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Chapter 3

Classification of Tile Patterns

3.1 Introduction

In this chapter we will show how tiles and bricks can be combined to form larger tilings and brick patterns, how the latter can be translated to crease patterns, and how we can determine their corresponding mountain-valley configurations. In particular, we formulate and solve the three combinatorial problems that govern tilings, brick patterns, and mountain-valley configurations.

An example of a 4×4 tile pattern is shown in Fig. 3.1.A. As all tiles fit, all vertex colors are consistently defined. This *tile pattern* or *tiling* can be converted into a *brick pattern*, as shown in Fig. 3.1.B. Here each tile has acquired an allowed supplementation pattern (Fig. 2.6, Fig. 2.7), and supplementations are consistent between adjacent bricks. All vertices are now uniquely defined, as we know their supplementation, their orientation, and their clockwise or counterclockwise character. If we then choose a set of angles α_i, as well as lengths t_i and l_i, we can convert this brick pattern into a crease pattern, as shown in Fig. 3.1.C. Note that an $m \times n$ brick pattern defines a $3 + m + n$-parameter family of crease patterns. Finally, we can determine a specific mountain-valley pattern for this crease pattern (Fig. 3.1.D). This mountain valley pattern is not unique as all crease patterns obtained by our method allow at least two different mountain valley patterns (not related by trivial mountain $\leftrightarrow$ valley symmetry).

In this chapter we will count the number of tilings, the number of supplementation patterns for each tiling, and finally, the number of mountain-valley patterns. In section 3.2 we start by showing how to define four distinct tiling classes (I, II, III, and IV). For a given class, each tiling contains one or more necessary tiles and an arbitrary number of optional

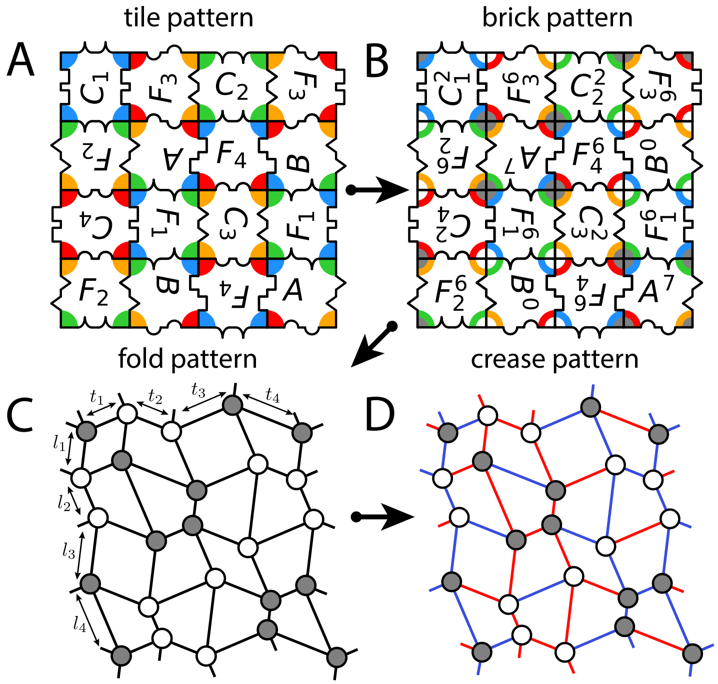

FIGURE 3.1: (A) A tile pattern (or tiling). (B) The same pattern with a specific supplementation pattern yields a definite brick pattern. (C) Crease pattern corresponding to the brick pattern in (B), with a choice of angles of $\alpha_1 = 60°, \alpha_2 = 90°, \alpha_3 = 135°, \alpha_4 = 75°$, and mesh lengths t_i and l_i. (D) One corresponding mountain (red) valley (blue) configuration.

tiles - other tiles do not fit. We show that an important property of this classification is that within each class, each L-shaped triplet of tiles admits precisely one fitting fourth tile. In section 3.3 we show how this last property allows us to exactly count the number of $m \times n$ tile patterns. We verify by brute force that the tilings in class-I–IV cover all possible tilings, up to $m = 6, n = 6$. For $m = 6, n = 6$ there are already to 4226048 distinct tilings. In section 3.4, we show in how many ways we can choose the supplementation of a pattern, i.e. in how many ways can we convert an $m \times n$ tile pattern into an $m \times n$ brick pattern. We show that there are *at least* two valid ways in which we can turn a tile-pattern into a brick pattern. However, some classes of tiling have exponentially many ways in which we can choose the supplemented vertices. Last, we show in section 3.5 how many valid mountain valley assignments each of the corresponding crease patterns have. Finally, we summarize these results in section 3.6.

3.2 Triplet Completion and Classification

While it is easy to identify tiles that can be connected in pairs, the problem of connecting multiple tiles is in general highly non-trivial. We will show however, that by considering all (L-shaped) triplets of tiles we can greatly simplify the tiling problem, by determining whether or not a fourth fitting tile exists. For example, in Fig. 3.2.A we show a triplet of tiles. The missing tile consists of four vertices, and we note that three of these are already completely specified by the tile triplet. Hence (up to supplementation), three of the four sector angles of the inner plate are specified. For generic sector angles, the sum rule implies at most one choice for the sector angle of the missing vertex[1]. Moreover, in each tile the number of clockwise and counterclockwise oriented vertices is even (see Fig. 2.4, and section 3), which specifies the orientation of the fourth vertex. These two simple rules

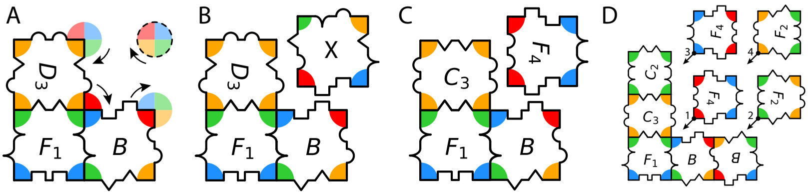

FIGURE 3.2: (A) A triplet of tiles. We can find the fourth vertex (dashed) of the fourth tile by inspecting the inner angles and orientation of the other three vertices. (B) The fourth tile, X, that fits this triplet, does not occur in the set of 34 tiles displayed in Fig. 2.4. Therefore, the three tiles D_3, F_1, B do not occur in the same class. (C) This triplet of tiles *does* admit a single, unique, fourth tile that is within the set of 34 tiles. Therefore, D_3, F_1, B, and F_4 occur in the same class. (D) Repeated application of the triplet 'completion rule' (for example, in the order shown in the figure) allows us to finish a tiling given only a single column and row.

[1] If the three inner angles are all equal, the fourth should be equal too; if the three inner angles are once α_i and twice α_j, the fourth should be α_i; if the three inner angles are all different, the fourth should be different from all of these.

specify the fourth vertex (up to supplementation) and thus *uniquely* specify a potential tile. In this case, this corresponds to tile 'X' shown in Fig. 3.2.B. However, tile X does not occur within the set of 34 tiles shown in Fig. 2.4, as the four corresponding operators do not satisfy the loop condition. In contrast, the triplet of tiles shown in Fig. 3.2.C *does* admit a tile that occurs within the set of 34 tiles shown in Fig. 2.4.

