Universiteit Leiden
The Netherlands

# Inverse Jacobian and related topics for certain superelliptic curves

Somoza Henares, A.

**Citation**
Somoza Henares, A. (2019, March 28). *Inverse Jacobian and related topics for certain superelliptic curves*. Retrieved from https://hdl.handle.net/1887/70564

| | |
|---|---|
| Version: | Not Applicable (or Unknown) |
| License: | Leiden University Non-exclusive license |
| Downloaded from: | https://hdl.handle.net/1887/70564 |

**Note:** To cite this publication please use the final published version (if applicable).

Cover Page

Universiteit Leiden

Leiden University Repository

The handle http://hdl.handle.net/1887/70564 holds various files of this Leiden University dissertation.

**Author**: Somoza Henares, A.
**Title**: Inverse Jacobian and related topics for certain superelliptic curves
**Issue Date**: 2019-03-28

# The family of Picard curves 1

A *Picard curve* over $\mathbb{C}$ is a genus-3 smooth, plane, projective curve given by $y^3 = f(x)$ where $f$ is a polynomial of degree 4. Such a curve has an automorphism $\rho$ of order 3 given by $(x, y) \mapsto (x, z_3 y)$ with $z_3 = \exp\left(\frac{2\pi i}{3}\right)$. It fixes the points $(t, 0)$ with $f(t) = 0$, the *affine branch points* of $C$. The curve $C$ also has a unique point at infinity, with projective coordinates $(0 : 1 : 0)$, which is also fixed by the automorphism $\rho$.

One can check that all isomorphisms between Picard curves are of the form

$$(x, y) \mapsto (ax + b, cy),$$

see Section 7.3 in Estrada [11, Appendix I] for details. Therefore, given a Picard curve $C$, every ordering of the affine branch points of $C$ gives rise to an isomorphic Picard curve given by an equation of the form

$$y^3 = x(x-1)(x-\lambda)(x-\mu) \tag{1.1}$$

with the first affine branch point at $(0, 0)$, the second at $(0, 1)$, the third at $(0, \lambda)$ and the forth at $(0, \mu)$. We refer to the form (1.1) as a *Legendre-Rosenhain equation of a Picard curve*.

In this chapter we present a method that, given the period matrix of the Jacobian of a Picard curve, gives a numerical approximation of the equation of the curve. This was initially done by Koike and Weng in [16], but their exposition presents some gaps and mistakes that we fix in this chapter, see Remarks 1.2.14, 1.3.8, and 1.4.2.

We start by introducing some concepts needed throughout this thesis in Section 1.1, such as principally polarized abelian variety, the Jacobian of a curve and the Riemann-Schottky problem.

In Section 1.2 we give a formula to approximate the $x$-coordinates of the affine branch points of a Picard curve in terms of theta constants on its Jacobian, see Theorem 1.2.13.

In Section 1.3 we develop an algorithm that given the Jacobian of a Picard curve $C$ returns the Legendre-Rosenhain equation of $C$, see Algorithm 1.3.9. The main step of the algorithm is applying the formula in Theorem 1.2.13, so we first identify the objects needed to apply said formula, such as the Riemann constant and the images by the Abel-Jacobi map of the affine branch points.

Finally, in Section 1.4 we characterize the polarized abelian varieties that arise as Jacobians of Picard curves, see Proposition 1.4.1, and in Section 1.5 we give some details on the implementation of Algorithm 1.3.9 and show examples of curves obtained using the algorithm.

This chapter is based on joint work with Joan-Carles Lario. In particular, Theorem 1.2.13 and the examples in Section 1.5 appeared before up to minor corrections in Joan-Carles Lario and Anna Somoza, *A note on Picard curves of CM-type*, arXiv:1611.02582 [21].

## 1.1 Preliminaries on abelian varieties

In this section we review some notions that will be needed throughout this thesis. We follow classical references such as Birkenhake-Lange [2], Lang [19], Milne [24, 25] or Mumford [30].

### 1.1.1 Polarized abelian varieties

An *abelian variety* $X$ over a field $k$ is a complete irreducible group variety defined over $k$, and it is smooth, projective and commutative. A *homomorphism of abelian varieties* is a morphism that respects the group structure. It is an *isogeny* if it is surjective and the abelian varieties have the same dimension. We say that an abelian variety is *absolutely simple* if it has no non-zero proper abelian subvarieties over the algebraic closure $\overline{k}$ of $k$.

Given an abelian variety $X$ defined over $k$, we define the *Picard group of* $X$ as the group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles on $X_{\overline{k}}$. Given a line bundle $\mathcal{L}$ on $X_{\overline{k}}$, we define the map

$$\begin{aligned} \phi_{\mathcal{L}} : X(\overline{k}) &\to \mathrm{Pic}(X) \\ x &\mapsto [T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}], \end{aligned}$$

where $T_x$ stands for the translation by $x$ on $X_{\overline{k}}$ and $[\mathcal{L}]$ stands for the isomorphism class of $\mathcal{L}$ in $\mathrm{Pic}(X)$. The map $\phi_{\mathcal{L}}$ is a homomorphism, see Corollary 4 in Mumford [30, Section 2.6].

We define $\mathrm{Pic}^0(X)$ as the subgroup of $\mathrm{Pic}(X)$ consisting of classes of line bundles $\mathcal{L}$ such that the map $\phi_{\mathcal{L}}$ is zero. It is the group of $\overline{k}$-points of an abelian variety over $k$ (see Section 2.8 in Mumford [30]), we call it the *dual variety of* $X$, and denote it by $\widehat{X}$.

A homomorphism of abelian varieties $f : X \to Y$ induces a map $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ that maps $\mathrm{Pic}^0(Y)$ to $\mathrm{Pic}^0(X)$, which gives us the dual homomorphism $\widehat{f} : \widehat{Y} \to \widehat{X}$.

We define a *polarization* on $X$ as an isogeny $\lambda = \phi_{\mathcal{L}}$ where $\mathcal{L}$ is an ample line bundle on $X_l$ for $l \supseteq k$ a finite separable extension of the field of definition $k$. It is called *principal* if it is an isomorphism. We say that a polarized abelian variety $(X, \lambda)$ is defined over $k$ if both $X$ and $\lambda$ are defined over $k$.

Two polarized abelian varieties $(X_1, \lambda_1)$ and $(X_2, \lambda_2)$ are *isomorphic* if there exists an isomorphism of abelian varieties $f : X_1 \to X_2$ that is compatible with the polarizations, meaning that it satisfies $\lambda_1 = \widehat{f} \circ \lambda_2 \circ f$.

Given a polarization $\lambda : X \to \widehat{X}$ and an endomorphism $f \in \mathrm{End}(X) \otimes \mathbb{Q}$, we define

$$f' := \lambda^{-1} \circ \widehat{f} \circ \lambda. \tag{1.2}$$

The map $\cdot' : \mathrm{End}(X) \otimes \mathbb{Q} \to \mathrm{End}(X) \otimes \mathbb{Q}$ given by $f \mapsto f'$ is an involution on $\mathrm{End}(X) \otimes \mathbb{Q}$, and we call it the *Rosati involution determined by* $\lambda$.

### 1.1.2 Polarized abelian varieties over $\mathbb{C}$ and complex tori

When considering an abelian variety $X$ defined over $\mathbb{C}$, the complex manifold $X(\mathbb{C})$ is (complex analytically isomorphic to) a *polarizable complex torus*, that is, a complex vector space $V$ modulo a lattice $\Lambda$ of full rank that admits a *Riemann form*. A Riemann form is an anti-symmetric form $E\colon V \times V \to \mathbb{R}$ that is $\mathbb{R}$-bilinear, satisfies $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$, such that for $u, v \in V$ we have $E(iu, v) = E(iv, u)$, and such that the associated hermitian form

$$H(u, v) = E(iu, v) + iE(u, v) \tag{1.3}$$

is positive definite. A polarization of an abelian variety $X$ defined over $\mathbb{C}$ determines a Riemann form $E$ on the complex torus $X(\mathbb{C})$, and the determinant of $E$ with respect to $\Lambda$ is $\det E = 1$ if and only if the polarization is principal. For more details on how the two are related see [19, Section 3.4].

Given a principally polarized complex torus $V/\Lambda$ of dimension $g$, we can choose bases $e_1, \ldots, e_g$ of $V$ and $\lambda_1, \ldots, \lambda_{2g}$ of $\Lambda$. Writing the latter in terms of the former, $\lambda_i = \sum_{j=1}^{g} l_{j,i} e_j$, defines a $g \times 2g$ matrix over $\mathbb{C}$,

$$\Pi = \begin{pmatrix} l_{1,1} & \cdots & \cdots & l_{1,2g} \\ \vdots & & & \vdots \\ l_{g,1} & \cdots & \cdots & l_{g,2g} \end{pmatrix}, \tag{1.4}$$

called a *big period matrix* of $V/\Lambda$, and we get $V/\Lambda \cong \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$. Moreover, the form $E$ is given with respect to the basis of $\Lambda$ by the matrix $M_E = (E(\lambda_i, \lambda_j))_{ij} \in \mathbb{Z}^{2g\times 2g}$. Analogously, the form $H$ is given with respect to the basis of $V$ by the matrix $M_H = (H(e_i, e_j))_{ij} \in \mathbb{C}^{g\times g}$. These matrices satisfy the relation

$$M_H = 2i(\Pi M_E^{-1}\, {}^t\Pi)^{-1}. \tag{1.5}$$

We say that the basis $(\lambda_i)_i$ is *symplectic* if the matrix $M_E$ of $E$ with respect to that basis is

$$\begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}. \tag{1.6}$$

In that case, the vectors $\lambda_{g+1}, \ldots, \lambda_{2g}$ form a basis of $V$ and if we choose this basis of $V$, then we obtain a big period matrix of the form $(\Omega, \mathbf{1}_g)$ with $\Omega \in \mathbb{C}^{g\times g}$ symmetric and with positive definite imaginary part. We call the matrix $\Omega$ a *period matrix*, and we define the *Siegel upper-half space* $\mathbf{H}_g$ to be the set of matrices $\Omega \in \mathbb{C}^{g\times g}$ symmetric and with positive definite imaginary part.

We say that a principally polarized complex abelian variety $X$ has period matrix $\Omega \in \mathbf{H}_g$ if $X(\mathbb{C})$ is isomorphic to $\mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$ and the Riemann form determined by the polarization of $X$ is given by the matrix (1.6).

