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Chapter I

Positive representations of algebras of continuous functions

Abstract

Let X be a locally compact Hausdorff space. We establish several Riesz representation theorems for positive linear maps from $\mathrm{C_c}(X)$ or $\mathrm{C_0}(X)$ to partially ordered vector spaces. Analogously to the case where the partially ordered vector space is the real numbers, the theorems state that such positive linear maps are given by integration with respect to vector-valued measures. If the partially ordered vector space is a partially ordered algebra, and if the positive linear map is also multiplicative, then the representing measure is shown to be a spectral measure. The resulting abstract spectral theorem is applied to positive representations of $\mathrm{C_0}(X)$ on KB-spaces and to representations of (the complexification of) $\mathrm{C_0}(X)$ on Hilbert spaces. The known results in the literature on the existence of spectral measures for such representations are thus seen to stem from one common underlying theorem.

A theory of measures with values in partially ordered vector spaces and of integration with respect to such measures is developed, since this needed to formulate the representation theorems.

§1. Introduction and overview

If X is a locally compact Hausdorff space, then we let $\mathrm{C}_0(X,\mathbb{C})$ denote the continuous complex-valued functions on X that vanish at infinity. If $\pi : \mathrm{C}_0(X,\mathbb{C}) \to \mathscr{B}(\mathscr{H})$ is a representation of the complex C*-algebra $\mathrm{C}_0(X,\mathbb{C})$ on a complex Hilbert space $\mathscr{H}$, then it is a classical result (see e.g. [**14**, Corollary 1.55]) that there exists a unique regular spectral measure P on the Borel σ-algebra $\mathscr{B}$ of X such that

$$\pi(f) = \int_X f \,\mathrm{d}P \tag{1.1}$$

for all $f \in \mathrm{C}_0(X,\mathbb{C})$. We recall that, in this Hilbert space context, a spectral measure is a map $P : \mathscr{B} \to \mathscr{B}(\mathscr{H})$ such that

(1) $P(A)$ is an orthogonal projection for every Borel set A;
(2) $P(\emptyset) = 0$;
(3) $P\left(\bigcup_{n=1}^{\infty} A_n\right) x = \sum_{n=1}^{\infty} P(A_n)x$ for all $x \in \mathscr{H}$ and every pairwise disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{B}$;
(4) $P(A_1 \cap A_2) = P(A_1)P(A_2)$ for all $A_1, A_2 \in \mathscr{B}$.

The classical result is actually for non-degenerate representations, in which case $P(X) = I$, but the statement for the general case follows easily from that for the non-degenerate case.

Associated with a spectral measure is a family of complex-valued measures $P_{x,x'} : \mathscr{B} \to \mathbb{C}$ for $x, x' \in \mathscr{H}$, defined by $P_{x,x'}(A) := \langle P(A)x, x' \rangle$ for $A \in \mathscr{B}$. The spectral measure P is then said to be a regular Borel measure if the measure $P_{x,x'}$ is a regular Borel measure for all $x, x' \in \mathscr{H}$. One of the ways to interpret the statement involving equation (1.1) is that

$$\left\langle \pi(f)x, x' \right\rangle = \int_X f \,\mathrm{d}P_{x,x'} \tag{1.2}$$

for all $f \in \mathrm{C}_0(X,\mathbb{C})$ and $x, x' \in \mathscr{H}$.

In [**12**], a similar result was obtained for a positive representation $\pi : \mathrm{C}_0(X) \to \mathrm{L}_{\mathrm{r}}(E)$ of the algebra $\mathrm{C}_0(X)$—the algebra of continuous real-valued functions on X that vanish at infinity—on a KB-space E. Again equations (1.1) and (1.2) hold, where now the spectral measure takes its values in the positive projections on E, and where the pairing in the definition of the $P_{x,x'}$ and in the left hand side of equation (1.2) is now between an element of E and an element of the dual E^* of E. Since the KB-spaces include, for example, all reflexive Banach lattices and also non-reflexive ones such as L^1-spaces, this result covers a reasonably large class of Banach lattices.

The approach in [**12**] for KB-spaces is strongly inspired by that in the literature for Hilbert spaces. It is a Banach space approach, with an order-theoretical component that, although crucial at one point, is otherwise not so prominent. However, after the existence of the spectral measure has been established, there are a few results in [**12**] that indicate that there may be a greater role for ordering than has become clear so far. For example, [**12**, Theorem 5.12] shows that

$$P(V) = \bigvee \{\, \pi(f) : f \in \mathrm{C_c}(X),\ \mathbf{0} \leq f \leq \mathbf{1},\ \operatorname{supp} f \subseteq V \,\} \tag{1.3}$$

for every non-empty open subset V of X. Here $\mathrm{C_c}(X)$ denotes the algebra of real-valued continuous functions on X with compact support, and the supremum is taken in the regular operators on the KB-space E. The proof of this uses the previous Banach space oriented results, but, in the end, the expression is a purely order-theoretical one, and it is also a perfect analogue of the corresponding formula in the classical context of positive linear functionals on $\mathrm{C}_0(X)$ (or $\mathrm{C}_\mathrm{c}(X)$). One could have written down this expression as an Ansatz at the very beginning. Could it, therefore, perhaps be the case that the Banach space structure is not essential, and that an order-theoretical approach to these spectral measures in the context of KB-spaces is also possible?

This question was the starting point for the research leading to the present body of work. The answer is affirmative. Investigating this order-theoretical approach has, however, led to a little bit more than a new proof for the existence of a spectral measure for a positive representation of $\mathrm{C}_0(X)$ on a KB-space. As has become clear, this result, as well as the classical result for spectral measures regarding representations of $\mathrm{C}_0(X,\mathbb{C})$ on Hilbert spaces, are immediate consequences of the same underlying more general theorem involving positive homomorphisms from $\mathrm{C}_0(X)$ or $\mathrm{C}_\mathrm{c}(X)$ into partially ordered algebras. We shall now explain how such a result (Theorem 15.1) is obtained through various stages, and how it specialises in the case of KB-spaces of Hilbert spaces.

The first step is to establish a Riesz representation theorem for a positive linear map T from $\mathrm{C}_0(X)$ (or $\mathrm{C}_\mathrm{c}(X)$) into a partially ordered vector space E. Under reasonably mild conditions, it can be shown that there exists a positive E-valued measure μ (one usually also has to allow μ to assume the value ∞) on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f)=\int_X f\,\mathrm{d}\mu \tag{1.4}$$

for all $f\in\mathrm{C}_0(X)$ (or $\mathrm{C}_\mathrm{c}(X)$). This measure μ is unique within the class of measures that represent T in this fashion and that have certain regularity properties.

The second step is to consider the case where E is not just a partially ordered vector space, but a partially ordered algebra $\mathscr{A}$, and where the map T is not just a positive linear map, but a positive algebra homomorphism π from $\mathrm{C}_0(X)$ (or $\mathrm{C}_\mathrm{c}(X)$) into $\mathscr{A}$. Under reasonably mild conditions, the representing measure for the positive linear map π now has an extra property: it is a spectral measure. Conforming to tradition, we shall then denote the representing measure by P rather than μ. The fact that P is a spectral measure then means, by definition, that $P(A_1\cap A_2)=P(A_1)P(A_2)$ for all $A_1,A_2\in\mathscr{B}$. This implies that $P:\mathscr{B}\to\mathscr{A}$ takes its values in the positive idempotents in $\mathscr{A}$.

In the end, an abstract spectral theorem is thus available, stating that there is a spectral measure that represents a positive algebra homomorphism $\pi:\mathrm{C}_0(X)\to\mathscr{A}$ or $\pi:\mathrm{C}_\mathrm{c}(X)\to\mathscr{A}$ into a partially ordered algebra $\mathscr{A}$, provided that $\mathscr{A}$ satisfies a few natural conditions.

Although the general result does not suppose this, the algebra $\mathscr{A}$ can be an algebra of operators that has a partial ordering. In that case, the spectral measures supplied by it are spectral measures in the classical sense, i.e. measures that take values in the positive projections.

In the case of a positive representation $\pi:\mathrm{C}_0(X)\to\mathrm{L}_\mathrm{r}(E)$ of $\mathrm{C}_0(X)$ on a KB-space E, one easily checks that the partially ordered algebra $\mathrm{L}_\mathrm{r}(E)$ of regular operators on a KB-spaces meets the requirements in the abstract spectral theorem. The main result of [**12**] on the existence of a spectral measure for π is then immediate. It takes its values in the positive idempotents in $\mathrm{L}_\mathrm{r}(E)$, i.e. in the positive projections on E.

In the case of a representation $\pi:\mathrm{C}_0(X,\mathbb{C})\to\mathscr{B}(\mathscr{H})$ of $\mathrm{C}_0(X,\mathbb{C})$ on a complex Hilbert space $\mathscr{H}$, one starts by noting that the self-adjoint elements of the von Neumann algebra that is generated by $\pi(\mathrm{C}_0(X,\mathbb{C}))$ form a partially ordered algebra that has the correct properties. An application of the abstract spectral theorem to the restricted positive homomorphism of $\mathrm{C}_0(X)$ into this (real) partially ordered algebra then immediately furnishes the classical spectral measure. It takes its values in the positive idempotents in the algebra of the self-adjoint elements of the von Neumann algebra that is generated by $\pi(\mathrm{C}_0(X,\mathbb{C}))$; these are orthogonal projections on $\mathscr{H}$.

There is one important aspect that we have not yet mentioned in the above outline, and that is the definition of integrals such as in equation (1.4) for a measure taking values in a partially ordered vector space. In general, the partially ordered vector spaces under consideration do not carry a topology, and one cannot use limits in the topological sense to define an integral. However, the spaces do carry a partial ordering, and this provides a means to introduce an integral after all. The construction of this (order) integral is as follows. The definition of the integral of a positive measurable function that takes only finitely many values is the natural one. Imposing a mild order completeness condition on the space, the integral of a positive measurable function can then be defined as the supremum of the integrals of an increasing sequence of positive elementary functions that approximate the function pointwise from below. From positive measurable functions one then proceeds to arbitrary integrable ones in the obvious fashion. This is all analogous to the introduction of the integral with respect to real-valued measures in many (though not all) text books, and if $E=\mathbb{R}$ the usual Lebesgue integral results. The notion of a positive measure with values in a partially ordered vector space has to be changed, though, because σ-additivity can no longer be expressed using a convergent or divergent series with positive terms. The sum of such a series has to be replaced with the supremum (possibly infinity) of the sequence of its partial sums.

It appears that Wright was the first to realise that such an adaptation of the notion of σ-additivity can profitably be made to yield a workable theory of measure and integration if the measure takes its values in a partially ordered vector space E. This was first done in [**24**], where E is a Stone algebra, i.e. a Riesz space of the form $\mathrm{C}(X)$ for an extremally disconnected compact Hausdorff space X. It is mentioned in [**26**, p. 194] that this can also be done if E is a σ-Dedekind complete Riesz space, but details are not provided. In [**27**, p. 678], a similar remark is made if E is a σ-monotone complete partially ordered vector space, again without providing the details or precise formulations of the results. It is also mentioned there that Fatou's lemma and the dominated convergence theorem, which are not stated, become more complicated. Since a good theory of measure and integration with respect to measures with values in a σ-monotone complete partially ordered vector space is essential to our work, we have decided to develop this theory in this particular context of σ-monotone complete partially ordered vector spaces from scratch. We supply full proofs and precise statements, including those of Fatou's lemma and the dominated convergence theorem. Needless to say, it was of considerable help to have [**24**] as a starting point. We shall comment a little more on this below.

We shall now combine the remainder of this introductory discussion with an overview of the arrangement of the material that is to follow. It will become clear how the abstract spectral theorem (Theorem 15.1) is obtained, but also which types of Riesz representation theorems for positive linear maps, which have interest of their own, are encountered along the way.

Section 2 contains basic notions for partially ordered vector space, with special attention to various types of order completeness. It is also explained how we can adjoin a point at infinity to the space, which is inevitable if one wants to establish Riesz representation theorems. It also contains a stockpile of small but important technical results that will allow us to argue with suprema and infima in the sequel.

Section 3 is concerned with operators between partially ordered vector spaces, and with duality. As will become clear in the sequel, the 'good' spaces for Riesz representation theorems are notably the normal and monotone complete spaces. The latter notion may speak for itself; a space is normal if there are sufficiently many (in a precise sense) order continuous linear functionals. Banach lattices with order continuous norms are normal and monotone complete, and so are the spaces that consist of the self-adjoint operators in a weakly closed subspace of the bounded operators on a Hilbert space. Spaces of operators between two partially ordered vector spaces will typically be normal and monotone complete again if the codomain is.

Section 4 is a short section on partially ordered algebras. A partially ordered algebra that has monotone continuous multiplication and that is also normal and monotone complete meets the requirements in the abstract spectral theorem. The algebra of regular operators on a Banach lattice with order continuous norm falls into this category, as does the algebra of the self-adjoint elements of a commutative von Neumann algebra.

In Section 5, measures on algebras of sets with values in the (extended) positive cone of a σ-monotone complete partially ordered vector space are introduced, and their basic properties are developed. The results are analogous to those for the real case, but most of the proofs need a slight adaptation when it comes to the handling of the infinite series that occur in the real case. We have taken care to supply the details, in order to convince the reader (and ourselves) that the proofs *can* be adapted to work in the more general context. This is perhaps less obvious than it may seem, because the real numbers have various properties that are used in the theory of real-valued measures and integration with respect to such measures in a very natural way, and there lies a certain danger in this. The real numbers are linearly ordered, they form a lattice, and they are a normed space with ensuing topology: these are all very convenient in proofs, but none of them can be used when passing to general partially ordered vector spaces. The only properties that *can* be used in the general case are the σ-monotone completeness and the accompanying Archimedean property, and one needs to make certain that one can indeed give a proof that uses these properties and no others. To give an example of the mistakes where a 'real mindset' could lead to: if S is a subset of a partially ordered vector space E that has an infimum x in E, and if $y \in E$ is positive, then it is tempting to 'choose an element $s \in S$ such that $x \leq s \leq y$', and proceed from there. However, whereas this is indeed possible if $E = \mathbb{R}$, it is already not generally possible if $E = \mathbb{R}^d$ with $d \geq 2$. One has to reconsider everything carefully and adapt the arguments where necessary.

Section 6 covers outer measures in the vector-valued context. One of our Riesz representation theorems for positive linear maps (Theorem 12.3) is established along the lines of one of the possible proofs of the classical case for the real numbers, using an outer measure. As in the classical case, one needs to know that the measurable subsets form a σ-algebra and that the restriction of the outer measure to this σ-algebra is σ-additive (in our ordered sense). We are not aware of earlier material in the literature in this direction.

Section 7 is concerned with vector-valued measures on the Borel σ-algebra of a locally compact Hausdorff spaces and their regularity properties.

In Section 8, cones of vector-valued measures are studied, both in the abstract setting of measures on a general algebra and in the context of measures on the Borel σ-algebra of a locally compact space that possess a certain degree of regularity. This material is not a part of the main line of the current development of the theory, but is intended as a stepping stone for further research. In the real case, there is a isometric lattice isomorphism between the norm dual of $\mathrm{C}_0(X)$ and the signed regular Borel measures on a locally compact Hausdorff space X, and an isomorphism of lattice cones between the order dual of $\mathrm{C}_\mathrm{c}(X)$ and the positive (possibly infinite) regular Borel measures on X. Presumably similar results can be obtained in a more general context, and we expect that the material in Section 8 will be helpful when investigating these matters.

Section 9 contains the basic theory of integration with respect to a measure with values in a σ-monotone complete partially ordered vector space. As with the part on measure theory, we have provided detailed proofs to make sure that there are no mistakes. As an example of what can go wrong, we consider the statement that an integrable function f that is almost everywhere zero has zero integral. For a measure μ with values in a Riesz space, one can presumably prove this for positive f and then use a triangle inequality for integrals in $|\int f\,\mathrm{d}\mu| \leq \int |f|\,\mathrm{d}\mu = 0$ to see that it is true for general f. For general vector-valued measures this is no longer a valid argument. One has to split f into its positive and negative part (for which it is almost immediate from the definition of the integral) to give a correct proof. Even though this is hardly a profound modification, it still shows that it is easy to make mistakes when thinking that results from the real case or Riesz space case generalise immediately to the general vector-valued case.

Compared to the seminal paper [**24**], we have a different starting point when it comes to the definition of the integral of a positive function. We use elementary functions that can have a support of infinite measure as pointwise approximants from below, whereas in [**24**] only finite measures of supports are allowed. The integral of a positive function that cannot be approximated pointwise from below by such functions is then defined to be infinity in [**24**]. Our approach is uniform. We also define the integral of a positive function that is allowed to take infinity as a value. We believe this is more natural to do, and it gives a more general version of Fatou's lemma and the monotone convergence theorem.

We also develop the theory a bit further than was done in [**24**], or than is needed to formulate the Riesz representation theorems later on. We hope that this extra material will prove to be useful in future applications. For example, we have included a systematic treatment of properties that hold almost everywhere, and how this is related to integrals. We have also included the definition and properties of the natural space of integrable functions, and of its quotient modulo the functions that vanish almost everywhere. Furthermore, Fatou's lemma and the dominated convergence theorem, which were said in [**27**, p. 678] to be more complicated in the case of σ-monotone complete partially ordered vector spaces that need not be lattices, are stated and proved.

Section 10 is a prelude to the proof of the abstract spectral theorem for positive homomorphisms from $\mathrm{C}_0(X)$ or $\mathrm{C}_\mathrm{c}(X)$ into partially ordered algebras. One can canonically construct a new measure from a given one and a positive measurable function, and in this section—which contains somewhat more material than is needed in the sequel—it is studied how the regularity properties of the original measure are inherited by the new measure, and how their integrals are related. These results, together with the uniqueness statements in our Riesz representation theorems, are a key ingredient in the proof of the abstract spectral theorem.

Section 11 serves a similar purpose as Section 10: it is a preparation for the proof of the abstract spectral theorem. If a measure takes its values in a partially ordered algebra, then one can define a new measure by (left or right) multiplication of the given measure by a fixed positive element of the algebra. We need to know that the regularity properties of the original measure are inherited by the multiplies ones, and how their integrals are related.

Section 12 contains our first Riesz representation theorem. It covers the case of a positive linear map $T : \mathrm{C_c}(X) \to E$, where E is a monotone complete partially ordered vector space that is also a normed space and that satisfies three reasonably mild additional requirements. It applies to every Banach lattice E with order continuous norm. The lengthy proof follows the outline of one of the existing proofs where $E = \mathbb{R}$, and uses the material from Section 6 on outer measures.

In Section 13, we move to the context of normal and monotone complete partially ordered vector spaces. Banach lattices with order continuous norm provide an important class of examples again, but this is also true for suitable spaces of self-adjoint operators on a Hilbert space that need not be lattices. Based on the classical Riesz representation theorem (which is a special case of the main result of Section 12), it is not so difficult to find a measure μ that represents a positive linear map $T : \mathrm{C_c}(X) \to E$, provided that the set $\{ T(f) : f \in \mathrm{C_c}(X),\, \mathbf{0} \leq f \leq \mathbf{1} \}$ is order bounded in E; the latter is trivially the case if X is compact. If this subset of X is not order bounded, then it is still possible to obtain a representing measure by including an extra step. Indeed, if U is a relatively compact open subset of X, then it is easy to see that the set $\{ T(f) : f \in \mathrm{C_c}(X),\, \mathbf{0} \leq f \leq \mathbf{1},\, \mathrm{supp}(f) \subseteq U \}$ is order bounded in E. By the previous result, there is a measure μ_U on the Borel σ-algebra of U that represents the restriction of T to $\mathrm{C_c}(U)$, and these local measures μ_U can be patched together to a measure that is defined on the Borel σ-algebra of X. This technique is already employed in [**26**, proof of Theorem 1].

In Section 14, we take up representation theorems for positive linear maps $T : \mathrm{C_0}(X) \to E$ that are defined on the larger space $\mathrm{C_0}(X)$. The strategy to obtain such theorems is clear: one restricts T to $\mathrm{C_c}(X)$ and applies the representation theorem to this restricted map. If the representing measure is finite, so that $\mathrm{C_0}(X)$ consists of integrable functions, then there is good hope that the measure will also represent T on $\mathrm{C_0}(X)$. This works in a number of cases. In each of them, the important step is to know a priori that $\{ T(f) : f \in \mathrm{C_c}(X),\, \mathbf{0} \leq f \leq \mathbf{1} \}$ is order bounded in E. For this, it is vital that $\mathrm{C_0}(X)$ is a Banach lattice. For example, if E is a Banach lattice with a Levi norm (such as a KB-space), then we know that T, being a positive linear map from a Banach lattice into a normed Riesz space, is norm continuous. This implies that the set $\{ T(f) : f \in \mathrm{C_c}(X),\, \mathbf{0} \leq f \leq \mathbf{1} \}$ is norm bounded in E. Since it is obviously upward directed and the norm is a Levi norm, it has a supremum in E. The desired order boundedness of the set is thus seen to be automatic in this case.

Variations on this theme of automatic norm continuity of positive linear maps with $\mathrm{C_0}(X)$ as domain are possible, for example, by composing T with positive linear functionals on the codomain E. This leads to representation theorems for positive linear maps defined on $\mathrm{C_0}(X)$ if E is, as we have called it, a quasi-perfect Riesz space. This is a reasonably large class of spaces that includes the perfect Riesz spaces, and also all normal Banach lattices with a Levi norm (such as KB-spaces).

