, we can assume $\phi_p = 0$ with no loss of generality.

Substituting this result into Eq. (9), it can be rewritten as

$$\frac{\partial A_{s(i)}}{\partial z} = i\Xi_{s(i)} |A_{p,0}|^2 A_{s(i)} + iX_{s(i)} |A_{p,0}|^2 A_{i(s)}^* e^{i(\Delta k + 2\Xi_p |A_{p,0}|^2)z}. \quad (14)$$

Furthermore, switching to a co-rotating frame such that $A_{s(i)} \mapsto A_{s(i)} e^{i\Xi_{s(i)} |A_{p,0}|^2 z}$, we can cast the equation in the form

$$\frac{\partial A_{s(i)}}{\partial z} = iX_{s(i)} |A_{p,0}|^2 A_{i(s)}^* e^{i(\Delta k + \Delta \Xi |A_{p,0}|^2)z}, \quad (15)$$

where $\Delta \Xi = 2\Xi_p - \Xi_s - \Xi_i$. This set of coupled differential equations can be solved analytically as [23]

$$A_{s(i)} = \left[A_{s(i),0} \left(\cosh g_z z - \frac{i\Delta K}{2g_z} \sinh g_z z \right) + \frac{iX_{s(i)} |A_{p,0}|^2}{g_z} A_{i(s),0}^* \sinh g_z z \right] e^{i\Delta K z/2}, \quad (16)$$

with $\Delta K = (\Delta k + \Delta \Xi |A_{p,0}|^2)$, which is related to $\Delta \Omega$ in Eq. (4) through $k_n + \Xi_n |A_{p,0}|^2 \mapsto (k_n + \Xi_n) \omega_n |A_{p,0}|^2 / k_n$; see Sec. VII. Here, $g_z = \sqrt{X_s X_i^* |A_{p,0}|^4 - (\Delta K/2)^2}$, from which the power gain of the signal for a TWPA of length l_T can be

¹Note that this is just the familiar form of k_n , $k_n = \omega_n \sqrt{LC}/a$, where $L \mapsto L/(1 - LC_{\parallel} \omega^2)$ as a result of the impedance $Z_J = Z_{L_J} \parallel Z_{C_J}$.

determined as

$$\begin{aligned}
 G_s &= \left| \frac{A_s}{A_{s,0}} \right|^2 \\
 &= \left| \cosh g_z l_T - \frac{i\Delta K}{2g_z} \sinh g_z l_T \right|^2 + \frac{|A_{i,0}|^2}{|A_{s,0}|^2} \left| \frac{X_s |A_{p,0}|^2}{g_z} \sinh g_z l_T \right|^2 \\
 &\quad + \frac{1}{|A_{s,0}|^2} \left[-iA_{s,0}A_{i,0} \left(\cosh g_z l_T - \frac{i\Delta K}{2g_z} \sinh g_z l_T \right) \frac{X_s^* |A_{p,0}|^2}{g_z^*} \sinh g_z^* l_T + \text{c.c.} \right].
 \end{aligned} \tag{17}$$

Effect of phase matching

As noted in Sec. II, the amplification process is most efficient if $\Delta\Omega = 0$. As will become clear from the quantum mechanical treatment of the problem in due course, $\Delta\Omega$ corresponds to ΔK in Eq. (16). However, due to dispersion [Eq. (10)] and self and cross modulation [Eq. (11)], this term cannot be zero in a transmission line embedded with Josephson junctions. In order to bring it closer to zero, we need dispersion engineering.

In Ref. [18], dispersion engineering is achieved by embedding resonators in each unit cell in Fig. 4, as depicted in Fig. 5. If the pump tone is chosen at a frequency close to the resonance frequency of the resonators, the result is that every resonator gives the tone a small phase kick (without diminishing the tone's amplitude too much) and $\Delta K \approx 0$ may be accomplished. The phase kick required per resonator depends on the density of resonators. Reference [18] puts a resonator in every unit cell, such that the phase kick per resonator only needs to be very small. This implies that only a little of the pump amplitude will be reflected. Contrarily, Ref. [12] puts a resonator after 17 unit cells, each containing three Josephson junctions, such that the required amount of phase shift per resonator is larger, resulting in a larger reflected pump amplitude accordingly.

If every unit cell contains a resonator, taking its effect on the theory into account is straightforward: we replace the capacitance C_g by an impedance

$$Z_{C_{\text{eff}},n} = Z_{C_g,n} \parallel Z_{r,n} = \frac{1}{i\omega_n C_g} + \frac{1 - L_r(C_r + C_c)\omega_n^2}{i\omega_n C_c(1 - L_r C_r \omega_n^2)}, \tag{18}$$

in which L_r and C_r are the inductance and capacitance of the resonator with a resonance at $\omega_r = 1/\sqrt{L_r C_r}$. C_c is the coupling capacitance between the resonator and the transmission

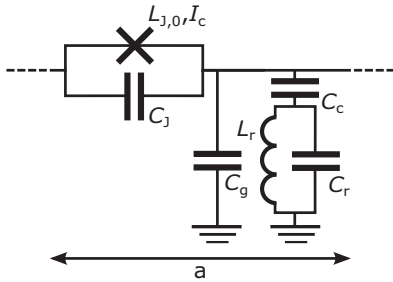


FIG. 5. Unit cell of a Josephson junction embedded transmission line with a resonator for achieving phase matching between the pump, signal, and idler mode in a TWPA.

line. Subsequently, we substitute $1/i\omega_n Z_{C_{\text{eff}},n}$ for C_g in the coupling constants Ξ_n and X_n in Eqs. (11) and (12), such that

$$\Xi_n^{\text{PM}} = \frac{ia^4 k_p^2 k_n^3 \omega_n Z_{C_{\text{eff}},n}}{16I_c^2 L_{J,0}^3 \omega_n^2} (2 - \delta_{pn}), \tag{19}$$

$$X_n^{\text{PM}} = \frac{ia^4 k_p^2 k_s k_i \omega_n Z_{C_{\text{eff}},n}}{16I_c^2 L_{J,0}^3 \omega_n^2} (k_n - \varepsilon_n \Delta k). \tag{20}$$

The effect of phase matching on the performance of the TWPA is depicted in Fig. 6. The red dash-dotted curve results from Eq. (17) without phase matching with $L_{J,0} = 100$ pH ($I_c = 3.29$ μA), $C_J = 329$ fF, and $C_g = 39$ fF with 2000 unit cells of 10 μm . The blue continuous curve results from adding resonators to the unit cells and evaluating Eq. (17). For the resonators, $C_c = 10$ fF, $L_r = 100$ pH, and $C_r = 7.036$ pF. In both calculations, the pump current, which is linked to the mode amplitude via the characteristic impedance of the TWPA as $I_p = -A_{p,0}\omega_p/Z_c$, is set to $0.5I_c$ at $\omega_p = 2\pi \times 5.97$ GHz. The initial idler current is set to zero.

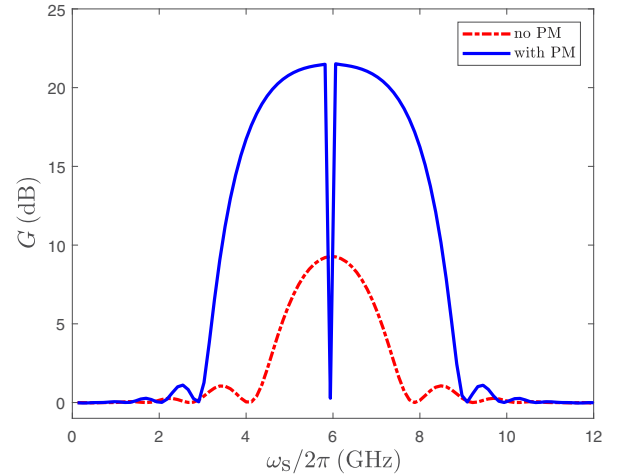


FIG. 6. Predicted power gain as a function of signal frequency without and with phase matching, with parameters taken from [18]. The pump frequency and current are 5.97 GHz and $0.5I_c$ and the initial idler current is 0. The transmission line parameters are $L_{J,0} = 100$ pH ($I_c = 3.29$ μA), $C_J = 329$ fF, and $C_g = 39$ fF and the resonator parameters are $C_c = 10$ fF, $L_r = 100$ pH, and $C_r = 7.036$ pF. The calculations have been performed taking into account 2000 unit cells of 10 μm length. The dip in the plot for a TWPA with phase matching actually contains two dips on closer inspection. They result from the signal and idler being on resonance with the phase-matching resonators, respectively.