We now define tiling classes as follows. If a triplet does not admit a fourth tile, the three tiles cannot be in the same class. If a triplet admits a fourth tile, all tiles are in the same class. By considering all triplets, we find that these two rules define four distinct classes, labeled I-IV. A given tiling can easily be identified as belonging to one of these classes by inspecting the presence of certain tiles (see Fig. 3.3). In tilings in each class, at least one tile out of a subset of tiles has to be present; in addition, some classes contain a group of *optional* tiles, which may or may not be present in a tiling.

Each tile is a necessary tile in precisely one class. It can be checked that for each triplet of tiles within a class, there always is a unique fourth fitting tile. This property, which we will refer to as *triplet completion*, greatly simplifies the construction and enumeration of tilings: once a single row and column of a $m \times n$ tiling is specified, the full tiling can trivially be constructed by iteratively applying the triplet completion rule (Fig. 3.2.D).

A

Class	Necessary tiles	Optional tiles
I	$\{A, B\}$	$\{C_i, F_i\}$
II	$\{C_i\}$	–
III	$\{E_i, F_i, G_i, H_i, I_i, K_i\}$	$\{C_i, D_i, J_i\}$
IV	$\{D_i, J_i\}$	$\{C_i\}$

B

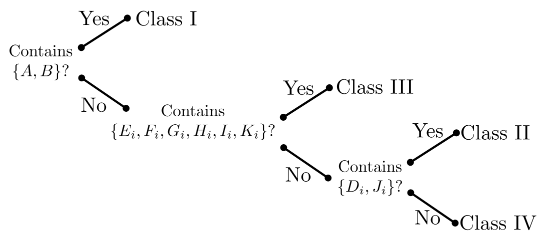

FIGURE 3.3: (A) Tiling classes. (B) Decision tree to determine whether a given tiling created using the tiles of Fig. 2.4 is a class-I, class-II, class-III, or class-IV tiling.

3.3 Counting Multiplicity of Tilings

In this section we will derive expressions for the number of tilings that can be constructed within each class. To do this, we first define the concept of *connection numbers*. In each class, we define for each side of a tile, the number of distinct sides of necessary tiles and the number of distinct sides of optional tiles that fit. To facilitate discussions about the tiles, we define the orientation of the tiles as shown Fig. 2.4 as the 'horizontal' orientation, and refer to their sides by the four cardinal directions (north, east, south, and west). We note that for all tiles, the connection numbers at opposite sides are equal, allowing us to capture the connections by four integers (Fig. 3.4.A). The necessary connection numbers along the north/south and east/west sides are, respectively, x and v (in blue). The optional connection along the north/south and east/west sides are, respectively y, w (in pink).

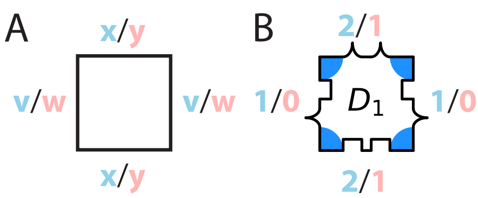

FIGURE 3.4: (A) For every side of every tile we can define a connection number within a given glass (blue: necessary, red optional tiles). (B) Example of connectivity of D_1 tile in class-IV.

An example of a tile and its connection numbers is given in Fig. 3.4.B for tile $\{D_1\}$, which is a necessary class-III tile. Within this class it connects to necessary tiles $\{D_4, J_2\}$ on its northside, necessary tile $\{J_4\}$ on the westside, necessary tiles $\{D_2, J_2\}$ on its southside, and necessary tile $\{J_4\}$ on its eastside. Additionally it connects to optional class-III tile $\{D_4\}$ on its southside, and optional tile $\{D_2\}$ on its northside. Therefore the connection numbers are as indicated in Fig. 3.4.B. We observe that opposing sides have identical *total* connection numbers. As a result, the total connection number at the sides within a row or column of tiles is conserved. This simplifies the counting of configurations.

Counting Class-I Tilings

All tiles in class-I and their connection numbers are shown in Fig. 3.5.A. Class-I tilings contain at least one *necessary* $\{A, B\}$-tile. Either of these can be used to form a periodic tiling which maps to the Huffman quadrilateral crease pattern (see Fig. 1.7.A). However, a vast number of additional tilings can be generated by mixing these necessary tiles with optional tiles $\{C_i\}$ and $\{F_i\}$.

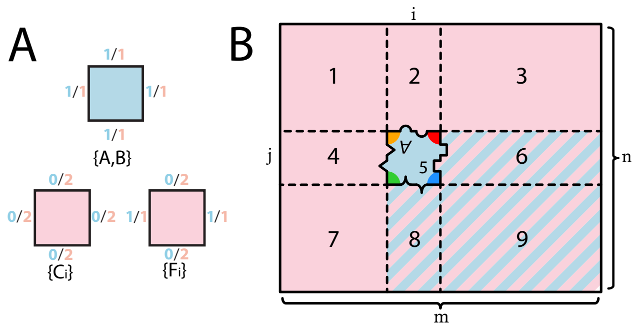

FIGURE 3.5: (A) Class-I tilings contain up to three different types tiles. The *necessary* class-I tiles are tiles $\{A, B\}$. Furthermore there are two different sets of optional class-I tiles: $\{C_i\}$, and $\{F_i\}$. The connection numbers of these sets of tiles are indicated. (B) When counting the number of class-I tilings, we divide a tiling into nine sectors, and assume the first *necessary* class-I tile we encounter is in sector 5, there are eight choices for this tile. Pink sectors contain only optional tiles, hatched blue-pink sectors can contain both optional as well as necessary tiles.

We now illustrate how to construct and count all possible $m \times n$, class I tilings. We make use of two general properties of the adjacencies of tiles within one class: *(I)* the number of necessary and optional adjacent tiles on opposite sites is equal, and *(II)* once a single row and column of a tiling are specified, the full tiling can trivially be constructed by triplet completion (Fig. 3.2.D). We label the columns and rows from $i = 1$ to $i = m$, and $j = 1$ to $j = n$ (Fig. 3.5). To construct and count the number of class I tilings, we define the first necessary tile as the necessary tile with the lowest value of $i + j$, denote its location as (i, j), and partition the tiling in nine sectors $1 - 9$ as indicated in Fig. 3.5.B. Sectors 1, 2, and 4 must consist of optional tiles. As the optional connection number for the necessary tile in sector 5 is 1 on all sides, this determines a unique pattern (of F_i tiles) for sectors 2 and 4. In turn, these F tiles uniquely determine sector 1 by applying triplet completion. For the tiles in sectors 6 and sector 8, there are two potential choices for each tile: these tiles can be either an optional type-1 tile, or a necessary type-1 tile. To indicate this, these positions are therefore pink-blue hatched, and these choices lead to $2^{m-i+n-j}$ options in total. For any given choice of tiles in sector 6 and sector 8, the sectors 3, 7 and 9 are again uniquely determined by triplet completion.