A *homomorphism* between complex tori is a holomorphic map $f$ from $V/\Lambda$ to $V'/\Lambda'$ that respects the group structure. In particular, it lifts to a $\mathbb{C}$-linear map $F : V \to V'$ that satisfies $F(\Lambda) \subseteq \Lambda'$.

This gives the map

$$\begin{aligned} \rho_a : \mathrm{Hom}(V/\Lambda, V'/\Lambda') &\to \mathrm{Hom}(V, V') \\ f &\mapsto F, \end{aligned}$$

the *analytic representation* of $\mathrm{Hom}(V/\Lambda, V'/\Lambda')$; and considering the restriction of $F$ to the lattice we obtain the map

$$\begin{aligned} \rho_r : \mathrm{Hom}(V/\Lambda, V'/\Lambda') &\to \mathrm{Hom}(\Lambda, \Lambda') \\ f &\mapsto F|_\Lambda, \end{aligned}$$

the *rational representation*.

Let now $\Pi \in \mathbb{C}^{g \times 2g}$ and $\Pi' \in \mathbb{C}^{g' \times 2g'}$ be big period matrices of $V/\Lambda$ and $V'/\Lambda'$ respectively. With respect to the chosen bases, the analytic representation $\rho_a(f)$ is a $g' \times g$ matrix over $\mathbb{C}$, and the rational representation $\rho_r(f)$ is a $2g' \times 2g$ matrix over $\mathbb{Z}$. They are related by the equation

$$\rho_a(f)\Pi = \Pi' \rho_r(f). \tag{1.7}$$

In the case where $f : (V/\Lambda, E) \to (V'/\Lambda', E')$ is an isomorphism of principally polarized abelian varieties we also have for all $u, v \in \mathbb{C}^g$ the equality $E(u,v) = E'(f(u), f(v)) =: f^*E'(u,v)$ . Assume now that the chosen bases are symplectic, so that the abelian varieties have respectively $\Omega, \Omega' \in \mathbf{H}_g$ as period matrices. In terms of matrices, the relation $f^*E' = E$ becomes

$$ {}^tN \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix} N = \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}, \tag{1.8}$$

for $N$ the matrix of $\rho_r(f)$ with respect to symplectic bases of $\Lambda$ and $\Lambda'$. We define the *symplectic group* $\mathrm{Sp}_{2g}(\mathbb{Z})$ as the group of matrices in $\mathbb{Z}^{2g \times 2g}$ that satisfy (1.8), so we have $\rho_r(f) \in \mathrm{Sp}_{2g}(\mathbb{Z})$.

Let $M$ be the transpose matrix of $\rho_r(f)$ and consider the subdivision in $g \times g$ blocks

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = {}^t\rho_r(f).$$

It follows from (1.7) and the symmetry of the period matrices that the matrix of $\rho_a(f)$ with respect to these bases for $\Lambda$ and $\Lambda'$ is

$$ {}^t(\gamma\Omega' + \delta) \tag{1.9}$$

and the period matrices $\Omega, \Omega'$ are related by the equation

$$\Omega = (\alpha\Omega' + \beta)(\gamma\Omega' + \delta)^{-1} =: M(\Omega'). \tag{1.10}$$

In particular, this relation gives an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbf{H}_g$. For details, see Section 8.2 in [2].

### 1.1.3 Jacobians and the Abel-Jacobi map

Let now $C$ be a *curve* of genus $g$ defined over a field $k$, that is, a smooth, projective, geometrically irreducible algebraic curve over $k$ of genus $g$. For such a curve $C$, let $\mathrm{Div}(C)$ (respectively $\mathrm{Div}^0(C)$) be the set of divisors on $C_{\overline{k}}$ (resp. degree-0 divisors on $C_{\overline{k}}$), let $\mathrm{Prin}(C)$ be the set of principal divisors and define $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/\mathrm{Prin}(C)$.

To the curve $C$ over $k$ one can associate in a natural way a principally polarized abelian variety of dimension $g$ over $k$, its *Jacobian* $J(C)$. We have $J(C)(\overline{k}) = \mathrm{Pic}^0(C)$, and denote by $\lambda_C$ its natural polarization. Its dimension is equal to the genus of $C$. Given a point $P \in C(\overline{k})$, we define the *Abel-Jacobi map with base point* $P$ as the morphism of varieties over $\overline{k}$ given by

$$\begin{aligned} \alpha : C &\to J(C) \\ Q &\mapsto [Q - P], \end{aligned} \tag{1.11}$$

and we extend it additively to divisors.

Given a morphism of curves $\varphi : C \to C'$, let $J(C)$ and $J(C')$ be respectively the Jacobians of $C$ and $C'$. The morphism $\varphi$ induces the homomorphisms $\varphi_* : \mathrm{Div}(C) \to \mathrm{Div}(C')$ given by $[P] \mapsto [\varphi(P)]$, and $\varphi^* : \mathrm{Div}(C') \to \mathrm{Div}(C)$ given by $[Q] \mapsto \sum_{P \in \varphi^{-1}(Q)} e_\varphi(P)[P]$, where $e_\varphi(P)$ is the order at $P$ of the function $t \circ \varphi$ for $t$ a uniformizer at $Q$. These homomorphisms map degree-0 divisors to degree-0 divisors and principal divisors to principal divisors, so they induce homomorphisms $\varphi_* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C')$ and $\varphi^* : \mathrm{Pic}^0(C') \to \mathrm{Pic}^0(C)$.

In particular, for $\alpha, \alpha'$ the Abel-Jacobi maps with base point $P \in C$ and $\varphi(P) \in C'$ respectively, the diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & C' \arrow[d, "\alpha'"] \\ J(C) \arrow[r, "\varphi_*"] & J(C') \end{tikzcd} \tag{1.12}$$

commutes. Conversely, an isomorphism of Jacobians determines an isomorphism between the corresponding curves, due to the following result:

**Theorem 1.1.1** (Torelli, see Milne [25, Section 12]). Let $C$ and $C'$ be curves over an algebraically closed field $k$, and let $\alpha, \alpha'$ be the Abel-Jacobi maps with base point $P \in C$, $\varphi(P) \in C'$ respectively. Let $\varphi : J(C) \to J(C')$ be an isomorphism of principally polarized abelian varieties.

(1) There exists an isomorphism $\rho : C \to C'$ that satisfies $\varphi = \pm\rho_*$.

(2) Assume that the curves have genus $\geq 2$. If $C$ is not hyperelliptic, then the map $\rho$ and the sign $\pm$ are uniquely determined by $\varphi$. If $C$ is hyperelliptic, then the sign can be chosen arbitrarily, and $\rho$ is uniquely determined by $\varphi$ and $\pm$. $\square$

Torelli's Theorem implies the injectivity of the map $J$, the *Torelli map*, from the set of curves of genus $g$ over $\overline{k}$ up to isomorphism to the set of isomorphism classes of principally polarized abelian varieties of dimension $g$ over $\overline{k}$. This motivates the *Riemann-Schottky problem*.

**The Riemann-Schottky problem.** Describe the image of $J$.

Our goal throughout this chapter is to give an inverse Jacobian algorithm restricted to the family $\mathcal{P}$ of Picard curves. We present Algorithm 1.3.9 which, given $X \in J(\mathcal{P})$, determines a curve $C$ with $X \cong J(C)$. Moreover, in Section 1.4 we also give a characterization of the absolutely simple principally polarized abelian varieties in $J(\mathcal{P})$.

**Proposition 1.1.2.** Every Picard curve is *non-hyperelliptic*, that is, the canonical map $C \to \mathbb{P}^2$ is an embedding.

*Proof.* One computes that a basis of regular differentials for a Picard curve is

$$\left(\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y}\right).$$

It follows that the canonical map is the embedding $(x : y : 1) : C \to \mathbb{P}^2$. □

### 1.1.4 Jacobians and the Abel-Jacobi map over $\mathbb{C}$

For a curve $C$ defined over $\mathbb{C}$, its Jacobian is also defined over $\mathbb{C}$ and therefore isomorphic to a principally polarized complex torus. We now construct this torus explicitly, as in Birkenhake-Lange [2, Section 11.1].

Let $H^0(\omega_C)$ be the complex vector space of regular differentials of $C$, and let $H^0(\omega_C)^*$ denote its dual. The homology $H_1(C, \mathbb{Z})$ of $C$ injects into $H^0(\omega_C)^*$ via the map $H_1(C, \mathbb{Z}) \to H^0(\omega_C)^*$ given by $\gamma \mapsto (\omega \mapsto \int_\gamma \omega)$, where the integral is taken for a representative of the class $\gamma \in H_1(C, \mathbb{Z})$.

The image of $H_1(C, \mathbb{Z})$ in $H^0(\omega_C)^*$ is a lattice of rank $2g$ in a complex vector space of dimension $g$. The Jacobian of $C$ is isomorphic to the $g$-dimensional complex torus given by the quotient $H^0(\omega_C)^*/H_1(C, \mathbb{Z})$, and the Riemann form is given by the oriented intersection paring on $H_1(C, \mathbb{Z})$.

**Theorem 1.1.3.** (Abel-Jacobi, see [2, Theorem 11.1.3]) Let $C$ be a curve and let $P \in C$. The map

$$\begin{aligned} C &\to H^0(\omega_C)^*/H_1(C, \mathbb{Z}), \\ Q &\mapsto \left\{\omega \mapsto \int_P^Q \omega\right\} \end{aligned} \tag{1.13}$$

induces a canonical isomorphism $\mathrm{Pic}^0(C) \to H^0(\omega_C)^*/H_1(C, \mathbb{Z})$, which does not depend on $P$. □

When we identify $J(C)$ with $H^0(\omega_C)^*/H_1(C, \mathbb{Z})$, the map (1.13) is the Abel-Jacobi map with base point $P$ as in (1.11).

## 1.2 A Thomae-like formula

In this section we present a formula that gives the $x$-coordinates of the affine branch points of a Picard curve $C$ given by a Legendre-Rosenhain equation as a quotient of Riemann theta functions evaluated at certain points of the Jacobian $J(C)$. We start by defining these functions.