In the case that one only adds resonators at specific points in the structure, these can be taken into account by evaluating the amplifier in parts. The resonators divide the structure in sections. Within each section, the evolution of the mode amplitudes follows Eq. (16), with the coupling constants as given by Eqs. (11) and (12). Then, between two sections, one evaluates the transmission coefficient due to the presence of the resonator, updates the mode amplitudes accordingly, and uses those amplitudes as input for the next section.

IV. QUANTUM THEORY OF PARAMETRIC AMPLIFICATION (4WM)

In this section, we derive the quantum Hamiltonian for the TWPA. In quantum theory, the evolution of the state vector, $|\psi\rangle$, describing the system is determined by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle, \tag{21}$$

where \hbar is the reduced Planck constant $h/2\pi$. Hence, in order to understand the quantum behavior of a parametric amplifier, we need to derive its Hamiltonian. In this section, the Hamiltonian for a 4WM parametric amplifier, where the nonlinearity is provided by Josephson junctions, is derived within Fock space for discrete modes. Although this may sound quite limiting, it should be noted that the same method can be easily applied to three-wave-mixing devices or devices with another source of the nonlinearity.

We derive the Hamiltonian in three steps. After covering the concept of energy in transmission lines, a concept which the rest of the derivation relies on, first a dispersionless LC -transmission line is quantized. As a second step, dispersion is added to this transmission line by adding an additional capacitance parallel to the inductance. As a final step, the inductance is replaced by a Josephson junction.

A. Energy in transmission lines

Typically, nondissipative transmission lines are quantized as electromagnetic circuits using currents (I), fluxes (Φ), voltages (V), and charges (Q) as quantum fields [24]. These give rise to a Hamiltonian via the inductors and capacitors that characterize the line. The energy stored in these elements is given by

$$U(t) = \int_{t_0}^t P dt' = \int_{t_0}^t V I dt', \tag{22}$$

the energy is given by the time-integrated power P , through the element, which equals the product of the voltage and current. Now the only task is to calculate the voltage over and current through the element, integrate, and sum over all the elements in the circuit. Specifically, for inductors,

$$U = \begin{cases} \int_{I(t_0)}^{I(t)} L I dI' = \frac{1}{2} L I^2 \\ \int_{\Phi(t_0)}^{\Phi(t)} \frac{1}{L} \Phi d\Phi' = \frac{1}{2L} \Phi^2, \end{cases} \tag{23}$$

using the current-voltage relation for inductors $V = L\partial I/\partial t$ in the first line and Faraday's induction law $V = \partial\Phi/\partial t$ along with $\Phi = LI$ in the second. Note that it is implicitly assumed that the current and flux are zero at $t = t_0$. This proves to be a

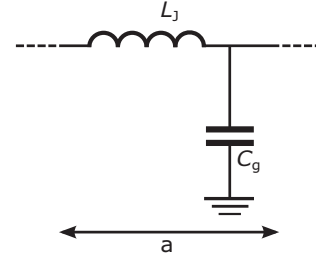


FIG. 7. Unit cell of a dispersionless LC -transmission line.

critical assumption of utmost importance, as will be shown in Sec. IV C. For the energy stored in capacitors, the same form of the energy arises if we interchange the current with voltage, the flux with charge, and the inductance with capacitance in Eq. (23).

B. Quantization of a nondispersive transmission line

Consider once more the transmission line in Fig. 4. For the moment, we replace the Josephson junction by an ordinary inductor L_J and neglect Josephson capacitance C_J , in which case the line is just an ordinary LC -transmission line without dispersion, as depicted in Fig. 7.

As suggested by the previous section, we postulate the following mesoscopic Hamiltonian for an electromagnetic (EM) field transmitting through the transmission line:

$$\hat{H} = \int_{l_q} \frac{1}{2} \mathcal{L}_J \hat{I}_{L_J}^2 + \frac{1}{2} C_g \hat{V}_{C_g}^2 dz. \tag{24}$$

Here, $\mathcal{L}_J = L_J/a$ is the inductance per unit length and $C_g = C_g/a$ is the capacitance per unit length. \hat{I}_{L_J} is the current through the inductor L_J and \hat{V}_{C_g} is the voltage over the capacitor C_g . l_q is the quantization length [25].

As in the classical theory, we assume sinusoidal waves passing through the line. In this case,

$$\hat{V}_{C_g} = \sum_n \hat{V}_{C_g,n} = \sum_n \sqrt{\frac{\hbar\omega_n}{2C_g l_q}} (\hat{a}_n e^{i(k_n z - \omega_n t)} + \text{H.c.}), \tag{25}$$

as suggested by [24], adapted for discrete-mode operators [25]. For waves traveling in a positive z direction, the wave number k_n is positive and will be labeled by a positive n . Waves traveling in a negative z direction have a negative wave number and will therefore be labeled by a negative n . In general, $k_{-n} = -k_n$. For frequencies, we have $\omega_{-n} = \omega_n$. The characteristic impedance of this line is given by $Z_c = \sqrt{\mathcal{L}_J/C_g}$ and the phase velocity of the traveling waves equals $v_{ph} = \omega_n/|k_n| = 1/\sqrt{\mathcal{L}_J C_g}$ [26].

From this voltage, we determine the current through the inductor by the telegrapher's equations [26]. Specifically,

$$\frac{\partial V_n}{\partial z} = -\mathcal{L} \frac{\partial I_n}{\partial t}. \tag{26}$$

Thus,

$$\hat{I}_{L_J} = \sum_n \hat{I}_{L_J,n} = \sum_n \text{sgn}(n) \sqrt{\frac{\hbar\omega_n}{2\mathcal{L}_J l_q}} (\hat{a}_n e^{i(k_n z - \omega_n t)} + \text{H.c.}). \tag{27}$$

We substitute relations (27) and (25) into Eq. (24) and use that [25]

$$\int_{l_q} e^{i(\Delta k_{nm})z} dz = \begin{cases} l_q \operatorname{sinc}(\Delta k_{nm} l_q / 2) & \text{if } -l_q/2 \leq z \leq l_q/2 \text{ (symmetric bounds)} \\ -i(e^{i\Delta k_{nm} l_q} - 1) / \Delta k_{nm} & \text{if } 0 \leq z \leq l_q \text{ (asymmetric bounds)} \end{cases} \quad (28)$$

$$\approx l_q \delta_{\Delta k_{nm}},$$

with $\Delta k_{nm} \equiv \pm k_n \pm k_m$. Here, the plus (minus) sign should be chosen if the wave number is associated to an annihilation (creation) operator. The approximation holds if $\Delta k_{nm} l_q \ll 1$, for which

$$\delta_{\Delta k_{nm}} = \begin{cases} 1 & \text{if } \Delta k_{nm} l_q = 0 \\ 0 & \text{else,} \end{cases} \quad (29)$$

such that we arrive at

$$\hat{H}_0 = \sum_n \frac{1}{2} \hbar \omega_n (\hat{a}_n^\dagger \hat{a}_n + \hat{a}_n \hat{a}_n^\dagger) = \sum_n \hbar \omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right), \quad (30)$$

taking into account the commutation relation $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}$ (see [25] for details).

C. The influence of the Josephson capacitance: Quantization of a dispersive transmission line

So far we have been neglecting the influence of the parallel capacitor C_J in the transmission line under consideration. Taking this capacitance into account leads to alterations to the theory presented so far because we now have a capacitor C_J parallel to the inductor L_J , as shown in Fig. 8. For frequencies $\omega_n < 1/\sqrt{L_J C_J}$, this can be taken into account by a frequency-dependent inductance,

$$L_{\text{eff}} = \frac{L_J}{1 - L_J C_J \omega_n^2} \equiv L_J \Lambda_n, \quad (31)$$

and, as a result, first

$$Z_{c,n} = \sqrt{\frac{\mathcal{L}_J \Lambda_n}{C_g}}, \quad v_{\text{ph},n} = \frac{1}{\sqrt{\mathcal{L}_J \Lambda_n C_g}} = \frac{\omega_n}{|k_n|}, \quad (32)$$

implying dispersion is added to the problem since the phase velocity is now frequency dependent. Second, we have to add an additional capacitive energy to the Hamiltonian.