Summing over all locations (i, j), and taking into account there are 8 choices for the first necessary tile (A or B, each in one of four rotations), we obtain that the number of class-I, $m \times n$ tilings equals,

$$N_{\mathrm{t}}^{\mathrm{I}}(m,n) = 8 \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} 2^{m-i} \cdot 2^{n-j} = 8 \cdot (2^n - 1) \cdot (2^m - 1). \tag{3.1}$$

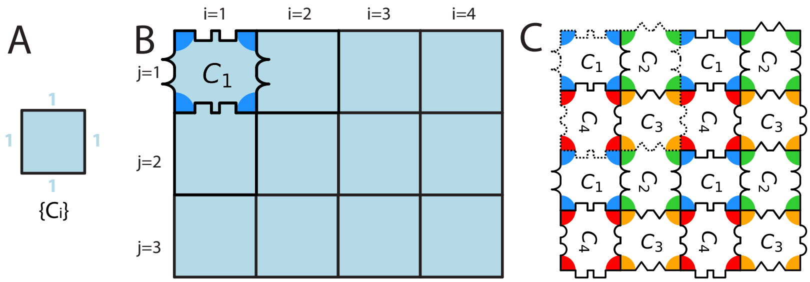

FIGURE 3.6: (A) Class-II tilings only contain tiles $\{C_i\}$. (B) Choosing the type and orientation of the (1,1) tile fixes the whole pattern. (C) Periodic 4×4 type-II tiling, where a 2×2 unit cell is indicated by the dashed line.

Counting Class-II Tilings

Class-II tilings consist exclusively of C_i tiles. These tiles are highly symmetric, and occur as optional tiles in all other classes (see table in Fig. 3.3.A). For class-II tilings, the connection number of the C_i tiles on all sides is 1 (see Fig. 3.6.A). This ensures that, when we determine the tile in position $(i, j) = (1, 1)$, all tiles in the first column, $i = 1$, and the first row, $j = 1$, are fixed. Repeated application of the triplet completion rule (see Fig.

3.2), then fixes the whole tiling (Fig. 3.6.B). The multiplicity of class-II tilings is thus specified by the four possible tiles (C_1,C_2,C_3,C_4), and the two orientations of the $(1,1)$ tile, yielding,

$$N_{\mathrm{t}}^{\mathrm{II}}(m,n) = 8, \tag{3.2}$$

class-II tilings, independent of m and n. We note that as the tiles form periodic patterns, all m by n tilings can be seen as subtilings of an infinite periodic tiling, where the number of choices of the $(1,1)$ tile corresponds to the translational and rotational symmetries of the periodic tiling. For an example of a four by four tile periodic class-II tiling, see Fig. 3.6.C.

Counting Class-III Tilings

Class-III tilings contain at least one *necessary* class-III tile $\{E_i, F_i, G_i, H_i, I_i, K_i\}$. Additionally, we can add two different sets of optional tiles, $\{C_i\}$, and $\{D_i, J_i\}$. Together, the connection numbers of these tiles are shown in Fig. 3.7.A. The necessary tiles in class-III only admit a single fitting tile along their east and west sides, and this tile is always a necessary tile. This significantly simplifies the construction of class-III tilings, as necessary tiles can therefore only occur as full columns or rows – but not both. Hence, class-III tilings come in two flavors. Either the necessary tiles are horizontally oriented, and occur in rows with the first one occurring in column 1 (Fig. 3.7.B), or the necessary tiles are vertically oriented, occur in columns, with the first one occurring in row 1 (Fig. 3.7.C).

We now first count the horizontally oriented tilings (Fig. 3.7). The first necessary tile at location $(1,j)$, sector 3, uniquely determines a pattern of necessary tiles in sector 4. There are 20 distinct necessary tiles in class-III, which can be in two horizontal orientations[2] at location $(1,j)$. For the optional tiles in sector 1 the relevant connection number is three, leading to 3^{j-1} choices, and once sector 1 and 4 are chosen, sector 2 is fixed. Finally for sector 5 we can use either optional or necessary tiles, with a combined connection number of 8, leading to 8^{n-j} options; sector 6 is then determined by triplet completion. Therefore the number of horizontally

[2]Either the tiles are oriented as in Fig. 2.4, or flipped upside down.

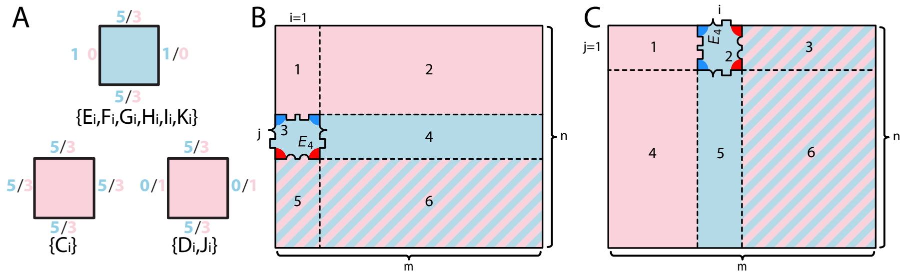

FIGURE 3.7: (A) Class-III tilings contain up to three different sets of tiles: a set of *necessary* class-III tiles: $\{E_i, F_i, G_i, H_i, I_i, K_i\}$, and two sets of optional class-III tiles: $\{C_i\}$, and $\{D_i, J_i\}$. The connection numbers of these three sets of tiles are indicated. (B) A horizontally oriented class-III pattern, divided into six sectors. (C) A vertically oriented class-III pattern, divided into six sectors.

oriented class-III tilings is,

$$N^{\text{III}}_{\text{t-horizontal}}(m,n) = 40 \cdot \sum_{j=1}^{n} 8^{n-j} \cdot 3^{j-1} = 8 \cdot (8^n - 3^n), \tag{3.3}$$

where m is the number of columns, and n the number of rows. The same holds for vertically oriented patterns, for which the necessary tiles are rotated a quarter turn (either clockwise or anticlockwise) with respect to their orientation as depicted in Fig. 3.7.C. A schematic for this scenario is shown in Fig. 3.7.C. The total number of class-III tilings is,

$$\begin{aligned} N^{\text{III}}_{\text{t}}(m,n) &= 40 \cdot \sum_{i=1}^{m} 8^{m-i} \cdot 3^{i-1} + 40 \cdot \sum_{i=1}^{n} 8^{n-i} \cdot 3^{i-1} \\ &= 8 \cdot (8^m - 3^m) + 8 \cdot (8^n - 3^n). \end{aligned} \tag{3.4}$$

Counting Class-IV Tilings

Class-IV tilings contain at least one *necessary* class-IV tile $\{D_i, J_i\}$, and optional tiles, $\{C_i\}$, see Fig. 3.8.A. As in class-III, the necessary tiles only admit one other necessary tile along their east- and westside. The counting is therefore very similar to class-III. In Fig. 3.8.B we show a class-IV tiling, divided into six sectors (1–6). Here we assume the necessary class-IV tiles

