Inverse Jacobian and related topics for certain superelliptic curves

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Moduli of abelian varieties with generalized CM-type 3

The goal for this chapter is to characterize the Jacobians of CPQ curves among the principally polarized abelian varieties of dimension 6, as an analogous result to Proposition 1.4.1 for Picard curves, which said that all simple principally polarized abelian threefolds over an algebraically closed field with order-3 automorphisms are Jacobians of Picard curves.

But when considering the case of CPQ curves, we have to take into account that not all principally polarized abelian varieties of dimension 6 are Jacobians of curves. Fortunately, the existence of the automorphism of CPQ curves given by $(x, y) \mapsto (x, \exp(2\pi i/5)y)$ and the corresponding automorphism on the Jacobian set some conditions on the structure of the Jacobian. We will show that these conditions are enough to determine a moduli space with the same dimension as the family of CPQ curves, and that said moduli space is connected. This will allow us to give a result analogous to Proposition 1.4.1, see Theorem 3.5.3.

In Sections 3.1 and 3.2 we introduce a generalization of the classical CM-theory due to Shimura to define the moduli space of principally polarized abelian varieties with given *generalized CM-type*. We follow [39], and also Birkenhake-Lange [2, Section 9.6].

We apply this theory in Section 3.3 to study the Jacobians of CPQ curves. We explicitly construct the complex torus and polarization, and study the structure of $\mathbb{Z}[\rho_*] \subseteq \mathrm{End}(J(C))$ for ρ_* the automorphism of the Jacobian induced by $\rho(x, y) = (x, z_5 y)$ with $z_5 = \exp(2\pi i/5)$. We show that the moduli space of principally polarized abelian varieties with the generalized CM-type $\mathfrak{Z}$ induced by ρ_* has dimension 2, as does the family of CPQ curves.

In Section 3.4 we introduce the concept of *polarized $\mathcal{O}_K$-lattice* and explain how the equivalence classes of certain polarized $\mathcal{O}_K$-lattices relate to the connected components of the moduli space of principally polarized abelian varieties with generalized CM-type $\mathfrak{Z}$. Using this relation we then prove, among other things, that the moduli space given in Section 3.3 is connected.

Finally, in Section 3.5 we put all the pieces together to prove the result analogous to Proposition 1.4.1, see Theorem 3.5.3.

3.1 CM-fields and m-CM-types

A *CM-field* is a totally imaginary quadratic extension K of a totally real number field K^+. The non-trivial element κ of $\mathrm{Aut}(K/K^+)$ satisfies $\phi \circ \kappa = \bar{\cdot} \circ \phi$ for every embedding $\phi : K \hookrightarrow \mathbb{C}$, where $\bar{\cdot}$ stands for the complex conjugation in $\mathbb{C}$. We call κ *complex conjugation* and denote it also by $\bar{\cdot}$.

Let K be a CM-field of degree $2e$. An *m-CM-type* of K is a multiset Ψ whose elements are elements of $\mathrm{Hom}(K, \mathbb{C})$ and such that for every homomorphism $\phi : K \to \mathbb{C}$ we have $\mathrm{mult}_\Psi(\phi) + \mathrm{mult}_\Psi(\overline{\phi}) = m$. We get $\#\Psi = em$. To it, we associate the representation

$$\rho_\Psi = \bigoplus_{\phi \in \Psi} \phi$$

of dimension em over $\mathbb{C}$.

Definition 3.1.1. With the notation above, a *polarized abelian variety with m-CM-type* (K, Ψ) is a triple (X, E, ι) with:

- ▷ $X \cong \mathbb{C}^{em}/\Lambda$ a complex torus of dimension em,
- ▷ E a Riemann form, and
- ▷ $\iota : K \hookrightarrow \mathrm{End}(X) \otimes \mathbb{Q}$ an embedding such that
  - – the analytic representation $\rho_a \circ \iota$ and the representation ρ_Ψ are equivalent, and
  - – the Rosati involution on $\mathrm{End}(X) \otimes \mathbb{Q}$ with respect to the polarization given by the Riemann form E extends the complex conjugation on K via ι.

Two polarized abelian varieties (X, E, ι) and (X', E', ι') with m-CM-type (K, Ψ) are *isomorphic* if there exists an isomorphism $f : X \to X'$ that satisfies $f^*E' = E$ and $f \circ \iota(a) = \iota'(a) \circ f$ for all $a \in K$.

Choose $\Phi = (\phi_1, \ldots, \phi_e)$ a sequence of e embeddings $K \to \mathbb{C}$ such that $\{\phi_1, \overline{\phi_1}, \ldots, \phi_e, \overline{\phi_e}\}$ is the set of all $2e$ embeddings. Then, by abuse of notation, an m-CM-type is a list $(\mathbf{r}, \mathbf{s}) = ((r_1, \ldots, r_e), (s_1, \ldots, s_e))$ of non-negative integers with $r_i + s_i = m$ for all $i = 1, \ldots, e$ via taking $r_i = \mathrm{mult}(\phi_i)$ and $s_i = \mathrm{mult}(\overline{\phi_i})$. In this case we denote the m-CM-type by $(\mathbf{r}, \mathbf{s})$, and the associated representation is given by

$$\rho_{\mathbf{r},\mathbf{s}}(a) = \mathrm{diag}(\phi_1(a)\mathbf{1}_{r_1}, \overline{\phi_1}(a)\mathbf{1}_{s_1}, \ldots, \phi_e(a)\mathbf{1}_{r_e}, \overline{\phi_e}(a)\mathbf{1}_{s_e}). \tag{3.1}$$

The choice of Φ also determines an embedding $\jmath : (K \otimes_{\mathbb{Q}} \mathbb{R})^m \to \mathbb{C}^{2em}$, given by

$$\mathbf{a} \mapsto \jmath(\mathbf{a}) = \begin{pmatrix} \phi_1(\mathbf{a}) \\ \overline{\phi_1}(\mathbf{a}) \\ \vdots \\ \phi_e(\mathbf{a}) \\ \overline{\phi_e}(\mathbf{a}) \end{pmatrix},$$

and it allows us to define a parametrization of the family of polarized abelian varieties with m-CM-type $(\mathbf{r}, \mathbf{s})$ as follows:

Let $\mathcal{H}_{r,s}$ be the set of matrices $Z \in \mathbb{C}^{r\times s}$ such that $\mathbf{1}_s - {}^t\overline{Z}Z$ is positive definite, which we write as $\mathbf{1}_s - {}^t\overline{Z}Z > 0$. Let $\Upsilon(\mathbf{r}, \mathbf{s})$ be the set of pairs $(\mathcal{M}, T)$ such that:

- ▹ $\mathcal{M}$ is a free $\mathbb{Z}$-submodule of K^m of rank $2em$,
- ▹ T is an $m \times m$ antihermitian matrix over K,
- ▹ the alternating bilinear form $\mathcal{M} \times \mathcal{M} \to \mathbb{Q}$ given by $(a, b) \mapsto \mathrm{tr}_{K/\mathbb{Q}}({}^t a T \overline{b})$ is integer-valued,
- ▹ T has *signature* $(\mathbf{r}, \mathbf{s})$, that is, for every $\nu = 1, \ldots, e$ there exists an invertible matrix $W_\nu \in \mathbb{C}^{m\times m}$ that satisfies

$$\phi_\nu(T) = {}^t\overline{W_\nu} \begin{pmatrix} i\mathbf{1}_{r_\nu} & 0 \\ 0 & -i\mathbf{1}_{s_\nu} \end{pmatrix} W_\nu. \tag{3.2}$$

Remark 3.1.2. One can check that $\mathbf{1}_r - Z\,{}^t\overline{Z} = (\mathbf{1}_r + Z(\mathbf{1}_s - {}^t\overline{Z}Z)^{-1}\,{}^t\overline{Z})^{-1}$ is also a positive-definite matrix. In particular, the map $Z \mapsto {}^t\overline{Z}$ gives a bijection between $\mathcal{H}_{r,s}$ and $\mathcal{H}_{s,r}$.

Remark 3.1.3. If $rs = 0$ we get $\mathcal{H}_{r,s} = \{0\}$, a space with a single point.

The choice of an m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ determines the product $\mathcal{H}_{\mathbf{r},\mathbf{s}} := \mathcal{H}_{r_1,s_1} \times \cdots \times \mathcal{H}_{r_e,s_e}$. In Section 3.2 we show how to associate a polarized abelian variety with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ to every element $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$ and $(\mathcal{M}, T) \in \Upsilon(\mathbf{r}, \mathbf{s})$ and vice versa.

3.2 Polarized abelian varieties with given m-CM-type

Our goal in this section is to give a correspondence between the set of polarized abelian varieties with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ and the space $\mathcal{H}_{\mathbf{r},\mathbf{s}} \times \Upsilon(\mathbf{r}, \mathbf{s})$.

We first construct a polarized abelian variety with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ and then prove that every such polarized abelian variety can be obtained through that construction.

This result for CM-fields is a particular case of the results in Shimura [39]. It is also explained in Birkenhake-Lange [2, Section 9.6], where some proofs for this case are left to the reader. We present them here for completeness, as we will use them in Section 3.3.

Fix a pair $(\mathcal{M}, T) \in \Upsilon(\mathbf{r}, \mathbf{s})$, matrices $W_1, \ldots, W_e$ as in (3.2), and an element $Z = (Z_1, \ldots, Z_e) \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$. We start by constructing a complex torus.

Consider the complex vector space homomorphism $\Gamma : \mathbb{C}^{2em} \to \mathbb{C}^{em}$ given by the block diagonal matrix

$$\Gamma = \operatorname{diag}(\Gamma_1, \ldots, \Gamma_e) \ \text{ with } \ \Gamma_\nu = \begin{pmatrix} (\mathbf{1}_{r_\nu} \ Z_\nu)\overline{W_\nu} & 0 \\ 0 & ({}^tZ_\nu \ \mathbf{1}_{s_\nu})W_\nu \end{pmatrix} \in \mathbb{C}^{m \times 2m}. \tag{3.3}$$

Remark 3.2.1. If we have $r_\nu = 0$ or $s_\nu = 0$, then we get $\Gamma_\nu = (0 \ W_\nu)$ or $\Gamma_\nu = (\overline{W_\nu} \ 0)$, respectively.

Lemma 3.2.2. Γ restricted to $\jmath((K \otimes_{\mathbb{Q}} \mathbb{R})^m) \subseteq \mathbb{C}^{2em}$ is an isomorphism of real vector spaces.

Proof. This is a particular case of Lemma 9.6.2 in Birkenhake-Lange [2]; we write the details of the proof for completeness. We proceed to prove it by blocks, hence assume $e = 1$ and omit the subindices.

Consider the map $\pi : \mathbb{C}^m \to \mathbb{C}^m$ given by

$$x \mapsto \pi(x) = \begin{pmatrix} \mathbf{1}_r & Z \\ {}^t\overline{Z} & \mathbf{1}_s \end{pmatrix} \overline{W}x.$$

Since W is non-singular by definition and the matrix

$$\begin{pmatrix} \mathbf{1}_r & Z \\ {}^t\overline{Z} & \mathbf{1}_s \end{pmatrix} \begin{pmatrix} \mathbf{1}_r & -Z \\ -{}^t\overline{Z} & \mathbf{1}_s \end{pmatrix} = \begin{pmatrix} \mathbf{1}_r - Z\,{}^t\overline{Z} & 0 \\ 0 & \mathbf{1}_s - {}^t\overline{Z}Z \end{pmatrix}$$

is positive definite, thus non-singular, the map π is a $\mathbb{C}$-isomorphism. Moreover, let $\kappa : \mathbb{C}^m \to \mathbb{C}^{2m}$ be given by $\kappa(x) = \begin{pmatrix} x \\ \overline{x} \end{pmatrix}$ and $\eta : \mathbb{C}^{r+s} \to \mathbb{C}^{r+s}$ be given by $\eta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \overline{y} \end{pmatrix}$. Then, as $\kappa \circ \phi$ is the map $\jmath$, the statement follows from the equality $\pi = \eta \circ \Gamma \circ \kappa$. $\square$

We conclude that the image $(\Gamma \circ \jmath)(\mathcal{M})$ is a lattice in $\mathbb{C}^{em}$ and the quotient $X := \mathbb{C}^{em}/(\Gamma \circ \jmath)(\mathcal{M})$ is a complex torus.

