Inverse Jacobian and related topics for certain superelliptic curves

Somoza Henares, A.

Author: Somoza Henares, A.
Title: Inverse Jacobian and related topics for certain superelliptic curves
Issue Date: 2019-03-28

Introduction

Given an *elliptic curve* E over $\mathbb{C}$, there exists a *lattice* $\Lambda \subseteq \mathbb{C}$ such that the group $E(\mathbb{C})$ of complex points on E is isomorphic to the complex analytic group $\mathbb{C}/\Lambda$. This link between elliptic curves and one-dimensional complex tori is called the Uniformization Theorem, and one can explicitly find the curve corresponding to a given lattice with the *Weierstrass* $\wp$*-function*, its derivative, and the *Eisenstein series.*

Similarly, given an algebraic curve C of genus g, one associates to it a *principally polarized abelian variety* $J(C)$, the *Jacobian of* C. Over $\mathbb{C}$, the Jacobian $J(C)$ is isomorphic to a g-dimensional complex torus $\mathbb{C}^g/\Lambda$ for a lattice Λ of full rank in $\mathbb{C}^g$.

This determines a map J from the set M_g of isomorphism classes of algebraic curves of genus g to the set A_g of principally polarized abelian varieties of dimension g, and one may wonder if there exists an explicit inverse to this map, as in the case of elliptic curves. We call this the *inverse Jacobian problem.*

This problem has been solved for curves of genus 2 [37, 50] and genus 3 [1, 9, 16, 21, 48, 52, 53]. However, for genus $g \geq 4$ there is the additional obstruction that not all principally polarized abelian varieties are Jacobians of curves, hence in order to solve the inverse Jacobian problem one needs to study the image by J of M_g in A_g. The problem of describing $J(\mathsf{M}_g)$ is known as the *Riemann-Schottky problem.* .

In this thesis we treat these two problems for two families of *superelliptic curves*, that is, curves of the form $y^k = \prod_{i=1}^{l}(x - \alpha_i)$. We focus on the family of *Picard curves*, with $(k, l) = (3, 4)$ and genus 3, where we solve the inverse Jacobian problem, and the family of *cyclic plane quintic curves* (CPQ curves), with $(k, l) = (5, 5)$ and genus 6, where we solve both problems.

In Chapter 1 we first introduce some background on abelian varieties, Jacobians of curves, and Riemann theta constants, and then we present an inverse Jacobian algorithm for Picard curves. Note that Picard curves have genus 3, hence there is no obstruction to the inverse Jacobian problem.

Since Picard curves are plane quartic curves, the inverse Jacobian problem for Picard curves could be solved using the formulas for plane quartics given in [52], but focusing on a smaller family of curves allows us to present a more efficient solution for the family of interest.

This was originally done by Koike and Weng in [16], but their exposition presents some mistakes that we address and correct here. This chapter is based on joint work with Joan-Carles Lario, see also [21].

In Chapter 2 we present an inverse Jacobian algorithm for CPQ curves. We follow a strategy analogous to the one in Chapter 1 for the case of Picard curves.

In Chapter 3 we address the Riemann-Schottky problem for CPQ curves, that is, we characterize the principally polarized abelian varieties that are Jacobians of CPQ curves. First we use a generalization of the classical theory of *complex multiplication* due to Shimura [39] to study how the existence of the automorphism of CPQ curves $(x, y) \mapsto (x, \exp(2\pi i/5)y)$ affects the structure of the Jacobians. Then we solve a class number one problem for higher-dimensional Hermitian lattices over $\mathbb{Z}[\zeta_5]$, which is key to solving the Riemann-Schottky problem for CPQ curves.

Finally, in Chapter 4 we present one application for the above algorithms: constructing curves such that their Jacobians have complex multiplication. This has previously been done for genus 2 [51, 47] and genus 3 [1, 13, 16, 21, 53]. Here we extend the methods of Kılıçer [12] to determine a complete list of CM-fields whose ring of integers occurs as the endomorphism ring over $\mathbb{C}$ of the Jacobian of a CPQ curve defined over $\mathbb{Q}$.

In particular, this allows us to list conjectural models for all CPQ curves over $\mathbb{Q}$ whose Jacobians have the maximal order of a degree-12 CM-field as endomorphism ring over $\mathbb{C}$. Our list contains the correct number of curves, which are defined over $\mathbb{Q}$ and numerically correct up to high precision.