Another opportunity to exploit automatic norm continuity is given by the fact that a positive linear map from the complexification of $\mathrm{C_0}(X)$, which is a C^*-algebra, into $\mathscr{B}(\mathscr{H})$ is also automatically norm continuous. Thus a number of representation theorems are obtained for $\mathrm{C_0}(X)$, including ones where E is the partially ordered vector space of all regular operators on a KB-space, and where E is the partially ordered vector space of the self-adjoint elements of a weakly closed linear subspace of $\mathscr{B}(\mathscr{H})$.

In Section 15, we reach our destination, and we take up the issue of a positive homomorphism π from $\mathrm{C_c}(X)$ or $\mathrm{C_0}(X)$ into a partially ordered algebra $\mathscr{A}$ that is normal, monotone complete, and that has monotone continuous multiplication. Using the accumulated material, it is now not so difficult to show that, if it is finite, the measure that represents π as a positive linear map from $\mathrm{C_c}(X)$ into $\mathscr{A}$ is, in fact, a spectral measure. It is thus that the abstract spectral theorem is obtained; see Theorem 15.1.

The natural algebras that come to mind for possible applications of this abstract spectral theorem are algebras of operators. We can now exploit the results from Section 14, stating that the representing measure of π is finite in a number of cases, to obtain spectral measures for two natural classes of positive representations of $\mathrm{C_0}(X)$: those on KB-spaces and those on Hilbert spaces. The two motivating theorems mentioned in the beginning of this section are now seen to be immediate consequences of the abstract spectral theorem. In the Hilbert space case, the results obtained seem even a little more precise than what is to be found in the literature.

We shall now discuss the relation of our work with the literature.

We start with earlier work on Riesz representation theorems for positive linear maps from spaces of continuous functions to various types of partially ordered vector spaces. While discussing this, we do not make any claim to completeness, and any omission is unintentional. The space X is always a locally compact Hausdorff space, unless otherwise stated.

To start with, there is the seminal paper by Wright [**24**] that was already mentioned earlier. It has a representation theorem for a positive linear map $T : \mathrm{C}(X)$, where X is compact, into a Stone algebra. A second paper [**25**] by the same author covers the case of a positive linear map from $\mathrm{C}(X)$, where X is compact, into a σ-Dedekind complete Riesz space. A third paper [**26**] contains a representation theorem for a positive linear map from $\mathrm{C_c}(X)$ into a Dedekind complete Riesz space. A fourth [**27**] is concerned with a positive linear map from $\mathrm{C}(X)$, where X is compact, into a σ-monotone complete partially ordered vector space.

In [**16**], one of Khurana's papers in this direction, it is shown that a positive linear map from $\mathrm{C_c}(X)$ into a monotone complete partially ordered vector space can be extended to a σ-order continuous linear map defined on the bounded Borel functions with compact support; this goes in the direction of a Riesz representation theorem. In a second paper [**17**], a representation theorem is established, under certain additional conditions, for a positive linear map from the bounded continuous functions on a completely regular Hausdorff space into a monotone complete partially ordered vector space. In [**17**], he proves a representation theorem for a positive linear map from the continuous functions on a completely regular $\mathrm{T_1}$ space into a Stone algebra.

In [**18**], Lipecki shows the existence of a representing finitely additive measure for positive linear maps from the bounded continuous functions on an arbitrary topological space into a monotone complete partially ordered vector space; there is a second result for the bounded continuous functions if the topological space is normal.

In [**10**], Coquand gives a new proof of Wright's result in [**27**] of the existence of a representing measure for a positive linear map from $\mathrm{C}(X)$, where X is compact, into a σ-monotone complete partially ordered vector space.

The results cited above also include regularity properties of the representing measures, but not always the same properties. It is, therefore, not clear if, in cases where more than one results applies, the representing measures are necessarily equal.

As may have become clear in the discussion preceding the literature overview, the condition that the subset $\{\, T(f) : f \in \mathrm{C_c}(X),\ \mathbf{0} \leq f \leq \mathbf{1} \,\}$ of E be order bounded in E is an important one. It is equivalent to the finiteness of the representing measure. For finite measures many difficulties disappear since there are no problems at all when manipulating with suprema and infima, or when one wants to compose the measure with positive linear functionals on

E. If the constant function $\mathbf{1}$ is in the domain of T (for example, if X is compact), then this is automatically the case: the subset is bounded from above by $T(\mathbf{1})$. If E is, in addition, a normed Riesz space, then this subset is also norm bounded. Most of the results cited above are in this relatively convenient context, but as our Theorem 13.5 (a representation theorem for a positive linear map from $\mathrm{C_c}(X)$ into a normal and monotone complete partially ordered vector space) shows, this is not a necessary prerequisite for representation theorems to hold. There are two papers in the above overview that 'avoid' this automatic order boundedness. Khurana's paper [**16**] is for positive linear maps defined on $\mathrm{C_c}(X)$, but the result obtained does not yet yield a measure. Wright's paper [**26**] contains a representation theorem for positive linear maps defined on $\mathrm{C_c}(X)$, and this is the existing result that appears to be most closely related to ours in Theorem 13.5. The main difference with our result is that Wright supposes that E is a Riesz space (which is used in the proof) and that we suppose that the space is normal (which is also used in the proof). It appears that both results are complementary. Ours is not applicable if the space E is not normal, whereas Wright's is not applicable if E is not a Riesz space, for example, if E is the set of self-adjoint elements of a weakly closed subspace of $\mathscr{B}(\mathscr{H})$.

It appears that the proofs of the existing representation theorems in the literature usually rely on the classical Riesz representation theorem for $E = \mathbb{R}$. The same is true for our proofs where the space is normal and monotone complete. There is an important difference, though: we are not aware of existing proofs in the literature that follow our approach of simply writing down the representing measure on open subsets of X as in equation (1.3), and proceeding from there.

We are not aware of a previous occurrence of an abstract spectral theorem such as Theorem 15.1.

Although they are beyond the current body of work, we mention two perspectives for possible applications of the present results.

Firstly, it should be mentioned that there are possible applications when studying a positive linear map (not necessarily a *-homomorphism) $T : A \to B$ between two complex C^*-algebras A and B, if we suppose that B has been embedded into $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. There are many commutative C^*-algebras in A, and to each of them Theorem 14.10 applies, yielding representing measures for the restricted maps that take their values in the von Neumann algebra generated by (the embedded) B.

Secondly, we expect that our results for positive representations can be used to obtain a better insight into the structure of (algebras of) orthomorphisms on Banach lattices. If E is a KB-space, then its center $Z(E)$ is isometrically lattice isomorphic to a space $\mathrm{C}(X)$ for some compact Hausdorff space X. The inverse of this map is a positive representation of $\mathrm{C}(X)$, and an application of the spectral theorem to this representation gives a spectral measure on X that takes its values in the order projections on E. This not only shows that every orthomorphism is built from order projections (something we already knew from Freudenthal's theorem), but also how this can be done in a uniform way for all orthomorphisms, and with an integration theory available. Of course, one can also do this for algebras of orthomorphisms other than $Z(E)$ itself, such as the algebra that is generated by one orthomorphism. It is hoped that in the latter case the spectral theorem will clarify the structure of a single orthomorphism in a way that is not dissimilar from that for a normal operator on a separable Hilbert space. Such an operator is built from multiplication operators on L^2-spaces (see [**9**, Theorem IX.10.1]), and the starting point for establishing this is the spectral theorem for the commuative C^*-algebra that is generated by the operator. Hopefully the same can be achieved in the ordered case now that we have a similar starting point.

§2. Partially ordered vector spaces

In this section, we establish some terminology and notation for partially ordered vector spaces, and collect a few basic facts about various types of order completeness, also in relation to the extended space that is obtained by adjoining a point at infinity.

All vector spaces we shall consider are over the real numbers, unless otherwise stated.

If E is a partially ordered vector space, then E^+ denotes its positive cone. We do not require that E^+ is generating, i.e. it need not be the case that $E = E^+ - E^+$. Stated equivalently, we not require E to be directed.

Subsets of a partially ordered vector space E of the form $\{ x \in E : a \leq x \leq b \}$ for $a, b \in E$ such that $a \leq b$, are called *order intervals* in E; they are denoted by $[a, b]$. A subset of E is called *order bounded* if it is contained in an order interval.

The order completeness properties that we are interested in are relevant when introducing the integral with respect to vector-valued measures, and also when establishing convergence theorems for integrals. They are also necessary to be able to define regularity properties of vector-valued measures on a locally compact Hausdorff space. We list them in the following definition, which also contains some self-evident notation that we shall use.

Definition 2.1. A partially ordered vector space E is called

(1) *σ-monotone complete* if every increasing sequence $\{a_n\}_{n=1}^{\infty}$ in E that is bounded from above has a supremum $\bigvee_{n=1}^{\infty} a_n$ in E;
(2) *monotone complete* if every increasing net $\{a_\lambda\}_{\lambda \in \Lambda}$ in E that is bounded from above has a supremum $\bigvee_{\lambda \in \Lambda} a_\lambda$ in E;
(3) *σ-Dedekind complete* if every non-empty at most countably infinite subset S of E that is bounded from above has a supremum $\bigvee \{ x : x \in S \}$ in E;
(4) *Dedekind complete* if every non-empty subset S that is bounded from above has a supremum $\bigvee \{ x : x \in S \}$ in E.

Equivalently, one can define these properties by requiring the existence of infima when replacing ‘increasing’ with ‘decreasing’ and ‘bounded from above’ with ‘bounded from below’.

There are evident logical implications between these four properties. For Riesz spaces, Dedekind completeness (resp. σ-Dedekind completeness) and monotone completeness (resp. σ-monotone completeness) are equivalent. If E has a generating cone and if E is σ-Dedekind complete, then, for every $x_1, x_2 \in E$, the subset $\{x_1, x_2\}$ is bounded from above, so that it has a supremum. Hence E is then a Riesz space.

A partially ordered vector space E is *Archimedean* if $\bigwedge \{ \varepsilon x : \varepsilon > 0 \} = 0$ for all $x \in E^+$. One can equivalently require that $\bigwedge \{ r_n x : n \in \mathbb{N} \} = 0$ for every $x \in E^+$ and every (or just one) sequence $\{r_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+ \setminus \{ 0 \}$ such that $r_n \downarrow 0$. Still equivalently, one can require that, whenever $y \in E^+$ and x are such that $nx \leq y$ for all $n \in \mathbb{N}$ (or such that $rx \leq y$ for all $r \in \mathbb{R}^+$), it follows that $x \leq 0$. As was observed in [**27**, Lemma 1.1], every σ-monotone complete partially vector space (and then also every monotone complete, σ-Dedekind complete, or Dedekind complete partially order vector space) is Archimedean. Indeed, if $x \geq 0$, then $\bigwedge \{ x/n : n \in \mathbb{N} \}$ exists since E is σ-Dedekind complete, and it satisfies $\bigwedge \{ x/n : n \in \mathbb{N} \} = \bigwedge \{ x/(2n) : n \in \mathbb{N} \} =$

$1/2 \bigwedge \{x/n : n \in \mathbb{N}\}$. Hence $\bigwedge \{x/n : n \in \mathbb{N}\} = 0$. We shall use this automatic Archimedean property a few times.

We record the following analogue of the well-known inequality for the limit inferior and the limit superior of a sequence of real numbers. We shall need it in the proof of the dominated convergence theorem (Theorem 9.13). The proof is completely analogous to the proof for the case where $E = \mathbb{R}$.

Lemma 2.2. *Let E be a σ-Dedekind complete partially ordered vector space, and let $\{x_n\}_{n=1}^{\infty}$ be an order bounded sequence in E. Then $\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k$ and $\bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k$ exist in E, and*

$$\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k \leq \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k.$$

In the sequel, we shall consider Riesz representation theorems for positive linear maps $T : \mathrm{C_c}(X) \to E$, where X is a locally compact Hausdorff space and E is a partially ordered vector space. Once could hope that, ideally, these theorems would state that this map T is given by integration of a scalar-valued function with respect to an E-valued measure. However, as the case where $E = \mathbb{R}$ already shows, we cannot expect this measure to be actually finite. In order to be able to develop an appropriate theory of measure and integration that incorporates this inevitable phenomenon, we need to adjoin an element ∞ to E^+ and let $\mathbb{R}^+$ act on the augmented structure. As is also done in e.g. [**24**, **26**], we shall actually adjoin ∞ to the whole space E, which is necessary for a formulation of some of the results (see e.g. part (4) of Lemma 5.4). The construction is as follows.

Firstly, we let $\overline{E} := E \cup \{\infty\}$ be a disjoint union, and we extend the partial ordering from E to $\overline{E}$ by declaring that $x \leq \infty$ for all $x \in \overline{E}$. The elements of $\overline{E}$ that are in E will be called *finite*. We set $\overline{E^+} := E^+ \cup \{\infty\}$; then $\overline{E^+}$ is the set of positive elements of $\overline{E}$. Secondly, we make $\overline{E}$ into an abelian additive semigroup with zero element by defining $\infty + x := \infty$ and $x + \infty := \infty$ for all $x \in \overline{E}$; then $\overline{E^+}$ is a sub-semigroup of $\overline{E}$. Thirdly, we define $r \cdot \infty := \infty$ for all $r \in \mathbb{R}^+ \setminus \{0\}$, and define $0 \cdot \infty := 0$. Thus the additive semigroup $\mathbb{R}^+$ acts as semigroup homomorphisms on $\overline{E}$. It is easily checked that, if $x, y \in \overline{E}$ are such that $x \leq y$, then $x + z \leq y + z$ for all $z \in \overline{E}$, and that $rx \leq sy$ for all $r, s \in \mathbb{R}^+$ such that $r \leq s$. If $x \in \overline{E}$ and $r, s \in \mathbb{R}^+$, then $(rs)x = r(sx)$. Hence the multiplicative semigroup $\mathbb{R}^+$ also acts as semigroup homomorphisms on $\overline{E^+}$.

We have now carried out the desired construction, but one can also go a little bit further, as follows. The construction in the previous paragraph can be applied to $\mathbb{R}$. The set of positive elements of $\overline{\mathbb{R}}$ is then the familiar extended positive real half line $\overline{\mathbb{R}^+}$. It is an abelian additive semigroup with zero element. The action of $\mathbb{R}^+$ on $\overline{E^+}$ can then be extended to an action of $\overline{\mathbb{R}^+}$ on $\overline{E^+}$ by defining $\infty \cdot 0 := 0$ and $\infty \cdot x := \infty$ for all $x \in \overline{E^+} \setminus \{0\}$. Then the additive semigroup $\overline{\mathbb{R}^+}$ acts as semigroup homomorphisms on $\overline{E^+}$. If $x, y \in \overline{E^+}$ are such that $x \leq y$ and $r, s \in \overline{\mathbb{R}^+}$ are such that $r \leq s$, then $rx \leq sy$.

The previous construction be applied with $E = \mathbb{R}$, which yields an action of $\overline{\mathbb{R}^+}$ on itself. Thus the familiar multiplicative structure on the extended positive real half line is obtained. It is compatible with the action of $\overline{\mathbb{R}^+}$ on $\overline{E^+}$: if $r, s \in \overline{\mathbb{R}^+}$ and $x \in \overline{E^+}$, then $r \cdot (s \cdot x) = (rs) \cdot x$. Hence the multiplicative semigroup $\overline{\mathbb{R}^+}$ also acts as semigroup homomorphisms on $\overline{E^+}$.

We shall employ the usual notation in which $a_\lambda \uparrow$ means that $\{a_\lambda\}_{\lambda \in \Lambda}$ is an increasing net in E (or in $\overline{E}$), and in which $a_\lambda \uparrow x$ means that $\{a_\lambda\}_{\lambda \in \Lambda}$ is an increasing net in E (or in $\overline{E}$) with supremum x in E (or in $\overline{E}$). The notations $a_\lambda \downarrow$ and $a_\lambda \downarrow x$ are similarly defined. If necessary we shall indicate explicitly whether we are working in E or in $\overline{E}$.

We collect a few basic technical facts that will be used repeatedly in the sequel. The proofs are elementary.

Lemma 2.3. *Let E be a partially ordered vector space, and let $S \subseteq \overline{E}$ be non-empty.*

(1) *If $S \subseteq E$ and $\bigvee \{ s : s \in S \}$ exists in E, then this supremum is also the supremum of S in $\overline{E}$; likewise for the infimum.*
(2) *If $\bigvee \{ s : s \in S \}$ exists in $\overline{E}$ and is finite, then S consists of finite elements, and the supremum of S exists in E and equals the supremum of S in $\overline{E}$.*
(3) *If $S \neq \{\infty\}$, then $\bigwedge \{ s : s \in S \}$ exists in $\overline{E}$ if and only if $\bigwedge \{ s : s \in S \cap E \}$ exists in E. If this is the case, then these infima are equal.*
(4) *$\bigvee \{ s : s \in S \} = \infty$ in $\overline{E}$ if and only if S is not bounded from above by a finite element.*
(5) *If $\bigvee \{ s : s \in S \}$ exists in $\overline{E}$, then, for all $x \in \overline{E}$, $\bigvee \{ x + s : s \in S \}$ exists in $\overline{E}$ and equals $x + \bigvee \{ s : s \in S \}$; likewise for the infimum.*
(6) *If $\bigvee \{ x + s : s \in S \}$ exists in $\overline{E}$ for some $x \in E$, then $\bigvee \{ s : s \in S \}$ exists in $\overline{E}$, and $\bigvee \{ x + s : s \in S \} = x + \bigvee \{ s : s \in S \}$; likewise for the infimum.*
(7) *If $\bigvee \{ s : s \in S \}$ exists in $\overline{E}$, then, for all $r \in \mathbb{R}^+$, $\bigvee \{ rs : s \in S \}$ exists in $\overline{E}$ and equals $r \bigvee \{ s : s \in S \}$; likewise for the infimum.*

Lemma 2.4. *Let E be a partially ordered vector space.*

(1) *If A and B are two non-empty subsets of $\overline{E}$ such that $\bigvee \{ a : a \in A \}$ and $\bigvee \{ b : b \in B \}$ exist in $\overline{E}$, then $\bigvee \{ a + b : a \in A, b \in B \}$ exists in $\overline{E}$ and equals $\bigvee \{ a : a \in A \} + \bigvee \{ b : b \in B \}$; likewise for the infima.*
(2) *If $\{a_\lambda\}_{\lambda \in \Lambda}$ and $\{b_\lambda\}_{\lambda \in \Lambda} \subseteq \overline{E}$ are two nets in $\overline{E}$ such that $\{a_\lambda\}_{\lambda \in \Lambda} \uparrow a$ and $\{b_\lambda\}_{\lambda \in \Lambda} \uparrow b$ in $\overline{E}$, then $\{a_\lambda + b_\lambda\} \uparrow (a + b)$ in $\overline{E}$; likewise for decreasing nets.*

Lemma 2.5. *Let E be a partially ordered vector space.*

(1) *If E is σ-monotone complete (resp. σ-Dedekind complete), then every increasing sequence in (resp. every at most countably infinite subset of) $\overline{E}$ has a supremum in $\overline{E}$. If the sequence (resp. set) is bounded from above by a finite element, then the supremum in $\overline{E}$ equals the supremum in E. If the sequence (resp. subset) is not bounded from above by a finite element, then the supremum in $\overline{E}$ equals ∞.*
(2) *If E is σ-monotone complete (resp. σ-Dedekind complete), then every decreasing sequence in (resp. every non-empty at most countably infinite subset of) $\overline{E}$ that is bounded from below in $\overline{E}$ has an infimum in $\overline{E}$. If all terms of the sequence are equal to ∞ (resp. if the set equals $\{\infty\}$), then this infimum in $\overline{E}$ equals ∞. If the sequence contains finite terms (resp. if the subset contains finite elements), then the infimum in $\overline{E}$ equals the infimum in E of the decreasing subsequence of finite terms (resp. the subset of finite elements of the subset), which is bounded from below by a finite element.*
(3) *If E is monotone complete (resp. Dedekind complete), then every increasing net in (resp. every non-empty subset of) $\overline{E}$ has a supremum in $\overline{E}$. If the set (resp. net) is bounded from above by a finite element, then the supremum in $\overline{E}$ equals the supremum in E. If the set (resp. net) is not bounded from above by a finite element, then the supremum in $\overline{E}$ equals ∞.*
(4) *If E is monotone complete (resp. Dedekind complete), then every decreasing net in (resp. every non-empty subset of) $\overline{E}$ that is bounded from below in $\overline{E}$ has an infimum in $\overline{E}$. If all terms of the net are equal to ∞ (resp. if the subset equals $\{\infty\}$), then this infimum in $\overline{E}$ equals ∞. If the net contains finite terms (resp. if the subset contains*

finite elements), then the infimum in $\overline{E}$ equals the infimum in E of the decreasing subnet of finite terms (resp. the subset of finite elements of the subset), which is bounded from below by a finite element.

§3. Operators and duality

It will become apparent in the sequel that the partially ordered vector spaces E for which Riesz representation theorems for positive linear maps $T : \mathrm{C}_0(X) \to E$ can be established, are notably the monotone complete partially ordered vector spaces that are also normal. The latter property will be defined below (see Definition 3.6); Proposition 3.8 shows its relevance for us.

This section is, therefore, primarily concerned with establishing that several common classes of partially ordered vector spaces (including partially vector spaces of operators) are normal and monotone complete. The pertinent results are contained in Propositions 3.10 to 3.12 and Theorem 3.13.

We start by recalling some of the usual notions and introducing the notations that we shall use.