**Definition 1.2.1.** The *Riemann theta function* is the function $\theta : \mathbb{C}^g \times \mathbf{H}_g \to \mathbb{C}$ given by

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z).$$

**Theorem 1.2.2** (Riemann's Vanishing Theorem, see [29, Corollary 3.6])**.** Let $C$ be a curve over $\mathbb{C}$ of genus $g$, let $J(C)$ be the Jacobian of $C$ with period matrix $\Omega \in \mathbf{H}_g$ and let $\alpha$ be an Abel-Jacobi map of $C$. There is an element $\Delta \in J(C)$, called a *Riemann constant* with respect to $\alpha$, such that the function $\theta(\,\cdot\,, \Omega)$ vanishes at $z \in \mathbb{C}^g$ if and only if there exist $Q_1, \dots, Q_{g-1} \in C$ that satisfy

$$z \equiv \alpha(Q_1 + \cdots + Q_{g-1}) - \Delta \mod \Omega\mathbb{Z}^g + \mathbb{Z}^g. \qquad \square$$

Next we prove that $\Delta$ is actually unique up to the choice of a base point for the Abel-Jacobi map $\alpha$. We will use the following lemma.

**Lemma 1.2.3.** Let $\Omega \in \mathbf{H}_g$ and let $\Theta \subseteq \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$ be the subset defined by $\theta(z, \Omega) = 0$. Then the map

$$\begin{aligned} \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g) &\to \{e + \Theta : e \in \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)\} \\ x &\mapsto \{z \in \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g) : \theta(z - x, \Omega) = 0\} = x + \Theta \end{aligned}$$

is injective.

*Proof.* See the proof of Theorem II.3.10(b) in Mumford [29]. $\square$

**Proposition 1.2.4.** Let $C$ be a curve over $\mathbb{C}$ of genus $g$, let $J(C)$ be the Jacobian of $C$ with period matrix $\Omega \in \mathbf{H}_g$, and let $\alpha$ be the Abel-Jacobi map with base point $P \in C$. The Riemann constant $\Delta$ with respect to $\alpha$ is uniquely determined by Theorem 1.2.2 and satisfies

$$2\Delta = \alpha(\kappa)$$

for $\kappa$ a canonical divisor of $C$.

*Proof.* For the first part of the statement, let $\Delta^1, \Delta^2 \in J(C)$ satisfy Theorem 1.2.2, that is, the equality $\Theta = \alpha(\mathrm{Sym}^{g-1} C) - \Delta^i$. We have

$$\Theta = \alpha(\mathrm{Sym}^{g-1} C) - \Delta^1 = \alpha(\mathrm{Sym}^{g-1} C) - \Delta^2 + \Delta^2 - \Delta^1 = \Theta + (\Delta^2 - \Delta^1)$$

thus it follows from Lemma 1.2.3 that $\Delta^2 - \Delta^1$ is zero, hence the Riemann constant is unique.

For the second part, consider an effective divisor $D = \sum_{i=1}^{g-1} P_i$ for $P_i \in C$. By the Riemann-Roch Theorem, there exist $g-1$ points $Q_1, \ldots, Q_{g-1}$ in $C$ that satisfy

$$\kappa - D \sim \sum_{i=1}^{g-1} Q_i,$$

or equivalently, $\alpha(\kappa - D) = \alpha(\sum_{i=1}^{g-1} Q_i)$. We get

$$\alpha(\kappa) - \alpha(\operatorname{Sym}^{g-1} C) \subseteq \alpha(\operatorname{Sym}^{g-1} C).$$

If we consider the translation $-\alpha(\operatorname{Sym}^{g-1} C) \subseteq \alpha(\operatorname{Sym}^{g-1} C) - \alpha(\kappa)$ and apply to it the bijection on $J(C)$ that maps a point $x$ to $-x$, then we obtain

$$\alpha(\operatorname{Sym}^{g-1} C) \subseteq -\alpha(\operatorname{Sym}^{g-1} C) + \alpha(\kappa),$$

hence the equality holds.

Observe now that the Riemann theta function is symmetric in $z$ via the map $n \mapsto -n$. In consequence the set $\Theta$ is symmetric, and we obtain

$$\alpha(\operatorname{Sym}^{g-1} C) - \Delta = -\alpha(\operatorname{Sym}^{g-1} C) + \Delta = \alpha(\operatorname{Sym}^{g-1} C) - \alpha(\kappa) + \Delta.$$

We conclude by the uniqueness of the Riemann constant that $\Delta$ satisfies the equality $\Delta = \alpha(\kappa) - \Delta$ and the result follows. $\square$

Next we introduce a theorem of Siegel that relates the values of a function on a curve $C$ at a *non-special* divisor with a quotient of Riemann theta functions evaluated at some points in the Jacobian.

**Definition 1.2.5.** We say that an effective divisor $D$ of degree $g$ is *special* if there exists a regular differential $\omega$ with $\operatorname{div}(\omega) \geq D$. Otherwise we call them *non-special* (called *general* in Siegel [44, pg. 154]).

**Theorem 1.2.6** (Theorem 11.3 in Siegel [44])**.** Let $C$ be a curve of genus $g$ over $\mathbb{C}$, and let $\phi$ be a function on $C$ with

$$\operatorname{div}(\phi) = \sum_{i=1}^{m} A_i - \sum_{i=1}^{m} B_i.$$

Let $P \in C$ and let $\omega$ be a basis of $H^0(\omega_C)$ for which the Jacobian $J(C)$ has period matrix $\Omega \in \mathbf{H}_g$. Let $\Delta$ be the Riemann constant with respect to the Abel-Jacobi $\alpha$ map with base point $P$.

Choose paths from the base point $P$ to $A_i$ and $B_i$ that satisfy

$$\sum_{i=1}^{m} \int_P^{A_i} \omega = \sum_{i=1}^{m} \int_P^{B_i} \omega.$$

Then, given an effective non-special divisor $D = P_1 + \cdots + P_g$ of degree $g$ that satisfies $P_j \notin \{A_i, B_i : 1 \leq i \leq m\}$, one has

$$\phi(D) := \phi(P_1) \ldots \phi(P_g) = E \prod_{i=1}^{m} \frac{\theta(\sum_{j=1}^{g} \int_P^{P_j} \omega - \int_P^{A_i} \omega - \Delta, \Omega)}{\theta(\sum_{j=1}^{g} \int_P^{P_j} \omega - \int_P^{B_i} \omega - \Delta, \Omega)},$$

where $E \in \mathbb{C}^\times$ is independent of $D$, and the integrals from $P$ to $P_j$ take the same paths both in the numerator and the denominator. □

Observe that the integrals at which we are evaluating the Riemann theta functions are representatives of the image by the Abel-Jacobi map of $C$ with base point $P$ of the points in the divisor, see Section 1.1.4.

But if a point in $J(C)$ is a torsion point, then we can write it as a rational vector with respect to the basis of the lattice. In fact, the bijection

$$\begin{aligned} \underline{\cdot} : J(C) &\to \mathbb{R}^{2g}/\mathbb{Z}^{2g} \\ \Omega x_1 + x_2 &\mapsto (x_1, x_2) \end{aligned}$$

maps the $m$-torsion of $J(C)$ to $\frac{1}{m}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$.

In this section we are interested in computing the $x$-coordinates of the affine branch points of a Picard curve $C$, so we will choose non-special divisors supported on these points. Note that for every affine branch point $P$ of a Picard curve, we have $\operatorname{div}(x - x(P)) = 3P - 3(0 : 1 : 0)$, so the image of $P$ via the Abel-Jacobi map with base point $(0 : 1 : 0)$ is a 3-torsion point.

Therefore, it is convenient for us to rewrite Theorem 1.2.6 in terms of the following modification of the Riemann theta function:

**Definition 1.2.7.** The *Riemann theta function with (real) characteristic* $x = (x_1, x_2) \in \mathbb{R}^{2g}$ is the function $\theta[x] : \mathbb{C}^g \times \mathbf{H}_g \to \mathbb{C}$ given by

$$\theta[x](z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i\, {}^t(n + x_1)\Omega(n + x_1) + 2\pi i\, {}^t(n + x_1)(z + x_2)). \quad (1.14)$$

It is a translate of the Riemann theta function as in Definition 1.2.1 times an exponential factor:

$$\theta[x](z, \Omega) = \exp(\pi i x_1^t \Omega x_1 + 2\pi i x_1^t (z + x_2))\theta(z + \Omega x_1 + x_2, \Omega). \quad (1.15)$$

A *Riemann theta constant* is a Riemann theta function evaluated at $z = 0$. For notational convenience, we denote it by $\theta[x](\Omega) := \theta[x](0, \Omega)$.

**Proposition 1.2.8** (Mumford [29, pg. 123])**.** The Riemann theta constants satisfy the following properties:

(1) They are symmetric with respect to $x$, that is

$$\theta[x](\Omega) = \theta[-x](\Omega)\,. \tag{1.16}$$

(2) They are quasi-periodic, meaning that for $m = (m_1, m_2) \in \mathbb{Z}^{2g}$ one has

$$\theta[x+m](\Omega) = \exp(2\pi i x_1 m_2)\theta[x](\Omega)\,. \tag{1.17}$$

$\square$

Note that, due to the quasi-periodicity of the Riemann theta constants, the domain for the characteristics is $\mathbb{R}^{2g}$, rather than $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Therefore we fix a representative for such elements. We define the map $\widetilde{\cdot} : \mathbb{R}^{2g}/\mathbb{Z}^{2g} \to [0,1)^{2g}$ that maps a class in $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ to its representative with entries in $[0,1)$.

For convenience, if the domain is clear we denote any composition of the maps

$$C \xrightarrow{\alpha} J(C) \xrightarrow{\underline{\cdot}} \mathbb{R}^{2g}/\mathbb{Z}^{2g} \xrightarrow{\widetilde{\cdot}} [0,1)^{2g}$$

by the last one. For example, for $P \in C$ we write $\widetilde{P}$ instead of $\widetilde{\underline{\alpha(P)}}$. Moreover, given a divisor $D = \sum n_P P$ we define $\widetilde{D} := \sum n_P \widetilde{P} \in \mathbb{R}^{2g}$.

**Warning 1.2.9.** Note that with our definition of $\widetilde{D}$, for most divisors $D$ we have $\widetilde{D} \neq \widetilde{\alpha(D)}$.