For didactic reasons, we now give two derivations of the Hamiltonian in which we take the parallel capacitor into account. In Eqs. (33) to (36), we present an erroneous approach, after which the correct manner is presented.

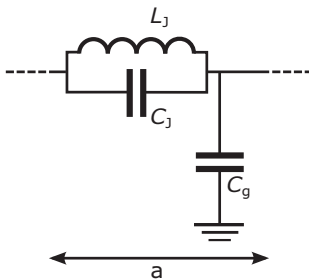


FIG. 8. Unit cell of an LC -transmission line in which dispersion is added due to the capacitor C_J parallel to the inductor L_J .

Intuitively, the energy contribution of C_J can be added to the Hamiltonian in the same way as the energy stored in C_g . That is,

$$\hat{H} = \int_l \frac{1}{2} \mathcal{L}_J \hat{I}_J^2 + \frac{1}{2} C_J \Delta \hat{V}_{C_J}^2 + \frac{1}{2} C_g \hat{V}_{C_g}^2 dz. \quad (33)$$

Realizing that $\Delta \hat{V}_{C_J} = \Delta \hat{V}_{L_J} = \Delta \hat{V}_{L_{\text{eff}}}$, it can be calculated by the current-voltage relationship for inductors. The current through L_{eff} equals the current in Eq. (27) with $L_J \mapsto L_{\text{eff}}$. Thus,

$$\begin{aligned} \Delta \hat{V}_{L_{\text{eff}}} &= \sum_n \Delta \hat{V}_{L_{\text{eff}},n} \\ &= L_J \sum_n \Lambda_n \frac{\partial \hat{I}_{L_{\text{eff}},n}}{\partial t} \\ &= \sum_n k_n a \sqrt{\frac{\hbar \omega_n}{2 C_g l_q}} (-i \hat{a}_n e^{i(k_n z - \omega_n t)} + \text{H.c.}). \end{aligned} \quad (34)$$

Using the same relationship, we can calculate \hat{I}_{L_J} as

$$\begin{aligned} \hat{I}_{L_J} &= \frac{1}{L_J} \int \Delta \hat{V}_{L_J} dt \\ &= \sum_n \frac{k_n}{\mathcal{L}_J \omega_n} \sqrt{\frac{\hbar \omega_n}{2 C_g l_q}} (\hat{a}_n e^{i(k_n z - \omega_n t)} + \text{H.c.}), \end{aligned} \quad (35)$$

and we can use the methods of the last section to find

$$\begin{aligned} \hat{H}_0 &= \sum_n \frac{1}{2} \hbar \omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right) [\Lambda_n + (\Lambda_n - 1) + 1] \\ &= \sum_n \hbar \omega_n \Lambda_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right). \quad (\text{Wrong}) \end{aligned} \quad (36)$$

This is an odd result: in the transmission line fed by a mode oscillating at a frequency ω_n , the mode seems to oscillate at $\omega_n \Lambda_n$. Indeed, the result is simply wrong by the exact reason pointed out in Sec. IV A. The voltage in Eq. (25) is “cosinlike,” whereas $\Delta \hat{V}_{C_J}$ is “sinlike.”² This implies that the energy cannot be zero in all elements at the same time, as we assumed in Eq. (23). Although the sinlike operator $\Delta \hat{V}_{C_J}$ is unsuitable to be used for the purpose of the derivation of the Hamiltonian, it should be noted that $\Delta \hat{V}_{C_J}$ is a “valid” operator in itself and thus is suitable for calculating expectation values from some quantum state $|\psi\rangle$.

²In the sense that the expectation value of the operator on a coherent state $|\alpha\rangle$ with $\alpha \in \mathbb{R}$ scales as either a cosine or a sine.

To solve this problem, consider once more the energy stored in C_J ,

$$U_{C_J} = \int_{t_0}^t V_{C_J} I_{C_J} dt' = \frac{1}{2} C_J V_{C_J}^2. \quad (37)$$

However, we can also cast this energy in terms of the flux, given by Faraday's induction law as

$$\Phi = \int V dt. \quad (38)$$

Since the flux is the time-integrated voltage, it will be cosine-like whenever the voltage is sinelike, and vice versa. From the definition,

$$U_{C_J} = \int_{t_0}^t V_{C_J} I_{C_J} dt' = \int_{t_0}^t C_J \frac{d\Phi_{C_J}}{dt'} \frac{d^2\Phi_{C_J}}{dt'^2} dt' = -\omega^2 C_J \Phi_{C_J}^2, \quad (39)$$

using the current-voltage relation for capacitors $I = C\partial V/\partial t$ and that $\partial^2\Phi/\partial t^2 = -\omega^2\Phi$. This suggests that a more fruitful approach is to start out with

$$\hat{H} = \frac{1}{2a^2} \int_{l_q} \frac{1}{\mathcal{L}_J} \Delta\hat{\Phi}_{L_J}^2 + C_J a^2 \Delta\hat{\Phi}_{C_J} \frac{\partial^2 \Delta\hat{\Phi}_{C_J}}{\partial t^2} + \frac{1}{C_g} \hat{Q}_{C_g}^2 dz. \quad (40)$$

In the above equation, we switched to flux and charge variables in all terms for aesthetic reasons. For the first and third terms, we might use the current and voltage variable just as well. The fluxes can be computed from either Eq. (34) or (35), and it follows that

$$\begin{aligned} \Delta\hat{\Phi}_{L_J} &= \Delta\hat{\Phi}_{C_J} = \Delta\hat{\Phi}_{L_{\text{eff}}} \\ &= \sum_n \frac{k_n a}{\omega_n} \sqrt{\frac{\hbar\omega_n}{2C_g l_q}} (\hat{a}_n e^{i(k_n z - \omega_n t)} + \text{H.c.}). \end{aligned} \quad (41)$$

Substituting, the Hamiltonian (40) yields

$$\hat{H}_0 = \sum_n \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right), \quad (42)$$

as expected.

This result can be generalized for any lossless, linear transmission line. From Eq. (40), we can infer that we can describe the same problem with just two terms in the Hamiltonian. Rewriting Eq. (40), we find

$$\begin{aligned} \hat{H} &= \frac{1}{2a} \int_{l_q} \left(\frac{1}{L_J} \Delta\hat{\Phi}_{L_J} + C_J \frac{\partial \Delta\hat{V}_{C_J}}{\partial t} \right) \Delta\hat{\Phi}_{C_J} + C_g \hat{V}_{C_g}^2 dz \\ &= \frac{1}{2a} \int_{l_q} \hat{L}_{L_{\text{eff}}} \Delta\hat{\Phi}_{L_{\text{eff}}} + C_g \hat{V}_{C_g}^2 dz, \end{aligned} \quad (43)$$

as $L_{L_{\text{eff}}} = L_J + I_{C_J}$. The same argument holds if C_g is replaced by a frequency-dependent effective capacitance C_{eff} , such that we may write

$$\hat{H} = \frac{1}{2a} \int_{l_q} \hat{L}_{L_{\text{eff}}} \Delta\hat{\Phi}_{L_{\text{eff}}} + \hat{V}_{C_{\text{eff}}} \hat{Q}_{C_{\text{eff}}} dz \quad (44)$$

for any lossless linear transmission line. This yields Eq. (42) after substitution of the quantum fields.

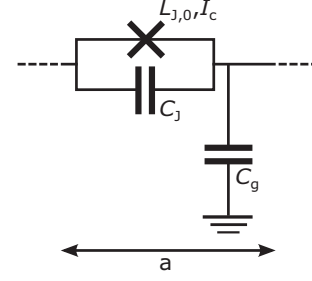


FIG. 9. Unit cell of a Josephson junction embedded transmission line in which the Josephson junction is modeled as a nonlinear inductor, L_J , with a parallel capacitor C_J .

