Positive representations of algebras of continuous functions

Jiang, X.

Author: Jiang, X.
Title: Positive representations of algebras of continuous functions
Issue Date: 2018-09-25

Propositions

ACCOMPANYING THE THESIS

Positive representations of algebras of continuous functions

BY

XINGNI JIANG

(1) A logical way to define the analogue of a real-valued signed measure in the context of partially ordered vector spaces is the following:

Definition. *Let Ω be an algebra of subsets of a set X, and let E be a σ-Dedekind complete partially ordered vector space. An E-*valued measure *on Ω is a set map $\mu : \Omega \to E$ such that*

(a) $\mu(\emptyset) = 0$;

(b) if $\{A_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω such that $\bigcup_{n=1}^{\infty} A_n \in \Omega$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigwedge_{M=1}^{\infty} \bigvee_{N=M}^{\infty} \sum_{n=1}^{N} \mu(A_n) = \bigvee_{M=1}^{\infty} \bigwedge_{N=M}^{\infty} \sum_{n=1}^{N} \mu(A_n).$$

(2) In Chapter I of this thesis, Theorem 12.3 and Theorem 13.5 show that Theorem 8.10 is still not entirely satisfactory.

(3) The proof of the classical Riesz representation theorem in [1] is not correct. There is a serious mistake on page 356.

[1] C.D. Aliprantis, O. Burkinshaw, *Principles of real analysis*, Academic Press, Inc., San Diego, CA, third edition, 1998.

(4) Let X be a locally compact Hausdorff space, let $\mathcal{A}$ be a Riesz algebra with identity element e, and let $\pi : \mathrm{C_c}(X) \to \mathcal{A}$ be a positive algebra homomorphism. If π has a representing spectral measure P such $P(X) = e$, then π is automatically a Riesz homomorphism.

(5) Let X be a locally compact Hausdorff space, let E be a normal and Dedekind complete Riesz space, and let $T : \mathrm{C_c}(X) \to E$ be a Riesz homomorphism such that $\{\, \pi(f) : \mathbf{0} \leq f \leq \mathbf{1},\ f \in \mathrm{C_c}(X) \,\}$ is bounded from above in E. Then the representing regular Borel measure μ of T satisfies $\mu(A \cap B) = \mu(A) \wedge \mu(B)$ for all Borel sets A and B.

(6) Let X be a locally compact Hausdorff space. For $x \in X$, set $\varphi_x(f) := f(x)$ for $f \in \mathrm{C}_0(X)$. Then φ_x is a Riesz homomorphism from $\mathrm{C}_0(X)$ into $\mathbb{R}$ such that $\|\varphi_x\| = 1$. Conversely, every Riesz homomorphism φ from $\mathrm{C}_0(X)$ into $\mathbb{R}$ with $\|\varphi\| = 1$ is of the form φ_x for a unique $x \in X$.

(7) For a locally compact Hausdorff space X, $\mathrm{C}_0(X)$ is Dedekind complete if and only if X is extremally disconnected.

(8) Let $d \geq 1$, and consider $\mathbb{R}^d$ in its usual Archimedean partial ordering. Suppose that, for $n \geq 1$, subsets S_n of $(\mathbb{R}^d)^+$ with infima ℓ_n are given with the property that $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} s_n = \infty$ in $\overline{(\mathbb{R}^d)^+}$ for all choices for $s_n \in S_n$. If $d = 1$, this implies that $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \ell_n = \infty$ in $\overline{\mathbb{R}^+}$. If $d \geq 2$, then the analogous implication is not valid.

In view of Remark 8.3 in Chapter I of this thesis, this shows that the existence of the infimum of two infinite measures with values in a partially ordered vector space requires more thought.

(9) If one is interested only in the existence of a simultaneous power factorization, and not in its further properties, then the conditions in Theorem 4.4 in Chapter II of this thesis are unnecessarily restrictive, because the following is true.

Theorem. *Let A be a real or complex Banach algebra with a bounded left approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$, and let π be a continuous representation of A on a Banach space X. If S is a non-empty subset of X with the property that $\lim_\lambda \pi(e_\lambda)s = s$ uniformly on S, then there exist $a \in A$ and maps $x_n : S \to X$ for $n \geq 1$ such that $s = \pi(a^n)x_n(s)$ for all $s \in S$ and $n \geq 1$.*

(10) The more you learn, the less you think you know.