If E and F are vector spaces, then $\mathrm{L}(E,F)$ denotes the vector space of linear operators from E into F. An operator $T \in \mathrm{L}(E,F)$ between two partially ordered vector spaces is *order bounded* if it maps order bounded subsets of E into order bounded subsets of F. The order bounded operators from E into F form a vector space that is denoted by $\mathrm{L_b}(E,F)$. An operator $T \in \mathrm{L}(E,F)$ is *positive* if $TE^+ \subseteq F^+$, and *regular* if it is the difference of two positive operators. The regular operators from E into F form a vector space that is denoted by $\mathrm{L_r}(E,F)$. A positive operator is order bounded, so that $\mathrm{L_r}(E,F) \subseteq \mathrm{L_b}(E,F) \subseteq \mathrm{L}(E,F)$. If E^+ is generating in E, then these three vector spaces are all partially ordered via their positive cones $\mathrm{L_r}(E,F)^+$. We shall write $E^\sim$ for $\mathrm{L_b}(E,\mathbb{R})$. If E is a Banach lattice, then $E^\sim$ coincides with the norm dual E^* of E.

Completeness properties of partially ordered vector spaces can be inherited by spaces of operators between them. For example, if E is a partially ordered vector space that has a generating cone and that has the Riesz decomposition property, and if F is a Dedekind complete Riesz space, then the spaces $\mathrm{L_r}(E,F)$ and $\mathrm{L_b}(E,F)$ coincide and are Dedekind complete Riesz spaces; see [**4**, Theorem 1.59]. In particular, they are monotone complete partially ordered vector spaces. The latter statement is also a consequence of the following result.

Proposition 3.1. *Let E be a partially ordered vector space with a generating positive cone, let F be a monotone complete (resp. σ-monotone complete) partially ordered vector space, and let V be a linear subspace of $\mathrm{L}(E,F)$ containing $\mathrm{L_r}(E,F)$.*

Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be an increasing net (resp. Let $\{T_n\}_{n=1}^\infty$ be an increasing sequence) in V. Then $\{T_\lambda\}_{\lambda\in\Lambda}$ (resp. $\{T_n\}_{n=1}^\infty$) is bounded from above in V if and only if, for all $x \in E^+$, $\{T_\lambda x\}_{\lambda\in\Lambda}$ (resp. $\{T_n x\}_{n=1}^\infty$) is bounded from above in F. In that case, $\{T_\lambda\}_{\lambda\in\Lambda}$ (resp. $\{T_n\}_{n=1}^\infty$) has a supremum T in V. For $x \in E^+$, it is given by $Tx = \bigvee_{\lambda\in\Lambda} T_\lambda x$ (resp. $Tx = \sup_{n\geq 1} T_n x$).

In particular, V is a monotone complete (resp. σ-monotone complete) partially ordered vector space.

Proof. We prove the statements for the monotone completeness and the σ-monotone completeness of V at the same time. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be an increasing net (possibly an increasing sequence) in V.

If the net is bounded from above by an element S of V, then $T_\lambda x \leq Sx$ for all $x \in E^+$, so that $\{T_\lambda x\}_{\lambda\in\Lambda}$ is bounded from above in F for all $x \in E^+$.

Conversely, suppose that $\{T_\lambda x\}_{\lambda\in\Lambda}$ is bounded from above in F for all $x \in E^+$. We shall show that the pointwise formula for T as in the statement of the theorem actually defines an element of V. This is then clearly the least upper bound of the net in V.

Choose any $\lambda_0 \in \Lambda$. Using that the nets $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{T_\lambda x\}_{\lambda\in\Lambda}$ for $x \in E^+$ are increasing, one easily sees, by considering the net $\{T_\lambda - T_{\lambda_0}\}_{\lambda\geqslant\lambda_0} \subseteq V^+$, that it is sufficient to prove that the pointwise formula defines an element of V when $T_\lambda \geq 0$ for all $\lambda \in \Lambda$.

Supposing, therefore, that $\{T_\lambda\}_{\lambda\in\Lambda} \subseteq V^+$, we define $T : E^+ \to F^+$ as in the statement of the theorem by

$$Tx := \bigvee_{\lambda\in\Lambda} T_\lambda x \tag{3.1}$$

for $x \in E^+$. Since $\{T_\lambda x\}_{\lambda\in\Lambda}$ is increasing and bounded from above, the supremum in the right hand side of equation (3.1) exists as a consequence of the pertinent completeness property of F. It is clear that $T(\lambda x) = \lambda T(x)$ for all $\lambda \geq 0$ and $x \in E^+$. Next we show that T is additive on E^+. Fix $x_1, x_2 \in E^+$. For all $\lambda \in \Lambda$, we have

$$T_\lambda(x_1 + x_2) = T_\lambda(x_1) + T_\lambda(x_2) \leq Tx_1 + Tx_2,$$

so $T(x_1 + x_2) \leq Tx_1 + Tx_2$. For the reverse inequality, consider arbitrary $\lambda_1, \lambda_2 \in \Lambda$. Choose $\lambda_3 \in \Lambda$ such that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$. Then, using that $\{T_\lambda\}_{\lambda\in\Lambda}$ is increasing, we have

$$T_{\lambda_1}x_1 + T_{\lambda_2}x_2 \leq T_{\lambda_3}x_1 + T_{\lambda_3}x_2 = T_{\lambda_3}(x_1 + x_2) \leq T(x_1 + x_2),$$

which easily implies that $T(x_1 + x_2) \geq Tx_1 + Tx_2$. Hence T is additive on E^+.

Next, if $x \in E$ is arbitrary, we choose $x_1, x_2 \in E^+$ such that $x = x_1 - x_2$, and we define $Tx := Tx_1 - Tx_2$. It is then easy to see that T is well-defined and that T is linear, so that $T \in \mathrm{L}(E,F)$. Since clearly $T \geq 0$, we also have $T \in \mathrm{L_r}(E,F)$. Since $\mathrm{L_r}(E,F) \subseteq V$, we have $T \in V$, as required.

□

The underlying spaces of the monotone (σ-)partially ordered vector spaces of operators in Proposition 3.1 are themselves partially ordered vector spaces, but there also exist monotone complete partially ordered vector spaces of operators where this is no longer the case. An important class of examples is provided by the following result, which is a direct consequence of [**11**, Lemma I.6.4].

Proposition 3.2. *Let $\mathscr{H}$ be a complex Hilbert space, and let* L *be a weakly closed (equivalently: strongly closed) complex linear subspace of $\mathscr{B}(\mathscr{H})$. Let $\mathrm{L_{sa}}$ be the real vector space that consists of the self-adjoint elements of* L*, supplied with the partial ordering that is inherited from the usual partial ordering on the self-adjoint elements of $\mathscr{B}(\mathscr{H})$. Then $\mathrm{L_{sa}}$ is a monotone complete partially ordered vector space.*

More precisely, if $\{T_\lambda\}_{\lambda\in\Lambda}$ is an increasing net in $\mathrm{L_{sa}}$ that is bounded from above in $\mathrm{L_{sa}}$, then $\{T_\lambda\}_{\lambda\in\Lambda}$ convergence with respect to the strong operator topology, i.e. $\mathrm{SOT}{-}\lim_\lambda T_\lambda$ *exists in $\mathscr{B}(\mathscr{H})$, is an element of $\mathrm{L_{sa}}$, and is equal to the supremum $\bigvee\{\, T_\lambda : \lambda \in \Lambda \,\}$ of the T_λ in $\mathrm{L_{sa}}$.*

Remark 3.3. In view of Kadison's anti-lattice theorem [**15**, Theorem 6], the space L_{sa} figuring in Proposition 3.2 will not generally be a Riesz space. For example, if $\dim \mathscr{H} \geq 2$, then, by Kadison's result, $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$ is not a Riesz space. The space $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$ for $\dim \mathscr{H} \geq 2$ also provides an example of a monotone complete partially ordered vector space that is not Dedekind complete. It is, in fact, not even σ-Dedekind complete. To see this, suppose that it is σ-Dedekind complete. Since $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$ is directed, every subset $\{T_1, T_2\}$ is bounded from above, so that it then has a supremum. Thus $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$ is a Riesz space, but we know this not to be the case.

Although many of the proofs in the sequel can be given entirely while working in the partially ordered vector space at hand, there are a few passages where it seems unavoidable to invoke the real numbers and their linear ordering as an aid; see e.g. the proof of Proposition 7.6. On other occasions (see e.g. the proof of Theorem 13.4), a proof using the real numbers can be given that runs so smoothly that the need to look for a possible direct proof is hardly felt anymore. The real numbers enter the picture by using certain linear functionals, which we are about to introduce. The relevant definition, where we also include the sequential case for future reference, can still be given in a more general context, as follows.

Definition 3.4. Let E and F be partially ordered vector spaces, and let $T : E \to F$ be a positive operator. Then T is called *order continuous* (resp. *σ-order continuous*) if $Tx_\lambda \downarrow 0$ in F whenever $x_\lambda \downarrow 0$ in E (resp. if $Tx_n \downarrow 0$ in F whenever $x_n \downarrow 0$ in E). A general operator in $\mathrm{L_r}(E,F)$ is order continuous (resp. σ-order continuous) if it is the difference of two positive order continuous operators. We shall write $\mathrm{L_n}(E,F)$ (resp. $\mathrm{L_c}(E,F)$) for the order continuous (resp. σ-order continuous) operators from E into F.

It is easy to see that the sum of two positive order continuous (resp. σ-order continuous) operators is again a positive order continuous (resp. positive σ-order continuous) operator, and it follows that $\mathrm{L_n}(E,F)$ (resp. $\mathrm{L_c}(E,F)$) is a linear subspace of $\mathrm{L_r}(E,F)$. If E is directed, then it is a partially ordered vector space with the positive order continuous operators (resp. the positive σ-order continuous operators) as positive cone, which is generating by definition. We shall write $E_{\mathrm{n}}^{\sim}$ for $\mathrm{L_n}(E,\mathbb{R})$ and $E_{\mathrm{c}}^{\sim}$ for $\mathrm{L_c}(E,\mathbb{R})$. It is the space $E_{\mathrm{n}}^{\sim}$ that will be our aid in the context of vector-valued Riesz representation theorems. It has $(E_{\mathrm{n}}^{\sim})^+$, the positive order continuous functionals, as generating positive cone.

Remark 3.5. If E and F are Riesz spaces, where F is Dedekind complete, then the above notion of order continuous operators agrees with the usual one in the literature. To see this, recall that $T : E \to F$ is order continuous in the sense of [**28**, p. 123] if $|Tx_\lambda| \downarrow 0$ in F whenever $x_\lambda \downarrow 0$ in E. Hence a positive T is order continuous in the sense of [**28**, p. 123] precisely when it is order continuous in the sense of our Definition 3.4. Furthermore, by [**28**, Lemma 84.1], $T : E \to F$ is order continuous in the sense of [**28**, p. 123] if and only if T^+ and T^- are order continuous in the sense of that same definition, i.e. if and only if they are positive order continuous operators in the sense of Definition 3.4. In addition, [**28**, Theorem 84.2] implies that the set of all $T : E \to F$ that are order continuous in the sense of [**28**, p. 123] form a vector space. It follows that the notions of (and notations for) order continuous operators coincide if E and F are Riesz spaces, where F is Dedekind complete. A similar argument shows that this is also the case for σ-order continuous operators.

The relation between the general case and the case when $E = \mathbb{R}$ is most profitably exploited if there are sufficiently many order continuous linear functionals in the sense of the first part of the following definition.

Definition 3.6. Let E be a partially ordered vector space. Then E is called *normal* if, for $x \in E$, $(x, x') \geq 0$ for all $x' \in (E_{\mathrm{n}}^{\sim})^+$ if and only if $x \in E^+$. We say that E is *σ-normal* if, for $x \in E$, $(x, x') \geq 0$ for all $x' \in (E_{\mathrm{c}}^{\sim})^+$ if and only if $x \in E^+$.

Clearly, if E is normal, then $(E_{\mathrm{n}}^{\sim})^+$ separates the points of E.

As with order continuous operators, if E is a Riesz space, then our notion of normality coincides with that in the literature. To see this, we include the following result.

Lemma 3.7. *Let E be a Riesz space, and let A be an order ideal of $E^{\sim}$. Then the following are equivalent:*

(1) *A^+ separates the points of E;*
(2) *A separates the points of E;*
(3) *For $x \in E$, $(x, x') \geq 0$ for all $x' \in A^+$ if and only if $x \in E^+$.*

If $A = E^\sim$, then the fact that part (2) implies part (3) can be found as [**3**, Theorem 1.66]). The following proof is an adaptation of the proof for that case.

PROOF. It is clear that part (1) implies part (2), and also that part (3) implies part (1). It remains to be shown that part (2) implies part (3).

One implication in (3) is trivial, so we turn to the non-trivial one. Let $x \in E$ be such that $(x, x') \geq 0$ for all $x' \in A^+$. If $x' \in A^+$, then [**3**, Theorem 1.23] shows that there exists $y' \in E^\sim$ such that $0 \leq y' \leq x'$ and

$$(x^-, x') = -(x, y'),$$

Since A is an order ideal, we have $y' \in A^+$. Hence $(x, y') \geq 0$ by assumption, and we see that $(x^-, x') \leq 0$. On the other hand, it is clear that $(x^-, x') \geq 0$. We conclude that $(x^-, x') = 0$ for all $x' \in A^+$, so that $(x^-, x') = 0$ for all $x' \in A$. Then $x^- = 0$ by the separation property of A, and therefore $x \in E^+$. □

As a consequence, a Riesz space is normal in the sense of our Definition 3.6 if and only if $E_{\mathrm{n}}^\sim$ separates the points of E, i.e. if and only if it is normal in the sense of [**1**, p. 21]. For Riesz spaces, therefore, our notion of normality coincides with the one in the literature.

The importance of normality for our work lies in the following observation.

Proposition 3.8. *Let E be a normal partially ordered vector space. Suppose that $\{x_\lambda\}_{\lambda \in \Lambda}$ is a net in E, and that $x \in E$.*

(1) *If $x_\lambda \downarrow$, then $x_\lambda \downarrow x$ if and only if $(x, x') = \inf_{\lambda \in \Lambda}(x_\lambda, x')$ for all $x' \in (E_{\mathrm{n}}^\sim)^+$.*
(2) *If $x_\lambda \uparrow$, then $x_\lambda \uparrow x$ if and only if $(x, x') = \sup_{\lambda \in \Lambda}(x_\lambda, x')$ for all $x' \in (E_{\mathrm{n}}^\sim)^+$.*

PROOF. We prove part (1) where $x_\lambda \downarrow$; part (2) follows from this.

If $x_\lambda \downarrow x$, so that $x_\lambda - x \downarrow 0$, and if $x' \in (E_{\mathrm{n}}^\sim)^+$, then, by definition, $(x_\lambda - x, x') \downarrow 0$. Hence $(x_\lambda, x') \downarrow (x, x')$, so that, in particular, $(x, x') = \inf_{\lambda \in \Lambda}(x_\lambda, x')$.

Conversely, suppose that $(x, x') = \inf_{\lambda \in \Lambda}(x_\lambda, x')$ for all $x' \in (E_{\mathrm{n}}^\sim)^+$.

With $\lambda \in \Lambda$ fixed, we then have $(x, x') \leq (x_\lambda, x')$, or $(x - x_\lambda, x') \leq 0$, for all $x' \in (E_{\mathrm{n}}^\sim)^+$. Since E is normal, we conclude that $x \leq x_\lambda$. Hence x is a lower bound of $\{ x_\lambda : \lambda \in \Lambda \}$.

If $\tilde{x}$ is a lower bound of $\{ x_\lambda : \lambda \in \Lambda \}$, and if $x' \in (E_{\mathrm{n}}^\sim)^+$ is fixed, then certainly $(\tilde{x}, x') \leq (x_\lambda, x')$ for all $\lambda \in \Lambda$. Hence $(\tilde{x}, x') \leq \inf_{\lambda \in \Lambda}(x_\lambda, x')$. Since the right hand side of this inequality equals (x, x') by assumption, we have $(\tilde{x}, x') \leq (x, x')$ for all $x' \in (E_{\mathrm{n}}^\sim)^+$. Since E is normal, we see that $\tilde{x} \leq x$.

We conclude that $x = \bigwedge \{ x_\lambda : \lambda \in \Lambda \}$. Hence $x_\lambda \downarrow x$. □

A similar proof establishes the following.

Proposition 3.9. *Let E be a σ-normal partially ordered vector space. Suppose that $\{x_n\}_{n=1}^\infty$ is a sequence in E, and that $x \in E$.*

(1) *If $x_n \downarrow$, then $x_n \downarrow x$ if and only if $(x, x') = \inf_{n \geq 1}(x_n, x')$ for all $x' \in (E_{\mathrm{c}}^\sim)^+$.*
(2) *If $x_n \uparrow$, then $x_n \uparrow x$ if and only if $(x, x') = \sup_{n \geq 1}(x_n, x')$ for all $x' \in (E_{\mathrm{c}}^\sim)^+$.*

As mentioned in the introduction of this section, the partially ordered vector spaces that are both normal and monotone complete are a convenient context for a vector-valued Riesz representation theorems. Therefore, we include a number of examples.

Proposition 3.10. *A Banach lattice with order continuous norm is a normal and monotone complete partially ordered vector space.*

Proof. The space E is Dedekind complete since the norm is order continuous (see [**3**, Corollary 4.10]). Hence E is certainly monotone complete. It follows easily from the fact that $E^{\sim} = E^*$ and the order continuity of the norm that $E^{\sim} = E_{\mathrm{n}}^{\sim}$. Hence $E_{\mathrm{n}}^{\sim} = E^*$. Since E^* separates the points of E, Lemma 3.7 shows that E is normal. □

Proposition 3.11. *Let E be a partially ordered vector space with a generating positive cone, let F be a normal and monotone complete partially ordered vector space, and let V be a linear subspace of $\mathrm{L}(E,F)$ that contains $\mathrm{L_r}(E,F)$.*

Then V is a normal and monotone complete partially ordered vector space.

Proof. Proposition 3.1 shows that V is monotone complete. In order to prove that it is normal, we define, for $x \in E^+$ and $x' \in (F_{\mathrm{n}}^{\sim})^+$, the linear functional $\varphi_{x,x'} : V \to \mathbb{R}$ by $(T, \varphi_{x,x'}) = (Tx, x')$ for $T \in V$. Then $\varphi_{x,x'}$ is evidently positive. We claim that it is order continuous. To see this, fix $x \in E$ and $x' \in (F_{\mathrm{n}}^{\sim})^+$, and suppose that $T_\lambda \downarrow 0$ in V. By Proposition 3.1, this implies that $T_\lambda x \downarrow 0$. Since x' is order continuous, it then follows that $(T_\lambda x, x') \downarrow 0$. Hence $\varphi_{x,x'}$ is order continuous, and we conclude that $\varphi_{x,x'} \in (V_{\mathrm{n}}^{\sim})^+$. Finally, suppose that $T \in V$ is such that $(T, \varphi_{x,x'}) \geq 0$ for all $x \in E^+$ and $x' \in (F_{\mathrm{n}}^{\sim})^+$. Since F is normal, this implies that $Tx \geq 0$ for all $x \in E^+$. Hence $T \geq 0$. This proves that V is normal. □

The following example is a continuation of Proposition 3.2.

Proposition 3.12. *Let $\mathscr{H}$ be a complex Hilbert space, and let L be a weakly closed complex linear subspace of $\mathscr{B}(\mathscr{H})$. Let L_{sa} be the real vector space that consists of the self-adjoint elements of L, supplied with the partial ordering that is inherited from the usual partial ordering on $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$. Then L_{sa} is a normal and monotone complete partially ordered vector space.*

Proof. In view of Proposition 3.2, we need to prove only that L_{sa} is normal. To this end, we define, for $x \in \mathscr{H}$, the linear functional $\varphi_x : L_{\mathrm{sa}} \to \mathbb{R}$ by $(T, \varphi_x) = \langle Tx, x\rangle$ for $T \in L_{\mathrm{sa}}$; here $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathscr{H}$. Then φ_x is clearly positive. We claim that φ_x is order continuous. To see this, suppose that $T_\lambda \downarrow 0$. By Proposition 3.2, this implies that $\mathrm{SOT-}\lim_\lambda T_\lambda = 0$. In particular, we have $(T_\lambda, \varphi_x) = \langle T_\lambda x, x\rangle \downarrow 0$. Hence φ_x is order continuous, and we conclude that $\varphi_x \in (V_{\mathrm{n}}^{\sim})^+$. Finally, suppose that $T \in L_{\mathrm{sa}}$ is such that $(T, \varphi_x) \geq 0$ for all $x \in \mathscr{H}$. Hence $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$, so that $T \geq 0$. This proves that L_{sa} is normal. □

Propositions 3.10 to 3.12 can be combined in various ways. The following immediate result seems worth recording explicitly.

Theorem 3.13. *Let E be a partially ordered vector space with a generating positive cone.*

If F is a Banach lattice with order continuous norm, and if V is a linear subspace of $\mathrm{L}(E,F)$ that contains $\mathrm{L_r}(E,F)$, then V is a normal and monotone complete partially ordered vector space. In particular, $\mathrm{L_r}(E)$ is a normal and Dedekind complete Riesz space for every Banach lattice E with order continuous norm.

If L is a weakly closed subspace of the linear operators on a complex Hilbert space, with self-adjoint part L_{sa}, and if V is a linear subspace of $\mathrm{L}(E, L_{\mathrm{sa}})$ that contains $\mathrm{L_r}(E, L_{\mathrm{sa}})$, then V is a normal and monotone complete partially ordered vector space. In particular, $\mathrm{L_r}(L_{\mathrm{sa}})$ is a normal and monotone complete partially ordered vector space.

§4. Partially ordered algebras

In this section, we establish some basic facts about partially ordered algebras. They will form the context for an abstract existence theorem for spectral measures representing positive homomorphisms from $\mathrm{C_c}(X)$ or $\mathrm{C_0}(X)$ into such algebras (see Theorem 15.1), that can later be specialised to (positive) representations on KB-spaces or Hilbert spaces (see Theorems 15.2 and 15.3).