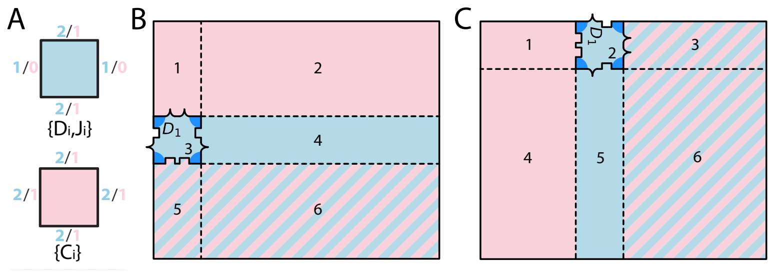

FIGURE 3.8: (A) Class-IV tilings consist out of two sets of tiles. The *necessary* class-IV tiles, $\{D_i, J_i\}$, and the *optional* class-IV tiles $\{C_i\}$. The connection numbers of these two sets of tiles are indicated in the figure. (B) Class-IV tilings are either horizontally or vertically oriented. In this case we show a horizontally oriented pattern, where the necessary tile is horizontally oriented. (C) Vertically oriented class-IV tiling. Pink sectors contain only optional class-IV tiles, blue sectors contain only necessary class-IV tiles, and hatched blue-pink sectors correspond to sectors in which we can find both optional and necessary tiles.

are oriented horizontally, so that the necessary tiles occur in rows. The first sector where we encounter necessary class-IV tiles is sector 3, at location $(1, j)$. The necessary connection number for all tiles in sector 4 is one, the whole row of tiles consisting of sector 3 and 4 together is fixed by choosing the tile (and its orientation) at position $(1, j)$, leading to 16 choices. For the optional tiles in sector 1, the connection number is one, which fixes sector 2. In sector 5 we do have a choice of tiles, as we can choose from two necessary, and one optional tile at every position, for a total of 3^{n-i} options. Summing over all possible initial positions of sector 3, we therefore find,

$$N^{\text{IV}}_{\text{t-hor}}(m, n) = 16 \cdot \sum_{i=1}^{n} 3^{n-i}, \tag{3.5}$$

horizontally oriented class-IV tilings, where m is the number of rows, and n the total number of columns. The same logic holds for vertically oriented patterns (Fig. 3.8.C), so in total we find,

$$N^{\text{IV}}_{\text{t-hor}}(m, n) = 16 \cdot \sum_{i=1}^{m} 3^{m-i} + 16 \cdot \sum_{i=1}^{n} 3^{n-i} = 8 \cdot (3^m - 1) + 8 \cdot (3^n - 1), \tag{3.6}$$

class-IV patterns.

Counting Total Number of Patterns

If we consider the total number of m, n tilings, $N_\mathrm{t}(m,n)$, we find, by summing the results in Eq.3.1, Eq.3.2, Eq.3.4, and Eq.3.6, that:

$$N_\mathrm{t}(m,n) = 2^{m+3}(4^m - 1) + 2^{n+3}(4^n - 1) + 8 \cdot 2^{m+n}, \tag{3.7}$$

Here we note that $N_\mathrm{t}(m,n)$ counts all configurations that are possible when placing the tiles of Fig. 2.4 on an m by n array. Hence, we double count tilings that are related by global rotations and translations[3]. We note in addition that class-III and class-IV could be combined in one super-class, that satisfies the triplet completion rule, and for which the counting is somewhat simpler, yielding a total of $N_\mathrm{t}^{\mathrm{III+IV}} = 8 \cdot (8^m + 8^n - 2)$ tilings (summing Eq. 3.4 and Eq. 3.6). However, class-III and class-IV are significantly different in their supplementation patterns, as we will see below.

We have numerically counted all tilings by brute force by using a backtracking algorithm where as only input we use a Boolean matrix that indicates which sides of which tiles fit to which other sides – without any knowledge of classes, edge characteristics etc – up to m = 6, n = 6. The resulting numbers exactly correspond to our analytical expression for the number of $m \times n$ tilings (see Table 3.1), thus illustrating that the tilings in class I–IV cover all possible tilings that can be made constructed out of our 34 tiles.

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
1	128					
2	592	1088				
3	4208	4768	8576			
4	32944	33632	37696	67328		
5	262448	263392	267968	298624	531968	
6	2097712	2099168	2104768	2137472	2374912	4226048

TABLE 3.1: Numerically obtained number of possible $m \times n$-tilings are consistent with our analytical expression (Eq. 3.7).

[3]The local rotation symmetry of the C_i tiles does not artificially increase the count N_t however, note that $N_\mathrm{t}^{\mathrm{II}} = 8$ and not 16.

3.4 Counting Supplementation Patterns

In this section we will count the number of different ways in which we can convert (supplement) tile patterns into brick patterns. We show that there are always at least two ways in which we can do this, but for classes II, III, IV there are exponentially many.

Counting Supplemented Angles for Class-I Tilings

Here, we will show that each class-I tiling has two valid supplementation patterns. We recall that class-I tilings contain tiles A, B, C_i and F_i. The supplementation patterns of tile A and B (0 and 7), and the supplementation patterns of tiles C_i (1-6) will be easy to deal with. However, the situation is more complex for tiles F_i, which admit patterns 1, 2, 4, 6, but not 3 or 5. Hence, the admissible supplementation patterns could potentially depend on the orientation of the F_i tiles, which requires a closer inspection of the structure of class-I tilings.

Assume that we specify a pattern of necessary and optional tiles in row 1 and column 1 as in Fig. 3.9.A[4]. First, using triplet completion we

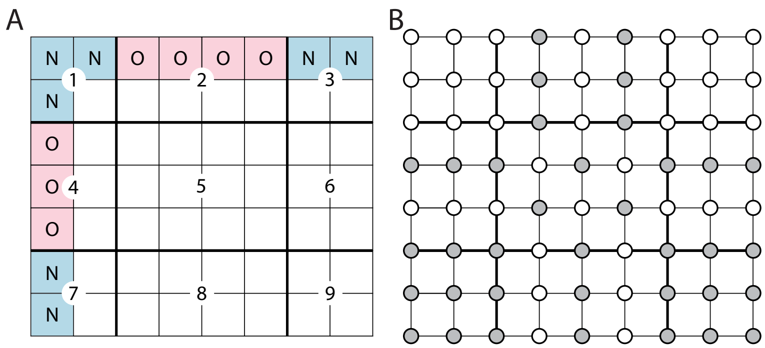

FIGURE 3.9: (A) A generic example of a class-I tiling-pattern: N indicate necessary $\{A, B\}$-tiles (in blue), O indicates indicate optional tiles $\{C_i, F_i\}$ (in pink). (B) Example of one of the two valid supplementation patterns for the tile pattern in (A).

can show that the missing tiles in sector 1 are necessary tiles, so that

[4]The case that these rows or columns are purely optional can easily be dealt with by focusing on the first rows and columns where necessary tiles occur.

the supplementation pattern of sector 1 is either fully empty (no vertices supplemented) or fully filled (all vertices supplemented); below we assume the former to be the case. In addition it is also easy to show that all necessary tiles in sector 1 are either horizontally or vertically oriented. Finally, triplet completion can be used to show that the tiles in sector 2,4,5,6 and 8 are all optional.

Second, the choice of the supplementation pattern in sector 1 fixes the supplementation pattern of the whole system, as demonstrated by the example in Fig. 3.9.B; i.e. the necessary sectors are 'monocolor', the sectors 2, 4, 6, and 8 that separate the necessary sectors are striped (as the optional bricks have always 2 supplemented vertices), and the sector 5 is checkerboard-like.