We can now rewrite Theorem 1.2.6 in terms of Riemann theta constants:

**Corollary 1.2.10.** With the notation in Theorem 1.2.6, let $a_i$ (resp. $b_i$) be the element in $\mathbb{R}^{2g}$ that satisfies $\int_P^{A_i} \omega = \Omega(a_i)_1 + (a_i)_2$ (resp. $\int_P^{B_i} \omega = \Omega(b_i)_1 + (b_i)_2$). We obtain

$$\phi(D) = E' \prod_{i=1}^{m} \frac{\theta\left[\sum_{j=1}^{g} \widetilde{P_j} - a_i - \widetilde{\Delta}\right](\Omega)}{\theta\left[\sum_{j=1}^{g} \widetilde{P_j} - b_i - \widetilde{\Delta}\right](\Omega)},$$

where $E' \in \mathbb{C}^\times$ is also independent of $D$.

*Proof.* Observe that the exponential factor in (1.15) for Riemann theta constants (that is, $z = 0$) can be written as $\exp(\pi i B(x,x))$ where $B$ is the symmetric bilinear form given by

$$B(u,v) = {}^t u \begin{pmatrix} \Omega & \mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} v.$$

Let $Q(u) = B(u,u)$ and let $c = \left(\sum_{j=1}^{g} \widetilde{P_j}\right) - \widetilde{\Delta}$. For $j = 1, \ldots, g$ let $x_j = \widetilde{P_j}$ and choose a path from $P$ to $P_j$ that satisfies $\int_P^{P_j} \omega = \Omega(x_j)_1 + (x_j)_2 \in \mathbb{C}^g$.

Let $E' \in \mathbb{C}^\times$ be defined by

$$E\prod_{i=1}^{m}\frac{\theta\left(\left(\sum_{j=1}^{g}\int_{P}^{P_j}\omega\right)-\int_{P}^{A_i}\omega-\Delta,\Omega\right)}{\theta\left(\left(\sum_{j=1}^{g}\int_{P}^{P_j}\omega\right)-\int_{P}^{B_i}\omega-\Delta,\Omega\right)}=E'\prod_{i=1}^{m}\frac{\theta\left[\left(\sum_{j=1}^{g}\widetilde{P_j}\right)-a_i-\widetilde{\Delta}\right](\Omega)}{\theta\left[\left(\sum_{j=1}^{g}\widetilde{P_j}\right)-b_i-\widetilde{\Delta}\right](\Omega)}.$$

We want to prove that $E'$ does not depend on $D=\sum_{j=1}^{g}P_j$. By (1.15) we get

$$\frac{E}{E'}=\exp\left(\pi i\sum_{i=1}^{m}(Q(c-a_i)-Q(c-b_i))\right),$$

so it suffices to show that $\sum_{i=1}^{m}(Q(c-a_i)-Q(c-b_i))$ does not depend on $D$. We have

$$\begin{aligned}\sum_{i=1}^{m}(Q(c-a_i)-Q(c-b_i)) &= \sum_{i=1}^{m}(Q(a_i)-Q(b_i)-2B(c,a_i-b_i))\\ &= \sum_{i=1}^{m}Q\left(a_i\right)-\sum_{i=1}^{m}Q\left(b_i\right)-2B\left(c,\sum_{i=1}^{m}(a_i-b_i)\right),\end{aligned}$$

but we know

$$\sum_{i=1}^{m}\int_{P}^{A_i}\omega=\sum_{i=1}^{m}\int_{P}^{B_i}\omega.$$

so in terms of characteristics we obtain $\sum_{i=1}^{m}(a_i-b_i)=0$ and then it follows that

$$\sum_{i=1}^{m}(Q(c-a_i)-Q(c-b_i))=\sum_{i=1}^{m}Q\left(a_i\right)-\sum_{i=1}^{m}Q\left(b_i\right)$$

does not depend on $D$. □

**Lemma 1.2.11.** Let $C$ be a Picard curve over $\mathbb{C}$ given by a Legendre-Rosenhain equation, and denote $P_0=(0,0)$ and $P_\infty=(0:1:0)$. Let $\alpha$ be the Abel-Jacobi map with base point $P_\infty$, let $\Omega\in\mathbf{H}_3$ be a period matrix of $J(C)$ and let $\Delta\in J(C)$ be the Riemann constant with respect to $\alpha$. Then, for every effective non-special divisor $D=R_1+R_2+R_3$ of degree 3 with $R_i\neq P_0,P_\infty$, we have

$$x(R_1)x(R_2)x(R_3)=E'\,\varepsilon(D)\left(\frac{\theta[\widetilde{D}-\widetilde{P_0}-\widetilde{\Delta}](\Omega)}{\theta[\widetilde{D}-\widetilde{\Delta}](\Omega)}\right)^3,$$

with $\varepsilon(D)=\exp(6\pi i(\widetilde{D}-\widetilde{P_0}-\widetilde{\Delta})_1(\widetilde{P_0})_2)$ and $E'\in\mathbb{C}^\times$ independent of $D$.

*Proof.* Let $\omega$ be the basis of holomorphic differentials for which $J(C)$ has period matrix $\Omega$. The divisor of the function $x$ on $C$ is $\operatorname{div}(x) = 3\,P_0 - 3\,P_\infty$, so in order to apply Corollary 1.2.10 for $\phi = x$ and $P = P_\infty$, we choose three times the zero path from $P_\infty$ to itself, the path $\gamma_1$ from $P_\infty$ to $P_0$ that for $a_1 = \widetilde{P_0}$ satisfies

$$\int_{\gamma_1} \omega = \Omega(a_1)_1 + (a_1)_2 \in \mathbb{C}^3,$$

and paths $\gamma_2$, $\gamma_3$ from $P_\infty$ to $P_0$ that satisfy

$$\sum_{k=1}^{3} \int_{\gamma_k} \omega = 0 \text{ in } \mathbb{C}^3. \tag{1.18}$$

Let $a_2, a_3$ be the elements in $\mathbb{R}^6$ that satisfy

$$\int_{\gamma_k} \omega = \Omega(a_k)_1 + (a_k)_2 \text{ for } k = 2, 3.$$

Then, by Corollary 1.2.10, we have

$$\phi(D) = E' \prod_{k=1}^{3} \frac{\theta[\widetilde{D} - a_k - \widetilde{\Delta}](\Omega)}{\theta[\widetilde{D} - \widetilde{\Delta}](\Omega)} \tag{1.19}$$

for some constant $E' \in \mathbb{C}^\times$ independent of $D$. Note that for $k = 1, 2, 3$ we have

$$\underline{P_0} = (a_k \bmod \mathbb{Z}^6),$$

so the differences $a_i - a_j$ for $i \neq j$ are integer vectors. Applying the quasi-periodicity property (1.17), equation (1.19) becomes

$$\phi(D) = E' \frac{\exp(2\pi i(\widetilde{D} - \widetilde{P_0} - \widetilde{\Delta})_1(a_1 - a_2 + a_1 - a_3)_2)\,\theta[\widetilde{D} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)^3}{\theta[\widetilde{D} - \widetilde{\Delta}](\Omega)^3}.$$

But it follows from (1.18) that the sum $a_1 + a_2 + a_3$ is zero, so we obtain $a_1 - a_2 + a_1 - a_3 = 3a_1 = 3\widetilde{P_0}$ and the statement follows. □

The final piece is to choose the right divisors and prove that they are non-special.

**Lemma 1.2.12** (Koike-Weng [16, pg. 506])**.** Let $C$ be a Picard curve and let $\mathcal{B}$ be the set of affine branch points of $C$. If $P, Q \in \mathcal{B}$ are distinct, then the divisor $P + 2Q$ is non-special. □

Now we have all the components to give a formula for the $x$-coordinates of the affine branch points of a Picard curve given by a Legendre-Rosenhain equation in terms of quotients of Riemann theta constants.

**Theorem 1.2.13.** Let $C$ be a Picard curve over $\mathbb{C}$ given by the Legendre-Rosenhain equation $y^3 = x(x-1)(x-\lambda)(x-\mu)$, let $\Omega \in \mathbf{H}_6$ be a period matrix of the Jacobian $J(C)$, let $\alpha$ be the Abel-Jacobi map with base point $(0:1:0)$, and let $\Delta$ be the Riemann constant with respect to $\alpha$. Let $P_t = (t, 0)$ for $t \in \{0, 1, \lambda, \mu\}$ and let $\eta \in \{\lambda, \mu\}$. Then we have

$$\eta = \varepsilon_\eta \left( \frac{\theta[\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)}{\theta[2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)} \right)^3, \tag{1.20}$$

with $\varepsilon_\eta = \exp(6\pi i((\widetilde{P_\eta} - \widetilde{P_1})_1(\widetilde{P_0})_2 + (\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{\Delta})_1(2\widetilde{\Delta} - 3(\widetilde{P_1} + \widetilde{P_\eta}))_2))$.

*Proof.* We apply Lemma 1.2.11 to the divisors $D_1 = P_1 + 2P_\eta$ and $D_2 = 2P_1 + P_\eta$, which are non-special by Lemma 1.2.12. We get

$$\begin{aligned}
\eta = \frac{x(P_1)x(P_\eta)^2}{x(P_1)^2 x(P_\eta)} &= \frac{E'\varepsilon(D_1)\left( \dfrac{\theta[\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)}{\theta[\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{\Delta}](\Omega)} \right)^3}{E'\varepsilon(D_2)\left( \dfrac{\theta[2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)}{\theta[2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{\Delta}](\Omega)} \right)^3} \\
&= \frac{\varepsilon(D_1)}{\varepsilon(D_2)} \left( \frac{\theta[\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)}{\theta[\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{\Delta}](\Omega)} \frac{\theta[2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{\Delta}](\Omega)}{\theta[2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)} \right)^3.
\end{aligned} \tag{1.21}$$