Let $\mathscr{A}$ be an associative algebra that is also a partially ordered vector space. Then $\mathscr{A}$ is an *partially ordered algebra* if $ab \in \mathscr{A}^+$ for all $a, b \in \mathscr{A}^+$. We do not suppose that $\mathscr{A}$ has an identity element or, if so, that the identity element is positive.

If $\mathscr{A}$ is a partially ordered algebra, then we shall say that the multiplication in $\mathscr{A}$ is *σ-monotone continuous* if, whenever $\{a_n\}_{n=1}^{\infty} \subseteq \mathscr{A}^+$ and $a \in \mathscr{A}^+$ are such that $a_n \uparrow a$, then $ba_n \uparrow ba$ and $a_n b \uparrow ab$ for all $b \in \mathscr{A}^+$. The multiplication is *monotone continuous* if, whenever $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq \mathscr{A}^+$ and $a \in \mathscr{A}^+$ are such that $a_\lambda \uparrow a$, then $ba_\lambda \uparrow ba$ and $a_\lambda b \uparrow ab$ for all $b \in \mathscr{A}^+$.

Remark 4.1. Suppose $\mathscr{A}$ is a Riesz algebra. Then the multiplication in $\mathscr{A}$ is σ-monotone continuous (resp. monotone continuous) in our sense if and only if, for all $b \in \mathscr{A}$, the multiplication operators $a \mapsto ba$ and $a \mapsto ba$ are σ-order continuous (resp. order continuous) operators on $\mathscr{A}$ in the sense of [**28**, p. 123]. This is an easy consequence of the fact that $\mathscr{A}^+$ generates $\mathscr{A}$ and the observation that linear combinations of positive operators that are order continuous in the sense of [**28**, p. 123] are again order continuous in the same sense.

The following easy observation is convenient.

Lemma 4.2. *Let $\mathscr{A}$ be a partially ordered algebra such that multiplication is monotone continuous.*

If $\{a_{\lambda_1}\}_{\lambda_1\in\Lambda_1}, \{b_{\lambda_2}\}_{\lambda_2\in\Lambda_2} \subset \mathscr{A}^+$, and $a, b \in \mathscr{A}^+$ are such that $a_{\lambda_1} \uparrow a$ and $b_{\lambda_2} \uparrow b$, then $a_{\lambda_1} b_{\lambda_2} \uparrow ab$.

If $\{a_\lambda\}_{\lambda\in\Lambda}, \{b_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{A}^+$, and $a, b \in \mathscr{A}^+$ are such that $a_\lambda \uparrow a$ and $b_\lambda \uparrow b$, then $a_\lambda b_\lambda \uparrow ab$.

If the multiplication is σ-monotone continuous the analogous statements for the term-wise product of two sequences are true.

Proof. We cover the general and the sequential case both statements at the same time.

Let $\{a_{\lambda_1}\}_{\lambda_1\in\Lambda_1}$ and $\{b_{\lambda_2}\}_{\lambda_2\in\Lambda_2}$ be increasing nets (possibly increasing sequences) in $\mathscr{A}^+$, and let $a, b \in \mathscr{A}^+$ be such that $a_{\lambda_1} \uparrow a$ and $b_{\lambda_2} \uparrow b$. Certainly $a_{\lambda_1} b_{\lambda_2} \leq ab$ for all $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Suppose that $u \in E$ is such that $a_{\lambda_1} b_{\lambda_2} \leq u$ for all $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Then the (σ-)monotone order continuity of the left multiplication in $\mathscr{A}$ implies that $a_{\lambda_1} b \leq u$ for all $\lambda_1 \in \Lambda_1$, and the (σ-)monotone continuity of the right multiplication then shows that $ab \leq u$. Hence $a_{\lambda_1} b_{\lambda_2} \uparrow ab$.

Let $\{a_\lambda\}_{\lambda\in\Lambda}, \{b_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{A}^+$ be increasing nets (possibly sequences) in $\mathscr{A}^+$, and let a, $b \in \mathscr{A}^+$ be such that $a_\lambda \uparrow a$ and $b_\lambda \uparrow b$. Certainly $a_\lambda b_\lambda \leq ab$ for all $\lambda \in \Lambda$. Suppose that $u \geq a_\lambda b_\lambda$ for all $\lambda \in \Lambda$. If $\lambda_1, \lambda_2 \in \Lambda$, then there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. This implies that $a_{\lambda_1} b_{\lambda_2} \leq a_{\lambda_3} a_{\lambda_3} \leq u$. As in the first part of the proof, the (σ-)monotone order continuity of the multiplication in $\mathscr{A}$ then yields that $u \geq ab$. Hence $a_\lambda b_\lambda \uparrow ab$. ☐

We include two important classes of partially ordered algebras with monotone continuous multiplication.

Proposition 4.3. *Let E be a Banach lattice with order continuous norm. Then every regular operator on E is order continuous, and $\mathrm{L_r}(E)$ is a normal and Dedekind complete Riesz algebra with monotone continuous multiplication.*

PROOF. To see that every regular operator on E is order continuous, it suffices to show this for a positive operator T on E. If $0 \leq x_\lambda \uparrow x$, then $x_\lambda \to x$ in norm since the norm is order continuous. Then $Tx_\lambda \to Tx$ in norm. As is well known, the fact that $Tx_\lambda \uparrow$ then implies that $Tx_\lambda \uparrow Tx$. Hence T is order continuous.

Suppose that $T \in \mathrm{L_r}(E)^+$ and that $0 \leq S_\lambda \uparrow S$ in $\mathrm{L_r}(E)$. If $x \in E^+$, then $0 \leq S_\lambda x \uparrow Sx$, and the order continuity of T implies that $TS_\lambda x \uparrow TS\mathrm{x}$. Hence $TS_\lambda \uparrow TS$ in $\mathrm{L_r}(E)$, so that the left multiplication in $\mathrm{L_r}(E)$ is monotone order continuous. Furthermore, since $Tx \geq 0$ for $x \geq 0$, we have $S_\lambda Tx = S_\lambda(Tx) \uparrow S(Tx) = STx$ for $x \geq 0$. Hence $S_\lambda T \uparrow ST$ in $\mathrm{L_r}(E)$, so that the right multiplication in $\mathrm{L_r}(E)$ is monotone order continuous.

We know from Theorem 3.13 that $\mathrm{L_r}(E)$ is a Dedekind complete Riesz space. □

Proposition 4.4. *Let $\mathscr{H}$ be a complex Hilbert space, and let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ be a commutative von Neumann algebra. Let $\mathscr{A}_{\mathrm{sa}}$ be the real vector space that consists of the self-adjoint elements of $\mathscr{A}$, supplied with the partial ordering that is inherited from the usual partial ordering on $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$. Then $\mathscr{A}_{\mathrm{sa}}$ is a normal and Dedekind complete Riesz algebra with monotone continuous multiplication.*

PROOF. There exists a compact Hausdorff space X such that $\mathscr{A}$ is isomorphic to $\mathrm{C}(X, \mathbb{C})$ as a C*-algebra. It is then clear that $\mathscr{A}_{\mathrm{sa}}$ is an algebra, and that it is a Riesz space. We know from Proposition 3.12 that it is normal and monotone complete.

We turn to the multiplication. Suppose that $T \in \mathscr{A}_{\mathrm{sa}}^+$ and that $0 \leq S_\lambda \uparrow S$ in $\mathscr{A}_{\mathrm{sa}}$. Then $S = \mathrm{SOT}{-}\lim_\lambda S_\lambda$ by Proposition 3.2, and this implies that $TS = \mathrm{SOT}{-}\lim_\lambda TS_\lambda$. Since $0 \leq TS_\lambda \uparrow\leq TS$ (this is clear in the $\mathrm{C}(X,\mathbb{C})$-model), Proposition 3.2 implies that $TS_\lambda \uparrow TS$ in $\mathscr{A}_{\mathrm{sa}}$. A similar argument shows that $S_\lambda T \uparrow ST$ in E_{sa}. Hence the multiplication in $\mathscr{A}_{\mathrm{sa}}$ is monotone order continuous.

□

§5. Positive $\overline{E}$-valued measures

This section is concerned with the basics of the theory of measures with values in the extended positive cone $\overline{E^+}$ of a partially ordered vector space E.

A *measurable space* is a pair (X, Ω), where X is a set and Ω is an algebra of subsets of X; i.e. Ω is a non-empty collection of subsets of X that is closed under taking complements and under taking finite unions. For the moment, we can still work with algebras of subsets, rather than σ-algebras. The elements of Ω are the *measurable subsets* of X.

The σ-additivity of a measure $\mu : \Omega \to \overline{\mathbb{R}^+}$ requires that $\mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$ in $\overline{\mathbb{R}^+}$, whenever $\{A_n\}_{n=1}^\infty$ is pairwise disjoint sequence in Ω such that $\bigcup_{n=1}^\infty A_n \in \Omega$. For a measure μ taking values in a vector space E that need not have a topology, such a requirement involving a series is meaningless. If E is a partially ordered vector space and μ is suppose to be positive, however, the requirement for the real-valued case can be rewritten in a way that does admit a generalisation. Indeed, if μ takes values in $\overline{\mathbb{R}^+}$, then one can equivalently require that $\mu(A) = \bigvee_{N=1}^\infty \sum_{n=1}^N \mu(A_n)$ in $\overline{\mathbb{R}^+}$. Since this involves only finite sums, this solves the convergence issue, but now one will want to guarantee that the supremum in the right hand side always exists: the topological requirement has to be replaced by an order completeness requirement. The natural minimal way to ensure this is by supposing that E is σ-monotone complete.

The following definition is, therefore, a natural one. We recall that the extension $\overline{E}$ of a partially ordered vector space E has been introduced in Section 2.

Definition 5.1. Let (X, Ω) be a measurable space, and let E be a σ-monotone complete partially ordered vector space. A *positive $\overline{E}$-valued measure* is a map $\mu : \Omega \to \overline{E^+}$ such that:

(1) $\mu(\emptyset) = 0$;
(2) whenever $\{A_n\}_{n=1}^\infty$ is a pairwise disjoint sequence in Ω with $\bigcup_{n=1}^\infty A_n \in \Omega$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) \tag{5.1}$$

in $\overline{E}$.

Since the partial sums form a increasing sequence in $\overline{E}$ as a consequence of $\mu(\Omega) \subseteq \overline{E^+}$, it follows from part (1) of Lemma 2.5 that, for a given enumeration of the A_n, the right hand side of equation (5.1) always exists in $\overline{E}$. A moment's thought shows that this supremum is actually independent of the choice for the enumeration.

A quadruple (X, Ω, μ, E), where X is a set, Ω is an algebra of subsets of X, E is a σ-monotone complete partially ordered vector space, and $\mu : \Omega \to \overline{E^+}$ is a positive $\overline{E}$-valued measure, will be called a *measure space*.

In view of the motivation above, we shall refer to the property under part (2) of Definition 5.1 as the *σ-additivity of μ*.

If $\mu(X) \in E$, then we say that μ is *finite*, or that it is *E-valued*. It will follow from part (2) of Lemma 5.4 that then $\mu(A) \in E^+$ for all $A \in \Omega$. if μ is not finite, i.e. if $\mu(X) = \infty$, then it is said to be *infinite*.

A measurable subset A of X is *σ-finite* if there exists a sequence $\{A_n\}_{n=1}^\infty$ in Ω such that $A \subseteq \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) \in E^+$ for all $n \geq 1$. If X is σ-finite, then we say that μ is a *σ-finite measure*.

A *null set* is a measurable subset of X with measure zero. A measure space is called *complete* if a subset of a null set is still a measurable subset. It will follow from part (2) of Lemma 5.4 that a measurable subset of a null set is again a null set.

A property that a point x in X may or may not have is said to hold *μ-almost everywhere*, or to hold *for μ-almost all x in X*, if the subset of X consisting of those points that do not have this property is contained in a null set. It is not required that this subset of exceptional points be measurable. If the measure is clear from the context, we shall simply write that the property holds almost everywhere, or that it holds for almost all x in X.

Before we proceed with the general theory, let us consider two particular contexts that we have in mind if E happens to be a space of operators. They show that, in various practical situations, our ordered requirement for the σ-additivity of a measure is the same as the classical σ-additivity in the strong operator topology for spectral measures.

Lemma 5.2. *Let (X, Ω) be a measurable space, let E be Dedekind complete Banach lattice, let $\mu : \Omega \to \mathrm{L_r}(E)^+$ be a set map such that $\mu(\emptyset) = 0$, and let $\{A_n\}_{n=1}^\infty \subseteq \Omega$ be such that $\bigcup_{n=1}^\infty A_n \in \Omega$.*

(1) *If $\mu\left(\bigcup_{n=1}^\infty A_n\right)x = \sum_{n=1}^\infty \mu(A_n)x$ in the norm topology on E for all $x \in E$, then $\mu\left(\bigcup_{n=1}^\infty A_n\right) = \bigvee_{N=1}^\infty \sum_{n=1}^N \mu(A_n)$ in $\mathrm{L_r}(E)$.*
(2) *If $\mu\left(\bigcup_{n=1}^\infty A_n\right) = \bigvee_{N=1}^\infty \sum_{n=1}^N \mu(A_n)$ in $\mathrm{L_r}(E)$ and if the norm on E is σ-order continuous. then $\mu\left(\bigcup_{n=1}^\infty A_n\right)x = \sum_{n=1}^\infty \mu(A_n)x$ in the norm topology of E for all $x \in E$.*

PROOF. We prove part (1). If $x \in E^+$, then $\sum_{n=1}^N \mu(A_n)x \uparrow$. Since this increasing sequence is norm convergent, its norm limit is also its supremum, i.e. $\sum_{n=1}^N \mu(A_n)x \uparrow \sum_{n=1}^\infty \mu(A_n)x = \mu\left(\bigcup_{n=1}^\infty A_n\right)x$. This shows that $\sum_{n=1}^N \mu(A_n) \uparrow \mu\left(\bigcup_{n=1}^\infty A_n\right)$.

We prove part (2). Let $x \in E^+$. Since $\sum_{n=1}^N \mu(A_n) \uparrow \mu\left(\bigcup_{n=1}^\infty A_n\right)$, we have $\sum_{n=1}^N \mu(A_n)x \uparrow \mu\left(\bigcup_{n=1}^\infty A_n\right)x$. Since the norm is σ-order continuous, we see that $\mu\left(\bigcup_{n=1}^\infty A_n\right)x = \sum_{n=1}^\infty \mu(A_n)x$ in the norm topology of E. By linearity, this is then also true for arbitrary $x \in E$. □

Lemma 5.3. *Let $\mathscr{H}$ be a complex Hilbert space, and let L be a weakly closed complex linear subspace of $\mathscr{B}(\mathscr{H})$. Let L_{sa} be the real vector space that consists of the self-adjoint elements of L, supplied with the partial ordering that is inherited from the usual partial ordering on $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$.*

Let (X,Ω) be a measurable space, and let $\mu : \Omega \to L_{\mathrm{sa}}^+$ be a set map such that $\mu(\emptyset) = 0$, and let $\{A_n\}_{n=1}^\infty \subseteq \Omega$ be such that $\bigcup_{n=1}^\infty A_n \in \Omega$.

Then the following are equivalent:

(1) $\mu\left(\bigcup_{n=1}^\infty A_n\right) = \bigvee_{N=1}^\infty \sum_{n=1}^N \mu(A_n)$ *in* L_{sa}*;*
(2) $\mu\left(\bigcup_{n=1}^\infty A_n\right)x = \sum_{n=1}^\infty \mu(A_n)x$ *in the norm topology of* $\mathscr{H}$ *for all* $x \in \mathscr{H}$.

PROOF. It is immediate from Proposition 3.2 that part (1) implies part (2).

We prove that part (2) implies part (1). For each $N \geq 1$, we have

$$\begin{aligned}\left\langle \sum_{n=1}^N \mu(A_n)x, x \right\rangle &= \sum_{n=1}^N \langle \mu(A_n)x, x\rangle \\ &\leq \sum_{n=1}^\infty \langle \mu(A_n)x, x\rangle \\ &= \left\langle \sum_{n=1}^\infty \mu(A_n)x, x \right\rangle \\ &= \left\langle \mu\left(\bigcup_{n=1}^\infty A_n\right)x, x \right\rangle.\end{aligned}$$

We thus see that $\sum_{n=1}^N \mu(A_n) \leq \mu\left(\bigcup_{n=1}^\infty A_n\right)$ for all $N \geq 1$. Since $\sum_{n=1}^N \mu(A_n) \uparrow$, Proposition 3.2 shows that the sequence has a supremum in L_{sa}, and that this supremum is also its SOT-limit. It is obvious from the validity of part (2) that this SOT-limit is $\mu\left(\bigcup_{n=1}^\infty A_n\right)$. □

We shall now collect a few basic facts, all to be expected, in the following result.

Lemma 5.4. *Let (X,Ω,μ,E) be a measure space.*

(1) *If $A_1,\ldots,A_n \in \Omega$ are pairwise disjoint measurable sets, then $\mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$ in $\overline{E}$.*
(2) *If $A_1,A_2 \in \Omega$ and $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$ in $\overline{E}$.*
(3) *If $A_1,A_2 \in \Omega$, then $\mu(A_1)+\mu(A_2) = \mu(A_1 \cap A_2)+\mu(A_1 \cup A_2)$ in $\overline{E}$.*
(4) *If $A_1,A_2 \in \Omega$, $A_1 \supseteq A_2$, and $\mu(A_2) \in E$, then $\mu(A_1 \setminus A_2) = \mu(A_1)-\mu(A_2)$ in $\overline{E}$.*
(5) *If $\{A_n\}_{n=1}^\infty$ is a sequence in Ω with $\bigcup_{n=1}^\infty A_n \in \Omega$, then*

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \bigvee_{N=1}^\infty \sum_{n=1}^N \mu(A_n),$$

in $\overline{E}$.

PROOF. Part (1) follows from the definitions when choosing $A_k = \emptyset$ for $k \geq n+1$.

Part (2) is immediate from Part (1).

For part (3), we use the disjoint decomposition $A_1 = (A_1 \cap A_2) \cup (A_1 \setminus A_2)$ and part (1) to see that

$$\mu(A_1) = \mu(A_1 \cap A_2) + \mu(A_1 \setminus A_2).$$

Similarly, we have

$$\mu(A_2) = \mu(A_2 \cap A_1) + \mu(A_2 \setminus A_1).$$

Hence

$$\mu(A_1) + \mu(A_2) = 2\mu(A_1 \cap A_2) + \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1).$$

Since $A_1 \cap A_2$, $A_1 \setminus A_2$, and $A_2 \setminus A_1$ are pairwise disjoint, and since their union equals $A_1 \cup A_2$, part (1) shows that the right hand side of the above equation equals $\mu(A_1 \cap A_2) + \mu(A_1 \cup A_2)$, as required.

For part (4), we use part (1) to see that $\mu(A_1) = \mu(A_2) + \mu(A_1 \setminus A_2)$. Since $\mu(A_2) \in E$, it has an inverse $-\mu(A_2)$ in the semigroup $\overline{E}$, and adding this inverse to both sides yields that $\mu(A_1 \setminus A_2) = \mu(A_1) - \mu(A_2)$, as required.

For part (5), we let $\widetilde{A}_1 = A_1$ and $\widetilde{A}_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ for $n \geq 2$. Then, using part (2), we see that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} \widetilde{A}_n\right) = \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(\widetilde{A}_n) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n).$$

□

We continue with a first rudimentary form of the monotone convergence theorem (Theorem 9.9).

Proposition 5.5. *Let (X, Ω, μ, E) be a measure space.*

If $\{A_n\}_{n=1}^{\infty}$ is a increasing sequence in Ω such that $\bigcup_{n=1}^{\infty} A_n \in \Omega$, then $\mu(A_n) \uparrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$ in $\overline{E}$.

Note that, since $\mu(A_n) \uparrow$ in $\overline{E}$, $\bigvee_{n=1}^{\infty} \mu(A_n)$ does indeed exists in $\overline{E}$ by part (1) of Lemma 2.5.

PROOF. Let us first suppose that $\bigvee_{n=1}^{\infty} \mu(A_n) \in E$. In that case, $\mu(A_n) \in E$ for all $n \geq 1$. Set $\widetilde{A}_1 := A_1$ and $\widetilde{A}_n := A_n \setminus A_{n-1}$ for $n \geq 2$. Then $\{\widetilde{A}_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω such that $\bigcup_{n=1}^{\infty} \widetilde{A}_n = \bigcup_{n=1}^{\infty} A_n \in \Omega$. Since $\mu(\widetilde{A}_n) = \mu(A_n) - \mu(A_{n-1})$ for $n \geq 2$ by part (4) of Lemma 5.4, we see that

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} \widetilde{A}_n\right) \\
&= \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(\widetilde{A}_n) \\
&= \bigvee_{N=2}^{\infty} \sum_{n=1}^{N} \mu(\widetilde{A}_n) \\
&= \bigvee_{N=2}^{\infty} \left[\mu(A_1) + \sum_{n=2}^{N} (\mu(A_n) - \mu(A_{n-1}))\right] \\
&= \bigvee_{N=2}^{\infty} \mu(A_n)
\end{aligned}$$

$$= \bigvee_{N=1}^{\infty} \mu(A_n).$$

If $\bigvee_{n=1}^{\infty} \mu(A_n) = \infty$, then certainly

$$\bigvee_{n=1}^{\infty} \mu(A_n) \geq \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

On the other hand, by the monotonicity of the measure, we obviously have $\mu(A_n) \leq \mu(\bigcup_{n=1}^{\infty} A_n)$ for all $n \geq 1$. Hence also

$$\bigvee_{n=1}^{\infty} \mu(A_n) \leq \mu(\bigcup_{n=1}^{\infty} A_n)$$

in $\overline{E}$. □

Naturally, Proposition 5.5 implies the usual counterpart for decreasing sequences of measurable subsets; it is a rudimentary form of the dominated convergence theorem (Theorem 9.13).

