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Chapter 32
Multi-digit Addition, Subtraction, Multiplication, and Division Strategies

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It has long been recognized that children's arithmetic is characterized by strategy variability. Children use a variety of different strategies to solve arithmetic problems. This variability is characterized by both interindividual variability, meaning that different individuals rely on different strategies to solve a given arithmetic task, and intraindividual variability, referring to one individual using different strategies to solve different tasks or even the same task at different moments and/or in different settings (e.g., Siegler, 2007). Furthermore, with increasing age and experience, children not only tend to develop from using less efficient to more efficient strategies but also become increasingly adaptive in their strategy choices, as described in Siegler's (1996) overlapping waves theory.

To optimally enhance children's arithmetic learning, it is important to know what strategies children use and what obstacles they encounter in acquiring these strategies. There are many studies on children's strategy use in single-digit arithmetic (for a review, see, for instance, Verschaffel, Greer, & De Corte, 2007), but research in the domain of multi-digit arithmetic is rather limited, in particular for multi-digit multiplication and division. This is problematic since the upper grades of primary school are usually devoted to instruction and practice in solving multi-digit arithmetic problems, and children may experience quite large difficulties in that domain.

The current chapter's aim is therefore to give an overview of what is known about primary school children's strategy use in multi-digit arithmetic, defined as

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addition, subtraction, multiplication, and division tasks in which at least one of the operands contains two or more digits. Furthermore, we aim to identify obstacles children encounter in developing, selecting, or executing these strategies, in the population of all learners as well as specifically in the group of children with mathematical difficulties.

Multi-digit Arithmetic Solution Strategies

Strategies for multi-digit arithmetic differ from those for single-digit arithmetic. In *single-digit arithmetic*, an important distinction is between computational strategies and retrieval. In computational strategies (also called backup strategies), the answer is computed in subsequent solution steps, for instance, by counting on from the larger integer (9 + 3 = 9, 10, 11, 12) or by reference to another easier or already known problem (derived facts: e.g., 9 + 3 = 10 + 3 – 1 = 13 – 1 or 12). Retrieval concerns recalling the answer from long-term memory as an arithmetic fact, without intermediate computational solution steps (e.g., 9 + 3 = (immediately) 12). Generally speaking, children's single-digit arithmetic development is characterized by the progression from concrete counting strategies via derived fact strategies to the final mastery of retrieval of the arithmetic fact (e.g., Verschaffel et al., 2007). By contrast, in *multi-digit arithmetic* retrieval of the outcome as an arithmetic fact is not feasible: the outcome needs to be computed. Hence, in multi-digit arithmetic the question is how the numbers are manipulated in order to find the answer. That is what we call a (solution) strategy.

An important characteristic of multi-digit strategies is how the numbers are operated on: respecting the place value the single digits of those numbers represent or not. This distinction yields two major types of strategies: number-based strategies and digit-based strategies (for reviews, see Fuson, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel et al., 2007).[1] In *number-based strategies*, the place value of the digits in the numbers is respected (e.g., the number 83 may be split into 80 and 3), whereas in *digit-based strategies*, the place value of the digits is ignored (e.g., 83 may be split in the digits 8 and 3, ignoring that the 8 actually stands for 8 tens = 80). The most common digit-based strategies are the written algorithms of long addition, subtraction, multiplication, and division, operating on single digits in a proceduralized way, usually from right to left. In the current chapter, we also dis-

[1] Some authors use the terms mental computation strategies and written arithmetic instead of number- and digit- based arithmetic, where mental computation strategies may refer to either operating on numbers *with* the head or entirely *in* the head, whereas written arithmetic refers to the execution of digit-based algorithms usually with paper and pencil (for more details, see Verschaffel et al., 2007). Since the most important distinguishing feature between the different types of multi-digit strategies is operating on numbers versus on digits (rather than mental versus written computation), we prefer the terms number-based versus digit-based strategies.

cuss so-called column-based strategies: a specific form of number-based strategies that in some reform-based mathematics curricula, such as in realistic mathematics education (RME) in the Netherlands, are instructed as an intermediate strategy to make the transition between number-based strategies and the digit-based algorithm smoother and more insightful (e.g., van den Heuvel-Panhuizen, 2008; van den Heuvel-Panhuizen & Drijvers, 2014). These column-based strategies have elements of the digit-based algorithmic approaches, since they also involve a structured, vertical, notation. However, they operate on whole numbers instead of digits, and they proceed from left to right, which are two characteristics that clearly distinguish them from the digit-based algorithms. Some authors (e.g., Buijs, 2008) therefore call these column-based strategies *stylized mental computation strategies* (where mental refers to computing *with* the head instead of entirely *in* the head; see also Footnote 1). Similar approaches can be found in other innovative mathematics learning-teaching methodologies, such as the open calculation based on numbers in Spain (Aragón, Canto, Marchena, Navarro, & Aguilar, 2017).

Given that there are several possible strategies to solve multi-digit arithmetic problems, the question arises how children select a particular strategy from their repertoire. This question has intrigued cognitive psychologists already since the 1950s (e.g., Siegler, 2007) and is also relevant from a mathematics education perspective: an important goal of contemporary mathematics education around the world is that children acquire the competence to solve mathematical problems efficiently, creatively, and flexibly or adaptively with an array of meaningfully acquired strategies (e.g., Hatano, 2003; Star et al., 2015). Scholars use different definitions of flexibility and adaptivity. In the current chapter, we use flexibility and adaptivity interchangeably as selecting the optimal strategy for a given problem in a given setting for a given person. Verschaffel, Luwel, Torbeyns, and Van Dooren (2009) discuss that adaptivity can be conceptualized with respect to task characteristics (i.e., Does the child select the strategy that is best for that problem given a rational task analysis?), subject characteristics (i.e., Does a child select the strategy (s)he performs best with?), and contextual characteristics (i.e., Does a child select the strategy that is optimal given the circumstances, such as the value of speed over accuracy?). According to that conceptualization, a child behaves adaptively if (s)he chooses the strategy that is the optimal one, taking into account the features of the task at hand, his/her mastery of the various strategies available in his/her strategy repertoire, and the sociocultural setting wherein (s)he is confronted with the task (Verschaffel et al., 2009)

In the following, we will discuss the research literature on children's strategy competencies in the additive domain (i.e., multi-digit addition and subtraction) and the multiplicative domain (i.e., multi-digit multiplication and division). In both parts, we start with presenting a comprehensive framework of the different number-based and digit-based strategies, followed by a review of empirical findings regarding children's use of these strategies and ending with a discussion of the obstacles in developing these strategies.