Next, we determine the polarization of X by determining a hermitian form. We define the map $H : \mathbb{C}^{em} \times \mathbb{C}^{em} \to \mathbb{C}$ as $H(x, y) = 2\,{}^tx \operatorname{diag}(H_1, \ldots, H_e)\overline{y}$ with

$$H_\nu = \begin{pmatrix} (\mathbf{1}_{r_\nu} - \overline{Z_\nu}\,{}^tZ_\nu)^{-1} & 0 \\ 0 & (\mathbf{1}_{s_\nu} - {}^t\overline{Z_\nu}Z_\nu)^{-1} \end{pmatrix} \in \mathbb{C}^{m \times m}, \tag{3.4}$$

which is positive definite and hermitian by definition. To see that it defines a polarization we need to see that the associated alternating form $E = \operatorname{Im} H$ is integer-valued on the lattice $(\Gamma \circ \jmath)(\mathcal{M})$. Given a linear map $f : A \to B$ and a real bilinear form $E : B \times B \to \mathbb{R}$, we define the real bilinear form $f^*E := E(f(\cdot), f(\cdot)) : A \times A \to \mathbb{R}$.

Lemma 3.2.3. For all $\mathbf{a}, \mathbf{b} \in (K \otimes \mathbb{R})^m$ we have

$$((\Gamma \circ \jmath)^* E)(\mathbf{a}, \mathbf{b}) = \operatorname{tr}_{K \otimes \mathbb{R}/\mathbb{R}}({}^t\mathbf{a} T \overline{\mathbf{b}}).$$

Proof. This is a particular case of Lemma 9.6.3 in Birkenhake-Lange [2]; we write the details of the proof for completeness. As before, we proceed to prove it by blocks, hence assume $e = 1$ and omit the subindices.

On the one hand we have

$$\begin{aligned}
(\Gamma \circ \jmath)^* E(\mathbf{a}, \mathbf{b}) &= 2 \operatorname{Im} \left({}^t\jmath(\mathbf{a})\, {}^t\Gamma H \overline{\Gamma \jmath(\mathbf{b})} \right) \\
&= 2 \operatorname{Im} \left({}^t\begin{pmatrix} \phi(\mathbf{a}) \\ \overline{\phi}(\mathbf{a}) \end{pmatrix} \begin{pmatrix} {}^t\overline{W} \begin{pmatrix} \mathbf{1}_r \\ {}^tZ \end{pmatrix} & 0 \\ 0 & {}^tW \begin{pmatrix} Z \\ \mathbf{1}_s \end{pmatrix} \end{pmatrix} \right. \\
&\qquad \left. \begin{pmatrix} (\mathbf{1}_r - \overline{Z}\, {}^tZ)^{-1} & 0 \\ 0 & (\mathbf{1}_s - {}^t\overline{Z} Z)^{-1} \end{pmatrix} \begin{pmatrix} (\mathbf{1}_r\ \overline{Z}) W & 0 \\ 0 & ({}^t\overline{Z}\ \mathbf{1}_s) \overline{W} \end{pmatrix} \overline{\begin{pmatrix} \phi(\mathbf{b}) \\ \overline{\phi}(\mathbf{b}) \end{pmatrix}} \right) \\
&= 2 \operatorname{Im} \left({}^t\phi(\mathbf{a})\, {}^t\overline{W} \begin{pmatrix} \mathbf{1}_r \\ {}^tZ \end{pmatrix} (\mathbf{1}_r - \overline{Z}\, {}^tZ)^{-1} (\mathbf{1}_r\ \overline{Z}) W \overline{\phi(\mathbf{b})} \right. \\
&\qquad \left. + {}^t\overline{\phi(\mathbf{a})}\, {}^tW \begin{pmatrix} Z \\ \mathbf{1}_s \end{pmatrix} (\mathbf{1}_s - {}^t\overline{Z} Z)^{-1} ({}^t\overline{Z}\ \mathbf{1}_s) \overline{W} \phi(\mathbf{b}) \right) \\
&= 2 \operatorname{Im} \left({}^t\phi(\mathbf{a})\, {}^t\overline{W} \begin{pmatrix} \mathbf{1}_r \\ {}^tZ \end{pmatrix} (\mathbf{1}_r - \overline{Z}\, {}^tZ)^{-1} (\mathbf{1}_r\ \overline{Z}) W \overline{\phi(\mathbf{b})} \right. \\
&\qquad \left. - {}^t\phi(\mathbf{a})\, {}^t\overline{W} \begin{pmatrix} \overline{Z} \\ \mathbf{1}_s \end{pmatrix} (\mathbf{1}_s - {}^tZ \overline{Z})^{-1} ({}^tZ\ \mathbf{1}_s) W \overline{\phi(\mathbf{b})} \right) \\
&= 2 \operatorname{Im} \left({}^t\phi(\mathbf{a})\, {}^t\overline{W} \left(\begin{pmatrix} \mathbf{1}_r \\ {}^tZ \end{pmatrix} (\mathbf{1}_r - \overline{Z}\, {}^tZ)^{-1} (\mathbf{1}_r\ \overline{Z}) \right. \right. \\
&\qquad \left. \left. - \begin{pmatrix} \overline{Z} \\ \mathbf{1}_s \end{pmatrix} (\mathbf{1}_s - {}^tZ \overline{Z})^{-1} ({}^tZ\ \mathbf{1}_s) \right) W \overline{\phi(\mathbf{b})} \right) \\
&= 2 \operatorname{Im} \left({}^t\phi(\mathbf{a})\, {}^t\overline{W} \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & -\mathbf{1}_s \end{pmatrix} W \overline{\phi(\mathbf{b})} \right) = 2 \operatorname{Im} \left(i\, {}^t\phi(\mathbf{a}) \phi(T) \overline{\phi(\mathbf{b})} \right) \\
&= 2 \operatorname{Re} \phi \left({}^t\mathbf{a} T \overline{\mathbf{b}} \right).
\end{aligned}$$

And on the other hand we get

$$\operatorname{tr}_{K\otimes\mathbb{R}/\mathbb{R}}({}^t\mathbf{a}T\overline{\mathbf{b}}) = \phi({}^t\mathbf{a}T\overline{\mathbf{b}}) + \overline{\phi}({}^t\mathbf{a}T\overline{\mathbf{b}}) = 2\operatorname{Re}\phi({}^t\mathbf{a}T\overline{\mathbf{b}}). \qquad \square$$

Lastly we determine an embedding $K \hookrightarrow \operatorname{End}(X)\otimes\mathbb{Q}$. Let $\mathcal{O}$ be the order $\{\alpha \in K : \alpha\mathcal{M} \subseteq \mathcal{M}\}$. Note that $\mathcal{O}$ acts on $\mathcal{M}$ via a natural action, which induces an $\mathbb{R}$-linear action on the lattice $(\Gamma\circ\jmath)(\mathcal{M})$ in $\mathbb{C}^{em}$. This gives an embedding $\mathcal{O} \hookrightarrow \operatorname{End}(X)$ which extends to an embedding

$$\iota : K \hookrightarrow \operatorname{End}(X)\otimes\mathbb{Q}.$$

We now have all the elements needed to determine a polarized abelian variety with m-CM-type, so we can state the result.

Proposition 3.2.4. Let $(\mathcal{M}, T) \in \Upsilon(\mathbf{r},\mathbf{s})$ and $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$. The triple $(X, \operatorname{Im} H, \iota)$ as defined above is a polarized abelian variety with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$.

Proof. This is a particular case of Lemma 9.6.4 in Birkenhake-Lange [2]; we write the details of the proof for completeness.

We need to prove that the analytic representation $\rho_a \circ \iota$ is equivalent to the representation $\rho_{\mathbf{r},\mathbf{s}}$ defined in (3.1) and that the Rosati involution on $\operatorname{End}(X)\otimes\mathbb{Q}$ with respect to the polarization extends the complex conjugation via ι.

The equivalence of representations follows from the equality

$$\rho_{\mathbf{r},\mathbf{s}}(a)(\Gamma\circ\jmath)(\mathbf{b}) = (\Gamma\circ\jmath)(a\mathbf{b}).$$

As before, we proceed to prove it by blocks, hence assume $e = 1$ and omit the subindices.

We have

$$\begin{aligned}
\rho_{\mathbf{r},\mathbf{s}}(a)(\Gamma\circ\jmath)(\mathbf{b}) &= \begin{pmatrix} \phi(a)\mathbf{1}_r & 0 \\ 0 & \overline{\phi}(a)\mathbf{1}_s \end{pmatrix} \begin{pmatrix} (\mathbf{1}_r\ Z)\overline{W} & 0 \\ 0 & ({}^tZ\ \mathbf{1}_s)W \end{pmatrix} \begin{pmatrix} \phi(\mathbf{b}) \\ \overline{\phi}(\mathbf{b}) \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{1}_r\ Z)\overline{W} & 0 \\ 0 & ({}^tZ\ \mathbf{1}_s)W \end{pmatrix} \begin{pmatrix} \phi(a)\mathbf{1}_m & 0 \\ 0 & \overline{\phi}(a)\mathbf{1}_m \end{pmatrix} \begin{pmatrix} \phi(\mathbf{b}) \\ \overline{\phi}(\mathbf{b}) \end{pmatrix} \\
&= (\Gamma\circ\jmath)(a\mathbf{b}),
\end{aligned}$$

so the equality holds.

That the Rosati involution on $\operatorname{End}(X)\otimes\mathbb{Q}$ with respect to the polarization extends the complex conjugation on K via ι is a consequence of the definition of ι as the unique extension of the natural action of K on $\mathcal{M}\otimes\mathbb{Q}$. $\square$

This construction defines a map A from $\mathcal{H}_{\mathbf{r},\mathbf{s}} \times \Upsilon(\mathbf{r},\mathbf{s})$ to the set of isomorphism classes of polarized abelian varieties with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ given by $A(Z, \mathcal{M}, T) = (X, \operatorname{Im} H, \iota)$. The next proposition shows that the map is surjective.

Proposition 3.2.5 (Shimura [39], see Birkenhake-Lange [2, Proposition 9.6.5])**.** Every polarized abelian variety (X, E, ι) with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ is isomorphic to $A(Z, \mathcal{M}, T)$ for some $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$ and $(\mathcal{M}, T) \in \Upsilon(\mathbf{r}, \mathbf{s})$.

Proof. We give the proof since it is omitted in Birkenhake-Lange [2] and because we will use it in Section 3.3.

Let the triple $(X = V/\Lambda, E, \iota)$ be a polarized abelian variety with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$. There exists a basis of V such that, for all $a \in K$, the analytic representation of $\iota(a)$ is the diagonal matrix $\rho_{\mathbf{r},\mathbf{s}}(a)$. We identify V with $\mathbb{C}^{em}$ via this choice. Since $\Lambda \otimes \mathbb{Q} \subseteq \mathbb{C}^{em}$ is a vector space over K of dimension m via $\rho_{\mathbf{r},\mathbf{s}}$, we choose a basis $b_1, \ldots, b_m \in \Lambda \otimes \mathbb{Q}$ and consider the isomorphism $\eta : K^m \to \Lambda \otimes \mathbb{Q}$ given by this basis. Then $\mathcal{M} = \eta^{-1}(\Lambda)$ is a $\mathbb{Z}$-module of rank $2em$ in K^m.