D. Adding the nonlinearity: Quantization of a nonlinear transmission line

As a last step, we replace the inductor L_J , which we considered as an inductor with a fixed value up to this point, by a Josephson junction. The unit cell for such a transmission line is depicted in Fig. 9. The current through a Josephson junction is [27]

$$I_J = I_c \sin\left(\frac{\Delta\Phi_J}{\varphi_0}\right), \quad (45)$$

with $\varphi_0 = \hbar/2e$ the reduced magnetic flux quantum, $\Phi_0/2\pi$, and e the elementary charge. From the current, we can calculate the Josephson energy in the usual fashion,

$$\begin{aligned} U_J &= \int_{t_0}^t V I dt' \\ &= \int_{t_0}^t \frac{d\Delta\Phi_J}{dt'} I_c \sin\left(\frac{\Delta\Phi_J}{\varphi_0}\right) dt' \\ &= I_c \varphi_0 \left[1 - \cos\left(\frac{\Delta\Phi_J}{\varphi_0}\right) \right]. \end{aligned} \quad (46)$$

Substituting this energy for the inductive energy in Eq. (40) yields

$$\begin{aligned} \hat{H} &= \frac{1}{2a^2} \int_{l_q} 2a I_c \varphi_0 \left[1 - \cos\left(\frac{\Delta\hat{\Phi}_J}{\varphi_0}\right) \right] \\ &\quad + C_J a^2 \Delta\hat{\Phi}_J \frac{\partial^2 \Delta\hat{\Phi}_J}{\partial t^2} + \frac{1}{C_g} \hat{Q}_{C_g}^2 dz \\ &= \frac{1}{2a^2} \int_{l_q} \left[\frac{1}{\mathcal{L}_{J,0}} \Delta\hat{\Phi}_J - \frac{1}{12\mathcal{L}_{J,0}\varphi_0^2} \Delta\hat{\Phi}_J^3 \right. \\ &\quad \left. + O(\Delta\hat{\Phi}_J^5) + C_J a^2 \frac{\partial^2 \Delta\hat{\Phi}_J}{\partial t^2} \right] \Delta\hat{\Phi}_J + \frac{1}{C_g} \hat{Q}_{C_g}^2 dz, \end{aligned} \quad (47)$$

in which we have Taylor expanded the cosine term and defined the Josephson inductance as $L_{J,0} = \varphi_0/I_c$. From this equation, it is clear immediately that the generalized Hamiltonian of Eq. (44) does not capture the nonlinear behavior.

To address the nonlinearity of the transmission line, we also calculate the nonlinear flux operator derived from Eq. (41). The dependence of L_J and thus Λ_n in the nonlinear flux operator on $\Delta\Phi_J$ is found from the Josephson current and

the flux $\Delta\Phi_J = L_J I_J$,

$$L_J(\Delta\Phi_J) = \frac{\varphi_0}{I_c} \frac{\Delta\Phi_J/\varphi_0}{\sin(\Delta\Phi_J/\varphi_0)} \equiv L_{J,0} \frac{\Delta\Phi_J/\varphi_0}{\sin(\Delta\Phi_J/\varphi_0)}. \quad (48)$$

Furthermore, as in the classical theory, we give an explicit time and spatial dependence to the creation and annihilation

operators, $\hat{a}_n^{(\dagger)} \mapsto \hat{a}_n^{(\dagger)}(z, t)$, in the voltage operator of Eq. (25). Invoking the slowly varying amplitude approximation, the time and spatial dependence of these operators is neglected in deriving the other field operators. Hence, from Eq. (41), we find for the nonlinear Josephson junction flux operator,

$$\begin{aligned} \Delta\hat{\Phi}_J &= \sum_n a \sqrt{\frac{\hbar\omega_n}{2l_q} \frac{\mathcal{L}_{J,0}\Delta\hat{\Phi}_J/\varphi_0}{\sin(\Delta\hat{\Phi}_J/\varphi_0) - \omega_n^2 L_J C_J \Delta\hat{\Phi}_J/\varphi_0}} (\hat{a}_n e^{i(k_n z - \omega_n t)} + \text{H.c.}) \\ &= \sum_n \sqrt{\frac{1}{1 - \Lambda_n \sum_{m,l} \Delta\hat{\Phi}_{J,m} \Delta\hat{\Phi}_{J,l} / 6\varphi_0^2 + O[(\Delta\hat{\Phi}_J/\varphi_0)^4]}} \Delta\hat{\Phi}_{J,n}^{(0)}, \end{aligned} \quad (49)$$

with $\Delta\hat{\Phi}_{J,n}^{(0)}$ given by Eq. (41). In the second line of this equation, we have explicitly written that $\Delta\hat{\Phi}_J^2 = \sum_{n,m} \Delta\hat{\Phi}_{J,n} \Delta\hat{\Phi}_{J,m}$. This recurrent relation can be solved iteratively, resulting in

$$\Delta\hat{\Phi}_J = \sum_n \left\{ 1 + \frac{\Lambda_n}{12} \left(\frac{\Delta\hat{\Phi}_J^{(0)}}{\varphi_0} \right)^2 + O \left[\left(\frac{\Delta\hat{\Phi}_J^{(0)}}{\varphi_0} \right)^4 \right] \right\} \Delta\hat{\Phi}_{J,n}^{(0)}. \quad (50)$$

Substitution of this expression in the Hamiltonian of Eq. (47) yields the Hamiltonian for a 4WM parametric amplifier, where the nonlinearity is due to Josephson junctions. Up to first nonlinear order (or fourth order in $\Delta\hat{\Phi}_J$), we find

$$\begin{aligned} \hat{H}_{\text{TWPA}} &= \sum_n \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right) + \sum_{n,m,l,k} \frac{-i\hbar^2 e^{-i\Delta\omega_{nmkl}t}}{96\mathcal{L}_{J,0} I_c^2 l_q^2 \Delta k_{nmkl}} (e^{i\Delta k_{nmkl} l_q} - 1) [(1 - 4L_{J,0} \Lambda_n C_J \omega_k^2) \{\hat{a} + \text{H.c.}\}_{n \times m \times l \times k} \\ &\quad + 4L_{J,0} \Lambda_n C_J (2\{\omega(-i\hat{a} + \text{H.c.})\}_{n \times m} \{\hat{a} + \text{H.c.}\}_{l \times k} + \{\hat{a} + \text{H.c.}\}_{n \times m} \{\omega(-i\hat{a} + \text{H.c.})\}_{l \times k})], \end{aligned} \quad (51)$$

where $\hat{a}_n \equiv \text{sgn}(n) \sqrt{\Lambda_n \omega_n} \hat{a}_n$ and we have chosen the asymmetric integral bounds of Eq. (28). The subscript $n \times m \times l \times k$ below the braces indicates multiplication, e.g., $\{\Lambda\omega\}_{n \times m} = \Lambda_n \omega_n \Lambda_m \omega_m$. $\Delta k_{nmkl} \equiv \pm k_n \pm k_m \pm k_l \pm k_k$ for the different terms resulting from expansion of the brackets. A plus (minus) sign refers to a corresponding annihilation (creation) operator, e.g., the term $\hat{a}_n^\dagger \hat{a}_m^\dagger \hat{a}_l^\dagger \hat{a}_k$ corresponds to $\Delta k_{nmkl} = k_n - k_m - k_l + k_k$. Similarly, $\Delta\omega_{nmkl} \equiv \pm\omega_n \pm \omega_m \pm \omega_l \pm \omega_k$.

This is the main result of this work. This Hamiltonian describes the full quantum behavior of Josephson TWPAs up to first nonlinear order. However, we will point out two remaining issues and how they may be dealt with. First, this Hamiltonian does not conserve energy *a priori*. For energy conservation, $\Delta\omega_{nmkl}$ must equal zero, which does not necessarily follow from the equation. At this point, we can demand energy conservation by considering only interactions between modes for which $\Delta\omega_{nmkl} = 0$. However, one could also reason that $\Delta\omega_{nmkl} \neq 0$ adds to the phase-mismatching term $\Delta\Omega$ in Eq. (4), which at small magnitudes (compared to ω_n) already greatly reduces the gain of the amplifier. From this argument, it follows that $\Delta\omega_{nmkl} \approx 0$. If the latter is the case, and $\Delta\omega_{nmkl}$ is not strictly zero, this might lead to line broadening of the modes. For now, we will assume strict energy conservation and demand $\Delta\omega_{nmkl} = 0$.