Multi-digit Addition and Subtraction Strategies

Strategies Framework

Table 32.1 shows an overview of the number-based and digit-based strategies for multi-digit addition and subtraction, based on earlier categorizations (Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012; Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). A first dimension along which the strategies can be categorized is the operation that underlies the solution process: addition or subtraction. In multi-digit addition there is only way of carrying out the operation, as *direct addition*: one operand is directly added to the other. By contrast, in multi-digit subtraction, there are three different ways in which the operation can be carried out: as *direct subtraction* in which the subtrahend is taken away from the minuend, as *indirect addition* in which one adds on from the subtrahend until the minuend is reached (also called adding-on strategy), and as *indirect subtraction* in which one determines the difference by how much has to be taken away from the minuend to reach the subtrahend.

A second, complementary, dimension concerns how the numbers are dealt with. In *sequential* strategies (also called jump or N10 strategies), the numbers are primarily seen as objects on the (mental) number line and the operations as forward or backward movements along this number line. By contrast, in *decomposition* strategies (also called split or 1010 strategies; e.g., Beishuizen, 1993;

Table 32.1 Overview of solution strategies for multi-digit addition and subtraction

		Number-based strategies				Digit-based algorithm
		Sequential	Decomposition	Varying	Column-based strategy	
Addition e.g., 38 + 46	Direct addition	38 + 40 = 78; 78 + 6 = **84**	30 + 40 = 70; 8 + 6 = 14; 70 + 14 = **84**	38 + 50 = 88; 88–4 = **84**	38 46+ 70 14+ **84**	1 38 46+ **84**
Subtraction e.g., 82 – 69	Direct subtraction	82 – 60 = 22 22 – 9 = **13**	80 – 60 = 20; 2 – 9 = −7; 20 –7 = **13**	For example, compensation 82 – 70 = 12; 12 + 1 = **13**	82 69– 20 −7 **13**	7 12 82 69– **13**
	Indirect addition	69 + 3 = 72; 72 + 10 = 82; 3 + 10 = **13**	9 + 3 = 12 60 + 10 = 70 3 + 10 = **13**	69 + 1 = 70; 70 + 12 = 82; 1 + 12 = **13**		
	Indirect subtraction	82 – 10 = 72; 72 – 3 = 69; 10 + 3 = **13**	80 – 10 = 70; 2 – 3 = −1; 10 + 3 = **13**	82 – 20 = 62 62 + 7 = 69 20 – 7 = **13**		

Blöte, van der Burg, & Klein, 2001), the numbers are primarily seen as objects with a decimal structure, and the operations involve partitioning or splitting the numbers. The category of *varying* strategies includes diverse strategies that involve the adaptation of the numbers and/or operations in the problem, such as in the compensation strategy where one of the operands is rounded up to a near round number (e.g., subtracting 70 instead of 69 and compensating back the 1 that was subtracted too much). Besides these three types of strategies, we distinguish – in line with Dutch (RME-based) mathematics educators – a fourth number-based strategy in Table 32.1: the column-based strategy, which essentially consists of the same numerical approach as the decomposition strategy. In the Dutch RME, this strategy is explicitly instructed as a separate strategy, functioning as an intermediate strategy bridging the gap between number-based strategies and digit-based algorithms, by its "hybrid" nature of, on the one hand, operating on numbers rather than digits but, on the other hand, doing so in a standardized step-by-step sequence accompanied by a structured vertical notation.

In most countries the *digit-based algorithms* fall in the category of *direct* addition or subtraction.[2] The main difference with the number-based strategies is that the integers are dealt with as digits, ignoring the place value they represent. For instance, in the digit-based addition strategy, one starts by adding the unit integers 8 + 6 = 14, then writes down the 4 and holds the 10 in memory as a 1, and then adds the tens integers 3 + 4 + 1 = 8. It is not before the 8 is combined with the 4 that the 8 turns out to represent 8 tens. The digit-based addition and subtraction algorithms proceed from the right to left (i.e., starting with the units, then the tens, etcetera).

Children's Strategy Use: Empirical Findings

As discussed in Verschaffel et al. (2007), studies on children's number-based and digit-based strategy competencies conducted in the 1900s and early 2000s revealed that children rely on different types of number-based strategies before the standard digit-based strategies are introduced at school. The level of strategy variety tends to depend on the nature of the provided instruction: Children who received instruction that focused on the mastery of a given number-based decomposition or sequential strategy with hardly any attention for strategy variety tend to rely on only the instructed strategy and to demonstrate less strategy variety than children who experienced instruction that focused on strategy variety.

Furthermore, Verschaffel et al. (2007) discuss how children's use of number-based strategies is typically challenged by the introduction of the digit-based algorithms for multi-digit addition and subtraction at school: Once the digit-based

[2] In some countries, such as Germany, the digit-based algorithm via indirect addition is used for subtraction; see Verschaffel et al. (2007) for an example. This is also called the Austrian algorithm.

algorithms are explicitly taught to the children, they tend to prefer these algorithms over the previously learnt number-based strategies, also on tasks for which the use of (specific) number-based strategies would be equally or even more efficient, such as 601 – 598 = _. However, children do not necessarily apply these newly learnt digit-based algorithms more efficiently, as illustrated by the frequent occurrence of errors due to the application of so-called "buggy procedures"–systematic erroneous procedures whereby one or more specific steps of the correct procedure are overlooked or executed wrongly (e.g., always subtracting the smaller from the larger digit when applying the digit-based subtraction algorithm, resulting in errors as 258 – 179 = 121).