Next, consider the maps $\pi_{ij} : K \to \mathbb{Q}$ given by $a \mapsto E(ab_i, b_j)$ for all $1 \leq i, j \leq m$. These are $\mathbb{Q}$-linear maps, hence there exist $t_{ij} \in K$ such that $E(ab_i, b_j) = \operatorname{tr}(at_{ij})$ holds for all $a \in K$. The matrix $T = (t_{ij})_{ij} \in K^{m\times m}$ satisfies

$$\eta^* E(\mathbf{a}, \mathbf{b}) = \operatorname{tr}({}^t\mathbf{a} T \overline{\mathbf{b}}) \in \mathbb{Z} \tag{3.5}$$

for all $\mathbf{a}, \mathbf{b} \in \mathcal{M}$. Shimura also proves that T is antihermitian and has signature $(\mathbf{r}, \mathbf{s})$ as a consequence of E being a Riemann form. For details see [39, pp. 158–160]. Let W_ν for $\nu = 1, \ldots, e$ be arbitrary matrices satisfying (3.2).

We have then a pair $(\mathcal{M}, T) \in \Upsilon(\mathbf{r}, \mathbf{s})$. We only need to find $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$ such that (X, E, ι) is isomorphic to $A(Z, \mathcal{M}, T)$.

A vector $b \in \mathbb{C}^{em}$ can be written as

$$b = \begin{pmatrix} u^1 \\ v^1 \\ \vdots \\ u^e \\ v^e \end{pmatrix}$$

with $u^\nu \in \mathbb{C}^{r_\nu}$ and $v^\nu \in \mathbb{C}^{s_\nu}$ for every $\nu = 1, \ldots, e$.

Consider such subdivision for the basis $b_1, \ldots, b_m$ of $\Lambda \otimes \mathbb{Q}$. We define the matrices

$$U_\nu = \begin{pmatrix} u_1^\nu & \cdots & u_m^\nu \end{pmatrix} \in \mathbb{C}^{r_\nu \times m}, \quad V_\nu = \begin{pmatrix} v_1^\nu & \cdots & v_m^\nu \end{pmatrix} \in \mathbb{C}^{s_\nu \times m},$$

$$X_\nu = \begin{pmatrix} U_\nu & 0 \\ 0 & V_\nu \end{pmatrix} \in \mathbb{C}^{m \times 2m}$$

and write

$$\begin{pmatrix} U_\nu & 0 \\ 0 & V_\nu \end{pmatrix} \begin{pmatrix} \overline{W}_\nu^{-1} & 0 \\ 0 & W_\nu^{-1} \end{pmatrix} = \begin{pmatrix} A_\nu & B_\nu & 0 & 0 \\ 0 & 0 & C_\nu & D_\nu \end{pmatrix}, \tag{3.6}$$

where we have $A_\nu \in \mathbb{C}^{r_\nu \times r_\nu}$, $B_\nu \in \mathbb{C}^{r_\nu \times s_\nu}$, $C_\nu \in \mathbb{C}^{s_\nu \times r_\nu}$, and $D_\nu \in \mathbb{C}^{s_\nu \times s_\nu}$.

Shimura proves that the matrices A_ν and D_ν are invertible and satisfy $A_\nu^{-1} B_\nu = {}^t(D_\nu^{-1} C_\nu)$. This follows from the same reasoning that gives the signature of T, see [39, (30) and the paragraph after]. Since it is not relevant for the use of the construction we omit the details here.

We define $Z_\nu = A_\nu^{-1} B_\nu = {}^t(D_\nu^{-1} C_\nu) \in \mathbb{C}^{r_\nu \times s_\nu}$ and change the basis of V by the matrix $\operatorname{diag}(A_1^{-1}, D_1^{-1}, \ldots, A_e^{-1}, D_e^{-1})$, so that without loss of generality (3.6) becomes

$$\begin{pmatrix} U_\nu & 0 \\ 0 & V_\nu \end{pmatrix} \begin{pmatrix} \overline{W}_\nu^{-1} & 0 \\ 0 & W_\nu^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{r_\nu} & Z_\nu & 0 & 0 \\ 0 & 0 & {}^tZ_\nu & \mathbf{1}_{s_\nu} \end{pmatrix} \in \mathbb{C}^{m \times 2m},$$

or equivalently,

$$X_\nu = \begin{pmatrix} U_\nu & 0 \\ 0 & V_\nu \end{pmatrix} = \begin{pmatrix} (\mathbf{1}_{r_\nu} \; Z_\nu)\overline{W_\nu} & 0 \\ 0 & ({}^tZ_\nu \; \mathbf{1}_{s_\nu})W_\nu \end{pmatrix} \in \mathbb{C}^{m \times 2m},$$

that is, with this basis the matrix X_ν is the ν-component of Γ as defined in (3.3).

Then, for all $\mathbf{a} \in (K \otimes \mathbb{R})^m$ we have

$$\begin{aligned} \eta(\mathbf{a}) &= \sum_{i=1}^m \rho_{\mathbf{r},\mathbf{s}}(a_i) b_i \\ &= \sum_{i=1}^m \begin{pmatrix} \phi_1(a_i)u_i^1 \\ \overline{\phi}_1(a_i)v_i^1 \\ \vdots \\ \phi_e(a_i)u_i^e \\ \overline{\phi}_e(a_i)v_i^e \end{pmatrix} = \begin{pmatrix} U_1 & 0 & \cdots & 0 & 0 \\ 0 & V_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & U_e & 0 \\ 0 & 0 & \cdots & 0 & V_e \end{pmatrix} \begin{pmatrix} \phi_1(\mathbf{a}) \\ \overline{\phi}_1(\mathbf{a}) \\ \vdots \\ \phi_e(\mathbf{a}) \\ \overline{\phi}_e(\mathbf{a}) \end{pmatrix} = (\Gamma \circ \jmath)(\mathbf{a}), \end{aligned}$$

hence we obtain $\eta = \Gamma \circ \jmath$. We claim that $Z = (Z_1, \ldots, Z_e)$ is in $\mathcal{H}_{\mathbf{r},\mathbf{s}}$ and the triple (X, H, ι) is isomorphic to $A(Z, \mathcal{M}, T)$.

In order to prove $\mathbf{1}_s - {}^t\overline{Z}Z > 0$, and hence $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$, we will use the definition of H in (3.4) and its positive-definiteness. Let $\mu_1, \ldots, \mu_{2em} \in \mathcal{M}$ be a $\mathbb{Z}$-basis and let $x_i = \eta(\mu_i)$ be the corresponding basis of Λ. The matrices of H and E satisfy the equality $M_H = 2i(\overline{\Pi} M_E^{-1}\, {}^t\Pi)^{-1}$ (see (1.5)) where Π is the big period matrix (see (1.4)) of the complex torus with respect to the chosen bases, so we start by computing Π and M_E.

The period matrix Π has as columns the vectors $x_i = \eta(\mu_i) = \Gamma\jmath(\mu_i)$, hence we write $\Pi = \Gamma M$ with $M = (\jmath(\mu_i))_i \in \mathbb{C}^{2em\times 2em}$. The matrix of E is $M_E = (E(x_i, x_j))_{i,j} \in \mathbb{Z}^{2em\times 2em}$ so we compute

$$\begin{aligned}
E(x_i, x_j) &= \eta^* E(\mu_i, \mu_j) \\
&= \operatorname{tr}({}^t\mu_i T \overline{\mu_j}) = \sum_{j=1}^{e} ({}^t\phi_j(\mu_i)\phi_j(T)\phi_j(\overline{\mu_j}) + {}^t\overline{\phi}_j(\mu_i)\overline{\phi}_j(T)\overline{\phi}_j(\overline{\mu_j})) \\
&= {}^t\jmath(\mu_i)\operatorname{diag}(\phi_1(T), \overline{\phi}_1(T), \dots, \phi_e(T), \overline{\phi}_e(T))\jmath(\overline{\mu_j}) \\
&= {}^t\jmath(\mu_i)\operatorname{diag}(\phi_1(T), \overline{\phi_1(T)}, \dots, \phi_e(T), \overline{\phi_e(T)})\overline{\jmath(\mu_j)},
\end{aligned}$$

hence we obtain

$$M_E = {}^tM\operatorname{diag}(\phi_1(T), \overline{\phi_1(T)}, \dots, \phi_e(T), \overline{\phi_e(T)})\overline{M}. \tag{3.7}$$

We can now compute M_H. It is again enough to compute M_H by blocks, hence we assume $e = 1$ and omit the subindices. Altogether it gives us

$$\begin{aligned}
M_H &= 2i(\overline{\Pi}M_E^{-1}\,{}^t\Pi)^{-1} = 2i[\overline{(\Gamma M)}(\overline{M}^{-1}\operatorname{diag}(\phi(T), \overline{\phi(T)})^{-1}\,{}^tM^{-1})\,{}^t(\Gamma M)]^{-1} \\
&= 2i(\overline{\Gamma}\operatorname{diag}(\phi(T), \overline{\phi(T)})^{-1}\,{}^t\Gamma)^{-1} \\
&= 2i\left[\begin{pmatrix} (\mathbf{1}_r\ \overline{Z})W & 0 \\ 0 & ({}^t\overline{Z}\ \mathbf{1}_s)\overline{W}\end{pmatrix}\begin{pmatrix}\phi(T)^{-1} & 0 \\ 0 & \overline{\phi(T)}^{-1}\end{pmatrix}\begin{pmatrix} {}^t\overline{W}\begin{pmatrix}\mathbf{1}_r \\ {}^tZ\end{pmatrix} & 0 \\ 0 & {}^tW\begin{pmatrix} Z \\ \mathbf{1}_s\end{pmatrix}\end{pmatrix}\right]^{-1} \\
&= 2\left[\begin{pmatrix}\mathbf{1}_r & \overline{Z} & 0 & 0 \\ 0 & 0 & {}^t\overline{Z} & \mathbf{1}_s\end{pmatrix}\begin{pmatrix}\mathbf{1}_r & 0 & 0 & 0 \\ 0 & -\mathbf{1}_s & 0 & 0 \\ 0 & 0 & -\mathbf{1}_r & 0 \\ 0 & 0 & 0 & \mathbf{1}_s\end{pmatrix}\begin{pmatrix}\mathbf{1}_r & 0 \\ {}^tZ & 0 \\ 0 & Z \\ 0 & \mathbf{1}_s\end{pmatrix}\right]^{-1} \\
&= 2\begin{pmatrix}\mathbf{1}_r - \overline{Z}\,{}^tZ & 0 \\ 0 & \mathbf{1}_s - {}^t\overline{Z}Z\end{pmatrix}^{-1},
\end{aligned}$$

and since H is positive definite and hermitian by definition, we obtain $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$. It follows that (X, H, ι) is isomorphic to $A(Z, \mathcal{M}, T)$ by construction. □

Observe that the equality (3.7) implies the following result.

Corollary 3.2.6. The pair $(\mathcal{M}, T) \in \Upsilon(\mathbf{r}, \mathbf{s})$ determines whether the polarization of $A(Z, \mathcal{M}, T)$ is principal. □

3.3 The endomorphism structure of the Jacobian of a CPQ curve

As we have seen in Chapter 2, every CPQ curve can be given by a Legendre-Rosenhain equation

$$y^5 = x(x-1)(x-\lambda)(x-\mu)$$

with $\lambda, \mu \in \mathbb{C}\setminus\{0,1\}$ distinct, and has an order-5 automorphism ρ given by $\rho(x,y) = (x, z_5 y)$.

In this section we give the Jacobian $J(C)$ of a CPQ curve C following the explicit construction explained in Section 1.1, together with its Riemann form, and we study the structure of the subring $\mathbb{Z}[\rho_*] \subseteq \mathrm{End}(J(C))$.

First we choose a $\mathbb{Z}$-basis for $H_1(C,\mathbb{Z})$. Note that the curve is a 5-cover of the projective line and the automorphism ρ cycles through the different sheets. Therefore, by studying the intersections in the x-plane of the paths in the $\mathbb{Q}(\zeta_5)$-basis of $H_1(C,\mathbb{Z})\otimes\mathbb{Q}$ appearing in Figure 3.1, we obtain the whole intersection matrix.