In order to simplify the formula we apply the symmetry (1.16) and quasi-periodicity (1.17) of the Riemann theta constants to obtain

$$\begin{aligned}
\theta[\widetilde{D_2} - \widetilde{\Delta}](\Omega) &= \theta[-\widetilde{D_2} + \widetilde{\Delta}](\Omega) \\
&= \theta[\widetilde{D_1} - \widetilde{\Delta} + (2\widetilde{\Delta} + 3(\widetilde{P_1} + \widetilde{P_\eta}))](\Omega) \\
&= \exp\left(2\pi i(\widetilde{D_1} - \widetilde{\Delta})_1(2\widetilde{\Delta} - 3(\widetilde{P_1} + \widetilde{P_\eta}))_2\right) \theta[\widetilde{D_1} - \widetilde{\Delta}](\Omega)
\end{aligned}$$

so that the formula (1.21) becomes

$$\eta = \varepsilon_\eta \left( \frac{\theta[\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)}{\theta[2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta}](\Omega)} \right)^3,$$

with

$$\begin{aligned}
\varepsilon_\eta &= \frac{\varepsilon(D_1)}{\varepsilon(D_2)} \exp(2\pi i(\widetilde{D_1} - \widetilde{\Delta})_1(2\widetilde{\Delta} - 3(\widetilde{P_1} + \widetilde{P_\eta}))_2)^3 \\
&= \frac{\exp(6\pi i(\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta})_1(\widetilde{P_0})_2)}{\exp(6\pi i(2\widetilde{P_1} + \widetilde{P_\eta} - \widetilde{P_0} - \widetilde{\Delta})_1(\widetilde{P_0})_2)} \exp(6\pi i(\widetilde{D_1} - \widetilde{\Delta})_1(2\widetilde{\Delta} - 3(\widetilde{P_1} + \widetilde{P_\eta}))_2) \\
&= \exp(6\pi i((\widetilde{P_\eta} - \widetilde{P_1})_1(\widetilde{P_0})_2 + (\widetilde{P_1} + 2\widetilde{P_\eta} - \widetilde{\Delta})_1(2\widetilde{\Delta} - 3(\widetilde{P_1} + \widetilde{P_\eta}))_2))
\end{aligned}$$

as desired. $\square$

**Remark 1.2.14.** Compare the formula for $\eta$ given in Theorem 1.2.13 with the ones given by Koike-Weng [16, Eq. 9]. The formulas in [16] are the same as (1.20) replacing $\varepsilon_\eta$ by 1, hence in general they do not hold due to the absence of the precise root of unity.

However, if we follow the original work by Picard [35, p. 131] where he constructs the period matrix of a Picard curve given by a Legendre-Rosenhain equation in a specific way (see also Shiga [38, Proposition I-3]), then we obtain that the factors $\varepsilon_\lambda$ and $\varepsilon_\mu$ are 1, so in that case the formulas in [16] remain correct.

But if $\Omega$ is not specifically constructed in that way, then we have to either be lucky (and get $\varepsilon_\lambda = \varepsilon_\mu = 1$) or use the formula for $\varepsilon_\eta$.

## 1.3 The inverse Jacobian algorithm

In this section we present an algorithm that, given the period matrix of the Jacobian of a Picard curve $C$ and the rational representation of the automorphism $\rho_*$ induced by $\rho(x, y) = (x, z_3 y)$, returns a numerical approximation of the $x$-coordinates of the affine branch points of $C$.

The main step of the algorithm uses Theorem 1.2.13. To apply that theorem we need to know the Riemann constant of $C$ with respect to the Abel-Jacobi map $\alpha$ with base point $(0 : 1 : 0)$ and the image by $\alpha$ of the affine branch points on $J(C)$.

We start by characterizing the Riemann constant of a Picard curve. We will do so by using both its uniqueness and the fact that the base point for $\alpha$ is fixed by the automorphism $\rho$.

First we show how a change of symplectic bases affects a Riemann theta function with characteristics.

**Definition 1.3.1.** For $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $c \in \mathbb{R}^{2g}$ we define

$$M[c] := {}^t M^{-1} c + \frac{1}{2} \begin{pmatrix} (\gamma\, {}^t\delta)_0 \\ (\alpha\, {}^t\beta)_0 \end{pmatrix},$$

where $X_0$ stands for the diagonal of the matrix $X$.

Note that the class $N[c] \bmod \mathbb{Z}^{2g}$ depends only on the class of $c \bmod \mathbb{Z}^{2g}$, so we denote it by $N[c \bmod \mathbb{Z}^{2g}]$. Moreover, for $x \in J(C)$ we denote the point that satisfies the equality $\underline{N[x]} = N[\underline{x}]$ by $N[x] \in J(C)$ .

**Proposition 1.3.2** (Proposition 8.6.1 in Birkenhake-Lange [2])**.** For a period matrix $\Omega \in \mathbf{H}_g$, a characteristic $c \in \mathbb{R}^{2g}$ and a symplectic matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}),$$

there exists a function $\kappa(M,\Omega,c,\cdot):\mathbb{C}^g\to\mathbb{C}^\times$ that satisfies for all $v\in\mathbb{C}^g$ the equality

$$\theta[M[c]]({}^t(\gamma\Omega+\delta)^{-1}v, M(\Omega)) = \kappa(M,\Omega,c,v)\theta[c](v,\Omega). \qquad \square$$

**Remark 1.3.3.** The factor $\kappa(M,\Omega,c,v)\in\mathbb{C}^\times$ is given explicitly in Birkenhake-Lange [2, Proposition 8.6.1].

**Proposition 1.3.4.** Let $C, C'$ be curves with equal genus $g$, let $\varphi: C\to C'$ be an isomorphism of curves, and let $\varphi_*: J(C)\to J(C')$ be the induced isomorphism on the Jacobians with period matrices $\Omega,\Omega'\in\mathbf{H}_g$ respectively. Define $N := {}^t\rho_r(\varphi_*)$. Let $P\in C$, let $\alpha$ be the Abel-Jacobi map with base point $P$, and let $\alpha'$ be the Abel-Jacobi map with base point $\varphi(P)$.

Let also $\Delta$ (resp. $\Delta'$) be the Riemann constant of $C$ (resp. $C'$) with respect to $\alpha$ (resp. $\alpha'$). The Riemann constants satisfy

$$N[\Delta'] = \Delta.$$

*Proof.* Recall that, given a curve $C$ and an Abel-Jacobi map $\alpha$ of $C$, the Riemann constant $\Delta$ is determined by Theorem 1.2.2, hence it satisfies

$$\alpha(\mathrm{Sym}^{g-1} C) = \left\{x\in J(C) : \theta[-\widetilde{\Delta}](x,\Omega) = 0\right\}. \tag{1.22}$$

To prove the proposition, we will use that the Riemann constant is uniquely defined by (1.22) (see Proposition 1.2.4). We start by applying the isomorphism $\varphi_*^{-1}$ to both sides of (1.22) in the case of the curve $C'$. We obtain

$$\varphi_*^{-1}\alpha'(\mathrm{Sym}^{g-1} C') = \left\{y\in J(C) : \theta[-\widetilde{\Delta'}](\varphi_*(y),\Omega') = 0\right\}. \tag{1.23}$$

Consider the subdivision in $g\times g$ blocks of the transpose of $\rho_r(\varphi_*)$

$$N = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}),$$

and recall that then the analytical representation of $\varphi_*$ is ${}^t(\gamma\Omega+\delta)$ and the period matrices satisfy the equality $N(\Omega')=\Omega$, see (1.9) and (1.10) respectively.

Let $y_0\in\mathbb{C}^g$ be a representative of $y\in J(C)$, that is, an element satisfying $y = (y_0 \mod \Omega\mathbb{Z}^g+\mathbb{Z}^g)$, thus also $\varphi_*(y) = ({}^t(\gamma\Omega+\delta)y_0 \mod \Omega'\mathbb{Z}^g+\mathbb{Z}^g)$. Then, by the theta transformation formula by $N$ given in Proposition 1.3.2, we get

$$\begin{aligned}
&\theta[-\widetilde{\Delta'}]({}^t(\gamma\Omega+\delta)y_0,\Omega') =\\
&\qquad = \kappa(N,\Omega',\Delta',{}^t(\gamma\Omega+\delta)y_0)^{-1}\theta[-N[\widetilde{\Delta'}]]({}^t(\gamma\Omega+\delta)^{-1}\,{}^t(\gamma\Omega+\delta)y_0, N(\Omega'))\\
&\qquad = \kappa(N,\Omega',\Delta',{}^t(\gamma\Omega+\delta)y_0)^{-1}\theta[-N[\widetilde{\Delta'}]](y_0,\Omega).
\end{aligned}$$

Recall that by definition of $\varphi_*$ we have $\varphi_* \circ \alpha = \alpha' \circ \varphi$. Therefore we obtain

$$\varphi_*^{-1}\alpha'(\operatorname{Sym}^{g-1} C') = \alpha(\operatorname{Sym}^{g-1} C),$$

and the equality of sets (1.23) becomes

$$\alpha(\operatorname{Sym}^{g-1} C) = \left\{ y \in J(C) : \theta[-N[\widetilde{\Delta'}]](y, \Omega) = 0 \right\}.$$

We conclude $N[\Delta'] = \Delta$ by the uniqueness of the Riemann constant. □

Now we can characterize the Riemann constant of a Picard curve with respect to the Abel-Jacobi map with base point $(0 : 1 : 0)$.

**Corollary 1.3.5.** Let $C$ be a Picard curve, let $\rho$ be the automorphism of $C$ given by $(x, y) \mapsto (x, z_3 y)$. The Riemann constant with respect to the Abel-Jacobi map with base point $P_\infty = (0 : 1 : 0)$ is the only point $\Delta \in J(C)$ with

(1) $\Delta \in J(C)[2]$, and
(2) ${}^t\rho_r(\rho_*)[\underline{\Delta}] = \underline{\Delta}$.

*Proof.* By Proposition 1.2.4 we have $2\Delta = \alpha(\kappa)$ for $\kappa$ a canonical divisor, and the computation $\operatorname{div}(dx/y^2) = 4P_\infty$ shows $\alpha(\kappa) = 0$, which proves (1). Moreover, since $P_\infty$ is fixed by $\rho$, we obtain by Proposition 1.3.4 that the point $\Delta$ satisfies (2).

To prove that it is the only point that satisfies (1) and (2), assume that there exist $\Delta^1, \Delta^2 \in J(C)$ that satisfy (1) and (2). By (2) we have

$$\underline{\Delta}^1 - \underline{\Delta}^2 = {}^t\rho_r(\rho_*)[\underline{\Delta}^1] - {}^t\rho_r(\rho_*)[\underline{\Delta}^2] = \rho_r(\rho_*)^{-1}(\underline{\Delta}^1 - \underline{\Delta}^2),$$

thus $\Delta^1 - \Delta^2$ is an element of $J(C)[1 - \rho_*^2] \subseteq J(C)[3]$. But by (1), the difference $\Delta^1 - \Delta^2$ is also a 2-torsion point, hence we conclude $\Delta^1 - \Delta^2 = 0$. □

Next, we identify the images on $J(C)$ of the affine branch points of $C$.