Proposition 5.6. *Let (X, Ω, μ, E) be a measure space.*

If $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence in Ω with $\mu(A_1) \in E$ such that $\bigcap_{n=1}^{\infty} A_n \in \Omega$, then $\mu(A_n) \downarrow \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$ in E.

Proof. Set $A := \bigcap_{n=1}^{\infty} A_n$. Since $A_1 \setminus A_n \uparrow \bigcup_{n=1}^{\infty}(A_1 \setminus A_n) = A_1 \setminus A$, Proposition 5.5, combined with part (4) of Lemma 5.4 and part (6) of Lemma 2.3, shows that

$$\begin{aligned}\mu(A_1) = \mu(A) + \mu(A_1 \setminus A) &= \mu(A) + \bigvee_{n=1}^{\infty} \mu(A_1 \setminus A_n) = \mu(A) + \bigvee_{n=1}^{\infty} (\mu(A_1) - \mu(A_n)) \\ &= \mu(A) + \mu(A_1) + \bigvee_{n=1}^{\infty}(-\mu(A_n)) = \mu(A) + \mu(A_1) - \bigwedge_{n=1}^{\infty} \mu(A_n)\end{aligned}$$

in $\overline{E}$. Since $\mu(A_1)$ and $\bigwedge_{n=1}^{\infty} \mu(A_n)$ have an inverse in $\overline{E}$, terms can be cancelled to conclude that $\bigwedge_{n=1}^{\infty} \mu(A_n) = \mu(A)$. □

The final result of this section is a generalisation of the Borel-Cantelli lemma. We refer to [**6**, Exercise 1.2.89] for this classical result for a probability measure. Note, however, that our measure need not be finite. This is the reason of the appearance of a finiteness condition in part (2) that is automatically satisfied for probability measures.

Lemma 5.7 (Borel-Cantelli lemma). *Let (X, Ω, μ, E) be a measure space, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence in Ω. Suppose that $B_k := \bigcup_{n=k}^{\infty} A_n \in \Omega$ for all $k \geq 1$, and that $B := \bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \in \Omega$.*

(1) *If $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) \in E$, then $\mu(B) = 0$.*
(2) *If $\mu((\bigcup_{n=1}^{\infty} A_n) \setminus B) \in E$, then $\mu(B) \geq x$ in $\overline{E}$ for all $x \in E$ with the property that $\mu(A_n) \geq x$ for all $n \geq 1$.*

Note that in part (2) it is not asserted that $\mu(B)$ is finite.

Proof. We start by proving part (1). It is clear from the fact that $B \subseteq B_k$ and Lemma 5.4 that

$$\mu(B) \leq \mu(B_k) \leq \bigvee_{N=k}^{\infty} \sum_{n=k}^{N} \mu(A_n) \tag{5.2}$$

in $\overline{E}$ for all $k \geq 1$.

Furthermore, for all $k \geq 2$, we have

$$\begin{aligned}\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) &= \bigvee_{N=k}^{\infty} \sum_{n=1}^{N} \mu(A_n) \\ &= \bigvee_{N=k}^{\infty} \left(\sum_{n=1}^{k-1} \mu(A_n) + \sum_{n=k}^{N} \mu(A_n) \right) \\ &= \sum_{n=1}^{k-1} \mu(A_n) + \bigvee_{N=k}^{\infty} \sum_{n=k}^{N} \mu(A_n)\end{aligned}$$

in $\overline{E}$. Since $\bigvee_{N=k}^{\infty} \sum_{n=k}^{N} \mu(A_n)$ is finite for all $k \geq 2$, combination with equation (5.2) yields that

$$\sum_{n=1}^{k-1} \mu(A_n) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) - \mu(B)$$

in $\overline{E}$ for all $k \geq 2$. Hence

$$\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) - \mu(B)$$

in $\overline{E}$. Since $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n)$ is finite, we see that $\mu(B) \leq 0$, and we conclude that $\mu(B) = 0$.

We turn to part (2). Since clearly $\mu(B_k) \geq \mu(A_k) \geq x$ for all $k \geq 1$, we have

$$\bigwedge_{k=1}^{\infty} \mu(B_k) \geq x, \tag{5.3}$$

where we note that the infimum in the left hand side exists since $\mu(B_k) \downarrow$.

Since $B_k \setminus B \downarrow \emptyset$ and since $\mu(B_1 \setminus B) = \mu(\bigcup_{n=1}^{\infty} A_n \setminus B) \in E$ by assumption, Proposition 5.6 implies that

$$\bigwedge_{k=1}^{\infty} \mu(B_k \setminus B) = 0. \tag{5.4}$$

Combining equations (5.3) and (5.4) with part (2) of Lemma 2.4, we see that

$$\mu(B) = \mu(B) + \bigwedge_{k=1}^{\infty} \mu(B_k \setminus B) = \bigwedge_{k=1}^{\infty} (\mu(B) + \mu(B_k \setminus B)) = \bigwedge_{k=1}^{\infty} \mu(B_k) \geq x$$

in $\overline{E}$. □

§6. Positive $\overline{E}$-valued outer measures

In the proof of one of the Riesz representation theorems (Theorem 12.3), we shall use an outer measure and the ensuing measure on the σ-algebra of measurable subsets in the vector-valued context. The present section contains the relevant material, which is a modest modification of that for the real case in [**2**, Section 11].

Throughout this section, X is a set and E is a σ-monotone complete partially ordered vector space.

We begin with the definition of a positive $\overline{E}$-valued outer measure.

Definition 6.1. A map $\mu^* : 2^X \to \overline{E^+}$ is called a *positive $\overline{E}$-valued outer measure* if

(1) $\mu^*(\emptyset) = 0$;
(2) $\mu^*(A) \leq \mu^*(B)$ for all $A,\ B \in 2^X$ such that $A \subseteq B$;
(3) for every sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu^*(A_n).$$

The combination of the parts (1) and (3) shows that μ^* is not only σ-sub-additive, but also finitely sub-additive.

Definition 6.2. A subset A of X is called *μ^*-measurable* if, for all $B \subseteq X$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^{\mathrm{c}}). \tag{6.1}$$

We let Ω denote the collection of all μ^*-measurable subsets of X. Obviously, $\emptyset,\ X \in \Omega$ and equally obviously Ω is invariant under taking complements. We shall proceed to show that Ω is a σ-algebra, and that the restriction of μ^* to Ω is a positive $\overline{E}$-valued measure.

Lemma 6.3. *Let A_1, A_2 be subsets of X such that $A_1 \in \Omega$ and $A_1 \cap A_2 = \emptyset$. Then, for any subset B of X,*

$$\mu^*\big(B \cap (A_1 \cup A_2)\big) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2).$$

Proof. Using that A_1 is μ^*-measurable and that $(A_1 \cup A_2) \cap A_1^{\mathrm{c}} = A_2$, we see that, for any $B \subseteq X$

$$\begin{aligned}\mu^*\big(B \cap (A_1 \cup A_2)\big) &= \mu^*\big([B \cap (A_1 \cup A_2)] \cap A_1\big) + \mu^*\big([B \cap (A_1 \cup A_2)] \cap A_1^{\mathrm{c}}\big)\\ &= \mu^*(B \cap A_1) + \mu^*(B \cap A_2).\end{aligned}$$

□

Applying Lemma 6.3 for $B = X$ yields the following.

Corollary 6.4. *Let A_1, A_2 be subsets of X such that $A_1 \in \Omega$ and $A_1 \cap A_2 = \emptyset$. Then $\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2)$.*

The following result shows that, also in the vector-valued context, everything is as expected.

Theorem 6.5. *Let X be a set, let E be a σ-monotone complete partially ordered vector space, and let $\mu^* : 2^X \to \overline{E^+}$ be a positive $\overline{E}$-valued outer measure.*

Then the set Ω of μ^*-measurable subsets of X is a σ-algebra. Furthermore, the restriction of μ^* to Ω is a positive $\overline{E}$-valued measure on Ω.

Proof. We start by proving that Ω is a σ-algebra. We have already observed that $\emptyset \in \Omega$ and that Ω is invariant under taking complements, so it remains to be shown that Ω is invariant under taking countable unions.

We show first that Ω is invariant under taking finite unions. Let A_1, A_2 in Ω. Set $A := A_1 \cup A_2$. Using the μ^*-measurability of A_1 for the first and the fourth equality, and that of A_2 for the third equality, we have, for any $B \subseteq X$,

$$\begin{aligned}\mu^*(B \cap A) + \mu^*(B \cap A^{\mathrm{c}}) &= \mu^*\big([B \cap A] \cap A_1\big) + \mu^*\big([B \cap A] \cap A_1^{\mathrm{c}}\big)\\ &\quad + \mu^*\big([B \cap A^{\mathrm{c}}] \cap A_1\big) + \mu^*\big([B \cap A^{\mathrm{c}}] \cap A_1^{\mathrm{c}}\big)\\ &= \mu^*(B \cap A_1) + \mu^*\big(\big[B \cap A_1^{\mathrm{c}}\big] \cap A_2\big)\end{aligned}$$

$$
\begin{aligned}
&\quad + \mu^*(\emptyset) + \mu^*\big(\big[B\cap A_1^{\mathrm{c}}\big]\cap A_2^{\mathrm{c}}\big)\\
&= \mu^*(B\cap A_1) + \mu^*(B\cap A_1^{\mathrm{c}})\\
&= \mu^*(B).
\end{aligned}
$$

Hence $A_1\cup A_2\in\Omega$, as desired. It follows that Ω is closed under taking finite unions.

Now suppose that $\{A_n\}_{n=1}^\infty$ is a sequence in Ω. We are to show that $\bigcup_{n=1}^\infty A_n$ is μ^*-measurable. Replacing A_n with $A_n\setminus\bigcup_{k=1}^n A_k$ for $n\geq 2$, which we now already know to be μ^*-measurable, we may and shall suppose that the A_n are pairwise disjoint. Set $F:=\bigcup_{n=1}^\infty A_n$, and $F_N:=\bigcup_{n=1}^N A_n$ for $N\geq 1$; then the F_N are μ^*-measurable. Using the μ^*-measurability of F_N in the first step, the monotonicity of μ^* in the second step, and Lemma 6.3 and the fact that Ω is closed under taking finite unions in the third step, we see that, for all $N\geq 1$ and $B\subseteq X$,

$$
\begin{aligned}
\mu^*(B) &= \mu^*(B\cap F_N) + \mu^*(A\cap F_N^{\mathrm{c}})\\
&\geq \mu^*(B\cap F_N) + \mu^*(A\cap F^{\mathrm{c}})\\
&= \sum_{n=1}^N \mu^*(B\cap A_n) + \mu^*(A\cap F^{\mathrm{c}}).
\end{aligned}
$$

Since this is true for each $N\geq 1$, combination with the σ-sub-additivity of μ^* implies that, for any $B\subseteq X$,

$$
\begin{aligned}
\mu^*(B) &\geq \bigvee_{N=1}^\infty\sum_{n=1}^N \mu^*(B\cap A_n) + \mu^*(B\cap F^{\mathrm{c}})\\
&\geq \mu^*\left(\bigcup_{n=1}^\infty (B\cap A_n)\right) + \mu^*(B\cap F^{\mathrm{c}})\\
&= \mu^*(B\cap F) + \mu^*(B\cap F^{\mathrm{c}})\\
&\geq \mu^*(B).
\end{aligned}
$$

Hence $F\in\Omega$, as desired.

We shall now show that μ^* is σ-additive on Ω. Let $\{A_n\}_{n=1}^\infty$ be a pairwise disjoint sequence in Ω. The monotonicity of μ^* and the finite additivity of μ^* on Ω that follows from Corollary 6.4 show that, for all $N\geq 1$,

$$
\begin{aligned}
\mu^*\left(\bigcup_{n=1}^\infty A_n\right) &\geq \mu^*\left(\bigcup_{n=1}^N A_n\right)\\
&= \sum_{n=1}^N \mu^*(A_n).
\end{aligned}
$$

Hence $\mu^*\left(\bigcup_{n=1}^\infty A_n\right)\geq\bigvee_{N=1}^\infty\sum_{n=1}^N\mu^*(A_n)$. Since the reverse inequality holds by the σ-sub-additivity of μ^*, we see that μ^* is σ-additive on Ω. $\square$

The following basic property carries over as well.

Lemma 6.6. *Let A be a subset of X such that $\mu^*(A)=0$. Then A is μ^*-measurable.*

Proof. If $\mu^*(A)=0$, then, for any subset B of X,

$$
\mu^*(B)\leq\mu^*(B\cap A)+\mu^*(B\cap A^{\mathrm{c}})\leq\mu^*(A)+\mu^*(B)=\mu^*(B).
$$

$\square$

§7. Measures on locally compact Hausdorff spaces

The uniqueness part of the classical Riesz representation involves the notion of regular Borel measures. There may be several measures on the Borel σ-algebra of a locally compact Hausdorff space X that represent a given positive linear map $T : \mathrm{C_c}(X) \to \mathbb{R}$, but there is only one that is a regular Borel measure. In our vector-valued context, we shall have similar results, stating that there is a measure with certain regularity properties that represents a positive linear map $T : \mathrm{C_c}(X) \to E$ for a partially ordered vector space E, and that it is the unique representing measure within the class of measures that also have these regularity properties. In this section, we shall investigate such properties. The situation is somewhat more involved than for the real case.

If X is a locally compact Hausdorff space, then we let $\mathscr{B}$ denote its Borel σ-algebra, i.e. $\mathscr{B}$ is the σ-algebra that is generated by the open subsets of X.

If E is a σ-monotone complete partially ordered vector space, then we shall let $\mathrm{M}(X, \mathscr{B}, \overline{E^+})$ denote the collection of all positive $\overline{E}$-valued measures $\mu : \mathscr{B} \to \overline{E^+}$.

We shall distinguish the following regularity properties.

Definition 7.1. Let X be a locally compact Hausdorff space, let E be a monotone complete partially ordered vector space, and let $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$. Then μ is called:

(1) a *Borel measure* (*on* X) if $\mu(K) \in E$ for all compact subset K;
(2) *inner regular at* $A \in \mathscr{B}$ if $\mu(A) = \bigvee \{\, \mu(K) : K \text{ is compact and } K \subseteq A \,\}$ in $\overline{E}$;
(3) *weakly inner regular at* $A \in \mathscr{B}$ if $\mu(A) = \bigvee \{\, \mu(A \cap K) : K \text{ is compact} \,\}$ in $\overline{E}$;
(4) *outer regular at* $A \in \mathscr{B}$ if $\mu(A) = \bigwedge \{\, \mu(V) : V \text{ is open and } A \subseteq V \,\}$ in $\overline{E}$;
(5) a *regular Borel measure* (*on* X) if μ is a Borel measure on X that is inner regular at all open subsets of X and outer regular at all Borel sets;
(6) a *quasi-regular Borel measure* (*on* X) if it is a Borel measure on X that is inner regular at all open subsets of X and weakly inner regular at all Borel sets.

Note that the three subsets of E occurring in the above definitions are all directed in the appropriate directions, so that the two suprema and the infimum exist in $\overline{E}$ as a consequence of the monotone completeness of E; σ-monotone completeness would not have sufficed here. In the sequel, there will be similar cases where the fact that a set is directed guarantees that a supremum or infimum exists, but we refrain from observing this explicitly each and every time.

To be entirely consistent with earlier terminology, it would be necessary to speak of, for example, a quasi-regular positive $\overline{E}$-valued Borel measure, but it seems clearer to shorten the terminology somewhat in this particular context, it being understood that the measure takes values in $\overline{E^+}$ for some fixed monotone complete partially ordered vector space E or, for finite measures, in E^+.

We let Υ denote the collection of relatively compact open subsets of X, i.e. the collection of open subsets of X with compact closure. Then Υ is closed under finite intersections and finite unions. Since every relatively compact open subset of X is contained in a compact one, and every compact subset of X is contained in a relatively compact open set, it is easy to see that μ is weakly inner regular at a Borel set A if and only if $\mu(A) = \bigvee \{\, \mu(A \cap V) : V \in \Upsilon \,\}$ in $\overline{E}$.

The next small result relates the two regularity properties of a quasi-regular Borel measure.

Lemma 7.2. *Let X be a locally compact Hausdorff space, let E be a monotone complete partially ordered vector space, let $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$, and let $A \in \mathscr{B}$. If μ is inner regular at A, then μ is weakly inner regular at A.*

PROOF. Using the inner regularity of μ at A in the first step, we merely need note that

$$\begin{aligned}\mu(A) &= \bigvee \{\, \mu(K) : K \text{ is compact and } K \subseteq A \,\}\\ &\leq \bigvee \{\, \mu(A \cap K) : K \text{ is compact} \,\}\\ &\leq \mu(A).\end{aligned}$$

□

Remark 7.3. The terminology 'quasi-regular' is admittedly not very descriptive, but it is hard to find a more appealing nomenclature. It is also used in [**24**, p. 108] as well as in [**26**, p. 195] by the same author. It denotes different notions in these two papers, and these are also both different from ours.

We shall write $\mathrm{M_B}(X, \mathscr{B}, \overline{E^+})$ for the Borel measures on X, $\mathrm{M_{rB}}(X, \mathscr{B}, \overline{E^+})$ for the regular Borel measures on X, $\mathrm{M_{qrB}}(X, \mathscr{B}, \overline{E^+})$ for the quasi-regular Borel measures on X, $\mathrm{M_{rB}}(X, \mathscr{B}, E^+)$ for the finite regular (Borel) measures on X, and $\mathrm{M_{qrB}}(X, \mathscr{B}, E^+)$ for the finite quasi-regular (Borel) measures on X. Every finite measure is, of course, a Borel measure.

If U is an open subset of X, then U is a locally compact Hausdorff space in the induced topology. We shall denote its Borel σ-algebra by $\mathscr{B}_U$. It is then easy to see that $\mathscr{B}_U = \{\, B \in \mathscr{B} : B \subseteq U \,\}$, so that we can identify $\mathscr{B}_U$ with a subset of $\mathscr{B}$. If $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$, then we let μ_U denote the restriction of μ to the subset $\mathscr{B}_U$ of $\mathscr{B}$. It is then a positive $\overline{E}$-valued measure on $\mathscr{B}_U$.

We collect a few technical observations in the following result. The proofs are elementary.

Lemma 7.4. *Let X be a locally compact Hausdorff space, let E be a monotone complete partially ordered vector space, let $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$, and let U be an open subset of X.*

(1) *If μ is a Borel measure on X, then μ_U is a Borel measure on U.*
(2) *If μ is inner regular at $A \in \mathscr{B}_U$, then so is μ_U; and conversely.*
(3) *If μ is outer regular at $A \in \mathscr{B}_U$, then so is μ_U; and conversely.*
(4) *If μ is a Borel measure on X and if $U \in \Upsilon$, then μ_U is a finite Borel measure on U.*
(5) *If μ is a Borel measure on U for all $U \in \Upsilon$, then μ is a finite Borel measure on U for all $U \in \Upsilon$.*

The following is then clear.

Lemma 7.5. *Let X be a locally compact Hausdorff space and let E be a monotone complete partially ordered vector space.*

If $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ is a regular Borel measure on X, then, for all open subsets U of X, μ_U is a regular Borel measure on U.

If $E = \mathbb{R}$, then every regular Borel measure on X is inner regular at the σ-finite Borel sets; see [**13**, Proposition 7.5]. If E is normal, an analogous result holds in our case for Borel sets of finite measure. The step from the case of finite measure to the σ-finite case as in the proof of [**13**, Proposition 7.5] uses the linear ordering of $\mathbb{R}$ and does not seem to have an obvious analogue in the general situation.

Proposition 7.6. *Let X be a locally compact Hausdorff space, let E be a normal and monotone complete partially ordered vector space, and let $\mu \in \mathrm{M}_{\mathrm{rB}}(X, \mathscr{B}, \overline{E^+})$.*

Then μ is inner regular at all Borel sets of finite measure.

PROOF. Let $A \in \mathscr{B}$ be such that $\mu(A) \in E$. We are to prove that

$$\mu(A) = \bigvee \{ \mu(K) : K \text{ is compact and } K \subseteq A \}.$$

It is clear that $\mu(A)$ is an upper bound of $\{ \mu(K) : K \text{ is compact and } K \subseteq A \}$. Using the normality of E, we shall show that it is the least upper bound.

Choose and fix $x' \in (E_{\mathrm{n}}^{\sim})^+$ and $\varepsilon > 0$.