Take the paths b_1, b_2, b_3 appearing in Figure 3.1 and consider the paths $\rho^j b_i := \rho^j \circ b_i$ for $i = 1,2,3$ and $j = 1,\dots,4$. We claim that

$$\gamma = (b_1, \rho b_1, \rho^2 b_1, \rho^3 b_1, b_2, \rho b_2, \rho^2 b_2, \rho^3 b_2, b_3, \rho b_3, \rho^2 b_3, \rho^3 b_3) \tag{3.8}$$

is a $\mathbb{Z}$-basis of $H_1(C,\mathbb{Z})$.

Recall $J(C) = H^0(\omega_C)^*/H_1(C,\mathbb{Z})$, and that the principal polarization attached to $J(C)$ is given by the oriented intersection pairing. We compute the oriented intersection between the paths in (3.8).

See for example the intersections corresponding to b_1. We can see in Figure 3.1 that the solid part of the path intersects b_2 and b_3 on the same sheet, and the dashed part crosses all three paths in the solid sheet. Therefore, the path b_1 intersects once the paths b_2, b_3, ρb_1, ρb_2 and ρb_3. The intersection sign is positive (resp. negative) if the angle going from the first path to the second is counterclockwise (resp. clockwise). For example, the five intersections $E(b_1,\cdot)$ listed here are $+1, +1, -1, -1$ and -1 respectively.

Working analogously for the other paths, we obtain the matrix of E with respect to γ

$$E_0 = \begin{pmatrix} A & B & B \\ -{}^tB & A & B \\ -{}^tB & -{}^tB & A \end{pmatrix},$$

with

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

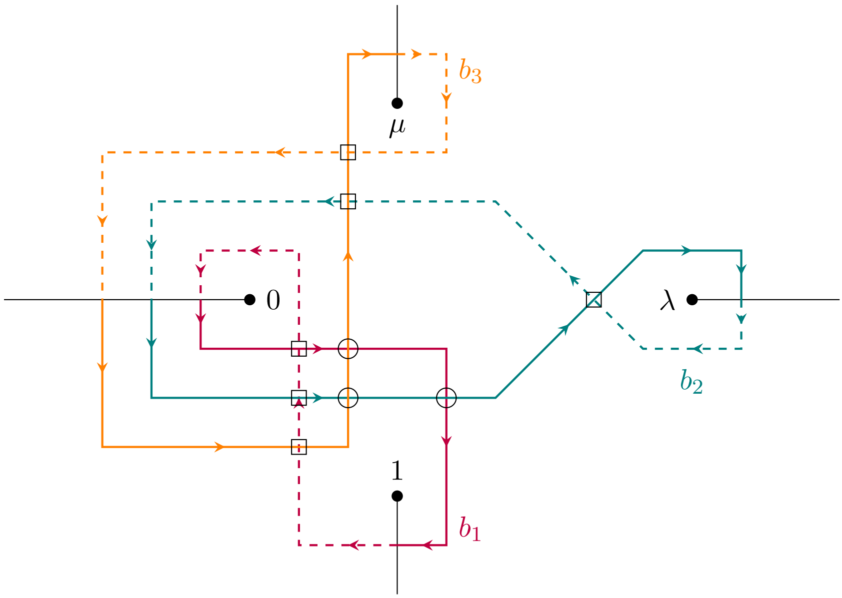

Figure 3.1: Representation on the x-plane of a $\mathbb{Q}(\zeta_5)$-basis of $H_1(C, \mathbb{Z}) \otimes \mathbb{Q}$ given by $\{b_1, b_2, b_3\}$. The solid black lines are branch cuts, which intersect at ∞. Crossing a branch cut clockwise around one affine branch point corresponds to switching to the next sheet, and it is represented by a change on the pattern of the path. Therefore circles indicate intersections on C and squares indicate intersections on the x-plane that are not intersections on C.

and since the determinant of E_0 is $\det E_0 = 1$, the choice (3.8) is indeed a $\mathbb{Z}$-basis of $H_1(C,\mathbb{Z})$. This basis, together with the basis of $H^0(\omega_C)$ given in Corollary 2.1.5, determines a big period matrix

$$\Pi = \begin{pmatrix} \int_{b_1} \frac{dx}{y^4} & \cdots & \int_{\rho^3 b_3} \frac{dx}{y^4} \\ \vdots & & \vdots \\ \int_{b_1} \frac{xdx}{y^2} & \cdots & \int_{\rho^3 b_3} \frac{xdx}{y^2} \end{pmatrix},$$

so that $J(C) \cong \mathbb{C}^6/\Pi\mathbb{Z}^{12}$.

Finally, we want to compute the analytic representation with respect to these bases of the automorphism ρ_* of the Jacobian induced by ρ.

The automorphism ρ induces a morphism $\rho^* : H^0(\omega_C) \to H^0(\omega_C)$ given by

$$\rho^*(fdg) = (f \circ \rho)d(g \circ \rho).$$

The morphism ρ^* acts on the basis chosen in Corollary 2.1.5 of $H^0(\omega_c)$ as $\rho^*(x^i y^{-j} dx) = z_5^{-j} x^i y^{-j} dx$, that is, as the diagonal matrix

$$A = \mathrm{diag}(z_5, z_5, z_5, z_5^2, z_5^2, z_5^3).$$

This basis has a dual basis of $H^0(\omega_C)^*$, and one can prove that ρ_* with respect to this dual basis acts as ${}^tA = A$ by using the definitions of α and ρ^*, so we get $\rho_a(\rho_*) = A$. We define the embedding $\iota : \mathbb{Q}(\zeta_5) \to \mathrm{End}(J(C)) \otimes \mathbb{Q}$ by taking $\iota(\zeta_5) = \rho_*$.

Proposition 3.3.1. Let C be a CPQ curve. Let $\phi_1, \phi_2 : \mathbb{Q}(\zeta_5) \to \mathbb{C}$ be the embeddings given by $\phi_1(\zeta_5) = z_5$ and $\phi_2(\zeta_5) = z_5^2$ respectively, and let $\iota : \mathbb{Q}(\zeta_5) \to \mathrm{End}(J(C)) \otimes \mathbb{Q}$ be the embedding that maps ζ_5 to ρ_*. Then $(J(C), \lambda_C, \iota)$ has 3-CM-type $\mathfrak{Z} = (\mathbb{Q}(\zeta_5), (\phi_1, \phi_2), (3, 2), (0, 1))$.

Proof. We just saw that ρ_3 and $\rho_a \circ \iota$ are equivalent representations, since they map ζ_5 to the same diagonal matrix.

All that is left to do is prove that the Rosati involution on $\mathrm{End}(J(C)) \otimes \mathbb{Q}$ with respect to the polarization λ_C extends complex multiplication on $\mathbb{Q}(\zeta_5)$ via ι.

Let α be an Abel-Jacobi map with a branch point as base point, hence fixed by ρ. Recall the diagram (1.12) that relates an automorphism of a curve ρ with its induced automorphism ρ_* on the Jacobian,

$$\begin{tikzcd} C \arrow[r, "\rho"] \arrow[d, "\alpha"'] & C \arrow[d, "\alpha"] \\ J(C) \arrow[r, "\rho_*"] & J(C). \end{tikzcd} \tag{3.9}$$

If we apply the functor Pic^0 to the diagram, then we obtain

$$\begin{tikzcd} \operatorname{Pic}^0(C) = J(C) & \operatorname{Pic}^0(C) = J(C) \arrow[l, "\rho^*"'] \\ \operatorname{Pic}^0(J(C)) = \widehat{J(C)} \arrow[u, "\alpha^*"] & \operatorname{Pic}^0(J(C)) = \widehat{J(C)}, \arrow[l, "\widehat{\rho_*}"'] \arrow[u, "\alpha^*"'] \end{tikzcd} \tag{3.10}$$

where α^* is the inverse of the polarization λ_C of $J(C)$, see Proposition 11.3.5 in Birkenhake-Lange [2]. Therefore ρ_* and ρ^* are dual to each other, in the sense that they satisfy $\widehat{\rho_*} = \lambda_C \rho^* \lambda_C^{-1}$, and by definition of the Rosati involution (see (1.2)) we obtain $\rho_*' = \rho^*$.

Since we have $\rho_* \rho^* = 1$, we conclude that $\iota(\zeta_5)' = \rho_*' = \rho_*^{-1} = \iota(\overline{\zeta_5})$ holds. Thus the statement follows. □

We have proven that $(J(C), \lambda_C, \iota)$ has 3-CM-type $\mathfrak{Z}$, so by Proposition 3.2.5 there exists a pair $(\mathcal{M}, T) \in \Upsilon(\mathfrak{Z})$ such that the triple $(J(C), \lambda_C, \iota)$ is of the form $A(Z, \mathcal{M}, T)$ for some $Z \in \mathcal{H}_{\mathfrak{Z}}$. The constructive proof of that proposition gives us the recipe to find that pair $(\mathcal{M}, T)$.

The dual of the $\mathbb{C}$-basis of $H^0(\omega_C)$ given in Corollary 2.1.5 already satisfies $\rho_a \circ \iota = \rho_{\mathbf{r},\mathbf{s}}$, and we choose $\{b_i\}_{i=1}^3$ as a $\mathbb{Q}(\zeta_5)$-basis of $H_1(C, \mathbb{Z}) \otimes \mathbb{Q}$. We obtain $\mathcal{M} = \eta^{-1}(H_1(C, \mathbb{Z})) = \mathcal{O}_K^3$.

Next we want to find a matrix $T \in \mathbb{Q}(\zeta_5)^{3\times 3}$ that satisfies

$$E(ab_i, b_j) = \operatorname{tr}(at_{ij}) \tag{3.11}$$

for all $a \in \mathbb{Q}(\zeta_5)$. For every $i, j = 1, \ldots, 3$ consider the condition (3.11) for $a \in \{\zeta_5^k : 1 \leq k \leq 4\}$. This gives a linear system whose solution determines t_{ij} uniquely, and we obtain

$$T = \frac{1}{5}\begin{pmatrix} \zeta_5 - \zeta_5^4 & 1 - \zeta_5^4 & 1 - \zeta_5^4 \\ -1 + \zeta_5 & \zeta_5 - \zeta_5^4 & 1 - \zeta_5^4 \\ -1 + \zeta_5 & -1 + \zeta_5 & \zeta_5 - \zeta_5^4 \end{pmatrix}. \tag{3.12}$$

We double-check that T has signature $((3,2),(0,1))$, which is consistent with Proposition 3.3.1.

We conclude that for every CPQ curve C there exists $Z \in \mathcal{H}_{\mathfrak{Z}}$ such that the triple $(J(C), \lambda_C, \iota)$ is of the form $A(Z, \mathcal{M}, T)$.

3.4 Equivalence of polarized lattices

We have seen in Corollary 3.2.6 that the pair $(\mathcal{M}, T)$ in $\Upsilon(\mathbf{r}, \mathbf{s})$ determines whether the polarization of a polarized abelian variety with m-CM-type $(K, \Phi, \mathbf{r}, \mathbf{s})$ is principal.

In this section we characterize the pairs that determine principal polarizations with the end goal of identifying the preimage of the set of principally polarized abelian varieties with 3-CM-type $\mathfrak{Z}$ and an order-5 automorphism by the map A defined in Section 3.2.

We start by presenting the concept of equivalent pairs $(\mathcal{M}, T) \in \Upsilon(\mathbf{r}, \mathbf{s})$ and how it relates to the map A.

