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Breaking of ensemble equivalence for complex networks

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Breaking of Ensemble Equivalence for Complex Networks

Andrea Roccaverde

Breaking of Ensemble Equivalence for Complex Networks

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Contents

1 Introduction **9**
§1.1 Gibbs Ensembles 10
§1.2 Equivalence of Ensembles 11
§1.3 Definition of Ensemble Equivalence 12
§1.3.1 Measure equivalence 13
§1.4 Statistical Ensembles for Complex Networks 14
§1.4.1 Microcanonical and Canonical Ensemble for Complex Networks 15
§1.4.2 α_n-Equivalence of Ensembles 16
§1.5 Summary of Chapter 2 17
§1.6 Summary of Chapter 3 18
§1.7 Summary of Chapter 4 19
§1.8 Summary of Chapter 5 19
§1.9 Summary of Chapter 6 20
§1.10 Development of the chapters 20
§1.11 Conclusions and Open Problems 21

2 Ensemble Nonequivalence in Random Graphs with Modular Structure **25**
§2.1 Introduction and main results 26
§2.1.1 Background and outline 26
§2.1.2 Microcanonical ensemble, canonical ensemble, relative entropy 28
§2.1.3 Main Theorems (Theorems 2.1.1-2.1.10) 30
§2.2 Discussion 37
§2.2.1 General considerations 37
§2.2.2 Special cases of empirical relevance 41
§2.3 Proofs of Theorems 2.1.1-2.1.10 48
§2.3.1 Proof of Theorem 2.1.1 48
§2.3.2 Proof of Theorem 2.1.4 49
§2.3.3 Proof of Theorem 2.1.5 50
§2.3.4 Proof of Theorem 2.1.6 52
§2.3.5 Proof of Theorem 2.1.7 54
§2.3.6 Proof of Theorem 2.1.8 55
§2.3.7 Proof of Theorem 2.1.9 58
§2.3.8 Proof of Theorem 2.1.10 60

3 Covariance structure behind breaking of ensemble equivalence in random graphs 65
§3.1 Introduction and main results . . . 66
§3.1.1 Background and outline . . . 66
§3.1.2 Constraint on the degree sequence . . . 67
§3.1.3 Relevant regimes . . . 68
§3.1.4 Linking ensemble nonequivalence to the canonical covariances . 69
§3.1.5 Discussion and outline . . . 71
§3.2 Proof of the Main Theorem . . . 75
§3.2.1 Preparatory lemmas . . . 75
§3.2.2 Proof (Theorem 3.1.5) . . . 77
§A Appendix . . . 79
§B Appendix . . . 80

4 Is Breaking of Ensemble Equivalence Monotone in the Number of Constraints? 85
§4.1 Introduction and main results . . . 86
§4.1.1 Background . . . 86
§4.1.2 Constraint on the full degree sequence . . . 87
§4.1.3 Constraint on the partial degree sequence . . . 88
§4.1.4 Linking ensemble nonequivalence to the canonical covariances . 90
§4.1.5 Discussion . . . 92
§4.2 Proof of the Main Theorem . . . 95
§4.2.1 Preparatory lemmas . . . 95
§4.2.2 Proof (Theorem 4.1.4) . . . 98
§A Appendix . . . 98
§B Appendix . . . 100

5 Ensemble Equivalence for dense graphs 103
§5.1 Introduction . . . 104
§5.1.1 Background and motivation . . . 104
§5.1.2 Relevant literature . . . 105
§5.1.3 Outline . . . 105
§5.2 Key notions . . . 106
§5.2.1 Microcanonical ensemble, canonical ensemble, relative entropy 106
§5.2.2 Graphons . . . 108
§5.2.3 Large deviation principle for the Erdős-Rényi random graph . . 109
§5.3 Variational characterisation of ensemble equivalence . . . 111
§5.3.1 Subgraph counts . . . 111
§5.3.2 From graphs to graphons . . . 112
§5.3.3 Variational formula for specific relative entropy . . . 113
§5.4 Main theorem . . . 115
§5.5 Choice of the tuning parameter . . . 117
§5.5.1 Tuning parameter for fixed n . . . 118
§5.5.2 Tuning parameter for $n \to \infty$ . . . 120

§5.6 Proof of the Main Theorem 5.4.1 123
§5.6.1 Proof of (I)(a) (Triangle model $T_2^* \geq \frac{1}{8}$) 123
§5.6.2 Proof of (I)(b) ($T_2^* = 0$) 123
§5.6.3 Proof of (II)(a) (Edge-Triangle model $T_2^* = T_1^{*3}$) 124
§5.6.4 Proof of (II)(b) ($T_2^* \neq T_1^{*3}$ and $T_2^* \geq \frac{1}{8}$) 124
§5.6.5 Proof of (II)(c) ($T_2^* \neq T_1^{*3}$, $0 < T_1^* \leq \frac{1}{2}$ and $0 < T_2^{*3} < \frac{1}{8}$) . . . 125
§5.6.6 Proof of (II)(d) ((T_1^*, T_2^*) on the scallopy curve) 126
§5.6.7 Proof of (II)(e) ($0 < T_1^* \leq \frac{1}{2}$ and $T_2^* = 0$) 127
§5.6.8 Proof of (III) (Star model $T[j]^* \geq 0$) 127
§A Appendix 128

6 Breaking of Ensemble Equivalence for Perturbed Erdős-Rényi Random Graphs 133
§6.1 Introduction 134
§6.2 Definitions and preliminaries 135
§6.3 Theorems 137
§6.4 Proofs of Theorems 6.3.1-6.3.3 142
§6.4.1 Proof of Theorem 6.3.1 142
§6.4.2 Proof of Theorem 6.3.2 144
§6.4.3 Proof of Theorem 6.3.3 145
§6.5 Proofs of Propositions 6.3.5–6.3.7 146
§6.5.1 Proof of Proposition 6.3.5 147
§6.5.2 Proof of Lemma 6.5.1 and Lemma 6.5.2 154

Bibliography 162

Samenvatting 169

Acknowledgements 171

Curriculum Vitae 173