A second problem in the expression above is the explicit dependence of the mixing term on the quantization length. This dependence arises both as a consequence of the dispersion in the line as well as an intrinsic dependence of the mixing term that scales as l_q^{-2} . Due to dispersion, Δk_{nmkl} and $\Delta\omega_{nmkl}$ cannot equal zero simultaneously, which introduces

an l_q dependence if we demand $\Delta\omega_{nmkl} = 0$ for the interacting modes.

Partly, the two contributions to the quantization-length dependence of the mixing term cancel each other as

$$\frac{-i(e^{i\Delta k_{nmkl} l_q} - 1)}{\Delta k_{nmkl}} = l_q \left\{ 1 - \frac{i}{2} \Delta k_{nmkl} l_q + O[(\Delta k_{nmkl} l_q)^2] \right\}. \quad (52)$$

The intrinsic dependence of the mixing term on l_q can be further resolved by introducing a classical pump; see Sec. V. To deal with the remaining l_q dependence due to dispersion, we can assume that the dispersion effects are small enough such that $\Delta k_{nmkl} \approx 0$, while $\Delta\omega_{nmkl} = 0$. We will make this assumption in the following sections. The problem is also resolved considering a transmission line of length l_q for quantization, of which just a part contains Josephson junctions and using continuous-mode quantization [21].

V. IMPLEMENTATIONS

Using Eq. (51), one can analyze the different implementations of an amplifier. In this section, we will study the nondegenerate amplifier with degenerate pump in detail, the same amplifier implementation that was studied classically in Sec. III. Treating the pump as a classical mode, we will solve the problem of the explicit appearance of the quantization length in the mixing coupling constants in Sec. V A. The section ends with a short discussion of other implementations of 4WM amplifiers in Sec. V B.

A. The nondegenerate parametric amplifier with undepleted degenerate classical pump: Quantum theory

As noted in Sec. III, for the nondegenerate parametric amplifier with degenerate pump, it is assumed that only three modes, i.e., the pump, signal, and idler, play a role. Then, from Eq. (51), we can determine the interaction Hamiltonian of the amplifier as

$$\begin{aligned} \hat{H}_{\text{int}} = & \sum_{n,m=p,s,i} \hbar \xi_{nm} \left(\hat{a}_n^\dagger \hat{a}_n \hat{a}_m^\dagger \hat{a}_m + \frac{c_m}{2} \hat{a}_n^\dagger \hat{a}_n + \frac{c_n}{2} \hat{a}_m^\dagger \hat{a}_m \right) \\ & + \hbar (\chi \hat{a}_p \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger + \text{H.c.}) \\ & + \sum_{n,m=p,s,i} \frac{c_n c_m \hbar \xi_{nm}}{2(2 - \delta_{nm})} - \frac{c_n c_m \hbar \xi_{nm} |_{\Lambda=0} (8 - 2\delta_{nm})}{3} \\ & \times \left[\frac{\omega_n}{\omega_m} (\Lambda_m - 1) + \frac{\omega_m}{\omega_n} (\Lambda_n - 1) \right], \end{aligned} \quad (53)$$

taking into account the commutation relations explicitly as $c_n \equiv [\hat{a}_n, \hat{a}_n^\dagger] = 1$. The first two lines in this equation determine the dynamics of the amplifier, whereas the third and fourth line represent the added zero-point energy. The coupling constants are found to be

$$\xi_{nm} = \frac{\hbar \Lambda_n \omega_n \Lambda_m \omega_m}{16 I_c^2 \mathcal{L}_{J,0} l_q} (2 - \delta_{nm}) (1 + \Lambda_{\xi_{nm}}), \quad (54)$$

$$\chi = \frac{\hbar \Lambda_p \omega_p \sqrt{\Lambda_s \omega_s \Lambda_i \omega_i}}{8 I_c^2 \mathcal{L}_{J,0} l_q} (1 + \Lambda_\chi), \quad (55)$$

where

$$\Lambda_{\xi_{nm}} \equiv \frac{2}{3} \left(\frac{\Lambda_n}{\Lambda_m} + \frac{\Lambda_m}{\Lambda_n} - 2 \right), \quad (56)$$

$$\begin{aligned} \Lambda_\chi \equiv & \frac{L_{J,0} C_J}{6} [\omega_p \omega_s (-2\Lambda_p + 5\Lambda_s - 3\Lambda_i) \\ & + \omega_p \omega_i (-2\Lambda_p - 3\Lambda_s + 5\Lambda_i) \\ & + \omega_s \omega_i (4\Lambda_p - 2\Lambda_s - 2\Lambda_i)], \end{aligned} \quad (57)$$

and $\xi_{nm}|_{\Lambda=0}$ implies that ξ_{nm} should be used without the contribution of $\Lambda_{\xi_{nm}}$.

In Eq. (53), the $\xi_{n=m}$ term represents the self modulation and the $\xi_{n \neq m}$ terms represent the cross modulation. The term in the equation with coupling constant χ is where the magic happens. This term represents the real amplification process in which two pump photons are scattered into a signal and an idler photon.

For a parametric amplifier to work effectively, the device must be driven into its nonlinear regime, which is generally achieved by applying a pump current close to the critical current of the device in addition to the much weaker signal current. In this case, we can approximate the Hamiltonian to second order in $\hat{a}_{s,i}^{(\dagger)}$. Moreover, as $\hbar \xi_{nm} \ll \hbar \omega_n$, we can neglect the terms resulting from the commutation relations as well. Hence, to a good approximation,

$$\begin{aligned} \hat{H}_{\text{TWPA}} \approx & \sum_{n=p,s,i} \hbar \omega_n \hat{a}_n^\dagger \hat{a}_n + \sum_{n=p,s,i} \hbar \xi_{pn} \hat{a}_p^\dagger \hat{a}_p \hat{a}_n^\dagger \hat{a}_n \\ & + \hbar (\chi \hat{a}_p \hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger + \text{H.c.}), \end{aligned} \quad (58)$$

where we have neglected the constant zero-point energy, which does not influence the dynamics of the amplifier, and we introduced

$$\xi_{pn} = \frac{\hbar \Lambda_p \omega_p \Lambda_n \omega_n}{16 I_c^2 \mathcal{L}_{J,0} l_q} (4 - 3\delta_{pn}) (1 + \Lambda_{\xi_{pn}}). \quad (59)$$

Here, the factor $4 - 3\delta_{nm}$ (instead of $2 - \delta_{nm}$) arises from converting the double sum into a single sum. Notably, the coupling constants corresponding to the self and cross modulation differ by a numerical factor of 4. In Ref. [21], this factor was found to be 2. However, as discussed below, both this work and Ref. [21] find identical operator equations of motion. In Eq. (58), χ is still given by Eq. (55).

Furthermore, we can make the approximation that the pump can be treated as a classical mode and we can replace the corresponding operators with amplitudes. In accordance with the classical treatment of the problem in Sec. III, we will choose the flux, $\hat{\Phi}_{C_g} = \int \hat{V}_{C_g} dt$, for the amplitude. Upon comparing this expression with its classical analog (cf. [25]), given by Eq. (7), we find

$$\hat{a}_p \mapsto -i \sqrt{\frac{\omega_p C_g l_q}{2\hbar}} A_p. \quad (60)$$

Then, for the signal and idler mode, which are still treated quantum mechanically, we find the classical pump Hamiltonian,

$$\hat{H}_{\text{TWPA}}^{(\text{CP})} \approx \sum_{n=s,i} \hbar (\omega_n + \xi'_n |A_p|^2) \hat{a}_n^\dagger \hat{a}_n - \hbar (\chi' A_p^2 \hat{a}_s^\dagger \hat{a}_i^\dagger + \text{H.c.}), \quad (61)$$

with

$$\begin{aligned} \xi'_n &= \frac{\Lambda_p \omega_p^2 \Lambda_n \omega_n}{32 I_c^2 \mathcal{L}_{J,0} Z_{c,p} v_{\text{ph},p}} (4 - 3\delta_{pn}) (1 + \Lambda_{\xi_{pn}}) \\ &= \frac{k_p^2 \Lambda_n \omega_n}{32 I_c^2 \mathcal{L}_{J,0}^2} (4 - 3\delta_{pn}) (1 + \Lambda_{\xi_{pn}}), \\ \chi' &= \frac{\Lambda_p \omega_p^2 \sqrt{\Lambda_s \omega_s \Lambda_i \omega_i}}{16 I_c^2 \mathcal{L}_{J,0} Z_{c,p} v_{\text{ph},p}} (1 + \Lambda_\chi) \\ &= \frac{k_p^2 \sqrt{\Lambda_s \omega_s \Lambda_i \omega_i}}{16 I_c^2 \mathcal{L}_{J,0}^2} (1 + \Lambda_\chi). \end{aligned} \quad (62)$$

Although ξ'_p does not appear in the classical pump Hamiltonian, it is still defined here for future reference.