Definition 3.4.1. Let $(K, \Phi, \mathbf{r}, \mathbf{s})$ be an m-CM-type. We say that two pairs $(\mathcal{M}_1, T_1)$ and $(\mathcal{M}_2, T_2)$ in $\Upsilon(\mathbf{r}, \mathbf{s})$ are *equivalent* if there exists $U \in \mathrm{GL}_m(K)$ that satisfies

$$U(\mathcal{M}_1, T_1) := (U\mathcal{M}_1, {}^tU^{-1}T_1\overline{U}^{-1}) = (\mathcal{M}_2, T_2).$$

Example 3.4.2. Consider the 3-CM-type $\mathfrak{Z}$ and define $\mathcal{M} = \mathcal{O}_K^3$ and T as in (3.12). The matrix

$$U = \begin{pmatrix} 1 & \zeta_5^3 & -\zeta_5^3 - \zeta_5 - 1 \\ 0 & -\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1 & 1 \\ 0 & \zeta_5 + 1 & 0 \end{pmatrix}$$

determines the equivalent pair $(\mathcal{M}_0, T_0) = U(\mathcal{M}, T)$ with $\mathcal{M}_0 = \mathcal{O}_K^3$ and

$$T_0 = \mathrm{diag}\left(\frac{1}{5}\zeta_5^3 + \frac{1}{5}\zeta_5^2 + \frac{2}{5}\zeta_5 + \frac{1}{5}, \frac{1}{5}\zeta_5^3 + \frac{1}{5}\zeta_5^2 + \frac{2}{5}\zeta_5 + \frac{1}{5}, -\frac{1}{5}\zeta_5^3 + \frac{1}{5}\zeta_5^2\right).$$

Proposition 3.4.3 (Proposition 4 in Shimura [39])**.** Let $(K, \Phi, \mathbf{r}, \mathbf{s})$ be an m-CM-type. Given two pairs $(\mathcal{M}, T), (\mathcal{M}', T') \in \Upsilon(\mathbf{r}, \mathbf{s})$ and two elements $Z, Z' \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$, the polarized abelian varieties $A(Z, \mathcal{M}, T)$ and $A(Z', \mathcal{M}', T')$ are isomorphic only if $(\mathcal{M}, T)$ and $(\mathcal{M}', T')$ are equivalent.

Conversely, if the pairs $(\mathcal{M}, T), (\mathcal{M}', T') \in \Upsilon(\mathbf{r}, \mathbf{s})$ are equivalent, then for every $Z \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$ there exists $Z' \in \mathcal{H}_{\mathbf{r},\mathbf{s}}$ such that the triple $A(Z, \mathcal{M}, T)$ is isomorphic to $A(Z', \mathcal{M}', T')$. □

Remark 3.4.4. If we consider the set $\Upsilon(\mathbf{r}, \mathbf{s})$ as a discrete topological space, then the map A coinduces a topology on the set of polarized abelian varieties with 3-CM-type $\mathfrak{Z}$. Moreover, it follows from Proposition 3.4.3 that the topological subspace of principally polarized abelian varieties with 3-CM-type $\mathfrak{Z}$ has as many connected components as equivalence classes of pairs $(\mathcal{M}, T) \in \Upsilon(\mathfrak{Z})$ determining principal polarizations.

In this section we present the majority of the original work in this chapter. We will prove the following theorem, which is key for the proof of the main result of the chapter.

Theorem 3.4.5. Let $\phi_1, \phi_2 : \mathbb{Q}(\zeta_5) \to \mathbb{C}$ be the embeddings that map ζ_5 to z_5 and z_5^2 respectively, and let $\mathfrak{Z}$ be the 3-CM-type $(\mathbb{Q}(\zeta_5), (\phi_1, \phi_2), (3, 2), (0, 1))$. Let $Z \in \mathcal{H}_3$, and consider a pair $(\mathcal{M}, T) \in \Upsilon(\mathfrak{Z})$ such that $\zeta_5 \mathcal{M} \subseteq \mathcal{M}$. Then the polarization of $A(Z, \mathcal{M}, T)$ is principal if and only if the pair $(\mathcal{M}, T)$ is equivalent to $(\mathcal{M}_0, T_0)$.

The end goal is then to prove that the pair $(\mathcal{M}_0, T_0)$ defined in Example 3.4.2 is the only pair in $\Upsilon(\mathfrak{Z})$ up to equivalence such that the abelian varieties $A(\,\cdot\,, \mathcal{M}_0, T_0)$ are principally polarized and such that $\iota(\zeta_5)$ is an endomorphism of the abelian variety. We focus on the algebraic structure of $(\mathcal{M}, T)$ letting go of its relation to m-CM-types as much as possible. We use some results by Shimura [40] to define and characterize these pairs.

Let K be a CM-field of degree $2e$, let K^+ be its maximal totally real subfield, and let m be a positive integer. An *$\mathcal{O}_K$-lattice* $\mathcal{M}$ in K^m is a finitely generated $\mathcal{O}_K$-module in K^m that spans K^m over $\mathcal{O}_K$.

We also define a *polarized $\mathcal{O}_K$-lattice* to be a pair $(\mathcal{M}, T)$ with $\mathcal{M}$ an $\mathcal{O}_K$-lattice and $T \in K^{m\times m}$ an antihermitian matrix such that the alternating bilinear form

$$\begin{aligned} E : \mathcal{M} \times \mathcal{M} &\to \mathbb{Q} \\ (u, v) &\mapsto \mathrm{tr}_{K/\mathbb{Q}}({}^t u T \overline{v}) \end{aligned}$$

satisfies $E(\mathcal{M}, \mathcal{M}) \subseteq \mathbb{Z}$. We say that it is *principally polarized* if the matrix of E with respect to a basis of $\mathcal{M}$ has determinant 1.

Two polarized $\mathcal{O}_K$-lattices $(\mathcal{M}_1, T_1)$ and $(\mathcal{M}_2, T_2)$ are *equivalent* if there exists $U \in \mathrm{GL}_m(K)$ that satisfies

$$U(\mathcal{M}_1, T_1) := (U\mathcal{M}_1, {}^tU^{-1} T_1 \overline{U}^{-1}) = (\mathcal{M}_2, T_2).$$

We denote it by $(\mathcal{M}_1, T_1) \sim (\mathcal{M}_2, T_2)$

Our goal is to characterize principally polarized $\mathcal{O}_K$-lattices and study their equivalence classes.

We define the *trace dual* as

$$\mathcal{M}^\vee = \{\alpha \in K^m : \mathrm{tr}({}^t\alpha\mathcal{M}) \subseteq \mathbb{Z}\}.$$

Proposition 3.4.6 (2.15 in Shimura [40])**.** Let $\mathcal{M}$ be an $\mathcal{O}_K$-lattice in K^m and consider $c \in K$, $\alpha \in K^{m\times m}$ and $\sigma \in \mathrm{Aut}(K)$. The trace dual satisfies:

(1) $(c\mathcal{M})^\vee = c^{-1}\mathcal{M}^\vee$,
(2) $(\alpha\mathcal{M})^\vee = {}^t\alpha^{-1}\mathcal{M}^\vee$,
(3) $(\mathcal{M}^\sigma)^\vee = (\mathcal{M}^\vee)^\sigma$. □

We define the *different of* K as the inverse as a fractional ideal of the trace dual of the ring of integers $\mathcal{D}_K^{-1} = \mathcal{O}_K^\vee$.

Proposition 3.4.7. A polarized $\mathcal{O}_K$-lattice $(\mathcal{M}, T)$ is principally polarized if and only if it satisfies

$$^{t}T\mathcal{M} = \overline{\mathcal{M}}^{\vee}.$$

Proof. Let $b_1, \ldots, b_{2em}$ be a $\mathbb{Q}$-basis of $V = K^m$, and let $b_1^*, \ldots, b_{2em}^*$ be the corresponding dual basis of $V^* = \mathrm{Hom}(V, \mathbb{Q})$. They satisfy $b_i^* b_j = \delta_{ij}$ for all $i, j = 1, \ldots, 2em$. Consider also the $\mathbb{Q}$-bilinear form

$$\begin{aligned} \mathrm{tr}_{K/\mathbb{Q}} : V \times V &\to \mathbb{Q} \\ (u, v) &\mapsto \mathrm{tr}_{K/\mathbb{Q}}(^{t}uv). \end{aligned}$$

It defines an isomorphism $\phi : V \to V^*$ given by $u \mapsto \mathrm{tr}_{K/\mathbb{Q}}(^{t}u\,\cdot)$, hence we can define a new $\mathbb{Q}$-basis of V as $b_i^{\vee} = \phi^{-1}(b_i^*)$, which satisfies $\mathrm{tr}_{K/\mathbb{Q}}(b_i^{\vee} b_j) = \delta_{ij}$.

Assume now that $(b_i)_i$ is a basis of $\mathcal{M}$. It follows that $(b_i^{\vee})_i$ is a basis of $\mathcal{M}^{\vee}$, and since $(\mathcal{M}, T)$ is a polarized $\mathcal{O}_K$-lattice, we also have $^{t}T\mathcal{M} \subseteq \overline{\mathcal{M}}^{\vee}$. Moreover, the index $[\overline{\mathcal{M}}^{\vee} : {}^{t}T\mathcal{M}]$ is equal to the determinant of E for the basis $(b_i)_i$ of $\mathcal{M}$, so the equality holds if and only if the determinant is 1. $\square$

Given $\mathcal{L}$ and $\mathcal{M}$ two $\mathcal{O}_K$-lattices in K^m for $m \in \mathbb{Z}_{>0}$, we define the *ideal index of* $\mathcal{M}$ *in* $\mathcal{L}$ as the fractional $\mathcal{O}_K$-ideal

$$[\mathcal{L}/\mathcal{M}]_K = (\det(\alpha) : \alpha \in K^{m\times m} \text{ such that } \alpha\mathcal{L} \subseteq \mathcal{M}).$$

Whenever the field is clear by context, we leave the subindex out of the notation.

Proposition 3.4.8 (2.15 in Shimura [40])**.** Let $\mathcal{L}, \mathcal{M}, \mathcal{N}$ be $\mathcal{O}_K$-lattices in K^m, let $\alpha \in \mathrm{GL}_m(K)$, and let $\sigma \in \mathrm{Aut}(K)$. Then we have

(1) $[\mathcal{L}/\mathcal{M}][\mathcal{M}/\mathcal{N}] = [\mathcal{L}/\mathcal{N}]$,
(2) $[\mathcal{L}/\alpha\mathcal{L}] = \det(\alpha)\mathcal{O}_K$, and
(3) If we have $\mathcal{L} \supseteq \mathcal{M}$ and there exists an $\mathcal{O}_K$-ideal $\mathfrak{b}$ that satisfies $\mathcal{L}/\mathcal{M} \cong \mathcal{O}_K/\mathfrak{b}$, then the ideal index of $\mathcal{M}$ in $\mathcal{L}$ is $[\mathcal{L}/\mathcal{M}] = \mathfrak{b}$. $\square$

Using the concepts introduced, we characterize now principally polarized $\mathcal{O}_K$-lattices.

Proposition 3.4.9. Let K be a CM-field with class number 1 and let K^+ be its maximal totally real subfield. Let $\mathcal{D}_K$ be the different of K and assume that there exists $\delta \in K$ generating $\mathcal{D}_K$ such that $\overline{\delta} = -\delta$. Every principally polarized $\mathcal{O}_K$-lattice $(\mathcal{M}, T)$ satisfies

$$\mathrm{N}_{K/K^+}([\mathcal{O}_K^m/\mathcal{M}]) = (\det(\delta T)^{-1})\mathcal{O}_{K^+}.$$

In order to prove Proposition 3.4.9 we need the following result.

Lemma 3.4.10. Let K be a number field with class number 1, let m be a positive integer, and let $\mathcal{M}$ be a $\mathcal{O}_K$-lattice in K^m. There exists $\gamma \in \mathrm{GL}_m(K)$ that satisfies

$$\mathcal{M} = \gamma \mathcal{O}_K^m.$$

Proof. It follows from the fact that K has class number one and the structure theorem of finitely generated modules over PIDs. □

Proof of Proposition 3.4.9. Consider the equality

$$[\mathcal{O}_K^m / {}^t T\mathcal{M}] = [\mathcal{O}_K^m/\mathcal{M}][\mathcal{M}/ {}^t T\mathcal{M}]. \tag{3.13}$$

Then we directly have

$$[\mathcal{M}/ {}^t T\mathcal{M}] = (\det T)\mathcal{O}_K.$$

By Lemma 3.4.10 there exists $\gamma \in \mathrm{GL}_m(K)$ that satisfies $\mathcal{M} = \gamma\mathcal{O}_K^m$, and hence we get

$$[\mathcal{O}_K^m/\mathcal{M}] = (\det \gamma)\mathcal{O}_K.$$

Moreover, it follows from Proposition 3.4.7 that if $(\mathcal{M}, T)$ is principally polarized, then it satisfies ${}^t T\mathcal{M} = \overline{\mathcal{M}}^\vee$. By definition of the different ideal we have $(\mathcal{O}_K^m)^\vee = (\mathcal{D}_K^{-1})^m = \delta^{-1}\mathcal{O}_K^m$.