Since Verschaffel et al.'s (2007) review, quite a number of researchers have continued to deepen our understanding of the variety, frequency, efficiency, and adaptiveness of children's number-based and digit-based multi-digit addition and subtraction strategies. One particularly interesting way they did so was by using a more sophisticated research paradigm than before: the so-called choice/no-choice method developed by Siegler and Lemaire (1997); see also Luwel, Onghena, Torbeyns, Schillemans, and Verschaffel (2009). In this method, children solve problems in two different condition types: the choice condition where they are free to select their strategy, and in two or more no-choice conditions where they have to use a particular strategy. The choice condition allows investigating children's strategy repertoire and variety, but strategy efficiency may be biased by selection effects. For instance, when a strategy is selected by weaker children and/or on more difficult problems, this strategy may seem less efficient than it actually is. The no-choice conditions overcome this because all children have to solve all problems with a particular strategy, allowing assessing the strategy's efficiency (accuracy and speed) in an unbiased way. These unbiased strategy efficiency data can be used to address adaptivity to individual mastery of the strategies, by investigating the extent to which children select the strategy (in the choice condition) that is most efficient for him/her (based on data from the no-choice conditions).

A first series of studies addressed (mid and upper) primary school children's number-based strategy competencies. Studies addressing children's number-based decomposition, sequential, and varying strategy use on multi-digit additions and subtractions generally confirm the results discussed above. Children who received instruction with primary focus on the mastery of one specific type of (decomposition or sequential) number-based strategy tend to consistently apply the instructed (decomposition or sequential) strategy on different types of multi-digit addition and subtraction problems (Csíkos, 2016; Heinze, Marschick, & Lipowsky, 2009). By contrast, reform-oriented instructional approaches stimulate children's efficient and adaptive use of different types of number-based strategies, including – although applied with limited frequency – varying strategies as compensation and indirect addition. Somewhat in contrast with this general finding, studies focusing on children's use of the number-based indirect addition strategy for multi-digit subtractions indicated that 9–12-year-olds frequently, efficiently, and adaptively rely on this indirect addition strategy, despite the strong instructional focus on and the frequent practice of (only) direct subtraction strategies (Peltenburg et al., 2012;

Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2013; Peters, De Smedt, Torbeyns, Verschaffel, & Ghesquière, 2014; Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2017). Although children were hardly confronted with indirect addition during mathematics instruction, they tended to frequently and highly efficiently apply this strategy, with accuracy and speed of strategy execution being at least as high as for direct subtraction (Torbeyns et al., 2018). Moreover, notwithstanding the absence of instruction in this strategy, they even adaptively took into account the numerical characteristics of the subtractions when selecting indirect addition versus direct subtraction strategies (Peltenburg et al., 2012; Peters et al., 2013, 2014; Torbeyns et al., 2018) as well as their individual mastery of the different types of strategies (Torbeyns et al., 2018). Importantly, these findings were observed for children of all mathematical achievement levels, including the lower-achieving children (Torbeyns et al., 2018) and children with mathematical difficulties (Peltenburg et al., 2012; Peters et al., 2014).

Other studies focused on (middle and upper) primary school children's use of number-based versus digit-based addition and subtraction strategies in different countries: the Netherlands, Belgium, Spain, and Taiwan (Hickendorff, 2013; Karantzis, 2010; Linsen, Torbeyns, Verschaffel, Reynvoet, & De Smedt, 2016; Torbeyns, Hickendorff, & Verschaffel, 2017; Torbeyns & Verschaffel, 2013, 2016; Yang & Huang, 2014). Confirming the results of previous studies in the domain, once being taught digit-based algorithms, many children tended to prefer them over number-based strategies (even applying the mental version of the digit-based algorithm when required to compute entirely in the head). But, contrasting previous findings, they applied the digit-based algorithms remarkably efficiently, with an accuracy and speed level that was at least as high as for the (previously taught and highly frequently practiced) number-based strategies. Finally, children demonstrated adaptive strategy choices, using number-based versus digit-based strategies in relation to the numerical characteristics of the problems (Torbeyns et al., 2018) and their individual mastery of the different types of strategies (Torbeyns & Verschaffel, 2013, 2016) but not the format of the problem (word problem versus symbolic problem; Hickendorff, 2013).

Obstacles in Development

Cumulative evidence indicates that the acquisition of multi-digit addition and subtraction strategies is a real challenge for many children, especially these of lower mathematical achievement levels. As discussed in Verschaffel et al. (2007), previous investigations point to children's limited conceptual understanding of number as one of the major sources of their difficulties in the acquiring and application of number-based and digit-based strategies. Linsen et al. (2016) recently provided further support for this claim, by analyzing the relation between 9–10-year-olds' magnitude understanding (i.e., insight into the magnitude or value of the numbers) and number-based and digit-based strategy efficiency in the domain of multi-digit subtraction. Their results revealed strong associations between children's magnitude understanding and their efficiency in both types of strategies. But the observed associations were stronger for number-based than for digit-based strategy use, suggesting a larger involvement of children's conceptual understanding of numbers in the execution of the former than in the execution of the latter type of strategies. Moreover, children's arithmetic fact knowledge for single-digit addition and subtraction was strongly related to their multi-digit strategy efficiency, which points to a second possible obstacle for children's multi-digit strategy acquisition in the domain of addition and subtraction, namely, their mastery of single-digit facts.

In addition to children's conceptual understanding of multi-digit numbers and their fluency with single-digit arithmetic facts, Selter, Prediger, Nührenbörger, and Hußmann (2012) discuss another possible obstacle for the development of fluency in multi-digit addition and subtraction, namely, their understanding of the arithmetic operations and their corresponding symbols (see also Robinson, 2017). For instance, using indirect addition on multi-digit subtractions relies on a broadened interpretation of the minus sign as indicating not only "taking away" (resulting in direct subtraction: taking away the smaller from the larger number) but also "bridging the difference" (enabling indirect addition). Likewise, when applying indirect addition, children have to understand the complementary relation between the addition and subtraction operation (i.e., understand that $a - b = ?$ can be solved via $b + ? = a$). For an extensive overview of the research on the role of understanding of the operations of addition and subtraction and their various arithmetical principles, see Baroody, Torbeyns, and Verschaffel (2009) and Robinson (2017).