Generalization of these equations to the case in which resonators are added for dispersion engineering is straightforward. Due to our results in Sec. IV C, this is as easy as making the substitution $C_g \mapsto 1/\omega_n Z_{c,\text{eff}}$ (implicit in $Z_{c,n}$, $v_{\text{ph},n}$, k_n , and Λ_n), as discussed in Sec. III.

To calculate the gain predicted by a parametric amplifier from the quantum theory, we calculate the Heisenberg equations of motion of the operators. By substituting Eq. (58) as the Hamiltonian and approximating the pump as a classical mode, this yields

$$\frac{\partial A_p}{\partial t} = -i(\omega_p + 2\xi'_p |A_p|^2 + c_p \xi_{pp}) A_p + 2i\chi^* A_p^* \hat{a}_s \hat{a}_i, \quad (64)$$

$$\frac{\partial \hat{a}_{s(i)}}{\partial t} = -i(\omega_{s(i)} + \xi'_{s(i)} |A_p|^2) \hat{a}_{s(i)} + i\chi' A_p^2 \hat{a}_{i(s)}^\dagger, \quad (65)$$

where again we showed the effect of the commutation relations explicitly. However, again we can neglect ξ_{pp} since $\xi_{pp} \ll \xi_p' |A_p|^2$. Under the undepleted pump approximation, we can neglect the last term in Eq. (64) as well and solve for the pump amplitude directly, as in the classical case. Hence,

in the co-rotating frame introduced in Sec. III,

$$\frac{\partial \hat{a}_{s(i)}}{\partial t} = i\chi' |A_{p,0}|^2 \hat{a}_{i(s)}^\dagger e^{-i\Delta\Omega t}. \quad (66)$$

Thus we find, similar to the classical theory,

$$\hat{a}_{s(i)} = \left[\hat{a}_{s(i),0} \left(\cosh g_t t + \frac{i\Delta\Omega}{2g_t} \sinh g_t t \right) + \frac{i\chi' |A_{p,0}|^2}{g_t} \hat{a}_{i(s),0}^\dagger \sinh g_t t \right] e^{-i\Delta\Omega t/2}, \quad (67)$$

with

$$\Delta\Omega = 2(\omega_p + 2\xi_p' |A_{p,0}|^2) - (\omega_s + \xi_s' |A_{p,0}|^2) - (\omega_i + \xi_i' |A_{p,0}|^2) = (4\xi_p' - \xi_s' - \xi_i') |A_{p,0}|^2, \quad (68)$$

$$g_t = \sqrt{|\chi'|^2 |A_{p,0}|^4 - (\Delta\Omega/2)^2}. \quad (69)$$

Then, if the state spends a time t_T in the TWPA,

$$G_s^q = \frac{\langle \hat{a}_s^\dagger \hat{a}_s \rangle}{\langle \hat{a}_{s,0}^\dagger \hat{a}_{s,0} \rangle} = \left| \cosh g_t t_T + \frac{i\Delta\Omega}{2g_t} \sinh g_t t_T \right|^2 + \frac{\langle \hat{a}_{i,0}^\dagger \hat{a}_{i,0} \rangle + 1}{\langle \hat{a}_{s,0}^\dagger \hat{a}_{s,0} \rangle} \left| \frac{\chi' |A_{p,0}|^2}{g_t} \sinh g_t t_T \right|^2 + \frac{1}{\langle \hat{a}_{s,0}^\dagger \hat{a}_{s,0} \rangle} \left(-i \langle \hat{a}_{s,0} \hat{a}_{i,0} \rangle \left(\cosh g_t t_T + \frac{i\Delta\Omega}{2g_t} \sinh g_t t_T \right) \frac{\chi'^* |A_{p,0}|^2}{g_t^*} \sinh g_t^* t_T + \text{c.c.} \right) \quad (70)$$

in which the term on the second line yields zero if the signal or the idler is initially in a number state.

One can also calculate the photon-number distribution in the output of a parametric amplifier from the theory in the limit of a classical undepleted pump. To this end, we calculate the evolution of the state vector from Eq. (21) in the interaction picture,

$$|\psi_I(t)\rangle = e^{-i\hat{H}_{\text{TWPA}}^{(\text{CP,rot})} t/\hbar} |\psi_I(0)\rangle, \quad (71)$$

where

$$\hat{H}_{\text{TWPA}}^{(\text{CP,rot})} = -\hbar(\chi' |A_p|^2 \hat{a}_s^\dagger \hat{a}_i^\dagger e^{-i\Delta\Omega t} + \text{H.c.}) \quad (72)$$

is the classical pump Hamiltonian rewritten in the co-rotating frame. Assuming $\Delta\Omega = 0$ and $\chi' \in \text{Re}$, we can rewrite the propagator in Eq. (71) using an ordering theorem [28],

$$e^{-i\hat{H}_{\text{TWPA}}^{(\text{CP,rot})} t/\hbar} \Big|_{\Delta\Omega=0} = e^{i \tanh(\kappa) \hat{a}_s^\dagger \hat{a}_i^\dagger} e^{-\ln(\cosh(\kappa)) (1 + \hat{a}_s^\dagger \hat{a}_s + \hat{a}_i^\dagger \hat{a}_i)} e^{i \tanh(\kappa) \hat{a}_s \hat{a}_i}, \quad (73)$$

where the amplification $\kappa \equiv \chi' |A_p|^2 t$.

For a single-photon input state $|1\rangle_s |0\rangle_i$, we calculate the output state as

$$|\psi_I(t)\rangle = \sum_{n=0}^{\infty} \frac{[i \tanh(\kappa)]^n}{\cosh^2(\kappa)} \sqrt{n+1} |n+1\rangle_s |n\rangle_i, \quad (74)$$

from which we easily compute that the probability of finding N signal photons in the output state equals

$$\text{Pr}(n_s = N) = |\langle N | \psi_I(t) \rangle|^2 = \frac{\tanh(\kappa)^{2(N-1)}}{\cosh^4(\kappa)} N. \quad (75)$$

For a coherent state $|\alpha\rangle_s |0\rangle_i$, we find

$$|\psi_I(t)\rangle = e^{-|\alpha|^2/2} \sum_{n,m=0}^{\infty} \frac{[i \tanh(\kappa)]^m}{[\cosh(\kappa)]^{1+n}} \frac{\alpha^n}{\sqrt{n!}} \sqrt{\binom{n+m}{n}} |n+m\rangle_s |m\rangle_i \quad (76)$$

and

$$\text{Pr}(n_s = N) = e^{-|\alpha|^2} \sum_{n=0}^N \frac{[\tanh(\kappa)]^{2(N-n)}}{[\cosh(\kappa)]^{2(1+n)}} \frac{|\alpha|^{2n}}{n!} \binom{N}{n}. \quad (77)$$

These probabilities are visualised in Fig. 10, in which can be observed how the photon-number distribution spreads out as a function of the TWPA gain.