Therefore, if $(\mathcal{M}, T)$ is principally polarized, then we have

$$\begin{aligned}
[\mathcal{O}_K^m/ {}^t T\mathcal{M}] &= [\mathcal{O}_K^m/\overline{\mathcal{M}}^\vee] = \overline{[\mathcal{O}_K^m/\mathcal{M}^\vee]} = \overline{[\mathcal{O}_K^m/ {}^t\gamma^{-1}(\mathcal{O}_K^m)^\vee]} \\
&= \overline{[\mathcal{O}_K^m/\delta^{-1}\, {}^t\gamma^{-1}\mathcal{O}_K^m]} = \overline{\det(\delta\gamma)^{-1}\mathcal{O}_K} = \overline{\det(\delta\gamma)}^{-1}\mathcal{O}_K
\end{aligned}$$

Altogether, (3.13) gives that if $(\mathcal{M}, T)$ is principally polarized, then we obtain

$$\overline{\det(\delta\gamma)}^{-1}\mathcal{O}_K = (\det\gamma)\mathcal{O}_K \cdot (\det T)\mathcal{O}_K,$$

or equivalently

$$(\det\gamma)\overline{(\det\gamma)}\mathcal{O}_K = \overline{\det(\delta T)}^{-1}\mathcal{O}_K = \det(\delta T)^{-1}\mathcal{O}_K.$$

We conclude

$$\begin{aligned}
\mathrm{N}_{K/K^+}([\mathcal{O}_K^m/\mathcal{M}]) &= \mathrm{N}_{K/K^+}((\det\gamma)\mathcal{O}_K) \\
&= (\det\gamma)\overline{(\det\gamma)}\mathcal{O}_{K^+} = (\det(\delta T)^{-1})\mathcal{O}_{K^+}.
\end{aligned}$$

□

Remark 3.4.11. The assumption that K has class number one is not necessary in Proposition 3.4.9, but it does simplify the proof and it is enough for our case $K = \mathbb{Q}(\zeta_5)$. One could also justify the equality by proving it locally for all primes $\mathfrak{p} \in \mathcal{O}_K$.

In the situation above we define the matrix $S = \delta T$, which is hermitian. We say that a hermitian matrix $S \in \mathrm{GL}_m(K)$ has *signature* $((r_1, \dots, r_e), (s_1, \dots, s_e))$ if for every $\nu = 1, \dots, e$, the matrix $\phi_\nu(S)$ has r_ν positive eigenvalues and s_ν negative ones. In the case above we obtain that the signature of S is completely determined by the signature of T and the image by ϕ_ν of δ. The following result characterizes the equivalence between hermitian matrices with the same signature.

Theorem 3.4.12 (Shimura)**.** Let K be a CM-field, let K^+ be its maximal totally real subfield and let m be an odd positive integer. If $S_1, S_2 \in \mathrm{GL}_m(K)$ are hermitian matrices with equal signature, then there exist a matrix $U \in \mathrm{GL}_m(K)$, and a constant $c \in (K^+)^{>>0}$ that satisfy $cS_2 = {}^tUS_1\overline{U}$.

Proof. This is a special case of Proposition 5.9 in Shimura [40]. The notation in [40] is introduced in paragraphs 5.0 and 5.7, and it is very different from ours. For reference, we now say how it is related.

The condition "$J_\lambda(V, \Phi) = J_\lambda(V, \Phi')$ for $1 \leq \lambda \leq t$" translates to the signature equality condition. The fact that in our case K is a CM-field means that this signature equality holds for all embeddings of K into $\mathbb{C}$, that is, for "$t = r$" in Shimura's notation.

Then [40, Proposition 5.9] states exactly that there exist $U \in \mathrm{GL}_m(K)$ and $c \in (K^+)^\times$ such that $cS_2 = {}^tUS_1\overline{U}$. In the proof of Proposition 5.9 in [40], Shimura concludes that the constant c in the statement satisfies "$c \equiv 1 \pmod{\prod_{\lambda=1}^t \mathfrak{p}_{\infty_\lambda}}$", which in our setting translates to c being totally positive. □

Proposition 3.4.13. Let $K = \mathbb{Q}(\zeta_5)$ and let $(\mathcal{M}, T)$ be a principally polarized $\mathcal{O}_K$-lattice with $\mathrm{sign}(T) = ((3, 2), (0, 1))$. There exists an $\mathcal{O}_K$-lattice $\mathcal{M}'$ that satisfies

$$(\mathcal{M}, T) \sim (\mathcal{M}', T_0)$$

for T_0 as defined in Example 3.4.2.

In order to prove the proposition we need the following easy lemmas, which can be easily proven with SageMath [49].

Lemma 3.4.14. Let $K = \mathbb{Q}(\zeta_5)$ and $K^+ = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$. We have

$$(\mathcal{O}_{K^+}^\times)^{>>0} = \mathrm{N}_{K/K^+}(\mathcal{O}_K^\times) \subseteq \mathrm{N}_{K/K^+}(K^\times). \qquad \square$$

Lemma 3.4.15. The element $\delta = -4\zeta_5^3 + 2\zeta_5^2 - 2\zeta_5 - 1$ is a generator of $\mathcal{D}_K$ that satisfies $\overline{\delta} = -\delta$. □

Proof of Proposition 3.4.13. Let δ be as in Lemma 3.4.15, and define $S = \delta T$ and $S_0 = \delta T_0$.

By Lemma 3.4.10 there exists $\gamma \in \mathrm{GL}_3(K)$ such that $\mathcal{M} = \gamma\mathcal{O}_K^3$, hence we can assume without loss of generality $\mathcal{M} = \mathcal{O}_K^3$, and by Proposition 3.4.8 we get $\det S \in \mathcal{O}_{K^+}^\times$.

Moreover, by Theorem 3.4.12 there exist $\alpha \in \mathrm{GL}_3(K)$ and $c \in (K^+)^{>>0}$ that satisfy

$$ {}^t\alpha S\overline{\alpha} = cS_0. \tag{3.14}$$

Taking determinants of (3.14) we obtain

$$\mathrm{N}_{K/K^+}(\det\alpha)\det S = c^3\det S_0, \tag{3.15}$$

so for $u = \det S/\det S_0$ and $\beta = \frac{c}{\det\alpha}\alpha$ we get

$$ {}^t\beta S\overline{\beta} = \frac{c^2}{N_{K/K^+}(\det\alpha)}\,{}^t\alpha S\overline{\alpha} = \frac{c^3}{N_{K/K^+}(\det\alpha)}S_0 = uS_0. \tag{3.16}$$

If we apply Proposition 3.4.9 to the principally polarized $\mathcal{O}_K$-lattice $(\mathcal{M}, T)$, then we obtain $\det S \in \mathcal{O}_{K^+}^\times$, and we can compute $\det S_0 \in \mathcal{O}_{K^+}^\times$, thus we have $u = \det S/\det S_0 \in \mathcal{O}_{K^+}^\times$. Moreover, the unit u is totally positive, because S and S_0 have the same signature. Then it follows from Lemma 3.4.14 that there exists an element $d \in K^\times$ which satisfies $u = \mathrm{N}_{K/K^+}(d)$, so by taking $\gamma = \beta/d$ we get ${}^t\gamma S\overline{\gamma} = S_0$. We conclude that $(\mathcal{M}, T)$ is equivalent to $(\gamma^{-1}\mathcal{M}, T_0)$. $\square$

Definition 3.4.16. Let $S \in \mathrm{GL}_m(K)$ be a hermitian matrix, and let $\mathcal{M}$ be an $\mathcal{O}_K$-lattice.

- ▷ The S-*norm* of $\mathcal{M}$ is the fractional $\mathcal{O}_{K^+}$-ideal $\mu^S(\mathcal{M}) = ({}^tuS\overline{u} : u \in \mathcal{M})$.
- ▷ The S-*scale* of $\mathcal{M}$ is the fractional $\mathcal{O}_K$-ideal $\mu_0^S(\mathcal{M}) = ({}^tuS\overline{v} : u, v \in \mathcal{M})$.
- ▷ An $\mathcal{O}_K$-lattice is S-*maximal* if it is inclusion-maximal among those with the same S-norm.
- ▷ We define the group of matrices $\mathcal{G}(S) = \{V \in \mathrm{GL}_m(K) : {}^tVSV = S\}$.

Theorem 3.4.17. Let K be a CM-field, let K^+ be its maximal totally real subfield and assume that their class numbers h_K, h_{K^+} are equal. Let m be an odd positive integer, let $S \in \mathrm{GL}_m(K)$ be a hermitian matrix and assume that there exists an embedding $\phi : K \to \mathbb{C}$ with respect to which the signature of S is neither $(m, 0)$ nor $(0, m)$. Then for every S-maximal $\mathcal{O}_K$-lattice $\mathcal{L}$ in K^m, the $\mathcal{G}(S)$-orbit of $\mathcal{L}$ consists of all the S-maximal $\mathcal{O}_K$-lattices $\mathcal{M}$ with the same S-norm.

Proof. In [40], Shimura introduces the concept of *genus* of $\mathcal{O}_K$-lattices with respect to $\mathcal{G}(S)$ as a local version of class with respect to $\mathcal{G}(S)$. Given a non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}_{K^+}$ we denote by $K_\mathfrak{p}^+$ the completition of K^+ with respect to $\mathfrak{p}$, and we write $K_\mathfrak{p} = K \otimes K_\mathfrak{p}^+$ and $\mathcal{L}_\mathfrak{p} = \mathcal{O}_{K_\mathfrak{p}^+}\mathcal{L}$. With this notation, two

$\mathcal{O}_K$-lattices belong to the same genus with respect to $\mathcal{G}(S)$ if for every $\mathfrak{p}$ there exists a matrix $U \in \mathcal{G}_\mathfrak{p}(S) = \{V \in \mathrm{GL}_3(K_\mathfrak{p}) : {}^tVSV = S\}$ such that $U\mathcal{L}_\mathfrak{p} = \mathcal{M}_\mathfrak{p}$.

By [40, Proposition 5.24(i)] we have that, since the respective class numbers of K and K^+ are equal, every genus with respect to $\mathcal{G}(S)$ consists of a single class. Moreover, by [40, Proposition 5.25(iv)] the genus of an S-maximal $\mathcal{O}_K$-lattice $\mathcal{L}$ with respect to $\mathcal{G}(S)$ is the set of all S-maximal $\mathcal{O}_K$-lattices $\mathcal{M}$ with the same S-norm, which completes the proof. □

We will use the following two easy lemmas, whose proofs are in Shimura [40].