The limited number of studies addressing the strategy competencies of children of the lower mathematical achievement levels and of children with mathematical difficulties did not yet provide unequivocal results about specific difficulties and the related foundational obstacles in their strategy development in the domain of multi-digit addition and subtraction (Peltenburg et al., 2012; Peters et al., 2014; Torbeyns, Hickendorff, et al., 2017; Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2017). Studies with children without diagnosed mathematical difficulties reported that children with higher general mathematical achievement level had higher levels of strategy variety, efficiency, and adaptivity (Torbeyns, Hickendorff, et al., 2017; Torbeyns, Peters, et al, 2017). However, the studies of Peltenburg et al. (2012) and Peters et al. (2013, 2014) indicated that children with mathematical difficulties are also able to frequently and adaptively apply various number-based strategies. Future studies in children of the lowest mathematical achievement levels, including children with mathematical difficulties, are needed to get a better view on the contribution of children's conceptual understanding of numbers, symbols, and operations (cf. Linsen et al., 2016; Selter et al., 2012; Torbeyns, Peters, De Smedt, Ghesquière, & Verschaffel, 2016), their arithmetic fact knowledge (cf. Linsen et al., 2016), and other child- and context-related characteristics to their strategy development in the domain of multi-digit addition and subtraction.

Multi-digit Multiplication and Division Strategies

Strategies Framework

There is much less consensus on the different types of strategies for multiplication and division than there is for addition and subtraction. Based on the existing frameworks (e.g., Buijs, 2008; Hickendorff, 2013; van Putten, van den Brom-Snijders, & Beishuizen, 2005; Zhang, Ding, Lee, & Chen, 2017), in the current chapter, we propose a comprehensive classification system with dimensions comparable to those for multi-digit addition and subtraction: one dimension characterizing which operation underlies the solution process (multiplication or division) and the other dimension how the numbers are dealt with; see Table 32.2. Regarding the first dimension, in multi-digit multiplication there is only *direct multiplication* in which the underlying process is multiplication. In multi-digit division one can start with dividend in *direct division.* An alternative way to solve division problems is by *indirect multiplication*, also called multiplying-on (van den Heuvel-Panhuizen, Robitzsch, Treffers, & Köller, 2009) or reversed multiplication (Ambrose, Baek, & Carpenter, 2003), where one starts with the divisor and determines how many times it has to be multiplied to reach the dividend.

With respect to the second dimension, within the number-based strategies, it is again possible to distinguish between sequential, decomposition, and varying strategies. *Sequential* strategies involve movements forward or backward on the (mental) number line. In multiplication and division strategies, the sequential strategies are repeated addition or subtraction strategies, based on additive reasoning (e.g., see Larsson, 2016). In repeated addition, the multiplication problem 23 × 19 is solved, for instance, by adding the number 23 for 19 times. Of course, it is also possible not to repeatedly add single 23 s but multiples of 23 (see Table 32.2). Repeated addition can also be used to solve division problems within the indirect multiplication approach. In repeated subtraction, a division problem is solved by subtracting the divisor repeatedly from the dividend until there is nothing left. Again, it is possible to do this with single divisors or multiples of the divisor. By contrast, in *decomposition* strategies the numbers are decimally split (one or both operands in multiplication and only the dividend in division – splitting the divisor leads to an incorrect procedure). These strategies are, according to Larsson (2016), based on two-dimensional multiplicative reasoning. *Varying* strategies involve the adaptation of number and/or operations, like in the compensation strategy examples in Table 32.2. As a final number-based strategy, we again distinguish the *column-based strategy*, inspired by Dutch (RME) mathematics educators. The column-based strategy is a vertically notated schematized version of the decomposition strategy in multiplication and of the repeated subtraction strategy in division (e.g., Buijs, 2008; Treffers, 1987; Van Den Heuvel-Panhuizen, 2008).

The digit-based strategies involve operating on the digits ignoring their place value. It is important to note that the digit-based multiplication algorithm proceeds from right to left, like the digit-based algorithms for addition and subtraction. By contrast, the digit-based division algorithm proceeds from left to right and does not work with only one digit at a time.

Table 32.2 Overview of solution strategies for multi-digit multiplication and division

		Number-based strategies				Digit-based algorithm
		Sequential	Decomposition	Varying	Column-based	
Multiplication, e.g., 23 × 19	Direct multiplication	23 + 23 + 23 + … + 23 = **437** *or* 5 × 23 = 115 4 × 23 = 92 115 + 115 + 115 + 92 = **437**	23 × 10 = 230 23 × 9 = 207 230 + 207 = **437** *or* 20 × 10 = 200 3 × 10 = 30 20 × 9 = 180 3 × 9 = 27 200 + 30 + 180 + 27 = **437**	For example, compensation 23 × 20 = 460; 460 – 23 = **437**	23 19× 200 30 180 27+ **437**	23 19× 207 230+ **437**
Division, e.g., 168 : 12	Direct division	168 – 12 = 156; 156 – 12 = 144; [subtracting 12s 14 times → **14**] *or* 168 – 120 (10x) = 48 48 – 48 (4x) = 0 10 + 4 = **14**	100 : 12 = 8,25 60 : 12 = 5 8 : 12 = 0,75 8,25 + 5 + 0,75 = **14**	For example, compensation 240 : 12 = 20 72 : 12 = 6 20 – 6 = **14**	168 : 12 = 120 – 10× 48 48 – 4× 0	12 / 168 \ 14 12 – 48 48 – 0
	Indirect multiplication	12 + 12 = 24; 24 + 12 = 36; [adding 12 s 14 times → **14**]	8,25 × 12 = 100; 5 × 12 = 60; 0,75 × 12 = 8 8,25 + 5 + 0,75 = **14**	20 × 12 = 240 6 × 12 = 72 20 – 6 = **14**		

Children's Strategy Use: Empirical Findings

Compared to multi-digit addition and subtraction, there is little research into children's solution strategies use in multi-digit multiplication and division. Verschaffel et al. (2007)'s summary of the (at that time) available studies showed that, as for addition and subtraction, children rely on different types of number-based strategies to solve multi-digit multiplication and division, before the digit-based algorithms were introduced at school. In multiplication, the use of number-based strategies seems to progress from sequential (i.e., additive) strategies to decomposition (i.e., multiplicative) strategies. In multi-digit division, children tend to progress from the sequential strategies repeated addition/subtraction with single divisors to more efficient approaches using multiples (also called chunks) of the divisor. There is some evidence that once the digit-based algorithm is instructed, children rely heavily on that, abandoning the number-based strategies they had been using before.