Since $\mu(A) \in E$, the outer regularity of μ at A can be written as

$$\mu(A) = \bigwedge \{ \mu(V) : V \text{ is open, } A \subseteq V \text{, and } \mu(V) \in E \}.$$

Proposition 3.8 then shows that

$$(\mu(A), x') = \inf \{ (\mu(V), x') : V \text{ is open, } A \subseteq V \text{, and } \mu(V) \in E \}.$$

We can then choose and fix an open subset V such that $A \subseteq V$, $\mu(V) \in E$, and

$$(\mu(V), x') < (\mu(A), x') + \varepsilon/3.$$

Using the outer regularity of μ at $V \setminus A$, which also has finite measure, we can likewise choose and fix an open subset W such that $V \setminus A \subseteq W$, $\mu(W) \in E$, and

$$(\mu(W), x') < (\mu(V \setminus A), x') + \varepsilon/3.$$

By replacing W with the open subset $W \cap V$ we may and shall suppose that $W \subseteq V$. We have

$$(\mu(W), x') < (\mu(V \setminus A), x') + \varepsilon/3 = (\mu(V), x') - (\mu(A), x') + \varepsilon/3 < 2\varepsilon/3.$$

Using the inner regularity of μ at the open subset V and the normality of E, an application of Proposition 3.8 enables us to choose and fix a compact subset K_1 such that $K_1 \subseteq V$ and

$$(\mu(V), x') < (\mu(K_1), x') + \varepsilon/3.$$

Set $K_2 := K_1 \setminus W$. Then K_2 is compact and

$$\begin{aligned} K_2 &= K_1 \cap W^{\mathrm{c}} \\ &\subseteq K_1 \cap (V \setminus A)^{\mathrm{c}} \\ &= (K_1 \cap V^{\mathrm{c}}) \cup (K_1 \cap A) \\ &= K_1 \cap A \\ &\subseteq A. \end{aligned}$$

We also have

$$\begin{aligned} V \setminus K_2 &= V \cap (K_1^{\mathrm{c}} \cup W) \\ &= (V \cap K_1^{\mathrm{c}}) \cup (V \cap W) \\ &= (V \setminus K_1) \cup W. \end{aligned}$$

Combining all of the above, we see that

$$\begin{aligned} (\mu(A) - \mu(K_2), x') &= (\mu(A \setminus K_2), x') \\ &\leq (\mu(V \setminus K_2), x') \\ &= (\mu[(V \setminus K_1) \cup W], x') \\ &\leq (\mu(V \setminus K_1), x') + (\mu(W), x') \end{aligned}$$

$$
\begin{aligned}
&= (\mu(V), x') - (\mu(K_1), x') + (\mu(W), x') \\
&< \varepsilon + 2\varepsilon/3 \\
&= \varepsilon.
\end{aligned}
$$

All in all, we have found a compact subset K_2 of A (depending on x' and ε) such that $(\mu(A), x') < (\mu(K_2), x') + 3\epsilon$. Hence we conclude, for our fixed $x' \in (E_{\mathrm{n}}^{\sim})^+$, that

$$(\mu(A), x') = \sup \{ (\mu(K), x') : K \text{ is compact and } K \subseteq A \}.$$

Since this is true for all $x' \in (E_{\mathrm{n}}^{\sim})^+$, an appeal to Proposition 3.8 completes the proof. □

We can use this to prove our next result.

Corollary 7.7. *Let X be a locally compact Hausdorff space, let E be a normal and monotone complete partially ordered vector space, and let $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ be such that, for all $U \in \Upsilon$, the restricted measure μ_U is a regular Borel measure on U.*

Suppose that A is a Borel set such that μ is weakly inner regular at A.

Then μ is inner regular at A.

Proof. If $U \in \Upsilon$, then $A \cap U \in \mathscr{B}_U$ has finite measure. Hence an application of Proposition 7.6 to the regular Borel measure μ_U on U shows that

$$\mu(A \cap U) = \bigvee \{ \mu(K) : K \text{ is compact and } K \subseteq A \cap U \}.$$

Hence

$$
\begin{aligned}
\mu(A) &= \bigvee \{ \mu(A \cap U) : U \in \Upsilon \} \\
&= \bigvee \{ \mu(K) : K \text{ is compact and } K \subseteq A \cap U \text{ for some } U \in \Upsilon \} \\
&= \bigvee \{ \mu(K) : K \text{ is compact and } K \subseteq A \}
\end{aligned}
$$

in $\overline{E}$. The last equation holds since the local compactness of X implies that, for any compact subset K with $K \subseteq A$, there exists a relatively compact open subset U such that $K \subseteq U$, and then $K \subseteq A \cap U$. □

This has the following consequence.

Corollary 7.8. *Let X be a locally compact Hausdorff space, let E be a normal and monotone complete partially ordered vector space, and let $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ be such that, for all $U \in \Upsilon$, the restricted measure μ_U is a regular Borel measure on U.*

(1) *If μ is weakly inner regular at all Borel sets, then μ is inner regular at all Borel sets.*
(2) *If μ is a finite measure that is weakly inner regular at all Borel sets, then μ is both inner regular and outer regular at all Borel sets.*

Proof. Part (1) is immediate from Corollary 7.7. In part (2), the inner regularity of μ at all Borel sets is clear from part (1). For the outer regularity, let $A \in \mathscr{B}$. Then the inner regularity of μ at A^{c} yields that

$$
\begin{aligned}
\mu(A) &= \mu(X) - \mu(A^{\mathrm{c}}) \\
&= \mu(X) - \bigvee \{ \mu(K) : K \text{ is compact and } K \subseteq A^{\mathrm{c}} \} \\
&= \mu(X) - \bigvee \{ \mu(X) - \mu(K^{\mathrm{c}}) : K \text{ is compact and } K \subseteq A^{\mathrm{c}} \} \\
&= \bigwedge \{ \mu(K^{\mathrm{c}}) : K \text{ is compact and } A \subseteq K^{\mathrm{c}} \}
\end{aligned}
$$

$$\geq \bigwedge \{\, \mu(U) : U \text{ is open and } A \subseteq U \,\}$$
$$\geq \mu(A).$$

Hence μ is outer regular at A. □

§8. Cones of measures

If X is a locally compact Hausdorff space, then the Banach lattice of real-valued signed regular Borel measures on X is isometrically lattice isomorphic to the norm dual of $\mathrm{C}_0(X)$. In our case, if E is, for example, a Dedekind complete Riesz space, then one would like to be able to analogously identify the Riesz space of all regular maps from $\mathrm{C}_0(X)$ into E with a Riesz space of sufficiently regular E-valued measures on the Borel σ-algebra of X. An attempt to establish such a result is not undertaken in the present body of work, but in this section we lay the foundations for further research in that direction by studying the properties of naturally occurring cones of measures. In Section 8.1 we start with measures on abstract spaces, and in Section 8.2 we specialise to measures on locally compact Hausdorff spaces.

8.1. Cones of general measures

If (X,Ω) is a measurable space and E is a σ-monotone complete partially ordered vector space, then we let $\mathrm{M}(X,\Omega,\overline{E^+})$ denote the collection of all positive $\overline{E}$-valued measures on Ω, and we write $\mathrm{M}(X,\Omega,E^+)$ for the subset of finite measures. We define a partial ordering on $\mathrm{M}(X,\Omega,\overline{E^+})$ and $\mathrm{M}(X,\Omega,E^+)$ by saying that $\mu_1 \leq \mu_2$ when $\mu_1(A) \leq \mu_2(A)$ in $\overline{E}$ for all $A \in \Omega$. Then $\mathrm{M}(X,\Omega,\overline{E^+})$ has a largest element, namely the measure μ such that $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ if $A \in \Omega$ and $A \neq \emptyset$. If $\mathrm{M}(X,\Omega,E^+) \neq \{0\}$, then $\mathrm{M}(X,\Omega,E^+)$ does not have a largest element. The zero measure is the smallest element of both $\mathrm{M}(X,\Omega,\overline{E^+})$ and $\mathrm{M}(X,\Omega,E^+)$.

We collect a few basic properties of $\mathrm{M}(X,\Omega,\overline{E^+})$ and $\mathrm{M}(X,\Omega,E^+)$ in our next result.

Theorem 8.1. *Let (X,Ω) be a measurable space, and let E be a σ-monotone complete partially ordered vector space.*

(1) *For μ, $\nu \in \mathrm{M}(X,\Omega,\overline{E^+})$ and $r \in \mathbb{R}^+$, define $\mu+\nu, r\mu : \Omega \to \overline{E^+}$ by*

$$(r\mu)(A) := r\mu(A),$$
$$(\mu+\nu)(A) := \mu(A) + \nu(A)$$

for $A \in \Omega$.

Then $\mu+\nu, r\mu \in \mathrm{M}(X,\Omega,\overline{E^+})$. Furthermore, $\mu+\nu \in \mathrm{M}(X,\Omega,E^+)$ if and only if $\mu \in \mathrm{M}(X,\Omega,E^+)$ and $\nu \in \mathrm{M}(X,\Omega,E^+)$, and, for $r>0$, $r\mu \in \mathrm{M}(X,\Omega,E^+)$ if and only if $\mu \in \mathrm{M}(X,\Omega,E^+)$.

(2) *Suppose that E is actually Dedekind complete.*

 (a) *For all μ and ν in $\mathrm{M}(X,\Omega,\overline{E^+})$, the supremum $\mu \vee \nu$ in $\mathrm{M}(X,\Omega,\overline{E^+})$ exists. Moreover, we have*

$$(\mu \vee \nu)(A) = \bigvee \{\, \mu(B) + \nu(A \setminus B) : B \in \Omega \text{ and } B \subseteq A \}$$

 in $\overline{E}$ for all $A \in \Omega$. Furthermore, $\mu \vee \nu \in \mathrm{M}(X,\Omega,E^+)$ if and only if $\mu \in \mathrm{M}(X,\Omega,E^+)$ and $\nu \in \mathrm{M}(X,\Omega,E^+)$.

(b) *For all μ, $\nu \in \mathrm{M}(X,\Omega,E^+)$, the infimum $\mu \wedge \nu$ in $\mathrm{M}(X,\Omega,\overline{E^+})$ exists. Moreover, we have*

$$(\mu \wedge \nu)(A) = \bigwedge \{\, \mu(B) + \nu(A \setminus B) : B \in \Omega \text{ and } B \subseteq A \}$$

in $\overline{E}$ for all $A \in \Omega$. The measure $\mu \wedge \nu$ is finite.

(c) *For all μ, $\nu \in \mathrm{M}(X,\Omega,E^+)$, we have*

$$\mu \vee \nu + \mu \wedge \nu = \mu + \nu.$$

The Dedekind completeness of E ensures that the suprema and infima occurring in part (2) exist in $\overline{E}$; see part (3) of Lemma 2.5.

Proof. Using part (7) of Lemma 2.3 (resp. part (2) of Lemma 2.4), one sees easily that $r\mu \in \mathrm{M}(X,\Omega,\overline{E^+})$ (resp. $\mu + \nu \in \mathrm{M}(X,\Omega,\overline{E^+})$). The statements about finiteness are obvious.

We turn to part (2), where we assume that E is Dedekind complete.

We start with part (2.a) concerning $\mu \vee \nu$. For this, we define a set map $\xi : \Omega \to \overline{E^+}$ by

$$\xi(A) := \bigvee \{\, \mu(B) + \nu(A \setminus B) : B \in \Omega \text{ and } B \subseteq A \}. \tag{8.1}$$

We shall verify that ξ is a positive $\overline{E}$-valued measure, i.e. that $\xi \in \mathrm{M}(X,\Omega,\overline{E^+})$.

It is clear that $\xi(\emptyset) = 0$.

Furthermore, ξ is monotone. Indeed, suppose that $A_1, A_2 \in \Omega$ and that $A_1 \subseteq A_2$. If $B \in \Omega$ is such that $B \subseteq A_1$, then $B \subseteq A_2$, and $\nu(A_1 \setminus B) \leq \nu(A_2 \setminus B)$. It follows from this that $\xi(A_1) \leq \xi(A_2)$.

We shall now establish the finite additivity of ξ. Suppose that $A_1, A_2 \in \Omega$ and that $A_1 \cap A_2 = \emptyset$. Then the measurable subsets B of $A_1 \cup A_2$ are precisely the subsets of the form $B_1 \cup B_2$, where $B_1, B_2 \in \Omega$ are such that $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. Using part (1) of Lemma 2.4 in the penultimate step, we therefore see that

$$\begin{aligned}
&\xi(A_1 \cup A_2) \\
&= \bigvee \{\, \mu(B) + \nu((A_1 \cup A_2) \setminus B) : B \in \Omega,\ B \subseteq A_1 \cup A_2 \,\} \\
&= \bigvee \{\, \mu(B_1 \cup B_2) + \nu((A_1 \cup A_2) \setminus (B_1 \cup B_2)) : B_1, B_2 \in \Omega,\ B_1 \subseteq A_1,\ B_2 \subseteq A_2 \,\} \\
&= \bigvee \{\, \mu(B_1) + \mu(B_2) + \nu(A_1 \setminus B_1) + \nu(A_2 \setminus B_2) : B_1, B_2 \in \Omega,\ B_1 \subseteq A_1,\ B_2 \subseteq B_2 \,\} \\
&= \bigvee \{\, \mu(B_1) + \nu(A_1 \setminus B_1) : B_1 \in \Omega,\ B_1 \subseteq A_1 \,\} \\
&\qquad + \bigvee \{\, \mu(B_2) + \nu(A_2 \setminus B_2) : B_2 \in \Omega,\ B_2 \subseteq A_2 \,\} \\
&= \xi(A_1) + \xi(A_2).
\end{aligned}$$

It follows that ξ is finitely additive.

We shall now show that ξ is σ-additive. Let $\{A_n\}_{n=1}^{\infty}$ be a pairwise disjoint sequence in Ω such that $A := \bigcup_{n=1}^{\infty} A_n$ is measurable. Since $A \supseteq \bigcup_{n=1}^{N} A_n$ for each $N \geq 1$, and since we already know that ξ is monotone and finitely additive, we see that $\xi(A) \geq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \xi(A_n)$. We need to establish the reverse inequality. For every $B \in \Omega$ with $B \subseteq A$, we have, using part (2) of Lemma 2.4 in the third step,

$$\begin{aligned}
\mu(B) + \nu(A \setminus B) &= \mu\left(\bigcup_{n=1}^{\infty} (A_n \cap B)\right) + \nu\left(\bigcup_{n=1}^{\infty} (A_n \setminus B)\right) \\
&= \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n \cap B) + \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \nu(A_n \setminus B)
\end{aligned}$$

$$= \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} (\mu(A_n \cap B) + \nu(A_n \setminus B))$$
$$= \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} (\mu(A_n \cap B) + \nu(A_n \setminus (A_n \cap B)))$$
$$\leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \xi(A_n).$$

Hence $\xi(A) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \xi(A_n)$, as required. This concludes the proof of the fact that $\xi \in \mathrm{M}(X, \Omega, \overline{E^+})$.

It remains to show that ξ is the least upper bound of $\{\mu, \ \nu\}$ in $\mathrm{M}(X, \Omega, \overline{E^+})$. By taking $B = \emptyset$ and $B = A$ in equation (8.1) it follows that $\xi \geq \nu$ and $\xi \geq \mu$. If $\xi' \in \mathrm{M}(X, \Omega, \overline{E^+})$ is any upper bound of $\{\mu, \ \nu\}$, and if $A \in \Omega$, then, for all $B \in \Omega$ such that $B \subseteq A$,

$$\xi'(A) = \xi'(B) + \xi'(A \setminus B) \geq \mu(B) + \nu(A \setminus B),$$

which implies that $\xi'(A) \geq \xi(A)$. Hence $\xi' \geq \xi$. This concludes the proof that $\xi = \mu \vee \nu$. The statement about finiteness is obvious.

We turn to part (2.b) concerning $\mu \wedge \nu$, where μ and ν are both finite. For this, we define the set map $\rho : \Omega \to \overline{E^+}$ by

$$\rho(A) := \bigwedge \{\mu(B) + \nu(A \setminus B) : B \in \Omega \text{ and } B \subseteq A\}. \tag{8.2}$$

Then ρ takes only finite values, $\rho(\emptyset) = 0$, and ρ is monotone again. To see the latter, suppose that $A_1, A_2 \in \Omega$ and that $A_1 \subseteq A_2$. If $B \in \Omega$ is such that $B \subseteq A_2$, then $A_1 \setminus (B \cap A_1) \subseteq A_2 \setminus B$. Note that

$$\mu(A_1 \cap B) + \nu(A_1 \setminus (A_1 \cap B)) \leq \mu(B) + \nu(A_2 \setminus B).$$

Since $A_1 \cap B \subseteq A_1$, it follows from this that $\rho(A_1) \leq \rho(A_2)$.

An argument completely analogous to that for $\mu \vee \nu$ shows that ρ is finitely additive. It remains to show that ρ is σ-additive.

As a preparation, we demonstrate that ρ has the usual behaviour for a decreasing sequence of measurable sets. Let $\{B_n\}_{n=1}^{\infty} \subseteq$ be a sequence in Ω such that $B_n \downarrow \emptyset$. We claim that $\rho(B_n) \downarrow 0$. To see this, note that $\rho(B_n) \leq \mu(B_n) + \nu(B_n)$. Since μ and ν are both finite, Proposition 5.6 shows that $\mu(B_n) \downarrow 0$ and $\nu(B_n) \downarrow 0$. Part (2) of Lemma 2.4 then yields that $\mu(B_n) + \nu(B_n) \downarrow 0$, and we conclude that $\rho(B_n) \downarrow 0$, as desired.

Suppose now that $\{A_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω such that $A := \bigcup_{n=1}^{\infty} A_n$ is measurable. Set $B_1 := A_1$ and $B_N := A \setminus \bigcup_{n=1}^{N-1} A_n$ for $N \geq 2$. Then $B_N \downarrow \emptyset$, so $\rho(B_N) \downarrow 0$ by what we have just seen. Since ρ is finitely additive, and since $A = B_{N+1} \cup \bigcup_{n=1}^{N} A_n$ is a disjoint union for $N \geq 1$, we see that

$$\rho(A) = \rho(B_{N+1}) + \sum_{n=1}^{N} \rho(A_n)$$

for $N \geq 1$, so that

$$\rho(A) - \rho(B_{N+1}) = \sum_{n=1}^{N} \rho(A_n)$$

for $N \geq 1$. Taking the supremum on each side, we have, since $-\rho(B_{N+1}) \uparrow 0$, that

$$\rho(A) = \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \rho(A_n).$$

Hence ρ is σ-additive.

The proof that ρ equals the infimum of $\{\mu,\ \nu\}$ is analogous to that above showing that ξ is the infimum of $\{\mu,\ \nu\}$.

We turn to part (2.c), where we suppose that μ and ν are both finite. We shall show that $\mu \vee \nu + \mu \wedge \nu = \mu + \nu$. Let $A \in \Omega$. Then, for all $B \in \Omega$ such that $B \subseteq A$, we have

$$\mu(B) + \nu(A \setminus B) + (\mu \vee \nu)(A) \geq \mu(B) + \nu(A \setminus B) + \mu(A \setminus B) + \nu(B) = \mu(A) + \nu(A).$$

Taking the infimum over all such B, we get

$$(\mu \wedge \nu)(A) + (\mu \vee \nu)(A) \geq \mu(A) + \nu(A).$$

Likewise, if $B \in \Omega$ and $B \subseteq A$, then

$$(\mu \wedge \nu)(A) + \mu(B) + \nu(A \setminus B) \leq \mu(A \setminus B) + \nu(B) + \mu(B) + \nu(A \setminus B) = \mu(A) + \nu(A),$$

and taking supremum over such B shows that

$$(\mu \wedge \nu)(A) + (\mu \vee \nu)(A) \leq \mu(A) + \nu(A).$$

Hence $\mu \vee \nu + \mu \wedge \nu = \mu + \nu$. □

Corollary 8.2. *Let (X, Ω) be a measurable space, and let E be a Dedekind complete partially ordered vector space.*

Then $\mathrm{M}(X, \Omega, E^+)$ is a lattice cone, and $\mu \vee \nu + \mu \wedge \nu = \mu + \nu$ for all $\mu,\ \nu \in \mathrm{M}(X, \Omega, E^+)$.

The lattice operations in $\mathrm{M}(X, \Omega, E^+)$ agree with those in $\mathrm{M}(X, \Omega, \overline{E^+})$.

Remark 8.3. For $E = \mathbb{R}$, it is a consequence of [**2**, Theorem 26.1]—stated in the context where Ω is a σ-algebra—that $\mathrm{M}(X, \Omega, \overline{\mathbb{R}^+})$ is a lattice cone. The infimum of two possibly infinite measures μ and ν always exists in $\mathrm{M}(X, \Omega, \overline{\mathbb{R}^+})$ and is again given by the right hand side of equation (8.2). Contrary to the real case, part (2) of Theorem 8.1 asserts the existence of the infimum only if both measures are finite. When attempting to prove this for possibly infinite measures, the following problem was encountered. Suppose that, for $n \geq 1$, subsets $S_n \subseteq \overline{E^+}$ with infima ℓ_n are given such that $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} s_n = \infty$ for all choices for $s_n \in S_n$. Is it then true that $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \ell_n = \infty$? For spaces where this holds, $\mu \wedge \nu$ also exists for infinite μ and ν, and it is given by the right hand side of equation (8.2). For $E = \mathbb{R}$, this is indeed the case. Arguing by contradiction, suppose that $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \ell_n = \sum_{n=1}^{\infty} \ell_n$ is finite. Then all ℓ_n are finite. For each n, we choose $s_n \in S_n$ such that $s_n \leq \ell_n + 1/2^n$. Then $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} s_n = \sum_{n=1}^{\infty} s_n \leq \sum_{n=1}^{\infty} (\ell_n + 1/2^n) = 1 + \sum_{n=1}^{\infty} \ell_n$ is finite. This is a contradiction, so that $\bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \ell_n$ is infinite. There seems to be no reason, however, why this property should hold in general. The above proof for $\mathbb{R}$ makes use of its one-dimensionality and the notion of a convergent series, and it does not seem to generalise naturally to other classes of spaces, especially infinite-dimensional ones.

We regard this once more as an illustration that one should be rather careful when surmising that results from the real case generalise to the vector-valued case, even when they seem to be of a purely order-theoretic nature.

We have the following upward order completeness result; the sequential case for $E = \mathbb{R}$ is [**2**, Theorem 26.2]. We are not aware of a downward version for $E = \mathbb{R}$.

