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## **Breaking of ensemble equivalence for complex networks**

Roccaverde, A.

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# Breaking of Ensemble Equivalence for Complex Networks

Andrea Roccaverde



# Breaking of Ensemble Equivalence for Complex Networks

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Samenstelling van de promotiecommissie:

**1<sup>e</sup> Promotor:**

Prof. dr. W. Th. F. den Hollander (Universiteit Leiden)

**2<sup>e</sup> Promotor:**

Dr. D. Garlaschelli (Universiteit Leiden)

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Prof. dr. A. Doelman (Universiteit Leiden, secretary)

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Dr. T. Squartini (IMT Institute for Advanced Studies in Lucca)

Prof. dr. H. Touchette (Stellenbosch University)

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