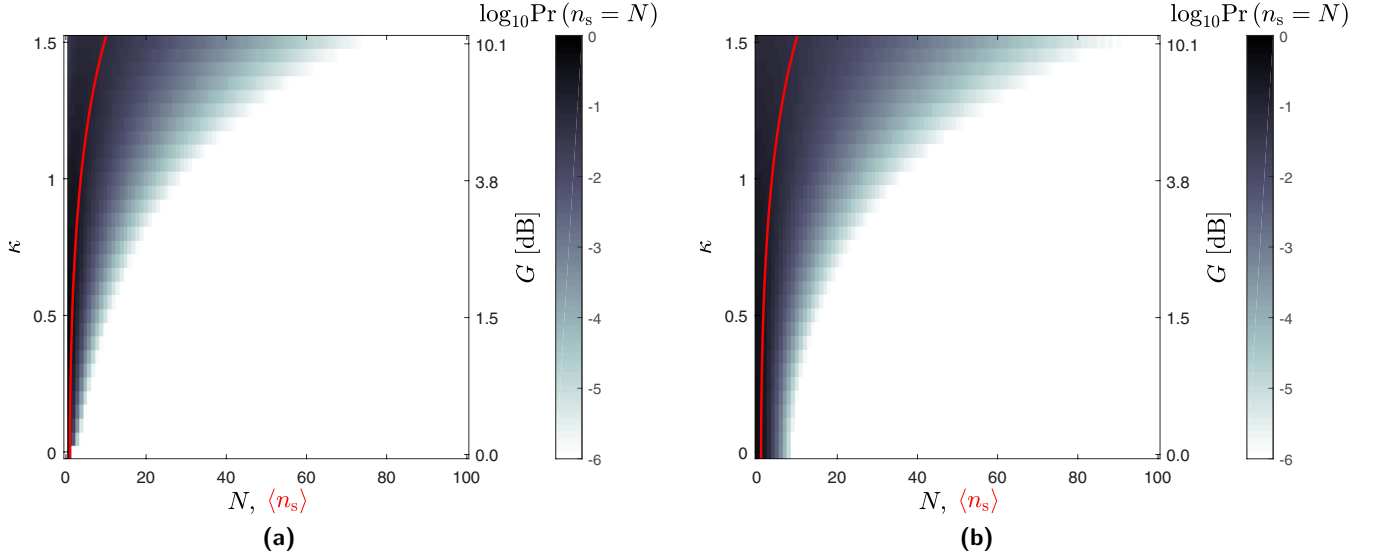


FIG. 10. Photon-number distribution in the output state of a TWPA for (a) a single-photon state and (b) a coherent state $\alpha = 1$ as a function of amplification $\kappa = \chi' |A_p|^2 t$ (left axis) or, equivalently, gain G (right axis), assuming $\Delta\Omega = 0$. The color bar, which is cut off at $\text{Pr} < 10^{-6}$, indicates the probability of finding N photons in the output state. The average number of photons in the output state is indicated in red.

B. Other implementations

Although it is most trivial to use the TWPA based on Josephson junctions as a nondegenerate amplifier with degenerate pump, there are other implementation schemes, which will be discussed briefly in this section. As in the last section, we assume that only the pump(s) is (are) a source of amplification.

(i) Nondegenerate pump, signal, and idler: Instead of feeding the TWPA with a single pump tone, we can apply two pump tones at different frequencies. In this case, we will find, in first order, three (primary) idler tones, one of which will arise as a result of mixing with the two different pump tones, $\omega_i = \omega_{p1} + \omega_{p2} - \omega_s$, which has been considered theoretically in [18]. Two other idler tones result from each pump working in a degenerate regime, for which $\omega_{i(2)} = 2\omega_{p(2)} - \omega_s$. However, in turn, each of these idler tones will be the source of (secondary) idler tones at yet different frequencies, such that we end up with a whole spectrum of idlers. A method to prevent this from happening is to engineer stop bands into the transmission line, such that only specific modes will transmit. However, the generation of many idler tones complicates the analysis for such devices. Still, in general, it can be said that such a device will work in the phase-preserving regime.

(ii) Nondegenerate pump, signal only: If we apply two pump tones at different frequencies, we can engineer a quasi-phase-sensitive amplifier, if the signal frequency is chosen at $2\omega_s = \omega_{p1} + \omega_{p2}$. It will work only in a quasi-phase-sensitive regime because each pump will also cause a primary idler tone to arise from a phase-preserving interaction with the signal. In turn, this again gives rise to a whole set of secondary idler tones. If the transmission line is engineered such that the primary idlers at $\omega_{i(2)} = 2\omega_{p(2)} - \omega_s$ fall into stop bands, the device will work as a real phase-sensitive device.

(iii) Pump, signal, and idler with dc current: If we put the TWPA between two bias Ts, we can add a dc current to the device. In this manner, we can use the device as quasi 3WM,

as has been demonstrated in Ref. [14]. Adding the current, we should insert $\Phi_J \mapsto \Phi_J + \Phi_{DC}$ into the Hamiltonian in Eq. (47), which yields, among others, a term proportional to $\hat{\Phi}_J^3 \Phi_{DC}$. Continuing the analysis, this yields a term proportional to $\Phi_{DC}(\hat{a}_p \hat{a}_s^\dagger \hat{a}_i^\dagger + \text{H.c.})$ in the Hamiltonian, which is a 3WM term. Of course, the full Hamiltonian will contain 4WM terms since the pump acts as a separate source as well. However, choosing the amplitude of the dc current to be large with respect to the amplitude of the pump tone, the latter terms can be made small. This implementation of the amplifier is phase preserving in general; however, it can be used as a phase-sensitive device as well by choosing $\omega_p = 2\omega_s$.

VI. TERMINOLOGY: REVISITED

This work started out with an introduction of the terminology used for parametric amplifiers. After this extensive excursion into 4WM TWPA theory, we have reached the point that we can fully understand the Hamiltonian in Eq. (4). The only difference between that Hamiltonian and our result in Eq. (58) is that the former uses a co-rotating frame, whereas the latter does not. If we cast Eq. (58) in a co-rotating frame, we can identify

$$\chi|_{\text{eq. (4)}} = \chi|_{\text{eq. (55)}}, \quad (78)$$

$$\Delta\Omega|_{\text{eq. (4)}} = (4\xi_{pp} - \xi_{ps} - \xi_{pi})\hat{a}_p^\dagger \hat{a}_p, \quad (79)$$

or, if we absorb a classical pump into the coupling constants,

$$\tilde{\chi}|_{\text{eq. (5)}} = -\chi'|_{\text{eq. (63)}} |A_p|^2, \quad (80)$$

$$\Delta\Omega|_{\text{eq. (5)}} = (4\xi'_p - \xi'_s - \xi'_i) |A_p|^2. \quad (81)$$

VII. MARRYING THE QUANTUM AND CLASSICAL THEORIES

Although the classical theory of the nonlinear wave equation and the quantum evolution described by Schrödinger's

equation seem to be a world apart, in fact the two descriptions can be mapped onto one another. This will be done in this section.

The marriage between the two theories runs via the Heisenberg equations of motion of the operators and the connection between the classical mode amplitudes on the one hand and creation and annihilation operators on the other. Starting from the Heisenberg equations of motion with a classical undepleted pump and neglecting small terms [cf. Eqs. (64) and (65)],

$$\frac{\partial A_p}{\partial t} = -i(\omega_p + 2\xi'_p |A_p|^2)A_p, \quad (82)$$

$$\frac{\partial \hat{a}_{s(i)}}{\partial t} = -i(\omega_{s(i)} + \xi'_{s(i)} |A_p|^2) \hat{a}_{s(i)} + i\chi' A_p^2 \hat{a}_{i(s)}^\dagger. \quad (83)$$

As the classical coupled-mode equations are defined in space, whereas the Heisenberg equations of motion are equations in time, the first step is to change coordinates from time to space, yielding the spatial Heisenberg equations of motion. From Eq. (25), we can infer that $-\omega_n \partial t = k_n \partial z$ by taking both the derivative to time and to space. Therefore,

$$\frac{\partial A_p}{\partial z} = i \left(k_p + 2 \frac{k_p \xi'_p}{\omega_p} |A_p|^2 \right) A_p, \quad (84)$$

$$\frac{\partial \hat{a}_{s(i)}}{\partial z} = i \left(k_{s(i)} + \frac{k_{s(i)} \xi'_{s(i)}}{\omega_{s(i)}} |A_p|^2 \right) \hat{a}_{s(i)} - i \frac{k_{s(i)} \chi'}{\omega_{s(i)}} A_p^2 \hat{a}_{i(s)}^\dagger. \quad (85)$$