**Theorem 1.3.6.** Let $J(C)$ be the Jacobian of a Picard curve $C$, let $\rho_*$ be the automorphism of $J(C)$ induced by the curve automorphism $\rho(x, y) = (x, z_3 y)$. Let $\mathcal{B}$ be the set of affine branch points of $C$, let $\alpha$ be the Abel-Jacobi map with base point $P_\infty = (0 : 1 : 0)$, let $\Delta$ be the Riemann constant with respect to $\alpha$ and define

$$\Theta_3 := \{x \in J(C)[1 - \rho_*] : \theta[x + \underline{\Delta}](\Omega) = 0\}.$$

Then $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ are the only subsets $\mathcal{T} \subseteq J(C)$ of four elements such that:

(i) the sum $\sum_{x \in \mathcal{T}} x$ is zero,
(ii) $\mathcal{T}$ is a set of generators of $J(C)[1 - \rho_*]$, and

(iii) the set $\mathcal{O}(\mathcal{T}) := \{\sum_{x\in\mathcal{T}} a_x x : a \in \mathbb{Z}^4_{\geq 0}, \sum_{x\in\mathcal{T}} a_x \leq 2\}$ satisfies

$$\mathcal{O}(\mathcal{T}) = \Theta_3.$$

*Proof.* We first show that $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ satisfy (i)–(iii), and then we prove that these are the only possibilities.

That $\alpha(\mathcal{B})$ satisfies (i) follows from $\operatorname{div}(y) = \sum_{P\in\mathcal{B}} P - 4P_\infty$. That $\alpha(\mathcal{B})$ satisfies (ii) is proven by Koike and Weng in [16, Remark 8]. Next we prove that $\alpha(\mathcal{B})$ satisfies (iii). On the one hand, given $Q_1, Q_2 \in \mathcal{B} \cup \{P_\infty\}$ we have $\alpha(Q_1 + Q_2) \in \Theta_3$ by Riemann's Vanishing Theorem 1.2.2, and since we have $\alpha(P_\infty) = 0$, this implies

$$\left\{\sum_{P\in\mathcal{B}} a_P \alpha(P) : a \in \mathbb{Z}^{\mathcal{B}}_{\geq 0}, \sum_{P\in\mathcal{B}} a_P \leq 2\right\} \subseteq \Theta_3.$$

On the other hand let $x \in \Theta_3$. Since $x$ satisfies $\theta[x + \underline{\Delta}](\Omega) = 0$, by Riemann's Vanishing Theorem 1.2.2 there exist $Q_1, Q_2 \in C$ such that we have $x = \alpha(Q_1 + Q_2)$. Moreover, since $x$ is a $(1-\rho_*)$-torsion point, we get

$$\alpha(Q_1 + Q_2) = \rho_*(\alpha(Q_1 + Q_2)) = \alpha(\rho(Q_1) + \rho(Q_2)),$$

hence there exists a function $h$ on $C$ such that $\operatorname{div}(h) = \rho(Q_1) + \rho(Q_2) - Q_1 - Q_2$. We conclude that $h$ is constant, since otherwise it has degree at most 2, hence the curve would be hyperelliptic, contradicting Proposition 1.1.2. Therefore we have $\rho(Q_1) + \rho(Q_2) = Q_1 + Q_2$, but since $\rho$ has order 3, the cardinality of the orbit of $Q_i$ has length 3 or 1, thus we obtain $\rho(Q_i) = Q_i$. Therefore $Q_1$ and $Q_2$ are branch points, so the other inclusion holds.

It is clear that $-\alpha(\mathcal{B})$ satisfies (i) and (ii). To see that it satisfies (iii), it is enough to prove that $\Theta_3$ is invariant under the map $x \mapsto -x$. But this follows from the symmetry $\theta[-x](\Omega) = \theta[x](\Omega)$ of the Riemann theta constants.

Next we prove that $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ are, in fact, all the subsets that satisfy (i)–(iii).

Let $B$ denote an ordering of $\alpha(\mathcal{B})$. Given a sequence $T = (t_1, t_2, t_3, t_4)$ in $J(C)^4$ such that the set $\{t_1, t_2, t_3, t_4\}$ has 4 elements and satisfies (i)–(iii), we define the map $\gamma[T] : \mathbb{F}_3^3 \to J(C)[1-\rho_*]$ given by $r \mapsto \sum_{i=1}^3 r_i t_i$. By Remark 8 in Koike-Weng [16] we have $\#J(C)[1-\rho_*] \cong (\mathbb{Z}/3\mathbb{Z})^3$, thus it follows from (i) and (ii) that $\gamma[T]$ is a bijection.

Consider the diagram

$$\begin{tikzcd} \mathbb{F}_3^3 \arrow[rr, "M(T)"] \arrow[dr, "\gamma[T]"'] & & \mathbb{F}_3^3 \arrow[dl, "\gamma[B]"] \\ & J(C)[1-\rho_*] & \end{tikzcd}$$

where $M(T)$ is the unique invertible matrix in $\mathbb{F}_3^{3\times 3}$ that makes the diagram commutative. Note that choosing a matrix $M(T)$ determines $T$ uniquely.

Let $e_1, e_2, e_3$ be the standard basis vectors of $\mathbb{F}_3^3$, and let $e_4 = -e_1 - e_2 - e_3$, so for $i = 1, \ldots, 4$ we have $\gamma[T](e_i) = t_i$. Consider

$$\mathcal{O}_0 = \left\{ \sum_{i=1}^{4} a_i e_i : a \in \mathbb{Z}_{\geq 0}^4, \sum_{i=1}^{4} a_i \leq 2 \right\} \subseteq \mathbb{F}_3^3.$$

One can check $\#\mathcal{O}_0 = 15$, and moreover we have $\gamma[T](\mathcal{O}_0) = \mathcal{O}(\{t_1, t_2, t_3, t_4\})$. If the set of elements of $T$ satisfies (iii), then we have

$$\gamma[T](\mathcal{O}_0) = \mathcal{O}(\{t_1, t_2, t_3, t_4\}) = \Theta_3 = \gamma[B](\mathcal{O}_0),$$

and thus $\mathcal{O}_0$ is stable under $M(T)$.

We checked with SageMath [49] that there are exactly 48 invertible matrices in $\mathbb{F}_3^{3\times 3}$ that map $\mathcal{O}_0$ to itself. Since a matrix $M(T)$ determines $T$ uniquely, there are 48 sequences $T \in J(C)^4$ that satisfy (i)–(iii). However, if we vary $\sigma$ in the symmetric group of 4 letters and $s \in \{\pm 1\}$, then $s\sigma(B)$ gives 48 sequences, which are different. We conclude that $\alpha(\mathcal{B})$ and $-\alpha(\mathcal{B})$ are the only subsets of $J(C)$ with 4 elements that satisfy (i)–(iii). □

From the proof above we obtain the following result.

**Corollary 1.3.7.** With the notation in Theorem 1.3.6, we get

$$\#\Theta_3 = 15.$$ □

**Remark 1.3.8.** With Theorem 1.3.6, we make precise the idea hinted at Corollary 11 in Koike-Weng [16]. There, they claim the existence of a 4-element set that satisfies (i) and (ii), prove that $\alpha(\mathcal{B})$ does satisfy (i) and (ii), and assume without further comments that when one finds such a set, it is $\alpha(\mathcal{B})$.

This is problematic not only because they disregard the case where the set is $-\alpha(\mathcal{B})$ but specially because they do not consider (iii) at all, but there exist 4-element sets in $J(C)$ that satisfy (i) and (ii) which are not $\alpha(\mathcal{B})$ or even $-\alpha(\mathcal{B})$.

In fact, there are $\#\,\mathrm{GL}_3(\mathbb{F}_3) = 11232$ possible sequences $T \in J(C)^4$ that satisfy (i) and (ii), hence the probability of finding one that corresponds to a permutation of $B$ is $1/468 \approx 0.002$.

We have now all the tools to state the algorithm.

**Algorithm 1.3.9**

**Input:** The Jacobian of a Picard curve $C$, given by a period matrix $\Omega \in \mathbf{H}_3$, and $\rho_*$ the automorphism on the Jacobian induced by the curve automorphism $\rho(x, y) = (x, z_3 y)$, given by its rational representation $N \in \mathbb{Z}^{6\times 6}$.

**Output:** The complex values $\lambda$ and $\mu$ in a Legendre-Rosenhain equation $y^3 = x(x-1)(x-\lambda)(x-\mu)$ of the Picard curve $C$.

1. Let $D$ be the unique solution of $N[D] = D$ in $\frac{1}{2}\mathbb{Z}^6/\mathbb{Z}^6$.
2. Compute the set

$$\underline{\Theta_3} = \left\{ x \in \frac{1}{3}\mathbb{Z}^6/\mathbb{Z}^6 : Nx = x \text{ and } \theta[x+D](\Omega) = 0 \right\}$$

of cardinality 15.
3. Let $T = \{t_1, t_2, t_3, t_4\} \subseteq \underline{\Theta_3}$ be a 4-element set that satisfies
   I. $\sum_{i=1}^4 t = 0$,
   II. $\{t_1, t_2, t_3\}$ are linearly independent, and
   III. $\{\sum_{i=1}^4 a_i t_i : (a_i)_i \in \mathbb{Z}_{\geq 0}^4, \sum_{i=1}^4 a_i \leq 3\} = \underline{\Theta_3}$.
4. Compute

$$\varepsilon_\lambda = \exp(6\pi i((\widetilde{t_3} - \widetilde{t_2})_1(\widetilde{t_1})_2 + (\widetilde{t_2} + 2\widetilde{t_3} - \widetilde{D})_1(2\widetilde{D} - 3(\widetilde{t_2} + \widetilde{t_3}))_2)),$$

$$\varepsilon_\mu = \exp(6\pi i((\widetilde{t_4} - \widetilde{t_2})_1(\widetilde{t_1})_2 + (\widetilde{t_2} + 2\widetilde{t_4} - \widetilde{D})_1(2\widetilde{D} - 3(\widetilde{t_2} + \widetilde{t_4}))_2)),$$

and

$$\lambda = \varepsilon_\lambda \left( \frac{\theta[\widetilde{t_2} + 2\widetilde{t_3} - \widetilde{t_1} - \widetilde{D}](\Omega)}{\theta[2\widetilde{t_2} + \widetilde{t_3} - \widetilde{t_1} - \widetilde{D}](\Omega)} \right)^3,$$

$$\mu = \varepsilon_\mu \left( \frac{\theta[\widetilde{t_2} + 2\widetilde{t_4} - \widetilde{t_1} - \widetilde{D}](\Omega)}{\theta[2\widetilde{t_2} + \widetilde{t_4} - \widetilde{t_1} - \widetilde{D}](\Omega)} \right)^3.$$

5. Return $\lambda$ and $\mu$.

---

**Warning 1.3.10.** Algorithm 1.3.9 is a *mathematical* algorithm, but, because it involves infinite sums, complex numbers and exponentials, it cannot be run on a Turing machine or a physical computer. To do so one needs to truncate the sum on the Riemann theta constants, approximate complex numbers and keep track of the error propagation. For more details on how to do this see Section 1.5.