Since the review of Verschaffel et al. (2007), few studies addressed children's number-based strategy competencies in the domain of multi-digit multiplication and division. Buijs (2008) followed Dutch 9–10-year-olds' strategy development in multi-digit multiplication. At each measurement point, children used the decomposition strategies most often, and the use increased over time. The frequency of repeated addition strategies was rather low, contrasting with Larsson's (2016) findings that Swedish 10–13-year-olds multi-digit multiplication strategy use remained to be heavily based on the repeated addition strategy.

Recent studies addressing (upper primary school) children's multi-digit number-based and digit-based strategy competencies in multiplication and division have primarily been conducted in the Netherlands. One exception is the study of Zhang et al. (2017), investigating the strategy use across single-digit and multi-digit multiplication problems in 8–11-year-old children from the USA. They found three distinct strategy use patterns, resembling different developmental levels: children who primarily used direct retrieval or the digit-based algorithm with high accuracy, children who primarily used number-based strategies (unitary counting, doubling, repeated addition, sequential, and decomposition strategies) with medium accuracy, and children who primarily used an incorrect operation or skipped the problems.

Before discussing the findings of the studies with Dutch children, it is important to note that due to the large influence of RME, the vast majority of the Dutch mathematics textbooks abandoned the digit-based algorithm for division for a long period of time (roughly mid-1990s–2010), because it was deemed very time-consuming to attain procedural mastery and at the same time rather meaningless and error-prone for children (Treffers, 1987; van den Heuvel-Panhuizen, 2008). Instead, the column-based strategy served as the standard written procedure. More recently, the digit-based division algorithm has returned in the latest version of the most common textbooks in the Netherlands (Royal Dutch Society of Arts and Sciences, 2009). A series of studies addressing Dutch 11–12-year-olds' strategy use in multi-digit multiplication and division showed, first, that strategy use was much less dominated by the digit-based algorithm than in addition and subtraction;

second, that in division children tended to use the column-based strategy as the preferred written procedure instead of the digit-based algorithm, in line with the instructional approach; and third, that the digit-based algorithms were as least as successful as the column-based strategies (Fagginger Auer, Hickendorff, Van Putten, Béguin, & Heiser, 2016; Hickendorff, 2013; Hickendorff, Heiser, Van Putten, & Verhelst, 2009). When analyzing the types of number-based strategies the children used, in multiplication, the decomposition strategies in which one or both of the operands were decimally split were the most often used number-based strategy in multiplication, whereas repeated addition was hardly used (Hickendorff, 2013), resembling the findings of Buijs' (2008) in 9–10-year-olds. In division, the column-based strategy was the most frequently used number-based strategy; repeated subtraction without the structured vertical notation, repeated addition, and decomposition were used rather infrequently (Hickendorff, 2013). Very recently, Hickendorff, Torbeyns, and Verschaffel (2017) investigated cross-national differences between 9–12-year-old children from the Netherlands and Flanders (Belgium) in solving multi-digit division problems. Children's strategy profiles were generally in line with differences in instruction between the two countries, as, for instance, reflected by the absence of the column-based strategy in Flemish children's strategy repertoire, although large intra- and interindividual strategy variety remained.

The few results regarding the adaptivity of strategy selection showed that, with respect using varying strategies in response to task characteristics, sixth graders' use of the compensation strategy on problems suitable for compensation (e.g., 2475: 25 via 2500: 25) was modest at most (Fagginger Auer, Hickendorff, & van Putten, 2016; Hickendorff, van Putten, Verhelst, & Heiser, 2010) but somewhat higher in Dutch children instructed according to RME principles than in Flemish children being taught in a more traditional way (Hickendorff et al., 2017).

Obstacles in Development

As in multi-digit addition and subtraction, the number-based strategies require sufficient conceptual knowledge of the place value system, and understanding of the arithmetic operations and symbols is also essential (e.g., Larsson, 2016; Robinson, 2017). Furthermore, children need to have sufficient knowledge and skills in elementary arithmetic to solve multi-digit multiplication and division problems. The example strategies in Table 32.2 illustrate that in multi-digit multiplication mastery of the single-digit addition and multiplication facts are essential in a multi-digit division strategies (multi-digit), subtraction is also involved.

As in addition and subtraction, there are some common systematic errors ("buggy procedures"), for instance, in the digit-based algorithms $N \times 0 = N$, errors with carries and errors in forgetting to write down zeros (Kilpatrick et al., 2001; Verschaffel et al., 2007). Larsson (2016) and Buijs (2008) identified a common error in number-based multiplication strategies: the incomplete factorization into partial products (e.g., $23 \times 19 = 20 \times 10 + 3 \times 9$). Larsson (2016) interpreted that "buggy" strategy as an overgeneralization of addition strategies forming a structural hindrance for the conceptualization of the two-dimensionality of multiplication.

The discussed research findings signal some specific obstacles children may encounter. Larsson (2016) found that children's understanding of multiplication was robustly rooted in repeated addition (and the associated understanding of multiplication in terms of equally sized groups). While this was found to be beneficial for their understandings of calculations and underlying arithmetical principles such as distributivity, it hindered them in making further steps in their multiplicative reasoning, for instance, in the fluent use of commutativity and in the proper conceptualization of decimal multiplication. In multi-digit division Dutch 11–12-year-olds seem to have difficulties making a choice between when and when not to write down their procedure and/or calculation steps: Substantial numbers of 11–12-year-olds solved the multiplication and/or division problems without writing anything down, and these nonwritten strategies were less accurate than written ones (Fagginger Auer, Hickendorff, & van Putten, 2016; Hickendorff et al., 2009). Two follow-up studies in division showed that demanding children who used nonwritten strategies to write down their working increased their performance: in all children (Hickendorff et al., 2010) or only in the children with lower mathematical achievement levels (Fagginger Auer, Hickendorff, & van Putten, 2016). Importantly, children with lower mathematical achievement levels were found to use nonwritten strategies just as often, or even more often, than their higher-achieving peers. This suggests that lower mathematical achievers have difficulties selecting their strategies, and multi-digit division problem-solving may be improved by promoting the use of written strategies. This is supported by the ideas that writing things down may both free up cognitive capacities and sequence the actions by schematizing (e.g., Buijs, 2008; Ruthven, 1998).