As a last step, we change the operators back into the classical amplitudes of the modes by virtue of Eq. (60). By substitution, we arrive at the classicalized spatial Heisenberg equations of motion,

$$\frac{\partial A_p}{\partial z} = i(k_p + \Xi_p^q |A_p|^2)A_p, \quad (86)$$

$$\frac{\partial A_{s(i)}}{\partial z} = i(k_{s(i)} + \Xi_{s(i)}^q |A_p|^2)A_{s(i)} + iX_{s(i)}^q A_p^2 A_{i(s)}^*, \quad (87)$$

where

$$\Xi_n^q = \frac{k_n \xi'_n}{\omega_n} (2 - \delta_{pn}) = \frac{a^4 k_p^2 k_n^3 (2 - \delta_{pn})}{16 C_g I_c^2 L_{j,0}^3 \omega_n^2} (1 + \Lambda_{\xi_{pn}}), \quad (88)$$

$$X_{s(i)}^q = \frac{k_{s(i)} \chi'}{\omega_{s(i)}} \sqrt{\frac{\omega_{i(s)}}{\omega_{s(i)}}} = \frac{a^4 k_p^2 k_s k_i}{16 C_g I_c^2 L_{j,0}^3 \omega_{s(i)}^2} (1 + \Lambda_\chi). \quad (89)$$

This set of equations is identical to Eqs. (8) and (9) after mapping $A_n \mapsto A_n e^{ik_n z}$ and removing small terms up to some details: Ξ_n^q contains a factor $(1 + \Lambda_{\xi_{pn}})$ which Ξ_n does not. Additionally, the factor $(1 - \Delta k/k_{s(i)})$ in $X_{s(i)}^q$ has been replaced by $(1 + \Lambda_\chi)$ in $X_{s(i)}^q$. The contributions $\Lambda_{\xi_{pn}}$ and Λ_χ in these factors result from taking into account the nonlinear Josephson inductance in Λ_n explicitly. The contribution scaling as $\Delta k/k_n$ in X_n , however, cannot arise from the quantum theory since it would need to arise from a coupling constant χ in the Hamiltonian, which is somehow different for the signal and idler mode. Such a difference is not permitted by the quantum theory. However, if we would not have neglected the contribution to χ , χ' and thus X_n^q due to dispersion, those coupling constants would have been multiplied, up to first order in Δk , by $(1 - i\Delta k l_q/2)$; see

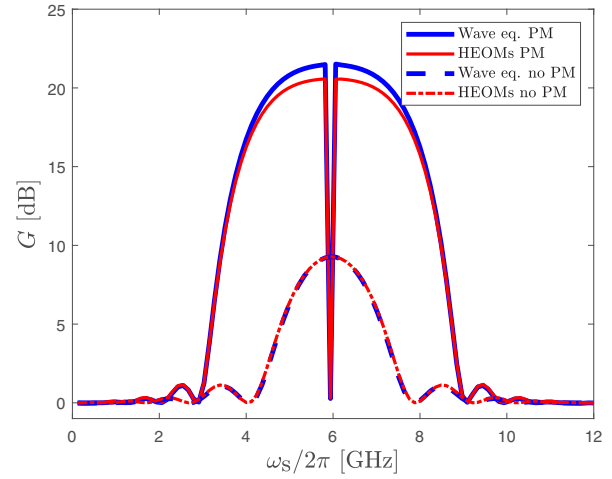


FIG. 11. Comparison of the predicted power gain from the coupled-mode equations derived from the classical nonlinear wave equation and the classicalized spatial Heisenberg equations of motion from the quantum theory. The comparison is made for the case with and without phase matching using the same parameters as in Fig. 6. The difference in gain with phase matching is due to the $\Delta k/k_n$ term present in the classical coupled-mode equation coupling constant X_n , but absent in the classicalized spatial Heisenberg equation of motion coupling constant X_n^q .

Eq. (52). This term resembles the factor $(1 - \Delta k/k_{s(i)})$ from the classical theory, although it depends on the unphysical quantization length. Still, the predictions of gain of both the classical coupled-mode equations and the classicalized spatial Heisenberg equations of motion agree well. As can be observed in Fig. 11, if we do not add dispersion engineering, there is hardly a difference in the predicted gain, whereas only the maximum gain differs in both approaches if we add dispersion engineering. This is solely due to the factor $\Delta k/k_{s(i)}$ in $X_{s(i)}$.

VIII. VALIDITY

In the presented theory, we made several assumptions. First, we only took the first nonlinear contribution of the Josephson energy into account and, second, it was assumed that the pump can be treated as a classical mode, which is undepleted. In this section, the implications of these assumptions will be presented.

The theory presented above is derived from a first-order Taylor expansion of the Josephson energy. This implies that at a certain magnitude of the flux through the junction, the theory becomes invalid as higher-order terms need to be taken into account. To estimate this flux, we inspect once more the Josephson energy,

$$U_J = I_c \varphi_0 \left[1 - \cos \left(\frac{\Delta \Phi_J}{\varphi_0} \right) \right] = I_c \varphi_0 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \left(\frac{\Delta \Phi_J}{\varphi_0} \right)^{2n}. \quad (90)$$

Thus, we find that the second-order ($n=3$) nonlinear effects are approximately $4!(\Delta \Phi_{J,p}/\varphi_0)^2/6!$ smaller than the lowest-order nonlinear terms. Hence, if we require that the contribution to the energy of the second-order terms is less

than 5% of the energy contribution of the first-order terms, we find that our theory breaks down at $\Delta\Phi_{J,p}/\varphi_0 \approx 1.2$ (or $I_p/I_c \approx 0.78$). The dominant second-order amplification term causes two secondary idler modes to appear at $\omega_{(i)'} = 4\omega_p - \omega_{(s)(i)}$, implying the general form of the Hamiltonian in Eqs. (4) and (61) becomes invalid. Moreover, the second-order terms cause additional modulation effects. It is only in the third-order nonlinear terms that $(\hat{a}_s^\dagger \hat{a}_i^\dagger)^2$ contributions start to play a role. Furthermore, additional secondary idlers are generated and the modulation effects are further increased. These terms have a maximal energy contribution of approximately $4!(\Delta\Phi_{J,p}/\varphi_0)^4/8! \approx 4 \times 10^{-3}$ at the critical flux. Therefore, they can be neglected for all practical purposes.

The undepleted pump approximation breaks down if the flux of the signal and idler photons in the amplifier becomes close to the flux of pump photons. Typically, this happens when $I_{s,0} \geq I_{p,0}/10$ [14,18], in which the case the full coupled-mode equations of Eqs. (8) and (9) need to be considered to calculate the output amplitudes. Alternatively, for the quantum case, the full Hamiltonian of Eq. (53) is to be considered to evaluate the evolution of the quantum state.

IX. CONCLUSIONS

After an introduction to the relevant terminology and the classical theory of the coupled-mode equations of Josephson traveling-wave parametric amplifiers, we derived the mesoscopic quantum Hamiltonian up to first nonlinear order describing the process using discrete-mode operators. We found that such a description is possible, even when taking into account dispersion effects in the transmission line, and showed that the classical coupled-mode equations can be derived from this Hamiltonian. Such a Hamiltonian description of TWPAs is necessary for treating the amplifier as a quantum device. From our theory, we obtain expressions for the coupling constants and can straightforwardly calculate averages, standard deviations, and higher-order moments of

measurement operators taking into account the commutation relations between operators explicitly. In contrast, from a classical theory, only average values follow and the effects of noncommuting operators are neglected.

In the derivation, however, there are a few remaining issues. First, it was found that in the nonlinear terms of the Hamiltonian, energy and momentum conservation could not be fulfilled simultaneously. Furthermore, the nonphysical quantization length is inherent to the theory. The latter can be solved solely under the approximation of a classical undepleted pump. However, the matter can also be resolved if one derives the Hamiltonian using continuous modes and a transmission line, of which only a part contains the nonlinearity. For the concurrent conservation of energy and momentum, we have not been able to find a satisfactory solution.

We found that our Hamiltonian, and therefore the coupled-mode equations, are valid to pump currents up to approximately $0.78I_c$. For larger pump currents, more nonlinear orders have to be taken into account, for which the same recipe can be followed as shown in this paper. The same recipe can also be followed to derive the Hamiltonian for TWPAs that have another source of nonlinear behavior, such as kinetic inductance.

To make the theories more applicable to experimental realizations of TWPAs, we suggest that the theories can be expanded by taking into account losses as well as reflections within the device, and reflections at the boundaries of the device at which it is coupled to its environment.

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