Lemma 3.4.18 (Proposition 2.11 in Shimura [40])**.** Let K be a CM-field, let K^+ be its maximal totally real subfield, let m be a positive integer and let $\mathcal{D}_{K/K^+}$ be the relative different of K/K^+. Let $\mathcal{M}$ be an $\mathcal{O}_K$-lattice in K^m and let $S \in \mathrm{GL}_m(K)$ be a hermitian matrix. The S-norm $\mu^S(\mathcal{M})$ and S-scale $\mu_0^S(\mathcal{M})$ of $\mathcal{M}$ satisfy the inclusions

$$\mu^S(\mathcal{M})\mathcal{O}_K \subseteq \mu_0^S(\mathcal{M}) \subseteq \mathcal{D}_{K/K^+}^{-1}\mu^S(\mathcal{M}). \qquad \square$$

Lemma 3.4.19 (Proposition 2.14 in Shimura [40])**.** Let $\mathcal{M}$ be an $\mathcal{O}_K$-lattice in K^m and let $S \in \mathrm{GL}_m(K)$ be a hermitian matrix. There exists an S-maximal $\mathcal{O}_K$-lattice $\mathcal{L}$ that contains $\mathcal{M}$. □

Proposition 3.4.20. Let $K = \mathbb{Q}(\zeta_5)$ and let δ be as in Lemma 3.4.15. Let $(\mathcal{M}, T)$ be a principally polarized $\mathcal{O}_K$-lattice and let $S = \delta T$. Then we have

$$\mu_0^S(\mathcal{M}) = \mathcal{O}_K \quad \text{and} \quad \mu^S(\mathcal{M}) = \mathcal{O}_{K^+}.$$

Moreover let $\mathcal{M}' \supsetneq \mathcal{M}$ be an S-maximal $\mathcal{O}_K$-lattice with $\mu^S(\mathcal{M}') = \mu^S(\mathcal{M})$. Then we have

$$\mu_0^S(\mathcal{M}') = (\zeta_5 - 1)^{-1}\mathcal{O}_K.$$

Proof. We start by computing the S-scale of $\mathcal{M}$. Since $(\mathcal{M}, T)$ is principally polarized, given $u, v \in \mathcal{M}$ we have

$$\mathrm{tr}_{K/\mathbb{Q}}(r\,{}^tuT\overline{v}) \in \mathbb{Z} \text{ for all } r \in \mathcal{O}_K.$$

In consequence we get ${}^tuT\overline{v} \in \mathcal{O}_K^\vee = \delta^{-1}\mathcal{O}_K$, hence ${}^tuS\overline{v} \in \mathcal{O}_K$ holds. Conversely, we want to show that there exist $u, v \in \mathcal{M}$ that satisfy ${}^tuS\overline{v} = 1$. By Lemma 3.4.10 we assume without loss of generality $\mathcal{M} = \mathcal{O}_K^3$, so we have $S \in \mathcal{O}_K^{3\times 3}$, and by Proposition 3.4.9 the matrix S has determinant $\det S \in \mathcal{O}_{K^+}^\times$. Then, for $v = {}^t(1,0,0)$ and $u = {}^tS^{-1}\,{}^t(1,0,0) \in \mathcal{M}$ we have ${}^tuS\overline{v} = 1$.

We compute now the S-norm of $\mathcal{M}$. The different of K/K^+ is the prime ideal $\mathfrak{p} = (\zeta_5 - 1)\mathcal{O}_K$, hence by Lemma 3.4.18 we have

$$\mathfrak{p} \subseteq \mu^S(\mathcal{M})\mathcal{O}_K \subseteq \mathcal{O}_K.$$

But $\mu^S(\mathcal{M})$ is an $\mathcal{O}_{K^+}$-ideal, and $\sqrt{5}\mathcal{O}_{K^+}$ ramifies in K/K^+ into $\mathfrak{p}^2$, so we conclude $\mu^S(\mathcal{M}) = \mathcal{O}_{K^+}$.

For the second part of the statement consider again Lemma 3.4.18 for $\mathcal{M}'$. By assumption we have $\mu^S(\mathcal{M}') = \mu^S(\mathcal{M}) = \mathcal{O}_{K^+}$, hence we obtain

$$\mathcal{O}_K \subseteq \mu_0^S(\mathcal{M}') \subseteq \mathfrak{p}^{-1}. \tag{3.17}$$

We have $\mathcal{M}' \supsetneqq \mathcal{O}_K^3$, hence given an element

$$u = (u_1, u_2, u_3) \in \mathcal{M}' \backslash \mathcal{O}_K^3,$$

we assume without loss of generality $u_1 \notin \mathcal{O}_K$. Take also $v = \overline{S}^{-1}\,{}^t(1,0,0) \in \mathcal{M}'$. Then we have

$$\mu_0^S(\mathcal{M}') \ni {}^t u S \overline{v} = (u_1, u_2, u_3) S S^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = u_1 \notin \mathcal{O}_K,$$

thus we obtain $\mu_0^S(\mathcal{M}') \supsetneqq \mathcal{O}_K$. The result then follows from (3.17), since $\mathfrak{p}$ is a prime ideal. $\square$

Using the properties above and SageMath [49] we have found that, for $\mathfrak{p} = \mathcal{D}_{K/K^+} = (\zeta_5 - 1)\mathcal{O}_K$ and $S_0 = \delta T_0$, the $\mathcal{O}_K$-lattice

$$\mathcal{L} = \mathcal{O}_K^3 + \mathfrak{p}^{-1}(1,2,0) \tag{3.18}$$

is an S_0-maximal $\mathcal{O}_K$-lattice with S_0-norm $\mathcal{O}_{K^+}$ strictly containing $\mathcal{M}_0$, thus $\mathcal{M}_0$ is not S_0-maximal. Therefore we cannot use Theorem 3.4.17 directly on $\mathcal{M}_0$, but we use it on $\mathcal{L}$.

Proposition 3.4.21. Let $K = \mathbb{Q}(\zeta_5)$, let $(\mathcal{M}, T)$ be a principally polarized $\mathcal{O}_K$-lattice with $\operatorname{sign}(T) = ((3,2),(0,1))$ and let $\mathcal{L}$ be as in (3.18). Then there exists an $\mathcal{O}_K$-lattice $\mathcal{M}'$ in K^3 that satisfies $(\mathcal{M}, T) \sim (\mathcal{M}', T_0)$, $\mathcal{M}' \subseteq \mathcal{L}$ and $\mathcal{L}/\mathcal{M}' \cong \mathcal{O}_K/\mathfrak{p}$.

Proof. By Proposition 3.4.13 we can assume $T = T_0$. Let δ be as defined in Lemma 3.4.15, and define the hermitian matrix $S_0 = \delta T_0$. By Lemma 3.4.19 there exists $\mathcal{N} \supseteq \mathcal{M}$ S_0-maximal with S_0-norm $\mu^{S_0}(\mathcal{N}) = \mu^{S_0}(\mathcal{M})$, which by Proposition 3.4.20 satisfies $\mu^{S_0}(\mathcal{M}) = \mathcal{O}_{K^+}$.

Since both $\mathcal{N}$ and $\mathcal{L}$ are S_0-maximal $\mathcal{O}_K$-lattices with S_0-norm $\mathcal{O}_{K^+}$, we have by Theorem 3.4.17 that they are in the same $\mathcal{G}(S_0)$-orbit. Therefore there exists $\gamma \in \mathcal{G}(S_0)$ such that $\gamma\mathcal{N} = \mathcal{L}$.

For $\mathcal{M}' = \gamma\mathcal{M}$ we have $(\mathcal{M}, S_0) \sim (\mathcal{M}', S_0)$, and $\mathcal{M}'$ satisfies $\mathcal{M}' \subseteq \mathcal{L}$ and $\mu^{S_0}(\mathcal{M}') = \mathcal{O}_{K^+}$. Next we prove $\mathcal{L}/\mathcal{M}' \cong \mathcal{O}_K/\mathfrak{p}$.

By Proposition 3.4.20 we have $\mu_0^S(\mathcal{L}) = \mathfrak{p}^{-1}$, which implies $\mathfrak{p}\mathcal{L} \subseteq \mathcal{M}'$, hence the quotient $\mathcal{L}/\mathcal{M}'$ is an $(\mathcal{O}_K/\mathfrak{p})$-module. Therefore, since $\mathfrak{p}$ is prime, it is enough to show $[\mathcal{L}/\mathcal{M}'] = \mathfrak{p}$ or, equivalently, $\mathrm{N}_{K/K^+}([\mathcal{L}/\mathcal{M}']) = \sqrt{5}\mathcal{O}_{K^+}$. We have

$$\mathrm{N}_{K/K^+}([\mathcal{L}/\mathcal{M}']) = \mathrm{N}_{K/K^+}([\mathcal{L}/\mathcal{O}_K^3])\,\mathrm{N}_{K/K^+}([\mathcal{O}_K^3/\mathcal{M}']),$$

and since $(\mathcal{M}', S_0)$ is principally polarized, by Proposition 3.4.9 we have $\mathrm{N}_{K/K^+}([\mathcal{O}_K^3/\mathcal{M}']) = \mathcal{O}_{K^+}$. The equality $\mathrm{N}_{K/K^+}([\mathcal{L}/\mathcal{O}_K^3]) = \sqrt{5}\mathcal{O}_{K^+}$ holds by Proposition 3.4.8.(2), since we have $\mathcal{L} = L\mathcal{O}_K^3$ for the matrix

$$L = \begin{pmatrix} -\frac{3}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^2 + \frac{1}{5}\zeta_5 - \frac{2}{5} & -\frac{1}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^2 + \frac{2}{5}\zeta_5 + \frac{1}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we compute $\det L = \sqrt{5}$. This completes the proof. □

Using Script 1 in [46] we computed all $\mathcal{O}_K$-lattices $\mathcal{M}$ such that $\mathcal{L}/\mathcal{M}$ is isomorphic to $\mathcal{O}_K/\mathfrak{p}$, and we obtained 6 different $\mathcal{O}_K$-lattices.

By Lemma 3.4.10, for every $\mathcal{O}_K$-lattice $\mathcal{M}$ there exists $\gamma \in \mathrm{GL}_3(K)$ that satisfies $\mathcal{M} = \gamma\mathcal{O}_K$, so we computed the equivalent pair $\gamma^{-1}(\mathcal{M}, T_0) = (\mathcal{M}_0, T_{\mathcal{M}})$.

Then, for every $\mathcal{O}_K$-lattice that we obtained with Script 1 we managed to find $\alpha \in \mathrm{GL}_3(\mathcal{O}_K)$ that satisfies ${}^t\alpha T_{\mathcal{M}}\overline{\alpha} = T_0$. For the explicit computations see Script 2 in [46].

We conclude that all polarized $\mathcal{O}_K$-lattices $(\mathcal{M}, T_0)$ such that $\mathcal{L}/\mathcal{M}$ is isomorphic to $\mathcal{O}_K/\mathfrak{p}$ are equivalent to $(\mathcal{M}_0, T_0)$, so we can now prove the main theorem of this section.

Proof of Theorem 3.4.5. By Example 3.4.2 the pair $(\mathcal{M}_0, T_0)$ is equivalent to a pair $(\mathcal{M}_1, T_1) \in \Upsilon(\mathfrak{Z})$ that we obtained in Section 3.3 from a Riemann form with determinant 1.

Therefore, by Proposition 3.4.3, if $(\mathcal{M}, T) \in \Upsilon(\mathfrak{Z})$ is equivalent to the pair $(\mathcal{M}_0, T_0) \sim (\mathcal{M}_1, T_1)$, then the polarized abelian varieties $A(Z, \mathcal{M}, T)$ for $Z \in \mathcal{H}_3$ are principally polarized.

For the other implication let $(\mathcal{M}, T) \in \Upsilon(\mathfrak{Z})$ such that $\zeta_5\mathcal{M} \subseteq \mathcal{M}$. Then the pair $(\mathcal{M}, T)$ is a polarized $\mathcal{O}_K$-lattice whose hermitian matrix T has signature $((3,2),(0,1))$. If $A(Z, \mathcal{M}, T)$ is principally polarized, then $(\mathcal{M}, T)$ is principally polarized as a polarized $\mathcal{O}_K$-lattice.

Therefore it follows from Proposition 3.4.21 that there exists an equivalent pair $(\mathcal{M}', T_0)$ with $\mathcal{M}' \subseteq \mathcal{L}$ and $\mathcal{L}/\mathcal{M}' \cong \mathcal{O}_K/\mathfrak{p}$. But we have seen that there are only 6 possibilities for such $\mathcal{M}'$ and they all satisfy $(\mathcal{M}', T_0) \sim (\mathcal{M}_0, T_0)$, hence the result follows. □

3.5 The Torelli locus of CPQ curves

In this section we solve the Riemann-Schottky problem for CPQ curves and give a result analogous to Proposition 1.4.1 for the family $\mathcal{S}$ of CPQ curves.

