Positive representations of algebras of continuous functions

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Summary

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If X is a locally compact Hausdorff space, then a representation of the complex C*-algebra $C_0(X, \mathbb{C})$ on a Hilbert space $\mathscr{H}$ is given by a spectral measure that takes its values in the orthogonal projections on $\mathscr{H}$. In an ordered context, is natural to ask whether something similar is true for a positive representation of the algebra $C_0(X)$ (we suppose that the functions are real-valued now) on a partially ordered vector space E, and, more specifically, on a (real) Banach lattice E.

It is known from earlier work by de Jeu and Ruoff that there is indeed a spectral measure for a positive representation of $C_0(X)$ on E if E is a KB-space. The KB-spaces form a reasonably large class of spaces, including, for example, all reflexive Banach lattices. The proof of this result follows primarily a Banach space approach, but some of the formulas thus obtained indicate that there may be a much larger role for the ordering than is apparent from their work. One starts to wonder if it may also be possible to prove this result from a purely order-theoretic viewpoint.

This question is taken up in the first part of the thesis, and the answer is affirmative. While developing the necessary theory, we obtain a bit more than an alternate approach to an already known result in the particular context of KB-spaces. As it turns out, there is an abstract spectral theorem underlying both the classical result for spectral measures in the context of Hilbert spaces and that in the context of KB-spaces.

Our approach towards this underlying theorem is divided into three steps, each covering several sections.

Step one is of a preparatory nature. We collect a number of results on partially ordered vector spaces, with special attention to partially ordered vector spaces that are normal. A partially ordered vector space E is said to be normal if, for $x \in E$, the fact that $(x, x') \geq 0$ for all positive order continuous linear functionals on E, implies that $x \geq 0$. As will become clear later on, the (monotone complete) normal spaces form a class of spaces in the context of which Riesz representation theorems can be established.

The formulation of these representation theorems is in terms of measures with values in partially ordered vector spaces and of integrals of real-valued functions with respect to such measures. The material developed in this first step also provides the very framework that is necessary to be able to state these results. We introduce σ-additive measures with values in a σ-monotone complete partially ordered vector space, as well as (order) integrals with respect to such measures. We also establish the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem. All this specialises to the usual results from Lebesgue theory if the partially ordered vector space equals the real numbers.

Still primarily as a preparation for what is to come, we also include material on measures with values in partially ordered algebras, and on vector-valued measures on the Borel σ-algebra of a locally compact Hausdorff space that have certain regularity properties.

Step two is concerned with Riesz representation theorems that generalise the classical ones. We start with theorems for a positive linear map T from $\mathrm{C_c}(X)$ to a partially ordered vector space E. Several representation theorems are then established. For example, if $T : \mathrm{C_c}(X) \to E$ is a positive linear map from $\mathrm{C_c}(X)$ into a normal and monotone complete partially ordered vector space E, then—we omit the statements on regularity and uniqueness for the sake of simplicity—there exists a measure m on the Borel σ-algebra of X, with values in the extended positive cone of E, such that

$$T(f) = {\int_X}^{\mathrm{o}} f \,\mathrm{d}m$$

for all $f \in \mathrm{C_c}(X)$. Here the superscript 'o' in the notation of the integral serves to denote that the integral is an order integral.

After that, we turn to a positive linear map $T : \mathrm{C_0}(X) \to E$ that is defined on the larger space $\mathrm{C_0}(X)$. If the measure that represents the restriction of T to $\mathrm{C_c}(X)$ is finite, i.e. if the subset $\{\, T(f) : f \in \mathrm{C_c}(X),\, \mathbf{0} \leq f \leq \mathbf{1} \,\}$ is bounded from above in E, then it will usually also represent the map T on the whole of $\mathrm{C_0}(X)$. Exploiting the fact that positive linear maps from the Banach lattice $\mathrm{C_0}(X)$ into normed Riesz spaces are automatically norm continuous, we can point out a number of cases where this boundedness is automatic, such as where E is the partially ordered vector space of the regular operators on a KB-space. A number of representation theorems for the map T, as defined on the full space $\mathrm{C_0}(X)$, thus result.

Step three consists of considering a positive algebra homomorphism π from $\mathrm{C_c}(X)$ or $\mathrm{C_0}(X)$ into a partially ordered algebra $\mathscr{A}$. If $\mathscr{A}$ is normal and monotone complete as a partially ordered vector space, then the results from the previous step provide a measure that represents π as a positive linear map. In this case, we know a bit more, namely, that the map π is also multiplicative. If the multiplication in $\mathscr{A}$ is monotone order continuous, it can then be shown that, if it is finite, the representing measure P (we switch to the customary notation here) for π is also spectral, i.e. that $P(A_1 \cap A_2) = P(A_1)P(A_2)$ for all Borel sets A_1 and A_2. An abstract spectral theorem has thus been obtained for positive algebra homomorphisms from $\mathrm{C_c}(X)$ into partially ordered algebras. The representing spectral measure P takes it values in the positive idempotents in $\mathscr{A}$.

As in the case of a positive linear map from $\mathrm{C_0}(X)$ into a partially ordered vector space, the measure that represents the restriction to $\mathrm{C_c}(X)$ of a positive algebra homomorphism π from $\mathrm{C_0}(X)$ into $\mathscr{A}$ is often automatically finite, so that it is then a spectral measure that represents π on the whole algebra $\mathrm{C_0}(X)$.

At the end of this step, we note that the algebra of regular operators on a KB-space fits into this picture, and that the same holds true for the algebra consisting of the self-adjoint elements of a commutative von Neumann algebra. The abstract spectral theorem is then seen to have the existence of a spectral measure in the two examples in the beginning of this summary as an immediate consequence.

There is earlier work, notably by J.D.M. Wright, on integration with respect to measures with values in partially ordered vector spaces, and on Riesz representation theorems formulated using the corresponding order integrals. A discussion of the relation between the present work and that in the literature is included.

Simultaneous power factorization

Let A be a Banach algebra with a bounded left approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$, let π be a continuous representation of A on a Banach space X, and let S be a non-empty subset of X such that $\lim_\lambda \pi(e_\lambda)s = s$ uniformly on S. If S is bounded, or if $\{e_\lambda\}_{\lambda\in\Lambda}$ is commutative, then we show that there exist $a \in A$ and maps $x_n : S \to X$ for $n \geq 1$ such that $s = \pi(a^n)x_n(s)$ for all $n \geq 1$ and $s \in S$. The properties of $a \in A$ and the maps x_n, as produced by the constructive proof, are studied in some detail. The results generalize previous simultaneous factorization theorems as well as Allan and Sinclair's power factorization theorem.

In an ordered context, we also consider the existence of a positive factorization for a subset of the positive cone of an ordered Banach space that is a positive module over an ordered Banach algebra with a positive bounded left approximate identity. Such factorizations are not always possible. In certain cases, including those for positive modules over ordered Banach algebras of bounded functions, such positive factorizations exist, but the general picture is still unclear.

Furthermore, simultaneous pointwise power factorizations for sets of bounded maps with values in a Banach module (such as sets of bounded convergent nets) are obtained.

A worked example for the left regular representation of $C_0(\mathbb{R})$ and unbounded S is included.