*Proof of Algorithm 1.3.9.* Let $\Delta \in J(C)$ be the Riemann constant with respect to $P_\infty = (0:1:0)$ and let $\mathcal{B}$ be the set of affine branch points of $C$. By Corollary 1.3.5, the point $\Delta$ is the only one that satisfies $N[\Delta] = \Delta$ and is a 2-torsion point, that is, it satisfies $\underline{\Delta} \in \frac{1}{2}\mathbb{Z}^6/\mathbb{Z}^6$. We conclude $D = \underline{\Delta}$.

By Theorem 1.3.6, the sequence $(t_1, t_2, t_3, t_4)$ is an ordering of either $\alpha(\mathcal{B})$ or $-\alpha(\mathcal{B})$. In the former case, the values $\lambda$, $\mu$ obtained in Step 4 are the $x$-coordinates of the affine branch points different from $(0,0)$ and $(0,1)$. A quasi-periodicity argument similar to those in the proofs of Lemma 1.2.11 or Theorem 1.2.13 yields that in the latter case that holds too. $\square$

As a consequence of the proof we obtain the following result.

**Corollary 1.3.11.** If the automorphism given in the input of Algorithm 1.3.9 is $\rho_*^2$, then the output is also correct.

*Proof.* Note that the automorphism in the input only plays a role in Steps 1 and 2 of Algorithm 1.3.9, to determine the Riemann constant and the $(1-\rho_*)$-torsion points in $J(C)$.

Note that both $\rho$ and $\rho^2$ fix the branch points on $C$. Therefore, by Proposition 1.3.4 the Riemann constant satisfies ${}^t\rho_r(\rho_*^2)[\Delta] = \Delta$. It follows that, for $M = {}^t\rho_r(\rho_*^2)$, the characteristic $D$ in Step 1 satisfies $M[D] = D$. We also get

$$\left\{x \in \frac{1}{3}\mathbb{Z}^6/\mathbb{Z}^6 : Mx = N^2x = x \text{ and } \theta[x+D](\Omega) = 0\right\} = \underline{\Theta_3}. \qquad \square$$

## 1.4 The Torelli locus of Picard curves

In the previous section we have seen how to reconstruct a Picard curve from its Jacobian. The following theorem characterizes the abelian varieties that arise as the Jacobian of a Picard curve. It is a variation of Lemma 1 in [16], see Remark 1.4.2.

**Proposition 1.4.1** (based on work of Koike-Weng and Estrada)**.** Let $X$ be a simple principally polarized abelian variety of dimension 3 defined over an algebraically closed field $k$. If $X$ has an automorphism $\varphi$ of order 3, then we have $X \in J(\mathcal{P})$. Furthermore, for the curve automorphism $\rho(x,y) = (x, z_3y)$, we get $\langle\varphi\rangle = \langle\rho_*\rangle$

*Proof.* Let $X$ be a simple principally polarized abelian variety of dimension 3 with an automorphism $\varphi$ of order 3. By Oort-Ueno [33], every simple principally polarized abelian variety of dimension $\leq 3$ over an algebraically closed field is the Jacobian of a curve, so let $C$ be a curve with $X \cong J(C)$.

By Torelli's Theorem 1.1.1, there is some non-trivial automorphism $\nu$ of $C$ that satisfies $\varphi = \pm\nu_*$. Then the automorphism $\eta = \nu^4$ satisfies $\eta_* = (\nu^4)_* = (\pm\nu)_*^4 = \varphi^4 = \varphi$, hence by the uniqueness in Torelli's Theorem 1.1.1 we obtain that $\eta$ has order 3.

We conclude that the automorphism $\eta$ has order 3, so the degree of the map $\pi : C \to C/\langle\eta\rangle$ is also 3, and by the Riemann-Hurwitz formula one obtains that $C/\langle\eta\rangle$ has either genus 0 or 1. But $X$ is simple, so the curve $C/\langle\eta\rangle$ is isomorphic to $\mathbb{P}^1$ and $\pi$ has 5 ramification points.

Then $k(C)/k(C/\langle\eta\rangle)$ is a Kummer extension of degree 3, hence $C$ is given by an equation of the form $y^3 = h(x)$. By Lemma 7.3 in Estrada [11, Appendix I], we obtain a model for $C$ given by $y^3 = f(x)$ where $f$ has degree 4 and distinct roots and $\eta$ is either the automorphism $\rho$ given by $(x, y) \mapsto (x, z_3 y)$ or its square. $\square$

**Remark 1.4.2.** While the idea behind the proof is the same in Proposition 1.4.1 and in [16, Lemma 1], the assumptions in [16] are in a way more restrictive, as Koike and Weng focus on maximal CM Picard curves (see page 33 for a definition). Moreover, the proof in [16] has a gap, which is fixed exactly by our reference to Estrada [11, Appendix I].

It follows from Proposition 1.4.1 that one can think of the input in Algorithm 1.3.9 as just a principally polarized abelian threefold with an order-3 automorphism.

## 1.5 Implementation and some CM examples

In this section we give some indications on how to implement Algorithm 1.3.9 so that it can run in a physical computer. In practice, in the implementation [45] we truncate the sums of the Riemann theta constants at some hypercube $[-B, B]^3 \subseteq \mathbb{Z}^3$ and use high precision floating point numbers and several checks through the implementation to make sure that the output is coherent.

If one of the checks fails or the final computation does not make sense, then we run the algorithm again for a larger bound $B \in \mathbb{Z}$. Alternatively, one could use interval arithmetic to keep track of the error propagation.

We use the following algorithm to truncate the Riemann theta constants:

---

**Algorithm 1.5.1**

---

**Input:** A real number $b \in (0, 1)$, a period matrix $\Omega \in \mathbf{H}_g$ to arbitrary precision, and a characteristic $c \in ([0, 1) \cap \mathbb{Q})^{2g}$.

**Output:** An approximation $\theta_b[c](\Omega)$ of $\theta[c](\Omega)$ that satisfies

$$|\theta[c](\Omega) - \theta_b[c](\Omega)| < b.$$

1. Compute $B \in \mathbb{Z}$ that satisfies
$$B > \sqrt{-\frac{\ln b + g \ln(1 - e^{-\pi\lambda(\Omega)}) - (g+1)\ln 2 - \ln g}{\pi\lambda(\Omega)}},$$
where $\lambda(\Omega)$ is the smallest eigenvalue of the imaginary part of $\Omega$.
2. Let $b' = (2B+1)^{-g} b/2$ and for $n \in [-B, B]^g$ compute $x_n$ that satisfies
$$|\exp(\pi i\, {}^t(n + c_1)\Omega(n + c_1) + 2\pi i\, {}^t(n + c_1)c_2) - x_n| < b'.$$
3. Return $\theta_b[c](\Omega) = \sum_{n \in [-B,B]^g} x_n$.

---

*Proof.* We will bound $|\theta_b[c](\Omega) - \theta[c](\Omega)|$. Let $X$ and $Y$ be respectively the real and imaginary part of $\Omega$, so that we write $\Omega = X + iY$. Every term in the sum $\theta[c](\Omega)$ consists of an oscillatory factor $F$ with $|F| = 1$ and a real exponential factor, hence we obtain

$$|\exp(\pi i\,{}^t(n+c_1)\Omega(n+c_1) + 2\pi i\,{}^t(n+c_1)c_2)| = \exp(-\pi\,{}^t(n+c_1)Y(n+c_1))$$

but since $Y$ is symmetric and positive definite we get

$$|\exp(\pi i\,{}^t(n+c_1)\Omega(n+c_1) + 2\pi i\,{}^t(n+c_1)c_2)| \leq \exp(-\pi\lambda(\Omega)||n+c_1||^2),$$

and, for $Q = \exp(-\pi\lambda(\Omega))$ we have

$$\begin{aligned} &|\theta[c](\Omega) - \theta_b[c](\Omega)| \\ &\quad\leq (2B+1)^g b' + \sum_{n\in\mathbb{Z}^g\setminus[-B,B]^g} |\exp(\pi i\,{}^t(n+c_1)\Omega(n+c_1) + 2\pi i\,{}^t(n+c_1)c_2)| \\ &\quad\leq \frac{b}{2} + \sum_{n\in\mathbb{Z}^g\setminus[-B,B]^g} Q^{||n+c_1||^2} \end{aligned}$$

Note that for $n \in \mathbb{Z}$ and $c \in [0,1)$ we have

$$(n+c)^2 \geq \begin{cases} n^2 & \text{if } n \geq 0, \\ (n+1)^2 & \text{if } n \leq -1. \end{cases} \tag{1.24}$$

Then, in order to bound the sum above, we deal with each "quadrant" of $\mathbb{Z}^g$ separately. Using the lowerbound in (1.24) we obtain that the sum at each "quadrant" is bounded by

$$\sum_{n_1\geq B}\sum_{n_2\geq 0}\cdots\sum_{n_g\geq 0}\prod_{j=1}^{g} Q^{n_j^2},$$

and we obtain

$$\begin{aligned} |\theta[c](\Omega) - \theta_b[c](\Omega)| &\leq \frac{b}{2} + 2^g g \sum_{n_1\geq B}\sum_{n_2\geq 0}\cdots\sum_{n_g\geq 0}\prod_{j=1}^{g} Q^{n_j^2} \\ &\leq \frac{b}{2} + 2^g g\left(\sum_{n_1\geq B} Q^{n_1^2}\right)\left(\sum_{n_2\geq 0} Q^{n_2^2}\right)\cdots\left(\sum_{n_g\geq 0} Q^{n_g^2}\right). \end{aligned} \tag{1.25}$$