Theorem 8.4. *Let (X, Ω) be a measurable space, and let E be a σ-monotone complete partially ordered vector space.*

If $\{\mu_n\}_{n=1}^{\infty}$ is an increasing sequence in $\mathrm{M}(X, \Omega, \overline{E^+})$, then the set function μ, defined by

$$\mu(A) = \bigvee \{ \mu_n(A) : n \geq 1 \}$$

for $A \in \Omega$, is a positive $\overline{E}$-valued measure, and $\mu_n \uparrow \mu$ in $\mathrm{M}(X, \Omega, \overline{E^+})$.

Suppose that E is actually monotone complete, and that $\{\mu_\lambda\}_{\lambda \in \Lambda}$ is an increasing net in $\mathrm{M}(X, \Omega, \overline{E^+})$. Then the set function μ, defined by

$$\mu(A) = \bigvee \{ \mu_\lambda(A) : \lambda \in \Lambda \}$$

for $A \in \Omega$, is a positive $\overline{E}$-valued measure, and $\mu_\lambda \uparrow \mu$ in $\mathrm{M}(X, \Omega, \overline{E^+})$.

It may look strange that the sequences and nets are not required to be bounded from above, since this is a consequence of the conclusion. Since $\mathrm{M}(X, \Omega, \overline{E^+})$ has a largest element this is, however, actually always the case.

Proof. We prove both statements at the same time. Let $\{\mu_\lambda\}_{\lambda \in \Lambda}$ be an increasing net (possibly an increasing sequence) in $\mathrm{M}(X, \Omega, \overline{E^+})$, and define $\mu : \Omega \to \overline{E^+}$ by

$$\mu(A) = \bigvee \{ \mu_\lambda(A) : \lambda \in \Lambda \}$$

for $A \in \Omega$. Note that the supremum exists in $\overline{E}$, as a consequence of either part (1) or part (3) of Lemma 2.5.

Clearly $\mu(\emptyset) = 0$ and $\mu_\lambda(A) \leq \mu(A)$ for all $\lambda \in \Lambda$ and $A \in \Omega$. Since the μ_λ are monotone, so is μ. Furthermore, part (2) of Lemma 2.4 implies that the finite additivity of the μ_λ is inherited by μ.

Suppose that $\{A_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω such that $A := \bigcup_{n=1}^{\infty} A_n$ is measurable. Then, for all $\lambda \in \Lambda$,

$$\mu_\lambda(A) = \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu_\lambda(A_n) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n).$$

Hence $\mu(A) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n)$. On the other hand, μ is monotone and finitely additive, so, for all $N \geq 1$, we have $\mu(A) \geq \mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu(A_n)$. Hence also $\mu(A) \geq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n)$. We conclude that μ is σ-additive. Now that we know that $\mu \in \mathrm{M}(X, \Omega, \overline{E^+})$, it is evident that μ is the supremum of the μ_λ in $\mathrm{M}(X, \Omega, \overline{E^+})$. □

Combining the previous results, we have the following order completeness property. The asymmetry between the finite and the infinite case regarding the upper bounds is again not really there, since $\mathrm{M}(X, \Omega, \overline{E^+})$ has a largest element.

Corollary 8.5. *Let (X, Ω) be a measurable space, and let E be a Dedekind complete partially ordered vector space. Suppose that $\{ \mu_\lambda : \lambda \in \Lambda \}$ is an arbitrary non-empty subset of $\mathrm{M}(X, \Omega, \overline{E^+})$.*

Then $\bigvee \{ \mu_\lambda : \lambda \in \Lambda \}$ exists in $\mathrm{M}(X, \Omega, \overline{E^+})$.

If there exists a finite measure μ such that $\mu_\lambda \leq \mu$ for all λ, then $\bigvee \{ \mu_\lambda : \lambda \in \Lambda \}$ is a finite measure.

Proof. Let Φ denote the collection of finite subsets of Λ. For $\phi \in \Phi$, set $\mu_\phi := \bigvee_{\lambda \in \phi} \mu_\lambda$; this supremum exists in view of Theorem 8.1. The sets $\{ \mu_\phi : \phi \in \Phi \}$ and $\{ \mu_\lambda : \lambda \in \Lambda \}$ evidently have the same upper bounds in $\mathrm{M}(X, \Omega, \overline{E^+})$. Since $\{ \mu_\phi : \phi \in \Phi \}$ is upward directed, Theorem 8.4 shows that it has a least upper bound in $\mathrm{M}(X, \Omega, \overline{E^+})$; this is then also the least upper bound of $\bigvee \{ \mu_\lambda : \lambda \in \Lambda \}$.

The final statement is clear. □

8.2. Cones of measures on locally compact Hausdorff spaces

The results on the existence of the supremum and infimum of two measures in Theorem 8.1 and Corollary 8.2 apply to $\mathrm{M}(X, \mathscr{B}, \overline{E^+})$ and $\mathrm{M}(X, \mathscr{B}, E^+)$. We shall now show that the various regularity properties are preserved under the cone and lattice operations. The supremum and infimum of two measures in Lemmas 8.6 to 8.9 are those in $\mathrm{M}(X, \mathscr{B}, \overline{E^+})$ from Theorem 8.1.

Lemma 8.6. *Let X be a locally compact Hausdorff space, and let E be a Dedekind complete partially ordered vector space.*

If $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ are Borel measures and if $r \in \mathbb{R}^+$, then $\mu + \nu$, $r\mu$, and $\mu \vee \nu$ are Borel measures.

If $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, E^+)$ are Borel measures, then $\mu \wedge \nu$ is a Borel measure.

Proof. Since $\mu \vee \nu \leq \mu + \nu$ and, for finite μ and ν, also $\mu \wedge \nu \leq \mu$, this is all clear. □

Lemma 8.7. *Let X be a locally compact Hausdorff space, and let E be a Dedekind complete partially ordered vector space.*

If $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ are inner regular at $A \in \mathscr{B}$ and if $r \in \mathbb{R}^+$, then $\mu + \nu$, $r\mu$, and $\mu \vee \nu$ are inner regular at A.

If $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, E^+)$ are inner regular at $A \in \mathscr{B}$, then $\mu \wedge \nu$ is inner regular at A.

Proof. Let $A \in \mathscr{B}$ be such that μ and ν are inner regular at A.

For the sum, we have, using part (1) of Lemma 2.4,

$$
\begin{aligned}
(\mu + \nu)(A) &= \mu(A) + \nu(A) \\
&= \bigvee \{ \mu(K_1) : K_1 \text{ is compact and } K_1 \subseteq A \} \\
&\qquad + \bigvee \{ \nu(K_2) : K_2 \text{ is compact and } K_2 \subseteq A \} \\
&= \bigvee \{ \mu(K_1) + \nu(K_2) : K_i \text{ is compact and } K_i \subseteq A \text{ for } i = 1, 2 \} \\
&\leq \bigvee \{ \mu(K_1 \cup K_2) + \nu(K_1 \cup K_2) : K_i \text{ is compact and } K_i \subseteq A \text{ for } i = 1, 2 \} \\
&= \bigvee \{ \mu(K) + \nu(K) : K \text{ is compact and } K \subseteq A \} \\
&= \bigvee \{ (\mu + \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&\leq (\mu + \nu)(A).
\end{aligned}
$$

We conclude that $\mu + \nu$ is inner regular at A.

The statement concerning $r\mu$ is immediate from part (7) of Lemma 2.4.

We turn to $\mu \vee \nu$.

Suppose that $(\mu \vee \nu)(A) = \infty$. Since $\mu \vee \nu \leq \mu + \nu$, we see that also $\mu(A) + \nu(A) = \infty$. So at least one of $\mu(A)$ and $\nu(A)$ equals ∞; suppose that $\mu(A) = \infty$. Then

$$\bigvee \{ \mu(K) : K \text{ is compact and } K \subseteq A \} = \infty.$$

Using $\mu \vee \nu \geq \mu$ we then also have that

$$\bigvee \{ \mu \vee \nu(K) : \text{ is compact and } K \subseteq A \} = \infty.$$

Hence $\mu \vee \nu$ is inner regular at A.

Suppose that $(\mu \vee \nu)(A)$ is finite. Then

$$\begin{aligned}
0 &\leq (\mu \vee \nu)(A) - \bigvee \{ (\mu \vee \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&= \bigwedge \{ (\mu \vee \nu)(A) - (\mu \vee \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&= \bigwedge \{ (\mu \vee \nu)(A \setminus K) : K \text{ is compact and } K \subseteq A \} \\
&\leq \bigwedge \{ \mu(A \setminus K) + \nu(A \setminus K) : K \text{ is compact and } K \subseteq A \} \\
&= \bigwedge \{ (\mu + \nu)(A) - (\mu + \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&= (\mu + \nu)(A) - \bigvee \{ (\mu + \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&= 0,
\end{aligned}$$

where the final equality holds since we already know that $\mu + \nu$ is inner regular at A. This shows that $\mu \vee \nu$ is inner regular at A,

We turn to $\mu \wedge \nu$ for $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, E^+)$. Using Theorem 8.1 and the inner regularity of $\mu + \nu$ at A that we have already demonstrated, we have

$$\begin{aligned}
(\mu \wedge \nu)(A) &\geq \bigvee \{ (\mu \wedge \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&= \bigvee \{ (\mu + \nu)(K) - (\mu \vee \nu)(K) : K \text{ is compact and } K \subseteq A \} \\
&\geq \bigvee \{ \mu + \nu)(K) - (\mu \vee \nu)(A) : K \text{ is compact and } K \subseteq A \} \\
&= \bigvee \{ (\mu + \nu)(K) : K \text{ is compact and } K \subseteq A \} - (\mu \vee \nu)(A) \\
&= (\mu + \nu)(A) - (\mu \vee \nu)(A) \\
&= (\mu \wedge \nu)(A).
\end{aligned}$$

We conclude that $\mu \wedge \nu$ is inner regular at A. $\square$

Lemma 8.8. *Let X be a locally compact Hausdorff space and let E be a Dedekind complete partially ordered vector space.*

If $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ are outer regular at $A \in \mathscr{B}$ and if $r \in \mathbb{R}^+$, then $\mu + \nu$, $r\mu$, and $\mu \vee \nu$ are outer regular at A.

If $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, E^+)$ are outer regular at $A \in \mathscr{B}$, then $\mu \wedge \nu$ is outer regular at A.

Proof. Let $A \in \mathscr{B}$ be such that μ and ν are outer regular at A.

For the sum, first suppose that $(\mu + \nu)(A) = \mu(A) + \nu(A) = \infty$. Then at least one of $\mu(A)$ and $\nu(A)$ equals ∞; suppose that $\mu(A) = \infty$. Then

$$\bigwedge \{ \mu(V) : V \text{ is open and } A \subseteq V \} = \infty.$$

That is, $\mu(V) = \infty$ for all open subsets V such that $A \subset V$. Then certainly $(\mu + \nu)(V) = \infty$ for such V, so that

$$\bigwedge \{ (\mu + \nu)(V) : V \text{ is open and } A \subseteq V \} = \bigwedge \{\infty\} = \infty.$$

This shows that $\mu + \nu$ is outer regular at A.

Suppose that $(\mu + \nu)(A)$ is finite. Then

$$\begin{aligned}
&(\mu + \nu)(A) \\
&\quad = \mu(A) + \nu(A) \\
&\quad = \bigwedge \{ \mu(V_1) : V_1 \text{ is open and } A \subseteq V_1 \} + \bigwedge \{ \nu(V_2) : V_2 \text{ is open and } A \subseteq V_2 \}
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge \{ \mu(V_1) + \nu(V_2) : V_i \text{ is open and } A \subseteq V_i \text{ for } i = 1,2 \} \\
&\geq \bigwedge \{ \mu(V_1 \cap V_2) + \nu(V_1 \cap V_2) : V_i \text{ is open and } A \subseteq V_i \text{ for } i = 1,2 \} \\
&= \bigwedge \{ \mu(V) + \nu(V) : V \text{ is open and } A \subseteq V \} \\
&= \bigwedge \{ (\mu + \nu)(V) : V \text{ is open and } A \subseteq V \} \\
&\geq (\mu + \nu)(A).
\end{aligned}
$$

This shows that $\mu + \nu$ is outer regular at A.

The statement concerning $r\mu$ is immediate from part (7) of Lemma 2.4.

We turn to $\mu \vee \nu$.

Suppose that $(\mu \vee \nu)(A) = \infty$. Since $(\mu \vee \nu)(A) \leq \mu(A) + \nu(A)$, at least one of $\nu(A)$ and $\nu(A)$ equals ∞; suppose that $\mu(A) = \infty$. Then the outer regularity of μ implies that $\mu(V) = \infty$ for all open subsets V such that $A \subseteq V$. Since $(\mu \vee \nu)(V) \geq \mu(V) = \infty$ for such V, we see that

$$\bigwedge \{ (\mu \vee \nu)(V) : V \text{ is open and } A \subseteq V \} = \bigwedge \{\infty\} = \infty.$$

Hence $\mu \vee \nu$ is outer regular at A.

Suppose that $(\mu \vee \nu)(A)$ is finite. Since $(\mu \vee \nu)(A) \geq \mu(A)$ and $(\mu \vee \nu)(A) \geq \nu(A)$, $\nu(A)$ and $\nu(A)$ are then also finite. The outer regularity of $\mu + \nu$ then implies that there exists an open subset V_0 such that $A \subseteq V_0$ and such that $(\mu + \nu)(V_0)$ is finite. Now that we have located an open neighbourhood V_0 of A such that $(\mu + \nu)(V_0)$ is finite, equivalently, $(\mu \vee \nu)(V_0)$ is finite. Using outer regularity of $\mu + \nu$ in the fourth step, we see that,

$$
\begin{aligned}
0 &\leq \bigvee \{ (\mu \vee \nu)(V_0) \colon V_0 \supseteq A,\ V_0 \text{ is open and } (\mu \vee \nu)(V_0) \in E \} - (\mu \vee \nu)(A) \\
&= \bigvee \{ (\mu \vee \nu)(V_0 \setminus A) \colon V_0 \supseteq A,\ V_0 \text{ is open and } (\mu \vee \nu)(V_0) \in E \} \\
&\leq \bigvee \{ (\mu + \nu)(V_0 \setminus A) \colon V_0 \supseteq A,\ V_0 \text{ is open and } (\mu + \nu)(V_0) \in E \} \\
&= 0.
\end{aligned}
$$

We therefore see that

$$(\mu \vee \nu)(A) = \bigwedge \{ (\mu \vee \nu)(V) : V \text{ is open and } A \subseteq V \}.$$

We conclude that $\mu \vee \nu$ is outer regular at A.

We turn to $\mu \wedge \nu$ for $\mu, \nu \in \mathrm{M}(X, \mathscr{B}, E^+)$. Using Theorem 8.1 and the outer regularity of $\mu + \nu$ at A that we have already demonstrated, we have

$$
\begin{aligned}
(\mu \wedge \nu)(A) &\leq \bigwedge \{ (\mu \wedge \nu)(U) : U \text{ is open and } A \subseteq U \} \\
&= \bigwedge \{ (\mu + \nu)(U) - (\mu \vee \nu)(U) : U \text{ is open and } A \subseteq U \} \\
&\leq \bigwedge \{ \mu + \nu)(U) - (\mu \vee \nu)(A) : U \text{ is open and } A \subseteq U \} \\
&= \bigwedge \{ (\mu + \nu)(U) : U \text{ is open and } A \subseteq U \} - (\mu \vee \nu)(A) \\
&= (\mu + \nu)(A) - (\mu \vee \nu)(A) \\
&= (\mu \wedge \nu)(A).
\end{aligned}
$$

We conclude that $\mu \wedge \nu$ is outer regular at A.

□

Lemma 8.9. *Let X be a locally compact Hausdorff space, and let E be a Dedekind complete partially ordered vector space.*

If μ, $\nu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ are weakly inner regular at $A \in \mathscr{B}$ and if $r \in \mathbb{R}^+$, then $\mu + \nu$, $r\mu$, and $\mu \vee \nu$ are weakly inner regular at A.

If μ, $\nu \in \mathrm{M}(X, \mathscr{B}, E^+)$ are weakly inner regular at $A \in \mathscr{B}$, then $\mu \wedge \nu$ is weakly inner regular at A.

PROOF. Let $A \in \mathscr{B}$ be such that μ and ν are weakly inner regular at A.

For the sum, we have, using part (1) of Lemma 2.4 and the fact that Υ is closed under finite unions,

$$\begin{aligned}
(\mu + \nu)(A) &= \mu(A) + \nu(A)\\
&= \bigvee \{\, \mu(A \cap U_1) : U_1 \in \Upsilon \,\} + \bigvee \{\, \mu(A \cap U_2) : U_2 \in \Upsilon \,\}\\
&= \bigvee \{\, \mu(A \cap U_1) + \nu(A \cap U_2) : U_i \in \Upsilon \text{ for } i = 1, 2 \,\}\\
&\leq \bigvee \{\, \mu(A \cap (U_1 \cup U_2)) + \nu(A \cap (U_1 \cup U_2)) : U_i \in \Upsilon \text{ for } i = 1, 2 \,\}\\
&= \bigvee \{\, (\mu + \nu)(A \cap U) : U \in \Upsilon \,\}\\
&\leq (\mu + \nu)(A).
\end{aligned}$$

We conclude that $\mu + \nu$ is weakly inner regular at A.

The statement concerning $r\mu$ is immediate from part (7) of Lemma 2.4.

We turn to $\mu \vee \nu$.

Suppose that $(\mu \vee \nu)(A) = \infty$. Since $\mu \vee \nu \leq \mu + \nu$, we see that also $\mu(A) + \nu(A) = \infty$. So at least one of $\mu(A)$ and $\nu(A)$ equals ∞; suppose that $\mu(A) = \infty$. Then

$$\bigvee \{\, \mu(A \cap U) : U \in \Upsilon \,\} = \infty.$$

Using $\mu \vee \nu \geq \mu$ we then also have that

$$\bigvee \{\, (\mu \vee \nu)(A \cap U) : U \in \Upsilon \,\} = \infty.$$

Hence $\mu \vee \nu$ is weakly inner regular at A.

Suppose that $(\mu \vee \nu)(A)$ is finite. Then

$$\begin{aligned}
0 &\leq (\mu \vee \nu)(A) - \bigvee \{\, (\mu \vee \nu)(A \cap U) : U \in \Upsilon \,\}\\
&= \bigwedge \{\, (\mu \vee \nu)(A) - (\mu \vee \nu)(A \cap U) : U \in \Upsilon \,\}\\
&= \bigwedge \{\, (\mu \vee \nu)(A \setminus U) : U \in \Upsilon \,\}\\
&\leq \bigwedge \{\, \mu(A \setminus U) + \nu(A \setminus U) : U \in \Upsilon \,\}\\
&= \bigwedge \{\, (\mu + \nu)(A) - (\mu + \nu)(A \cap U) : U \in \Upsilon \,\}\\
&= (\mu + \nu)(A) - \bigvee \{\, (\mu + \nu)(U) : U \in \Upsilon \,\}\\
&= 0,
\end{aligned}$$

where the final equality holds since we already know that $\mu + \nu$ is weakly inner regular at A. This shows that $\mu \vee \nu$ is weakly inner regular at A,

We turn to $\mu \wedge \nu$ for μ, $\nu \in \mathrm{M}(X, \mathscr{B}, E^+)$. Using Theorem 8.1 and the fact that $\mu + \nu$ is weakly inner regular at A that we have already demonstrated, we have

$$\begin{aligned}
(\mu \wedge \nu)(A) &\geq \bigvee \{\, (\mu \wedge \nu)(A \cap U) : U \in \Upsilon \,\}\\
&= \bigvee \{\, (\mu + \nu)(A \cap U) - (\mu \vee \nu)(A \cap U) : U \in \Upsilon \,\}\\
&\geq \bigvee \{\, \mu + \nu)(A \cap U) - (\mu \vee \nu)(A) : U \in \Upsilon \,\}
\end{aligned}$$

$$\begin{aligned}
&= \bigvee \{ (\mu+\nu)(A\cap U) : U \in \Upsilon \} - (\mu\vee\nu)(A) \\
&= (\mu+\nu)(A) - (\mu\vee\nu)(A) \\
&= (\mu\wedge\nu)(A).
\end{aligned}$$

We conclude that $\mu\wedge\nu$ is weakly inner regular at A.

□

The following is immediate from Lemmas 8.6 to 8.9.

Theorem 8.10. *Let X be a locally compact Hausdorff space, and let E be a monotone complete partially ordered vector space.*

Then $\mathrm{M}_{\mathrm{rB}}(X,\mathscr{B},\overline{E^+})$ and $\mathrm{M}_{\mathrm{qrB}}(X,\mathscr{B},\overline{E^+})$ are cones.

Furthermore, if E is Dedekind complete then the supremum of two measures in any of these two cones exists and equals the supremum of the two measures in $\mathrm{M}(X,\mathscr{B},\overline{E^+})$ from Theorem 8.1. Moreover, $\mathrm{M}_{\mathrm{rB}}(X,\mathscr{B},E^+)$ and $\mathrm{M}_{\mathrm{qrB}}(X,\mathscr{B},E^+)$ are lattice cones. The supremum (resp. the infimum) of two measures in any of these two cones equals their supremum (resp. their infimum) in $\mathrm{M}(X,\mathscr{B},\overline{E^+})$ from Theorem 8.1.

Remark 8.11. For $E=\mathbb{R}$, if μ and ν are real-valued σ-finite regular Borel measures on X, it is stated in [**2**, Theorem 28.5] that the right hand side of equation (8.2) is again a regular Borel measure on X. Hence the real-valued σ-finite regular Borel measures on X form a lattice cone. In Theorem 8.10, this is, for general E, asserted only for the finite regular Borel measures.