To the best of our knowledge, there is hardly any research addressing multi-digit multiplication and division strategies in children with mathematical difficulties. Only Zhang, Xin, Harris, and Ding (2014) investigated the effectiveness of strategy training interventions for children struggling with multiplication in a small-scale study with three 8–9-year-old children. Their results imply that children may experience difficulties in multiplication because their strategy development lags behind and that targeting (strategy) instruction to the individual child's current level of strategy knowledge may be beneficial.

Discussion

The current chapter focused on number-based and digit-based solution strategies for multi-digit addition, subtraction, multiplication, and division problems. Based on the strategy classifications used in the literature, we presented two similar, comprehensive frameworks for addition/subtraction and multiplication/division strategies. These frameworks are based on two complementary dimensions: first, the way the numbers are manipulated to compute the outcome, as whole numbers in number-based strategies or as single digits ignoring their place value in digit-based strategies, and second, the kind of operation that is underlying the strategy. Within the number-based strategies, we distinguished sequential strategies, in which the operation involves movements along a (mental) number line, from decomposition strategies, in which the numbers are primarily seen as objects with a decimal structure and split and processed accordingly. Varying strategies involve the flexible adaptation of numbers and/or operations. Finally, the column-based strategy is an intermediate strategy between number-based and digit-based strategies due to its hybrid nature and its position in the RME-based instructional pathway.

Starting from these two frameworks, we discussed the empirical findings on the (development of) children's solution strategies in multi-digit arithmetic. Compared to single-digit arithmetic, the research body is rather small, and in particular, studies addressing multi-digit multiplication and division remain remarkably scarce (see also Larsson, 2016). Further research addressing multiplication and division simultaneously is necessary, since these two operations and the relations between them are more difficult for children to understand and may require explicit instruction (Robinson, 2017). Relatedly it is interesting to note that the four multi-digit arithmetic operations are very rarely addressed simultaneously, see Hickendorff (2013) for an exception, whereas mathematically, psychologically, and educationally, the operations are clearly interrelated. For instance, the work of Larsson (2016) signals the overgeneralization of aspects of additive reasoning to multiplication. In order to increase our understanding of (the development of) multi-digit solution strategies, further research into children's multi-digit strategy competencies in the four operations and their interrelations is called for. Finally, an important remark is that the majority of the studies discussed were carried out in the USA or Europe, whereas cultural differences in preferred strategies have been reported which may be related to the curriculum (e.g., abacus instruction enhancing visualization strategies) as well as extracurricular culture-specific factors (e.g., language for numbers) (e.g., Campbell, Xue, & Campbell, 2001; Cantlon & Brannon, 2006). Future cross-cultural research would allow a broader perspective on children's strategy development in multi-digit arithmetic in different curricula and cultures.

The empirical findings show that children use a variety of number-based strategies efficiently and adaptively, before the introduction of the digit-based algorithms. The introduction of the digit-based algorithms seems a critical instructional event: children show a large tendency to use the digit-based algorithms once they are instructed, and recent findings indicate that they do so rather efficiently. Furthermore, in the Dutch RME approach, the column-based strategies are introduced as a smooth transitory path between number-based strategies and the digit-based algorithm, or even – more radically – as a more insightful, more conceptually based alternative for these digit-based algorithms. Studies show that Dutch children perform equally well with column-based strategies as with the digit-based algorithm in division. Moreover, Flemish and Dutch children with rather different instructional settings perform equally well in the domain of subtraction and division. These results may indicate that the column-based strategy may act as a fruitful stepping stone, or even alternative, to the digit-based algorithms. However, further research into the value of the column-based strategies, in particular for children with mathematical difficulties, is necessary.

All these results combined are relevant for the debate between proponents of different mathematics educational theories on the position and value of digit-based algorithms (e.g., Kamii & Dominick, 1997; Treffers, 1987). As noted before (e.g., Verschaffel et al., 2007), strategy efficiency may be at odds with other components of mathematical competence, such as insightful and adaptive computations. The focus of mathematics education on these latter aspects is not only because these are expected to increase computational efficiency but also because mathematics education targets other goals as well, such as conceptual understanding of mathematical operations and the disposition to choose flexibly from a repertoire of strategies. These elements in particular may form a challenge for children with mathematical difficulties.

The acquisition of multi-digit arithmetic strategies is a real challenge for many children, especially those with mathematical difficulties. The major obstacles these children may encounter in multi-digit arithmetic seem to be their conceptual understanding, procedural fluency, and adaptive/flexible strategy selection. Children's limited understanding of multi-digit numbers is likely one of the major obstacles in multi-digit arithmetic, since it is essential in the execution of both number-based and digit-based strategies. Moreover, children may have difficulties understanding the arithmetic operations and their corresponding symbols. Regarding procedural fluency, not having mastered single-digit arithmetic facts is an obstacle for children with mathematical difficulties in acquiring multi-digit strategies competence. Lastly, the research findings suggest that the adaptive selection of strategies from a repertoire of candidate strategies, and choosing when to write down the solution procedure instead of calculating in the head, may be challenging for children with lower levels of mathematical achievement.

Finally, this brings us to the issue of the effective strategy instruction for children with mathematical difficulties. Although the available studies show that at the group level there are differences in children's strategy use that can be related to differences in the instruction they received, at the level of an individual child there are a lot of variety and manifestations of strategy preference and use that do not coincide with the nature of the instruction received. Given this complex relation between strategy instruction and strategy development, we plead for instruction that (a) acknowledges that children develop their own strategies and stimulates children to use them, (b) diagnoses strategic development by ongoing assessment and progress monitoring, (c) assigns tasks based on children's current strategy level, (d) stimulates children to (self-)explain their strategies, and (e) provides explicit strategy instruction for struggling children (Zhang et al., 2014). Evidently, more research has to be done to optimize strategy instruction in the domain of multi-digit arithmetic for children with mathematical difficulties.