Third, the supplementation patterns in the necessary sectors are clearly compatible with the necessary tiles, and the supplementation pattern in sector 5 is clearly compatible with all optional tiles - we note in passing that sector 5 exclusively consists of C_i tiles. The potential mismatch between tile pattern and supplementation pattern might occur in the striped sectors: vertically (horizontally) oriented F_i tiles in sector 2, 8 (4, 6) would be incompatible with the supplementation pattern. However, the conservation of edge characteristic prevents this: in rows or columns where A or B tiles are present, all edges have opposite bumps, and this immediately orients the F tiles in sector 2 and 8 horizontally, and in sector 4 and 6 vertically.

Hence: once the supplementation of *one* necessary tile is specified, a unique and compatible supplementation for the whole system arises. Since there are two choices for the supplementation pattern of necessary tiles, this construction yields precisely two (complementary) supplementation patterns for each class-I tiling. Therefore, the number of supplementation patterns in class I, $N_{\mathrm{s}}^{\mathrm{I}}$, equals:

$$N_{\mathrm{s}}^{\mathrm{I}} = 2. \tag{3.8}$$

Counting Supplemented Angles for Class-II Tilings

Class-II tilings contain only $\{C_i\}$ tiles, which can all be supplemented in six different ways, as is shown in Fig. 2.6. In Fig. 3.10.A we show a class-II brick pattern, where we choose the supplementation of the left-most column of vertices. Doing so fixes the supplementation of all the vertices in the whole pattern, as every column has the opposite pattern of

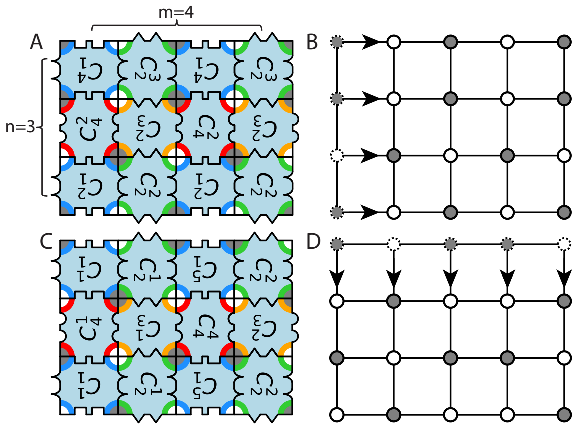

FIGURE 3.10: (A) A 4 by 3 class-II tiling, where we arbitrarily choose the supplementation of the leftmost column of vertices. (B) Supplementation pattern of the brick pattern in (A), the arrows indicate how the left column uniquely determines the adjacent columns. (C) The same 4 by 3 tile pattern, but with a different supplementation pattern. (D) Supplementation pattern of the brick pattern in (C).

its neighbors. In total there are 2^{n+1} ways to choose a supplementation pattern of the leftmost column. The same holds when we choose the supplementation pattern of the top row of vertices, as we did in the example shown in Fig. 3.10.C. In that case every row of vertices has the opposite supplementation pattern of its neighbor (Fig. 3.10.D), and there are 2^{m+1} ways to choose a supplementation pattern on the top row.

We therefore find that we can supplement the vertices in class-II tilings in

$$N_{\mathrm{s}}^{\mathrm{II}} = 2^{m+1} + 2^{n+1} - 2, \tag{3.9}$$

different ways. Here the -2 is necessary to prevent double counting patterns where both the columns and the rows follow alternating patterns.

Counting Supplemented Angles for Class-III Tilings

Class-III tilings can either be horizontally or vertically oriented (see section 3.3), and contain necessary tiles $N = \{E_i, F_i, G_i, H_i, I_i, K_i\}$, and optional tiles $O = \{C_i, D_i, J_i\}$. Let us assume that the tiling is horizontally oriented, as in Fig. 3.7.B. We conjecture that we can choose the supplementation pattern freely on the left edge of the left column, and then take the supplementation of adjacent columns to alternate. To allow this, we require that tiles with only supplementation patterns $1, 2, 4, 6$ (all necessary tiles $\{E_i, F_i, G_i, H_i, I_i, K_i\}$-tiles, and optional $\{J_i\}$-tiles) are horizontally oriented. Optional $\{C_i, D_i\}$-tiles have supplementation pattern $1, 2, 3, 4, 5, 6$ and can be oriented arbitrarily.

To show that all $\{E_i, F_i, G_i, H_i, I_i, K_i\}$-tiles are horizontally oriented in a horizontally oriented class-III tiling, we start by noting that the necessary tile at sector 3 (Fig. 3.7.B) is by definition horizontally oriented, and as all N-tiles have connection numbers $c_n = 1$ and $c_o = 0$ along their East/West edges, sector 4 consists solely of N tiles as well. The orientation of the tiles in sector 4 is also horizontal, which can be seen by considering the edge characteristics of the necessary $\{E_i, F_i, G_i, H_i, I_i, K_i\}$-tiles, which are $\{oeee, oeoo, oeeo, oeee, oeee, oeee\}$ respectively (see Table 2.1). Inspection reveals that none of their North/South and East/West sides are compatible, and since the tile in sector 3 is horizontally oriented and these edge characteristics are conserved in every row (and column), all tiles in sector 4 are as well.

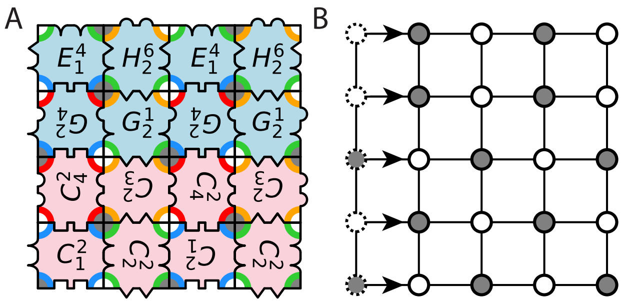

FIGURE 3.11: (A) A horizontally oriented class-III tiling, where we choose the supplementation of the vertices on the left boundary. (B) Supplementation pattern of the brick pattern shown in (A).

When we now look at the North/South edge characteristic of the horizontally oriented tiles in sector 3 and 4, we see that these are all equal to oe. As these edge characteristics are conserved in every column, that means all tiles in sectors-1,2,5,6 also have North/South edge characteristic

oe. Comparison with Table. 2.1 then shows that all optional and necessary tiles in sectors 1,2,5,6 must also be horizontally oriented, except for the C_i-tiles, which can also be vertically oriented. We conclude that the resulting orientations are compatible with the conjectured supplementation pattern, where we can arbitrarily choose the supplementation of the vertices on the right boundary (see Fig. 3.11). This therefore yields,

$$N^{\text{III}}_{\text{s-hor}} = 2^{n+1}, \tag{3.10}$$

supplementation patterns for horizontally oriented class-III patterns. Likewise, we find

$$N^{\text{III}}_{\text{s-ver}} = 2^{m+1}, \tag{3.11}$$

possible supplementations for vertically oriented class-III patterns.