If we now apply the bound

$$\sum_{m\geq M} Q^{m^2} \leq \sum_{m\geq M^2} Q^m = \frac{Q^{M^2}}{1-Q} \text{ if } |Q| < 1$$

to (1.25), then we obtain

$$|\theta[c](\Omega) - \theta_b[c](\Omega)| \leq \frac{b}{2} + 2^g g \frac{Q^{B^2}}{(1-Q)^g},$$

which for $B$ as in the statement implies $|\theta[c](\Omega) - \theta_b[c](\Omega)| < b$. □

Then one replaces Step 2 in Algorithm 1.3.9 by the following substeps:

I. For $b = 2^{-5}$ compute

$$\underline{\Theta_{3,b}} = \left\{ x \in \frac{1}{3}\mathbb{Z}^6/\mathbb{Z}^6 : Nx = x \text{ and } \theta_b[x+D](\Omega) < b \right\}.$$

II. If $\underline{\Theta_{3,b}}$ has more than 15 points, then square $b$ and repeat steps I and II.

By Algorithm 1.5.1 we have

$$\underline{\Theta_3} \subseteq \underline{\Theta_{3,b}},$$

and for small enough $b > 0$ we obtain the equality. By Corollary 1.3.7, we obtain $\#\underline{\Theta_{3,b}} = 15$ in a finite number of steps.

For efficiency, we would like the smallest eigenvalue of the imaginary part of $\Omega$ to be as big as possible, due to its role in the computation of $B$ in Algorithm 1.5.1. Since the isomorphism class of a principally polarized abelian variety only depends on the orbit of $\Omega$ under the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$, this can be achieved by choosing a representative in a certain fundamental domain of $\mathbf{H}_g$. For this we use the implementation due to Kılıçer–Streng [14] of Algorithm 2 in Labrande-Thomé [18, Section 4.1] on our period matrix before applying Algorithm 1.3.9.

**Remark 1.5.2.** This was enough to obtain the examples given in this section, but it might take too long for other cases. Alternatively, one could use Labrande's method [17], which computes Riemann theta functions with characteristics in quasi-linear time.

After numerically approximating the $x$-coordinates of the branch points of a Picard curve with Algorithm 1.3.9, we obtain a polynomial

$$f(x) = x(x-1)(x-\lambda)(x-\mu) \in \mathbb{C}[x]$$

up to some precision, while maybe the curve is actually isomorphic to $y^3 = h(x)$ for a certain polynomial $h$ over a number field.

Given the quartic polynomial

$$p(x) = x^4 + g_2x^2 + g_3x + g_4 \text{ with } g_2 \neq 0$$

we define the *absolute invariants of* $p$ as

$$j_1 = \frac{g_3^2}{g_2^3}, \qquad j_2 = \frac{g_4}{g_2^2}.$$

In order to find $h$ from $f$ we compute the absolute invariants of $C$ by computing $j_1$ and $j_2$ for an isomorphic curve of the form $y^3 = x^4 + g_2x^2 + g_3x + g_4$. We then recognize $j_1$ and $j_2$ as algebraic numbers and reconstruct $h$ from the exact absolute invariants, obtaining

$$y^3 = h(x) = x^4 + j_1x^2 + j_1^2x + j_1^2j_2.$$

Note that in order to be able to recognize $j_1$ and $j_2$ as algebraic numbers we have to compute $\lambda$ and $\mu$ with enough precision.

Next we include a list of Picard curves computed with our algorithm. We define a *maximal CM Picard curve* as a Picard curve such that its Jacobian has endomorphism ring isomorphic to the maximal order of a sextic number field $K$. Since $\rho_*$ is an automorphism of order 3, the field $K$ contains a primitive 3rd root of unity $\zeta_3 \in K$. In fact, the field $K$ is determined by a totally real cubic field $K_0$ that satisfies $K = K_0(\zeta_3)$.

In Section 4.1 we explain how to obtain, for a given sextic field $K = K_0(\zeta_3)$, a complete list of period matrices of principally polarized abelian varieties with endomorphism ring isomorphic to $\mathcal{O}_K$, together with the rational representation of the corresponding order-3 automorphism $\varphi$.

Using Algorithm 1.3.9 on the resulting list of pairs $(\Omega, N)$, we computed numerical approximations of some maximal CM curves. Here we present the resulting Picard curves which are numerically close (and conjecturally equal) to the maximal CM curves. In Chapter 4 we will see that, in particular, this list contains conjectural models for all Picard curves defined over $\mathbb{Q}$ with maximal CM over $\mathbb{C}$. The curves (1)–(5) also appear in [16, Section 6.1].

We obtained the following curves:

(1) $y^3 = x^4 - x$, with $K_0$ defined by $\nu^3 - 3\nu - 1$.

(2) $y^3 = x^4 - 2 \cdot 7^2\, x^2 + 2^3 \cdot 7^2\, x - 7^3$, with $K_0$ defined by $\nu^3 - \nu^2 - 2\nu + 1$.

(3) $y^3 = x^4 - 2 \cdot 7^2 \cdot 13\, x^2 + 2^3 \cdot 5 \cdot 13 \cdot 47\, x - 5^2 \cdot 13^2 \cdot 31$, with $K_0$ defined by $\nu^3 - \nu^2 - 4\nu - 1$.

(4) $y^3 = x^4 - 2 \cdot 7 \cdot 31 \cdot 73\, x^2 + 2^{11} \cdot 31 \cdot 47\, x - 7 \cdot 31^2 \cdot 11593$, with $K_0$ defined by $\nu^3 + \nu^2 - 10\nu - 8$.

(5) $y^3 = x^4 - 2 \cdot 7 \cdot 43^2 \cdot 223\, x^2 + 2^7 \cdot 11 \cdot 41 \cdot 43^2 \cdot 59\, x - 11^2 \cdot 43^3 \cdot 419 \cdot 431$, with $K_0$ defined by $\nu^3 - \nu^2 - 14\nu - 8$.

(6) $y^3 = x^4 - 2\cdot 3^2\cdot 5^2\cdot 7^2\,x^2 + 2^9\cdot 7^2\cdot 71\,x - 3^2\cdot 5\cdot 7^3\cdot 2621$, with $K_0$ defined by $\nu^3 - 21\nu - 28$.

(7) $y^3 = x^4 - 2^2\cdot 3^2\cdot 7^2\cdot 37\,x^2 + 5\cdot 7^2\cdot 149\cdot 257\,x - 2\cdot 3^2\cdot 5^2\cdot 7^3\cdot 2683$, with $K_0$ defined by $\nu^3 - 21\nu + 35$.

(8) $y^3 = x^4 - 2\cdot 3^2\cdot 5^2\cdot 7\cdot 11\cdot 13\,x^2 + 2^7\cdot 11\cdot 13\cdot 59\cdot 149\,x - 3^2\cdot 5\cdot 7\cdot 13^2\cdot 17\cdot 17669$, with $K_0$ defined by $\nu^3 - 39\nu + 26$.

(9) For $K_0$ defined by $\nu^3 - \nu^2 - 6\nu + 7$, and $w^3 = 19$,

$$y^3 = x^4 + (10w^2 - 2w - 70)\,x^2 + (96w^2 - 7w - 496)\,x + (235w^2 - 215w - 1101).$$

(10) For $K_0$ defined by $\nu^3 - \nu^2 - 12\nu - 11$, and $w^3 = 37$,

$$\begin{aligned} y^3 = x^4 &+ (-2366w^2 + 490w + 24626)\,x^2 + (-257958w^2 - 686928w \\ &+ 5152928)\,x + (1226851w^2 - 56922233w + 176054907). \end{aligned}$$

(11) For $K_0$ defined by $\nu^3 - 109\nu - 436$, and $w^3 = 109$,

$$\begin{aligned} y^3 = x^4 &+ \big(1115888872w^2 - 4007074778w - 6321528472\big)\,x^2 \\ &+ \big(-39141169182336w^2 + 294349080537984w - 512926132238464\big)\,x \\ &+ 8163420095545193050w^2 - 92763246224286050480w \\ &+ 2568408685549314429 6. \end{aligned}$$

(12) For $K_0$ defined by $\nu^3 - \nu^2 - 42\nu - 80$, and $w^3 = 127$,

$$\begin{aligned} y^3 = x^4 &+ \big(-92075757704w^2 + 319193013538w + 721950578888\big)\,x^2 \\ &+ \big(-49404281036538240w^2 - 182817463505393280w + \\ &21671832943051936 00\big)x + 216905110270037364330 25w^2 - \\ &1188030290867222054498 00w + 4913488212848348562 7800. \end{aligned}$$

(13) For $K_0$ defined by $v^3 - 61v - 183$, we have four curves. The first one is defined over $\mathbb{Q}$.

$$\begin{aligned} y^3 = x^4 &- 2\cdot 3\cdot 7\cdot 61^2\cdot 1289\,x^2 + 2^3\cdot 3^7\cdot 11\cdot 41\cdot 53\cdot 61^2\,x \\ &- 3^2\cdot 7\cdot 11^2\cdot 61^3\cdot 419\cdot 4663 \end{aligned}$$

$$\begin{aligned} y^3 = x^4 &+ \big(89264v^2 - 547484v - 4059720\big)\,x^2 + \big(-29558196v^2 + 49526073v \\ &+ 772138494\big)x + 88325678v^2 - 16281030326v - 72348132021 \end{aligned}$$

(14) For $K_0$ defined by $v^3 - v^2 - 22v - 5$, similarly one gets:

$$y^3 = x^4 + 2\cdot 7\cdot 67\cdot 179\,x^2 + 2^3\cdot 3^3\cdot 5\cdot 67\cdot 137\,x + 5^2\cdot 7\cdot 67^2\cdot 71\cdot 89$$

$$\begin{aligned} y^3 = x^4 &+ \big(12222v^2 - 263088v - 1290744\big)\,x^2 + \big(-19721880v^2 + 232016400v \\ &+ 1277237160\big)x + 11453819175v^2 - 62791404525v - 447679991475\,. \end{aligned}$$