References

Ambrose, R., Baek, J.-M., & Carpenter, T. P. (2003). Children's invention of multiplication and division algorithms. In *The development of arithmetic concepts and skills: Constructive adaptive expertise* (pp. 305–336). https://doi.org/10.4324/9781410607218

Aragón, E., Canto, M. C., Marchena, E., Navarro, J. I., & Aguilar, M. (2017). Cognitive profile in learning mathematics with open calculation based on numbers (ABN) algorithm. *Revista de Psicodidactica/Journal of Psychodidactics, 22*(1), 1–14. https://doi.org/10.1387/RevPsicodidact.16396

Baroody, A. J., Torbeyns, J., & Verschaffel, L. (2009). Young children's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning, 11*(1–2), 2–9. https://doi.org/10.1080/10986060802583873

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4), 294. https://doi.org/10.2307/749464

Blöte, A. W., van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *Journal of Educational Psychology, 93*(3), 627–638. https://doi.org/10.1037/0022-0663.93.3.627

Buijs, K. (2008). *Leren vermenigvuldigen met meercijferige getallen [Learning to multiply with multidigit numbers]*. Utrecht: Freudenthal Institute for Science and Mathematics Education.

Campbell, J. I. D., Xue, Q., & Campbell, I. D. (2001). Cognitive arithmetic across cultures. *Journal of Experimental Psychology: General, 130*(2), 299–315. https://doi.org/10.1037//0096-3445.130.2.299

Cantlon, J. F., & Brannon, E. M. (2006). Adding up the effects of cultural experience on the brain. *Trends in Cognitive Sciences, 11*(1), 1–4. https://doi.org/10.1016/j.tics.2006.10.008

Csíkos, C. (2016). Strategies and performance in elementary students??? Three-digit mental addition. *Educational Studies in Mathematics, 91*(1), 123–139. https://doi.org/10.1007/s10649-015-9658-3

Fagginger Auer, M. F., Hickendorff, M., & van Putten, C. M. (2016). Solution strategies and adaptivity in multidigit division in a choice/no-choice experiment: Student and instructional factors. *Learning and Instruction, 41*, 52–59. https://doi.org/10.1016/j.learninstruc.2015.09.008

Fagginger Auer, M. F., Hickendorff, M., Van Putten, C. M., Béguin, A. A., & Heiser, W. J. (2016). Multilevel latent class analysis for large-scale educational assessment data: Exploring the relation between the curriculum and students' mathematical Strategies. *Applied Measurement in Education, 29*(2), 144–159. https://doi.org/10.1080/08957347.2016.1138959

Fuson, K. C. (2003). Developing mathematical power in whole number operations. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 68–94). Reston, VA: National Council of Teachers of Mathematics.

Hatano, G. (2003). Foreword. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. xi–xiii). Mahwah, NJ: Lawrence Erlbaum Associates.

Heinze, A., Marschick, F., & Lipowsky, F. (2009). Addition and subtraction of three-digit numbers: Adaptive strategy use and the influence of instruction in German third grade. *ZDM, 41*(5), 591–604. https://doi.org/10.1007/s11858-009-0205-5

Hickendorff, M. (2013). The effects of presenting multidigit mathematics problems in a realistic context on sixth graders' problem solving. *Cognition and Instruction, 31*(3), 314–344. https://doi.org/10.1080/07370008.2013.799167

Hickendorff, M., Heiser, W. J., Van Putten, C. M., & Verhelst, N. D. (2009). Solution strategies and achievement in dutch complex arithmetic: Latent variable modeling of change. *Psychometrika, 74*(2), 331–350. https://doi.org/10.1007/s11336-008-9074-z

Hickendorff, M., Torbeyns, J., & Verschaffel, L. (2017). Grade-related differences in strategy use in multi-digit division in two instructional settings. *Paper Submitted for Publication*.

Hickendorff, M., van Putten, C. M., Verhelst, N. D., & Heiser, W. J. (2010). Individual differences in strategy use on division problems: Mental versus written computation. *Journal of Educational Psychology, 102*(2), 438–452. https://doi.org/10.1037/a0018177

Kamii, C., & Dominick, A. (1997). To teach or not to teach algorithms. *The Journal of Mathematical Behavior, 16*(1), 51–61. https://doi.org/10.1016/S0732-3123(97)90007-9

Karantzis, I. (2010). Mental arithmetic calculation in the addition and subtraction of two-digit numbers: The case of third and fourth grade elementary school pupils. *International Journal for Mathematics in Education, 3*, 3–24 Retrieved from https://eclass.upatras.gr/modules/document/file.php/PDE1308/3oΆρθρο.pdf

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learning mathematics. Igarss 2014*. https://doi.org/10.1007/s13398-014-0173-7.2

Larsson, K. (2016). *Students â€™ understandings of multiplication* (Doctoral dissertation). Stockholm University, Sweden.

Linsen, S., Torbeyns, J., Verschaffel, L., Reynvoet, B., & De Smedt, B. (2016). The association between symbolic and nonsymbolic numerical magnitude processing and mental versus algorithmic subtraction in adults. *Acta Psychologica, 165*, 34–42. https://doi.org/10.1016/j.actpsy.2016.01.008

Luwel, K., Onghena, P., Torbeyns, J., Schillemans, V., & Verschaffel, L. (2009). Strengths and weaknesses of the choice/no-choice method in research on strategy use. *European Psychologist, 14*(4), 351–362. https://doi.org/10.1027/1016-9040.14.4.351

Peltenburg, M., van den Heuvel-Panhuizen, M., & Robitzsch, A. (2012). Special education students' use of indirect addition in solving subtraction problems up to 100—A proof of the didactical potential of an ignored procedure. *Educational Studies in Mathematics, 79*(3), 351–369. https://doi.org/10.1007/s10649-011-9351-0

Peters, G., De Smedt, B., Torbeyns, J., Ghesquière, P., & Verschaffel, L. (2013). Children's use of addition to solve two-digit subtraction problems. *British Journal of Psychology, 104*(4), 495–511. https://doi.org/10.1111/bjop.12003