In Proposition 1.4.1 we focused on principally polarized abelian threefolds, which all are Jacobians of curves, so we only needed to prove that the curves were Picard curves. The main obstacle to obtain an analogous result for CPQ curves is that, since CPQ curves have genus 6, their Jacobians have dimension 6, but when considering 6-dimensional principally polarized abelian varieties, not all of them are Jacobians of curves.

However, we have seen in Proposition 3.3.1 that Jacobians of CPQ curves have 3-CM-type

$$\mathfrak{Z} = (\mathbb{Q}(\zeta_5), (\phi_1, \phi_2), (3,2), (0,1)) \text{ with } \phi_i : \mathbb{Q}(\zeta_5) \to \mathbb{C} \text{ given by } \phi_i(\zeta_5) = z_5^i. \tag{3.19}$$

Therefore we focus on the set of principally polarized abelian varieties with 3-CM-type $\mathfrak{Z}$ and order-5 automorphisms. Let $\mathcal{M}_0 = \mathcal{O}_K^3$ and consider T_0 as in Example 3.4.2. Let A_6 be the smooth algebraic variety as in [28], which has complex points

$$\mathsf{A}_6(\mathbb{C}) = \mathrm{Sp}_{12}(\mathbb{Z})\backslash\mathbf{H}_6,$$

and let $\mathsf{A}_3 \subseteq \mathsf{A}_6$ be the image of $\mathcal{H}_3$ by the map

$$\begin{aligned} \mathcal{H}_3 &\to \mathrm{Sp}_{12}(\mathbb{Z})\backslash\mathbf{H}_6, \\ Z &\mapsto (\text{the class of a period matrix } \Omega \text{ of } A(Z, \mathcal{M}_0, T_0)). \end{aligned}$$

Let M_6 be the moduli space of genus-6 curves. We define the *open Torelli locus* T_6° as the image $J(\mathsf{M}_6) \subseteq \mathsf{A}_6$ of M_6 by the Torelli map J, and we call its Zariski closure $\mathsf{T}_6 = \overline{\mathsf{T}_6^\circ}$ the *Torelli locus*. The points $X \in \mathsf{T}_6\backslash\mathsf{T}_6^\circ$ correspond to decomposable principally polarized abelian varieties. For more details, see Section 1 in Moonen-Oort [28].

In order to prove our result, we need to assume the following conjecture.

Conjecture 3.5.1. The set A_3 is an algebraic subset of the variety A_6, that is, a Zariski-closed subset.

Remark 3.5.2. We are convinced that this follows from the basics of Shimura varieties. However, since we are not familiar enough with the theory and due to time restrictions we have not been able to prove the conjecture yet.

Theorem 3.5.3. Assume that Conjecture 3.5.1 holds and let X be a simple principally polarized abelian variety of dimension 6 over $\mathbb{C}$. The following are equivalent:

(1) The principally polarized abelian variety X has an automorphism φ of order 5 such that the eigenvalues of $\rho_a(\varphi)$ are z_5, z_5^2 and z_5^3 with multiplicity 3, 2 and 1 respectively.
(2) There exists a cyclic plane quintic curve C that satisfies $X \cong J(C)$ and for $\rho \in \mathrm{Aut}(C)$ given by $\rho(x, y) = (x, \zeta_5 y)$, we get $\varphi = \rho_*$.

For the last step of the proof we will need the following result:

Lemma 3.5.4. If C is a curve given by $y^5 = f(x)$ where the x-map $C \to \mathbb{P}_1$ has 5 ramification points, then C is isomorphic to a curve with one of the following forms:

(1) $y^5 = x(x-1)(x-\lambda)(x-\mu)$,
(2) $y^5 = x^3(x-1)(x-\lambda)(x-\mu)$, or
(3) $y^5 = x^2(x-1)^2(x-\lambda)(x-\mu)$.

Moreover if ρ is the automorphism of C given by $\rho(x, y) = (x, z_5 y)$, then the eigenvalues of $\rho_a(\rho_*)$ are in each case

(1) z_5, z_5^2 and z_5^3 with multiplicity 3, 2 and 1 respectively;
(2) z_5, z_5^2, z_5^3 and z_5^4 with multiplicity 2, 2, 1 and 1 respectively; or
(3) z_5, z_5^2, z_5^3 and z_5^4 with multiplicity 2, 1, 2 and 1 respectively.

Proof. Let C be given by

$$y^5 = f(x) := (x-a_1)^{e_1}(x-a_2)^{e_2}(x-a_3)^{e_3}(x-a_4)^{e_4}(x-a_5)^{e_5},$$

for $a_i \in \mathbb{C}$, $e_i \in \mathbb{Z}_{\geq 0}$ and $a_i \neq a_j$ if $i \neq j$.

Since the curve is ramified exactly at the points $(a_i, 0)$, we have

$$e_i \not\equiv 0 \pmod 5 \quad \text{and} \quad \sum_{i=1}^{5} e_i \equiv 0 \pmod 5.$$

Furthermore, we only need to consider the class $\overline{e_i} = (e_i \bmod 5)$, and the polynomials f^k with $k \not\equiv 0 \pmod 5$ all give the same curve, thus we can consider the vector $(\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}, \overline{e_5}) \in (\mathbb{Z}/5\mathbb{Z})^5$ up to multiplication by $(\mathbb{Z}/5\mathbb{Z})^\times$.

With these conditions we obtain three possible exponent vectors, which are precisely $(1,1,1,1,1)$, $(1,1,1,3,4)$ and $(1,1,2,2,4)$. We can then consider an isomorphic curve where the points with larger exponents are at $(1:0:0)$, $(0:0:1)$ and $(1:0:1)$, thus obtaining the models in the statement.

The first case corresponds to the family of CPQ curves, so we have already computed the eigenvalues of $\rho_a(\rho_*)$ in that case. For the remaining two cases we computed a basis of differentials using the *algcurves* library [8] in Maple [27]. We obtained respectively the bases

$$\left(\frac{dx}{y}, \frac{xdx}{y^2}, \frac{xdx}{y^3}, \frac{x^2dx}{y^3}, \frac{x^2dx}{y^4}, \frac{x^3dx}{y^4}\right)$$

and

$$\left(\frac{dx}{y}, \frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{x(x-1)dx}{y^3}, \frac{x(x-1)dx}{y^4}, \frac{x(x^2-1)dx}{y^4}\right).$$

Then we computed the eigenvalues of $\rho_a(\rho_*)$ by considering the action of ρ^* on the basis of $H^0(\omega_C)$ for each case, as we did for CPQ curves in Section 3.3. $\square$

We prove now the main result of the chapter.

Proof of Theorem 3.5.3. That (2) implies (1) follows from Proposition 3.3.1. We will now prove the converse.

Suppose that X satisfies (1) and let $\mathfrak{Z}$ be the 3-CM-type defined in (3.19). We start by proving that X is in A_3. Then we show that A_3 is an irreducible subvariety of the Torelli locus, hence there exists $C \in \mathsf{M}_6$ whose Jacobian is isomorphic to X, and finally we prove that C is a CPQ curve.

Let λ be the principal polarization of X and consider the embedding $\iota : \mathbb{Q}(\zeta_5) \to \mathbb{C}$ given by $\iota(\zeta_5) = \varphi$. As φ is an automorphism of the principally polarized abelian variety X, it satisfies $\lambda = \widehat{\varphi} \circ \lambda \circ \varphi$. We obtain

$$\iota(\zeta_5)' = \varphi' = \lambda^{-1} \circ \widehat{\varphi} \circ \lambda = \varphi^{-1} = \iota(\zeta_5)^{-1} = \iota(\overline{\zeta_5}),$$

thus the Rosati involution on $\mathrm{End}(X) \otimes \mathbb{Q}$ with respect to the polarization λ extends the complex conjugation on K via ι.

Then the triple (X, λ, ι) has 3-CM-type $\mathfrak{Z}$, hence by Proposition 3.2.5 there exist $(\mathcal{M}, T) \in \Upsilon(\mathfrak{Z})$ and $Z \in \mathcal{H}_3$ that satisfy $A(Z, \mathcal{M}, T) \cong (X, \lambda, \iota)$. Since we have $\iota(\zeta_5) = \varphi \in \mathrm{End}(X)$, the lattice $\mathcal{M}$ satisfies $\zeta_5\mathcal{M} \subseteq \mathcal{M}$.

By Theorem 3.4.5 the pair $(\mathcal{M}, T)$ is equivalent to $(\mathcal{M}_0, T_0)$, so it follows from Proposition 3.4.3 that there exists $Z' \in \mathcal{H}_3$ such that $A(Z', \mathcal{M}_0, T_0)$ is isomorphic to $A(Z, \mathcal{M}, T) \cong (X, \lambda, \iota)$. We conclude that the class of X is in A_3.

Next we prove that A_3 is an irreducible subvariety of the Torelli locus. On the one hand, the complex manifold $\mathcal{H}_3 = \mathcal{H}_{3,0} \times \mathcal{H}_{2,1} \cong \mathcal{H}_{2,1}$ is irreducible and has dimension 2, hence if A_3 is an algebraic subset of A_6, then it is an irreducible closed subvariety of A_6. On the other hand, the family of CPQ curves $\mathcal{S} \subseteq \mathsf{M}_6$ also has dimension 2, as can be seen from the Legendre-Rosenhain equation $y^5 = x(x-1)(x-\lambda)(x-\mu)$. The closure S of its image $J(\mathcal{S})$ by the Torelli map is then a dimension-2 closed subvariety, and since by Proposition 3.3.1 we have $J(S) \subseteq \mathsf{A}_3$, we get that S is also contained in A_3.

It follows that S is a closed irreducible subvariety of the irreducible variety A_3. Therefore, by definition of dimension (see Definition 2.48 in Milne [26]), we obtain $\mathsf{S} = \mathsf{A}_3$.

We conclude that the class of X is in $\mathsf{S} \subseteq \mathsf{T}_6$ and, as X is simple, it is in fact in T_6°, so there is a curve C that satisfies $J(C) \cong X$.

Finally we prove that C is a CPQ curve and $\varphi = \rho_*$.

By Torelli's Theorem 1.1.1, there is some non-trivial $\nu \in \mathrm{Aut}(C)$ such that $\varphi = \pm\nu_*$. Then the automorphism $\eta = \nu^6$ satisfies $\eta_* = (\nu^6)_* = (\pm\nu)_*^6 = \varphi^6 = \varphi$, hence by the uniqueness in Torelli's Theorem 1.1.1 we get that η has order 5.

It follows that the projection $\pi : C \to C/\langle\eta\rangle$ is a cyclic Galois covering map of degree 5, hence all the ramification indices of π are either 1 or 5. Therefore by the Riemann-Hurwitz formula one obtains that $C/\langle\eta\rangle$ has either genus 0 or 2. But X is simple, so the curve $C/\langle\eta\rangle$ is isomorphic to $\mathbb{P}^1$ and the map π has 5 ramification points.

Then $k(C)/k(C/\langle\eta\rangle)$ is a Kummer extension of degree 5, hence C is given by an equation of the form $y^5 = f(x)$, the x-map π has 5 ramification points, and η is a power of the automorphism ρ given by $(x, y) \mapsto (x, z_5 y)$. We conclude by Lemma 3.5.4 that, as the eigenvalues of $\rho_a(\varphi)$ are z_5, z_5^2 and z_5^3 with multiplicity 3, 2 and 1 respectively, the curve C is isomorphic to a curve of the form $y^5 = x(x-1)(x-\lambda)(x-\mu)$, that is, the curve C is a CPQ curve, and η is equal to ρ. $\square$

It follows from Theorem 3.5.3 that if Conjecture 3.5.1 holds, then one can think of the input in Algorithm 2.2.6 as just a principally polarized abelian variety of dimension 6 with an order-5 automorphism whose analytic representation has the right eigenvalues.