Counting Supplementation Patterns for Class-IV Tilings

Class-IV tilings contain necessary D_i and J_i tiles, and optional C_i tiles. The necessary tiles are either horizontally or vertically oriented, similar to class-III. However, unlike the necessary class-III tiles, which can all be supplemented in only four different ways, the D_i and C_i tiles allow six different supplementations, whereas the J_i tiles are only compatible with the four supplementation patterns where the E and W sides have opposite supplementations (see Fig. 2.6). As a result, the location of the J_i tiles determines the number of allowed supplementation patterns for type-IV tilings.

We now first consider the location and orientation of the various tiles in horizontally oriented class-IV tilings. Consider the leftmost column of such a tiling, filled with a combination of necessary and optional tiles. As shown in Fig. 3.8, the necessary tiles only connect to a single other, necessary, tile along their E/W sides: a necessary tile in the leftmost column thus uniquely determines a row of necessary tiles. Specifically, D_i tiles only connect to J_i tiles and vice versa, so each row of necessary tiles consists of alternating D_i and J_i tiles, each of these oriented horizontally. In addition, optional rows consist of C_i tiles only. As a result, we can distinguish two types of columns: those with J_i tiles, and those without. This allows us to distinguish four subclasses of class-IV tilings, depending on which columns contain J_i tiles:

- Subclass 1: Here J_i tiles occur in all columns. This is the most common situation, and arises whenever the leftmost column has at least one D_i and J_i tile.

- Subclass 2-4: Here J_i tiles occur in alternating columns; subclass 2 corresponds to even m, where J tiles occur in either the leftmost or rightmost column; subclass 3 and 4 correspond to odd m, with J tiles occurring in neither the left nor rightmost column (subclass 3) or in both columns (subclass 4).

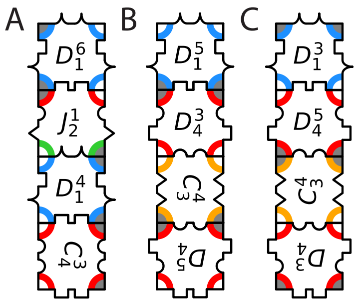

FIGURE 3.12: (A) A column containing at least *one* J_i-tile leads to an alternating supplementation pattern in the horizontal direction. This scenario occurs in subclass-1. (B,C) A column containing no J-tiles allows identical supplementation patterns on the left and right side, as long as the supplementation pattern alternates in the vertical direction. This scenario occurs in subclasses-2,3,4.

We first consider the supplementation pattern of individual columns. For columns containing J_i tiles, the four allowed supplementation patterns of J_i tiles correspond to opposite supplementations at their E and W sides. It is easy to show that adjacent tiles therefore also need opposite supplementation patterns (Fig. 3.12.A), and by iteration, we find that the only allowed supplementation patterns are precisely opposite at E and W sides, leading to 2^{n+1} allowed supplementation patterns for such a column. We note that two of these correspond to patterns where the supplementation of the left and right columns are strictly alternating in the vertical direction, and $2^{n+1} - 2$ where they do not strictly alternate in the vertical direction. In contrast, columns that are free of J_i tiles, allow two additional 'ladder' configurations, where the vertices on the left and right side have identical supplementation and are strictly alternating, see Fig. 3.12.B,C. This leads to $2^{n+1} + 2$ supplementation patterns; 4 of these correspond to patterns where the supplementation of vertices is strictly alternating in the vertical direction, and $2^{n+1} - 2$ to patterns where this is not the case. Hence, the presence of J_i tiles, both determines the subclass and the number of supplementation patterns:

- Subclass 1: J_i tiles occur in each column. This occurs when the left column contains at least one J_i and one D_i tile. Once the supplementations of the left most column of vertices are fixed, adjacent vertex columns have alternating signs, yielding precisely,

$$N_{\text{s}}^{\text{IV-1}} = 2^{n+1}, \tag{3.12}$$

  supplementation patterns.

- Subclass 2: Every second column is free of J_i tiles and m is even. This occurs when the left column does not contain both D_i and J_i tiles. To count the number of supplementations, suppose only the odd columns contains J_i, and the even columns do not. Then the left column allows $2^{n+1} - 2$ non-alternating supplementation patterns, and 2 alternating patterns. For each of the non-alternating patterns, the supplementation pattern of all other columns are fixed, yielding $2^{n+1} - 2$ configurations. For each of the 2 alternating patterns, each of the $m/2$ J_i-free columns allow 2 strictly alternating supplementation patterns, yielding $2 \cdot 2^{m/2}$ supplementation patterns. Hence the total number of horizontally oriented subclass-2 supplementation patterns yields:

$$N_{\text{s-hor}}^{\text{IV-2}} = 2^{n+1} - 2 + 2^{m/2+1}. \tag{3.13}$$

- Subclass 3: For odd m with J_i tiles absent from the left and right column, we find $(m+1)/2$ columns with ladder configurations, leading to:

$$N_{\text{s-hor}}^{\text{IV-3}} = 2^{n+1} - 2 + 2^{(m+1)/2+1}. \tag{3.14}$$

- Subclass 4: For odd m with J_i tiles present from the left and right column, we find $(m-1)/2$ columns with ladder configurations, leading to:

$$N_{\text{s-hor}}^{\text{IV-4}} = 2^{n+1} - 2 + 2^{(m-1)/2+1}. \tag{3.15}$$

We now count the number of horizontally oriented tilings of each subclass. For even m, we encounter both subclass 1 and subclass 2, for a total of:

$$N^{\text{IV-1,2}}_{\text{t-hor}} = 8 \cdot (3^n - 1), \tag{3.16}$$

tilings, see Eq. 3.6. The left column of a subclass-2 tiling contains either D_i and C_i tiles, or J and C tiles. Suppose we only have D and C-tiles, then (following the same argument as that leads to Eq. 3.5) we find a total of:

$$8 \cdot \sum_{i=1}^{n} 2^{n-i} = 8 \cdot (2^n - 1) \tag{3.17}$$

tilings; and the same amount when we have only J_i and C_i tiles in the first column. Hence the total number of horizontally oriented subclass-2 tilings is:

$$N^{\text{IV-2}}_{\text{t-hor}} = 16 \cdot (2^n - 1). \tag{3.18}$$

As the sum of the number of subclass-1 and subclass-2 tilings is given by Eq. 3.16, we readily obtain that:

$$N^{\text{IV-1}}_{\text{t-hor}} = 8 \cdot (3^n - 1) - 16 \cdot (2^n - 1) = 8(3^n + 1 - 2^{n+1}). \tag{3.19}$$

For odd m, we encounter subclass 1, 3 and 4. The left column of a subclass-3 tiling can only contain D_i and C_i tiles, with at least one D_i-tile, leading to:

$$N^{\text{IV-3}}_{\text{t}-\text{hor}} = 8 \cdot \sum_{i}^{n} 2^{n-i} = 8 \cdot (2^n - 1). \tag{3.20}$$

Similarly, the left column of a subclass-4 tiling only contains J_i tiles and C_i-tiles, and cannot consist of C_i tiles only, leading to:

$$N^{\text{IV-4}}_{\text{t}-\text{hor}} = 8 \cdot (2^n - 1). \tag{3.21}$$

Hence, the number of subclass-1 tilings for odd m equals:

$$N^{\text{IV-1}}_{\text{t-hor}} = 8 \cdot (3^n - 1) - 2 \cdot 8 \cdot (2^n - 1) = 8 \cdot (3^n + 1 - 2^{n+1}). \tag{3.22}$$