Peters, G., De Smedt, B., Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2014). Subtraction by addition in children with mathematical learning disabilities. *Learning and Instruction, 30*, 1–8. https://doi.org/10.1016/j.learninstruc.2013.11.001

Robinson, K. M. (2017). The understanding of additive and multiplicative arithmetic concepts. In D. C. Geary, D. B. Berch, R. J. Ochsendorf, & K. M. Koepke (Eds.), *Acquisition of complex arithmetic skills and higher-order mathematics concepts* (pp. 21–46). Elsevier Inc. https://www.elsevier.com/books/acquisition-of-complex-arithmetic-skills-and-higher-order-mathematics-concepts/geary/978-0-12-805086-6

Royal Dutch Society of Arts and Sciences. (2009). *Rekenonderwijs op de basisschool. Analyse en sleutels tot verbetering [Mathematics education in primary school. Analysis and recommendations for improvement]*. Amsterdam: KNAW.

Ruthven, K. (1998). The use of mental, written and calculator strategies of numerical computation upper primary pupils within a 'calculator-aware' number curriculum. *British Educational Research Journal, 24*(1), 21–42.

Selter, C., Prediger, S., Nührenbörger, M., & Hußmann, S. (2012). Taking away and determining the difference–a longitudinal perspective on two models of subtraction and the inverse relation to addition. *Educational Studies in Mathematics, 79*(3), 389–408. https://doi.org/10.1007/s10649-011-9305-6

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking*. New York: Oxford University Press Retrieved from https://books.google.nl/books?hl=nl&lr=&id=lb-hjI0Et8kC&oi=fnd&pg=PR9&dq=siegler+emerging+minds&ots=0JyIMyFrGu&sig=7OGtuG8rkwfaZCoYaNqQodK_GtI#v=onepage&q=sieglere merging minds&f=false

Siegler, R. S. (2007). Cognitive variability. *Developmental Science, 10*(1), 104–109. https://doi.org/10.1111/j.1467-7687.2007.00571.x

Siegler, R. S., & Lemaire, P. (1997). Older and younger adults' strategy choices in multiplication: Testing predictions of ASCM using the choice/no-choice method. *Journal of Experimental Psychology: General, 126*(1), 71–92. https://doi.org/10.1037/0096-3445.126.1.71

Star, J. R., Newton, K., Pollack, C., Kokka, K., Rittle-Johnson, B., & Durkin, K. (2015). Student, teacher, and instructional characteristics related to students' gains in flexibility. *Contemporary Educational Psychology, 41*, 198–208. https://doi.org/10.1016/j.cedpsych.2015.03.001

Torbeyns, J., De Smedt, B., Stassens, N., Ghesquière, P., & Verschaffel, L. (2009). Solving subtraction problems by means of indirect addition. *Mathematical Thinking and Learning, 11*(1–2), 79–91. https://doi.org/10.1080/10986060802583998

Torbeyns, J., Hickendorff, M., & Verschaffel, L. (2017). The use of number-based versus digit-based strategies on multi-digit subtractions: 9–12-year-olds' strategy use profiles and task performances. *Learning and Individual Differences, 58*(June 2016), 64–74. https://doi.org/10.1016/j.lindif.2017.07.004

Torbeyns, J., Peters, G., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2016). Children's understanding of the addition/subtraction complement principle. *British Journal of Educational Psychology, 86*(3), 382–396. https://doi.org/10.1111/bjep.12113

Torbeyns, J., Peters, G., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2017). Subtraction by addition strategy use. Paper Submitted for Publication.

Torbeyns, J., Peters, G., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2018). Subtraction by addition strategy use. *Journal for Numerical Cognition, 4*(1), 215–234. https://doi.org/10.5964/jnc.v4i1.77

Torbeyns, J., & Verschaffel, L. (2013). Efficient and flexible strategy use on multi-digit sums: A choice/no-choice study. *Research in Mathematics Education*. https://doi.org/10.1080/14794802.2013.797745

Torbeyns, J., & Verschaffel, L. (2016). Mental computation or standard algorithm? Children's strategy choices on multi-digit subtractions. *European Journal of Psychology of Education, 31*(2), 99–116. https://doi.org/10.1007/s10212-015-0255-8

Treffers, A. (1987). Integrated column arithmetic according to progressive schematisation. *Educational Studies in Mathematics, 18*(2), 125–145. https://doi.org/10.1007/BF00314723

van den Heuvel-Panhuizen, M. (2008). *Children learn mathematics*. Rotterdam, the Netherlands: Sense Publishers.

van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 521–525). Dordrecht, Heidelberg/New York: Springer.

van den Heuvel-Panhuizen, M., Robitzsch, A., Treffers, A., & Köller, O. (2009). Large-scale assessment of change in student achievement: Dutch primary school students' results on written division in 1997 and 2004 as an example. *Psychometrika, 74*(2), 351–365. https://doi.org/10.1007/s11336-009-9110-7

Van Putten, C. M., van den Brom-Snijders, P. A., & Beishuizen, M. (2005). Progressive Mathematization of long division strategies in Dutch primary schools. *Journal for Research in Mathematics Education, 36*(1), 44–73 Retrieved from http://www.jstor.org/stable/10.2307/30034920

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning pages* (pp. 557–628). Greenwich: Information Age Publishing.

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335–359. https://doi.org/10.1007/BF03174765

Yang, D.-C., & Huang, K.-L. (2014). An intervention study on mental computation for second graders in Taiwan. *The Journal of Educational Research, 107*(1), 3–15. https://doi.org/10.1080/00220671.2012.753854

Zhang, D., Ding, Y., Lee, S., & Chen, J. (2017). Strategic development of multiplication problem solving: Patterns of students' strategy choices. *The Journal of Educational Research, 110*(2), 159–170. https://doi.org/10.1080/00220671.2015.1060928

Zhang, D., Xin, Y. P., Harris, K., & Ding, Y. (2014). Improving multiplication strategic development in children with math difficulties. *Learning Disability Quarterly, 37*, 15–30. https://doi.org/10.1177/0731948713500146