We finally combine the results for the number of tilings and supplementation patterns per subclass to obtain $H_{\text{b-hor}} = N_{\text{t-hor}} \cdot N_{\text{s-hor}}$, the number of horizontally oriented brick patterns in each subclass:

- subclass-1 (even m),

$$H_{\text{b-hor}} = 8 \cdot (3^n + 1 - 2^{n+1}) \times 2^{n+1}; \tag{3.23}$$

- subclass-1 (odd m),

$$H_{\text{b-hor}} = 8 \cdot (3^n + 1 - 2^{n+1}) \times 2^{n+1}; \tag{3.24}$$

- subclass-2 (even m),

$$H_{\text{b-hor}} = 16 \cdot (2^n - 1) \times (2^{n+1} - 2 + 2^{m/2+1}); \tag{3.25}$$

- subclass-3 (odd m),

$$H_{\text{b-hor}} = 8 \cdot (2^n - 1) \times (2^{n+1} - 2 + 2^{(m+1)/2+1}); \tag{3.26}$$

- subclass-4 (odd m),

$$H_{\text{b-hor}} = 8 \cdot (2^n - 1) \times (2^{n+1} - 2 + 2^{(m-1)/2+1}). \tag{3.27}$$

The total number of class-IV $m \times n$ tilings and brick patterns can be obtained by adding the horizontal and vertically oriented patterns, distinguishing different subtypes depending on the parity of both m and n.

3.5 Counting Folding Branches

A single flat 4-vertex has two distinct folding branches, which each have a single, continuous degree-of-freedom. On each of these folding branches one of the four fold angles is opposite in sign to the other three. These two folds are called 'odd folds', and they straddle a common 'odd plate', for which the corresponding sector angle satisfies the inequality: $\alpha_i + \alpha_{i+1} < \alpha_{i+2} + \alpha_{i+3}$ [28]. Analytical expressions for the relations between the fold angles on these two fold branches are given in appendix A.

In this section we will determine how the two folding branches of a single vertex determine the number of independent folding branches per tile, and ultimately, the number of folding branches of crease patterns in class I–IV. We start by counting the folding branches per tile, by reconsidering the underlying operator quads (Eq. 2.9–2.19). So far, we have assumed that all operators refer to folding motion on the same branch, but now dress these operators with a superscript I or II, to indicate their respective folding branch. The number of folding branches per tile is now equal to the number of combinations of I and II labels in the operator quads that lead to identities. We can group all operator quads in three groups: the first group contains the C_i-tiles (Eq. 2.11) which combine the operators $\rho^{\mathrm{I,II}}_{i,i-1}$. To obtain an identity, we need to pair adjacent operators. For example,

$$\boldsymbol{\rho}^{I}_{14}\boldsymbol{\rho}^{I}_{41}\boldsymbol{\rho}^{II}_{14}\boldsymbol{\rho}^{II}_{41} = I, \tag{3.28}$$

which corresponds to tile $\{C_1\}$, which represents a rigidly foldable configuration as $\boldsymbol{\rho}^{I}_{14}\boldsymbol{\rho}^{I}_{41} = \pm I$, and $\boldsymbol{\rho}^{II}_{14}\boldsymbol{\rho}^{II}_{41} = \pm I$.[5] Conversely,

$$\boldsymbol{\rho}^{I}_{14}\boldsymbol{\rho}^{II}_{41}\boldsymbol{\rho}^{I}_{14}\boldsymbol{\rho}^{II}_{41} \neq I \tag{3.29}$$

does not represent a rigidly foldable configuration, as the folding operators on different branches do not 'annihilate' in pairs. There are six distinct choices for the folding branches of C_i tiles where 'adjacent operators' are on the same branch - tiles $\{C_i\}$ can therefore be folded into six different configurations. We show these six configurations in Fig. 3.13.A.

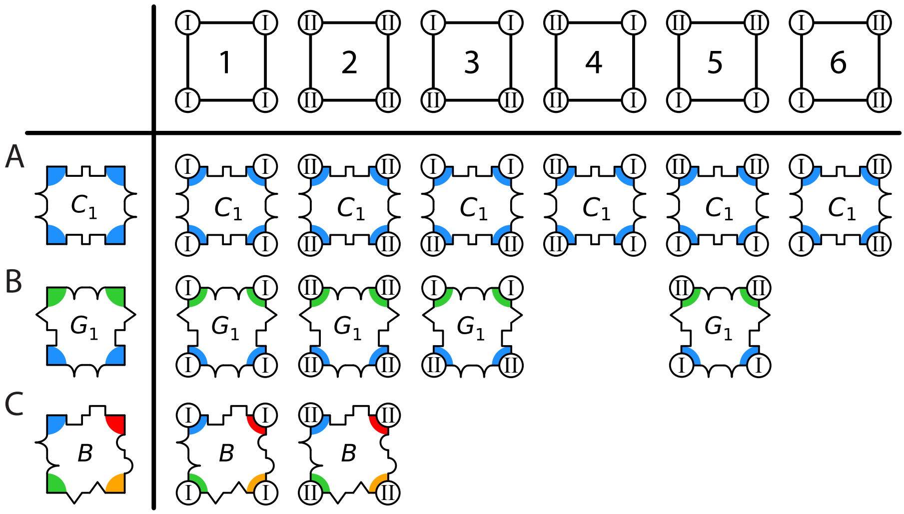

FIGURE 3.13: The allowed folding branches for all tiles. (A) The branches of tile C_1 can take six different configurations. (B) For tiles $\{D_i, E_i, F_i, G_i, H_i, I_i, J_i, K_i\}$ the branches of their vertices take four different configurations. (C) Tiles $\{A, B\}$ can only take two configurations.

[5]Here the $\pm$ sign reflect the fact we left out which operators are supplemented, we will come back to this on the next page.

Second, there are tiles $\{D_i, E_i, F_i, G_i, H_i, I_i, J_i, K_i\}$, which contain pairs of distinct operators, and which only allow four choices of the vertex branches. These correspond to the choices of branch I or II for each pair of vertices. These Kokotsakis meshes can thus be folded into four different configurations. For example, Eq. 2.15 with $i = 1$,

$$\boldsymbol{\rho}_{41}^{II}\boldsymbol{\rho}_{14}^{II}\boldsymbol{\rho}_{21}^{I}\boldsymbol{\rho}_{12}^{I} = I, \tag{3.30}$$

but,

$$\boldsymbol{\rho}_{41}^{II}\boldsymbol{\rho}_{14}^{I}\boldsymbol{\rho}_{21}^{I}\boldsymbol{\rho}_{12}^{II} \neq I. \tag{3.31}$$

The four possible configurations are shown in Fig. 3.13.B.

Third, there remain the tiles $\{A, B\}$ related to Eq. 2.9 and Eq. 2.10, which can only be folded into two different configurations for the identity of Eq. 2.9 or Eq. 2.10 to hold, as all of the vertices need to be in the same folding branch. For example,

$$\boldsymbol{\rho}_{43}^{I}\boldsymbol{\rho}_{32}^{I}\boldsymbol{\rho}_{21}^{I}\boldsymbol{\rho}_{14}^{I} = I, \tag{3.32}$$

see Fig. 3.13.C.

We note here that the supplementation pattern is not relevant for counting the number of branches, although it is important for the corresponding mountain valley pattern. Similarly, the choice of the odd folds (i.e. the choice of the numerical values of $\alpha_i - \alpha_4$) determines the specific M-V patterns. Examples of these are shown in Fig. 3.14. This illustrates the power of separately solving for the choice of branches for each vertex that solve the loop condition, Eq. 2.8, and the supplementation patterns which satisfy the sum rule, Eq. 2.1.

Counting Folding Branches for Class-I Tilings

Class-I tilings exhibit two distinct folding branches:

$$N_{\mathrm{b}}^{\mathrm{I}} = 2. \tag{3.33}$$

This is because the choice of supplementation patterns and the choice of branches are identical combinatorial problems in class I. Hence, the branches of one of the necessary class-I tiles ($\{A\}$ or $\{B\}$) immediately determines the folding branch of all other vertices.

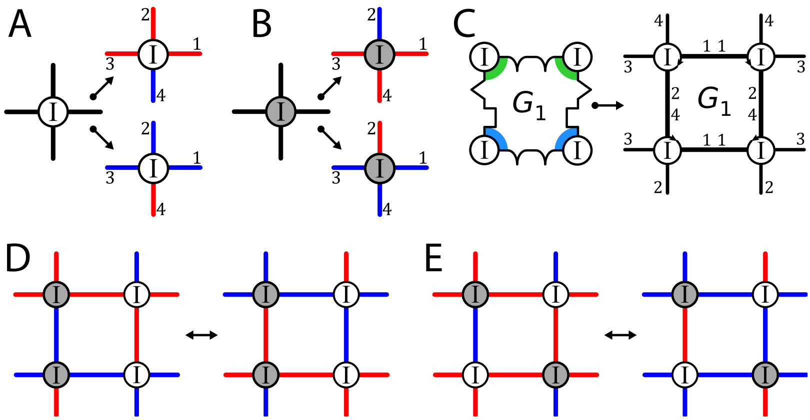

FIGURE 3.14: Mountain-Valley patterns. (A) Assuming that ρ_4 is the odd fold at branch I, a single vertex can be in two M-V configurations. (B) The odd fold on branch I of the supplemented vertex is ρ_2 (opposite to ρ_4). (C) The folding angle configuration for tile G_1. (D) For a given supplementation pattern, two corresponding and opposite M-V patterns exist. (E) A different supplementation pattern yields two different M-V patterns.

Counting Folding Branches for Class-II Tilings

The total number of folding branches of an $m \times n$ class-II tiling is:

$$N_{\text{b}}^{\text{II}} = 2^{m+1} + 2^{n+1} - 2. \tag{3.34}$$

This is because the choice of supplementation patterns and branches are identical combinatorial problems in class II: C_i tiles have identical configurations for the choice of supplementation and the choice of folding branch at each vertex.

Counting Folding Branches for Class-III and Class-IV Tilings

For class-III we again observe that the number of supplementation patterns equals the number of folding branches. We find,

$$N_{\text{b-hor}}^{\text{III}} = 2^{m+1}, \tag{3.35}$$

for a horizontally oriented class-III pattern, analogous to Eq. 3.10.

Class-IV contains the only tiles for which the number of supplementation patterns differs from the number of branch patterns: tiles D_i. These can be supplemented in six different ways, but have only four possible branch configurations. The combinatorial problem of choosing the branches of a class-IV tiling is therefore identical to the problem of counting branches for class-III tilings. This means that for an $m \times n$ class-IV pattern we have:

$$N_{\text{b-hor}}^{\text{IV}} = 2^{m+1}. \tag{3.36}$$

Conversely, when a class-III or class-IV pattern is vertically oriented, we have:

$$N_{\text{b-ver}}^{\text{III}} = N_{\text{b-ver}}^{\text{IV}} = 2^{n+1}. \tag{3.37}$$

3.6 Summary and Outlook

We summarize the results of this chapter, in Table 3.2. Here we show the classification of the 34 tiles into four classes, the number of tile patterns N_t within each class, the number of possible supplementations into brick patterns N_s, and the number of possible folding branches N_b. Note that for class-III and class-IV, the expressions in the table are for horizontally oriented patterns. Expressions for vertically oriented patterns can be found by interchanging m and n.

In the next chapter, we aim to design bipotent crease patterns where we can change the folded shape of two folding branches *independently*. Table 3.2 shows that class-II patterns can not be used with this goal in mind, as the number of tilings is fixed at 8. The number of supplemented angles $N_s = 2^n + 2^m - 2$ also does not form a large enough design space to facilitate this, as any changes in the supplementation pattern always occur along either the horizontal, or the vertical direction. Class-III and class-IV patterns have a larger design space, where the number of possible patterns scales exponentially with n (m). However, this still only allows us to change the tiles on the left (top) side of the pattern, which does not allow us to independently tune the folding shape of two or more branches. The only remaining class is therefore class-I, where we see that the number of tilings scales as $N_t \sim 2^{m+n}$. This reflects the fact that we can independently choose the tiles on the top row and left column of the pattern; where the choice of tiles in these locations directly changes the shape of the two folding branches. In the next chapter we will show how to design the two folded shapes of class-I patterns, by changing the composition of the top row and left column.

Class	Necessary Tiles	Optional Tiles	Tilings (N_t)	Sup.Angles (N_s)	Branches (N_b)
I	$\{A, B\}$	$\{C_i, F_i\}$	$8(2^m-1)(2^n-1)$	2	2
II	$\{C_i\}$	–	8	$2^{m+1}+2^{n+1}-2$	$2^{m+1}+2^{n+1}-2$
III†	$\{E_i, F_i, G_i, H_i, I_i, K_i\}$	$\{C_i, D_i, J_i\}$	$8(8^n-3^n)$	2^{n+1}	2^{n+1}
IV-1 (m even)†	$\{D_i, J_i\}$	$\{C_i\}$	$8(3^n+1-2^{n+1})$	2^{n+1}	2^{n+1}
IV-1 (m odd)†	$\{D_i, J_i\}$	$\{C_i\}$	$8(3^n+1-2^{n+1})$	2^{n+1}	2^{n+1}
IV-2 (m even)†	$\{D_i, J_i\}$	$\{C_i\}$	$16(2^n-1)$	$2^{n+1}-2+2^{\frac{m+2}{2}}$	2^{n+1}
IV-3 (m odd)†	$\{D_i, J_i\}$	$\{C_i\}$	$8(2^n-1)$	$2^{n+1}-2+2^{\frac{m+3}{2}}$	2^{n+1}
IV-4 (m odd)†	$\{D_i, J_i\}$	$\{C_i\}$	$8(2^n-1)$	$2^{n+1}-2+2^{\frac{m+1}{2}}$	2^{n+1}

TABLE 3.2: Table summarizing the results of sections 3.3, and 3.4, 3.5. The symbol † indicates the pattern is horizontally oriented, expressions for vertically oriented patterns can be obtained by $m \leftrightarrow n$