§9. Integration with respect to a positive $\overline{E}$-valued measure

In this section, we start by introducing the integral with respect to positive $\overline{E}$-valued measures, and conclude by establishing three basic convergence theorems: the monotone convergence theorem (Theorem 9.9), Fatou's lemma (Theorem 9.12), and the dominated convergence theorem (Theorem 9.13). Each of these has the corresponding theorem for $E=\mathbb{R}$ as a special case.

As mentioned in Section 1, Wright was the first to establish an integration theory in a related context in [**24**], but we go a bit further than the results in [**24**].

9.1. Integrals

Consider a measure space (X,Ω,μ,E), where Ω is a σ-algebra, E is a σ-monotone complete partially ordered vector space, and $\mu:\Omega\to\overline{E^+}$ is a positive $\overline{E}$-valued measure. In this section, we shall introduce an integral of real-valued functions corresponding to these data. For some of the convergence theorems involving non-negative functions, such as the monotone convergence theorem, it is, in fact, more convenient (even more natural) to work with functions taking values in the extended positive real numbers. This will, therefore, be our starting point.

Also introducing some notation, we start with the usual definitions and elementary results, referring to e.g. [**5**, p. 49-52] for details.

We supply $\overline{\mathbb{R}^+}$ with the topology of the one-point compactification of $\mathbb{R}^+$. Then a function $f:X\to\overline{\mathbb{R}^+}$ is *measurable* if $f^{-1}(A)\in\Omega$ for every Borel subset A of $\overline{\mathbb{R}^+}$. One can equivalently require that $\{x\in X : f(x)\geq r\}\in\Omega$, that $\{x\in X : f(x)>r\}\in\Omega$, that $\{x\in X : f(x)\leq r\}\in$

Ω, or, finally, that $\{ x \in X : f(x) < r \} \in \Omega$ for all (finite) $r \in \mathbb{R}^+$. It is (already) here that we use that Ω is not just an algebra, but actually a σ-algebra. If f takes only finite values, then f is measurable if and only if $f^{-1}(A) \in \Omega$ for every Borel subset A of $\mathbb{R}^+$.

We let $\mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ denote the set of measurable functions $f : X \to \overline{\mathbb{R}^+}$. Then $\mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ is a lattice cone, and $fg \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ for all $f, g \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$; here the operations are defined pointwise in $\overline{\mathbb{R}^+}$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$, then $\inf_{n\geq 1} f_n$, $\sup_{n\geq 1} f_n$, $\liminf_{n\to\infty} f_n$, and $\limsup_{n\to\infty} f_n$ are all measurable; here again the operations are defined pointwise in $\overline{\mathbb{R}^+}$. We shall write $f_n \uparrow$ if $f_n(x) \uparrow$ in $\overline{\mathbb{R}^+}$ for all $x \in X$. Since the function $f : X \to \overline{\mathbb{R}^+}$, defined by $f(x) = \sup_{n\geq 1} f_n(x)$ for $x \in X$, is then actually measurable, it is the supremum of the f_n in $\mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$. We can thus write $f_n \uparrow f$ to express a pointwise condition, while at the same time being consistent with our earlier notations.

An element φ of $\mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ is an *elementary function* if it takes only finitely many values, which are all finite. It can be written (generically non-uniquely) as a finite sum $\varphi = \sum_{i=1}^{n} r_i \chi_{A_i}$ for some $r_1, \ldots, r_n \in \mathbb{R}^+$ and $A_1, \ldots, A_n \in \Omega$. Here the r_i are all finite, but it is allowed that $\mu(A_i) = \infty$ for some of the A_i. We let $\mathscr{E}(X, \Omega; \mathbb{R}^+)$ denote the set of elementary functions. It is a lattice cone.

If $\varphi = \sum_{i=1}^{n} r_i \chi_{A_i}$ is an elementary function, where the A_i are pairwise disjoint, then we define its (order) integral, which is an element of $\overline{E^+}$, by

$$\int_X^{\mathrm{o}} \varphi \, \mathrm{d}\mu := \sum_{i=1}^{n} r_i \mu(A_i),$$

where $r_i\mu(A_i)$ refers to the action of the semigroup $\mathbb{R}^+$ as semigroup homomorphisms of $\overline{E}$. We have added a superscript to indicate that the integral that we shall introduce for more general functions is defined using order properties. In contexts where E is also a Banach space, integrals with respect to vector-valued measures can be defined using norm convergence; this notation keeps the distinction clear.

The facts that μ is finitely additive and that $\mathbb{R}^+$ acts as semigroup homomorphisms on $\overline{E}$ imply that the integral does not depend on the choice for the pairwise disjoint A_i; the proof is exactly as for [**5**, Lemma 10.2]. These two facts, combined with the fact that $(rs)x = r(sx)$ for all $r, s \in \mathbb{R}^+$ and $x \in \overline{E}$, also yield the following, where the equalities and the inequality are in $\overline{E}$. At this stage, Ω can actually still be just an algebra.

Lemma 9.1. *Let (X, Ω, μ, E) be a measure space. Let $\varphi, \psi \in \mathscr{E}(X, \Omega; \mathbb{R}^+)$, and let $r \geq 0$. Then:*

(1) $\int_X^{\mathrm{o}} r\varphi \, \mathrm{d}\mu = r\int_X^{\mathrm{o}} \varphi \, \mathrm{d}\mu$;
(2) $\int_X^{\mathrm{o}} (\varphi + \psi) \, \mathrm{d}\mu = \int_X^{\mathrm{o}} \varphi \, \mathrm{d}\mu + \int_X^{\mathrm{o}} \psi \, \mathrm{d}\mu$;
(3) *if* $\varphi \leq \psi$, *then* $\int_X^{\mathrm{o}} \varphi \, \mathrm{d}\mu \leq \int_X^{\mathrm{o}} \psi \, \mathrm{d}\mu$.

It follows from this that, for $\varphi \in \mathscr{E}(X, \Omega; \mathbb{R}^+)$, $\int_X^{\mathrm{o}} \varphi \, \mathrm{d}\mu = \sum_{i=1}^{n} r_i \mu(A_i)$ whenever $\varphi = \sum_{i=1}^{n} r_i \chi_{A_i}$ for not necessarily disjoint $A_i \in \Omega$. The proofs for all this are exactly as in [**5**, p. 55-56].

We shall define the (order) integral of an arbitrary $f \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ in the usual way. In order to know that it is well-defined, we need the following preparatory result. The proof is rather analogous to that of [**5**, Theorem 11.1], but we still include it since it illustrates how the Archimedean property of E underlies the well-definedness of the integral.

Lemma 9.2. *Let (X,Ω,μ,E) be a measure space. Let $\varphi \in \mathscr{E}(X,\Omega;\mathbb{R}^+)$ and let $\{\varphi_n\}_{n=1}^\infty \subseteq \mathscr{E}(X,\Omega;\mathbb{R}^+)$ be such that $\phi_n \uparrow$ and $\varphi \leq \sup_{n\geq 1}\varphi_n$ in $\overline{\mathbb{R}^+}$. Then*

$$\int_X^{\mathrm{o}} \varphi \,\mathrm{d}\mu \leq \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu$$

in $\overline{E}$.

Proof. We can write $\varphi = \sum_{i=1}^m r_i\chi_{A_i}$ for some $r_i \in \mathbb{R}^+$ and $A_i \in \Omega$. Let ε be fixed such that $0<\varepsilon<1$. For $n\geq 1$, set $B_n := \{\, x \in X : \varphi_n(x) \geq (1-\varepsilon)\varphi(x)\,\}$. Then B_n is measurable and $B_n \uparrow X$, since $\varphi \leq \sup_{n\geq 1}\varphi_n$ and $\varphi_n\uparrow$. This implies that $B_n\cap A_i \uparrow A_i$ for $i=1,\cdots,m$, and then Proposition 5.5 shows that $\mu(A_i\cap B_n)\uparrow\mu(A_i)$ in $\overline{E}$ for $i=1,\ldots,m$. Since $\varphi_n \geq (1-\varepsilon)\chi_{B_n}\varphi$, Lemma 9.1 shows that

$$\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu \geq (1-\varepsilon)\int_X^{\mathrm{o}} \chi_{B_n}\varphi \,\mathrm{d}\mu \tag{9.1}$$

for $n\geq 1$. Furthermore,

$$\int_X^{\mathrm{o}} \chi_{B_n}\varphi \,\mathrm{d}\mu = \int_X^{\mathrm{o}} \sum_{i=1}^m r_i\chi_{B_n}\chi_{A_i}\,\mathrm{d}\mu = \int_X^{\mathrm{o}} \sum_{i=1}^m r_i\chi_{B_n\cap A_i}\,\mathrm{d}\mu = \sum_{i=1}^m r_i\mu(B_n\cap A_i).$$

Since $B_n\cap A_i\uparrow A_i$ for $i=1,\cdots,m$, part (7) of Lemma 2.3 and part (2) of Lemma 2.4 then yield that

$$\begin{aligned}\bigvee_{n=1}^\infty \int_X^{\mathrm{o}} \chi_{B_n}\varphi\,\mathrm{d}\mu &= \bigvee_{n=1}^\infty \sum_{i=1}^m r_i\mu(B_n\cap A_i) = \sum_{i=1}^m r_i \bigvee_{n=1}^\infty \mu(B_n\cap A_i)\\ &= \sum_{i=1}^m r_i\mu(A_i) = \int_X^{\mathrm{o}} \varphi\,\mathrm{d}\mu.\end{aligned}$$

Combining this with equation (9.1), we see that

$$\bigvee_{n=1}^\infty \int_X^{\mathrm{o}} \varphi_n\,\mathrm{d}\mu \geq (1-\varepsilon)\int_X^{\mathrm{o}} \varphi\,\mathrm{d}\mu.$$

in $\overline{E}$. If $\int_X^{\mathrm{o}}\varphi\,\mathrm{d}\mu = \infty$, then we take $\varepsilon = 1/2$ to see that $\bigvee_{n=1}^\infty \int_X^{\mathrm{o}}\varphi_n\,\mathrm{d}\mu = \infty$; we then have equality in the lemma. If $\int_X^{\mathrm{o}}\varphi\,\mathrm{d}\mu$ is finite, then we can write

$$\bigvee_{n=1}^\infty \int_X^{\mathrm{o}} \varphi_n\,\mathrm{d}\mu - \int_X^{\mathrm{o}} \varphi\,\mathrm{d}\mu \geq -\varepsilon\int_X^{\mathrm{o}} \varphi\,\mathrm{d}\mu.$$

Since this is true for every ε such that $0<\varepsilon<1$, this implies that

$$\bigvee_{n=1}^\infty \int_X^{\mathrm{o}} \varphi_n\,\mathrm{d}\mu - \int_X^{\mathrm{o}} \varphi\,\mathrm{d}\mu \geq -\bigwedge_{k=2}^\infty \frac{1}{k}\int_X^{\mathrm{o}} \varphi\,\mathrm{d}\mu.$$

Since E is Archimedean, the right hand side is zero. This concludes the proof. □

Supposing that Ω is a σ-algebra, let $f \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$. Then (see e.g. [**8**, proof of Proposition 2.1.7]) there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subseteq \mathscr{E}(X,\Omega;\mathbb{R}^+)$ such that $\varphi_n\uparrow f$; if f is bounded we may choose the φ_n such that, in addition $\varphi_n \to f$ uniformly on X. We define the (order) integral of f, which is an element of $\overline{E^+}$, by

$$\int_X^{\mathrm{o}} f \,\mathrm{d}\mu := \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu.$$

Since $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu \uparrow$, the σ-monotone completeness of E guarantees that this supremum exists in $\overline{E}$; see part (1) of Lemma 2.5. In order to show that this definition is independent of the choice for the sequence $\{\varphi_n\}_{n=1}^{\infty}$, let $\{\psi_n\}_{n=1}^{\infty} \subset \mathscr{E}(X, \Omega; \mathbb{R}^+)$ be a second sequence such that $\psi_n \uparrow f$. Then $\varphi_k \le f = \sup_{n\ge1} \psi_n$ in $\overline{\mathbb{R}^+}$ pointwise for all $k \ge 1$, so that Lemma 9.2 shows that $\int_X^{\mathrm{o}} \varphi_k \,\mathrm{d}\mu \le \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \psi_n \,\mathrm{d}\mu$ in $\overline{E}$. Hence $\bigvee_{k=1}^{\infty} \int_X^{\mathrm{o}} \varphi_k \,\mathrm{d}\mu \le \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \psi_n \,\mathrm{d}\mu$ in $\overline{E}$. The reverse inequality is likewise true, and we conclude that $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ is well-defined as an element of $\overline{E^+}$.

The integral has the usual properties. We include the proofs, which are completely analogous to that for the real case, for the sake of completeness.

Lemma 9.3. *Let (X, Ω, μ, E) be a measure space, where Ω is a σ-algebra, let $f_1, f_2 \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$, and let $r_1, r_2 \in \mathbb{R}^+$. Then:*

(1) $\int_X^{\mathrm{o}} (r_1 f_1 + r_2 f_2) \,\mathrm{d}\mu = r_1 \int_X^{\mathrm{o}} f_1 \,\mathrm{d}\mu + r_2 \int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$ *in* $\overline{E}$*;*
(2) *If* $f_1 \le f_2$ *in* $\overline{\mathbb{R}^+}$ *pointwise, then* $\int_X^{\mathrm{o}} f_1 \,\mathrm{d}\mu \le \int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$ *in* $\overline{E}$.

Proof. Choose a sequence $\{\varphi_n\}_{n=1}^{\infty} \subseteq \mathscr{E}(X, \Omega; \mathbb{R}^+)$ such that $\varphi_n \uparrow f_1$ in $\overline{\mathbb{R}^+}$ pointwise, and a sequence $\{\psi_n\}_{n=1}^{\infty} \subseteq \mathscr{E}(X, \Omega; \mathbb{R}^+)$ such that $\psi_n \uparrow f_2$ in $\overline{\mathbb{R}^+}$ pointwise. Then $r_1 \varphi_n + r_2 \psi \uparrow r_1 f + r_2 f$ in $\overline{\mathbb{R}^+}$ pointwise, so that part (1) follows from the definition of the integral, combined with part (7) of Lemma 2.3 and part (2) of Lemma 2.4.

Since, for all $n \ge 1$, $\varphi_n \le f_2 = \sup_{n\ge1} \psi_n$ in $\overline{\mathbb{R}^+}$ pointwise, Lemma 9.2 yields that $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu \le \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \psi_n \,\mathrm{d}\mu = \int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$. Hence $\int_X^{\mathrm{o}} f_1 \,\mathrm{d}\mu = \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu \le \int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$, which is part (2). ☐

Lemma 9.4. *Let (X, Ω, μ, E) be a measure space, where Ω is a σ-algebra, and let $f \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ be such that $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ is finite. Then:*

(1) *f is almost everywhere finite;*
(2) *the subset $\{x \in X : f(x) > 0\}$ is σ-finite.*

Proof. We prove part (1). For $n \ge 1$, set $A_n := \{x \in X : f(x) \ge n \text{ in } \overline{\mathbb{R}^+}\}$. Then $n\chi_{A_n} \le f$, so that $n\mu(A_n) = \int_X^{\mathrm{o}} n\chi_{A_n} \,\mathrm{d}\mu \le \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ by part (2) of Lemma 9.3. Hence $\mu(A_n) \le 1/n \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for all $n \ge 1$. From $\{x \in X : f(x) = \infty\} = \bigcap_{n=1}^{\infty} A_n$, we see that $\mu\{x \in X : f(x) = \infty\} \le 1/n \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for all $n \ge 1$. Since $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ is finite, the Archimedean property of E then implies that $\mu\{x \in X : f(x) = \infty\} \le 0$, which shows that $\mu\{x \in X : f(x) = \infty\} = 0$.

For part (2), set $B_n := \{x \in X : f(x) \ge 1/n\}$ in $\overline{\mathbb{R}^+}$ for $n \ge 1$. Then one sees similarly that $\mu(B_n) \le n \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$, which is finite. Since $\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} B_n$, part (2) is clear. ☐

Lemma 9.5. *Let (X, Ω, μ, E) be a measure space, where Ω is a σ-algebra. Let $f \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$. Then the following are equivalent:*

(1) $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu = 0$*;*
(2) $f(x) = 0$ *for almost all* $x \in X$

Proof. Choose a sequence $\{\varphi_n\}_{n=1}^{\infty} \subset \mathscr{E}(X,\Omega;\mathbb{R}^+)$ such that $\varphi_n \uparrow f$ in $\overline{\mathbb{R}^+}$ pointwise.

Suppose that part (1) holds. Then part (2) of Lemma 9.3 shows that $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu = 0$ for all $n \geq 1$. For the elementary functions φ_n, however, it is a direct consequence of the definition of their integrals that then $\mu(\{x \in X : \varphi_n(x) \neq 0\}) = 0$. Since $\{x \in X : f(x) \neq 0\}$ can be written as the union $\bigcup_{n=1}^{\infty} \{x \in X : \varphi_n(x) \neq 0\}$, the σ-sub-additivity of μ then implies that $\mu(\{x \in X : f(x) \neq 0\} = 0$. Hence part (1) implies part (2).

Suppose that part (2) holds. Since $0 \leq \varphi_n(x) \leq f(x)$ for all $x \in X$, we see that $\varphi_n(x) = 0$ for almost all $x \in X$. For the elementary functions φ_n, however, it is a direct consequence of the definition of their integrals that then $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu = 0$ for all n. Since $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu \uparrow \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$, it follows that $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu = 0$. Hence part (2) implies part (1). $\square$

Corollary 9.6. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra. Let $f_1, f_2 \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ and suppose that $f_1(x) = f_2(x)$ for almost all $x \in X$. Then $\int_X^{\mathrm{o}} f_1 \,\mathrm{d}\mu = \int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$ in $\overline{E}$.*

Proof. Set $A := \{x \in X : f_1(x) \neq f_2(x)\}$. Then (X,Ω) is a measurable subset of measure zero. There exists $g \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ such that $f_1 = g + f_1\chi_A$ and $f_2 = g + f_2\chi_A$. Then Lemma 9.3 and Lemma 9.5 show that $\int_X^{\mathrm{o}} f_1 \,\mathrm{d}\mu$ and $\int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$ are both equal to $\int_X^{\mathrm{o}} g \,\mathrm{d}\mu$. $\square$

A function $f : X \to \mathbb{R}$ is measurable if $f^{-1}(A)$ is a measurable subset of X for every Borel subset A of $\mathbb{R}$. If $f(x) \geq 0$ for all $x \in X$, then f is measurable in the present sense precisely if it is measurable in the earlier sense as a map from X into $\overline{\mathbb{R}^+}$. We shall write $\mathscr{M}(X,\Omega;\mathbb{R})$ for the Riesz space of all such finite-valued measurable functions and $\mathscr{M}(X,\Omega;\mathbb{R}^+)$ for its positive cone of all positive measurable functions. Set

$$\mathscr{N}(X,\Omega,\mu;\mathbb{R}) := \{f \in \mathscr{M}(X,\Omega;\mathbb{R}) : f(x) = 0 \text{ for almost all } x \in X\}.$$

Then $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is an order ideal in $\mathscr{M}(X,\Omega;\mathbb{R})$ and, by Lemma 9.5, we also have that $\mathscr{N}(X,\Omega,\mu;\mathbb{R}) = \{f \in \mathscr{M}(X,\Omega;\mathbb{R}) : \int_X^{\mathrm{o}} |f| \,\mathrm{d}\mu = 0\}$.

We let $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ denote the set of all $f \in \mathscr{M}(X,\Omega;\mathbb{R})$ such that $\int_X^{\mathrm{o}} |f| \,\mathrm{d}\mu$ is finite, and write $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ for the set of all positive measurable f with finite integral. Obviously, $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is an order ideal in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$. It is clear from the triangle inequality in $\mathbb{R}$ and Lemma 9.3 that $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ is a vector space under pointwise operations. It also follows from Lemma 9.3 and the fact that $f^+, f^- \leq |f| \leq f^+ + f^-$ that $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ if and only if $f^+ \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ and $f^- \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$. In particular, $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R}) \subseteq \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+) - \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$. Since $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ is a vector space, the reverse inclusion likewise holds, and we see that $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R}) = \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+) - \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$.

If $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$, we choose $f_1, f_2 \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ such that $f = f_1 - f_2$, and we define the (order) integral of f as $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu := \int_X^{\mathrm{o}} f_1 \,\mathrm{d}\mu - \int_X^{\mathrm{o}} f_2 \,\mathrm{d}\mu$, it is an element of E. Invoking Lemma 9.3, the usual arguments show that this is well-defined, and that it defines a linear map.

If $f \in \mathscr{N}(X,\Omega,\mu;\mathbb{R})$, then also $f^+, f^- \in \mathscr{N}(X,\Omega,\mu;\mathbb{R})$. Lemma 9.5 and the linearity of the integral then show that $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu = 0$.

We collect a few basic properties in the following result.

Proposition 9.7. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra.*

(1) *$\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ is an order ideal in the Riesz space $\mathscr{M}(X,\Omega;\mathbb{R})$. As a consequence, it is a σ-Dedekind complete Riesz space.*

(2) *$\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is a σ-order ideal in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$.*

(3) $\int_X^{\mathrm{o}} f \, \mathrm{d}\mu = 0$ *for all* $f \in \mathscr{N}(X,\Omega,\mu;\mathbb{R})$.
(4) *If* $f_1,\, f_2 \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ *and* $r_1,\, r_2 \in \mathbb{R}$*, then* $\int_X^{\mathrm{o}} (r_1 f_1 + r_2 f_2)\, \mathrm{d}\mu = r_1 \int_X^{\mathrm{o}} f_1 \, \mathrm{d}\mu + r_2 \int_X^{\mathrm{o}} f_2 \, \mathrm{d}\mu$.
(5) *If* $f_1,\, f_2 \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ *and* $f_1 \leq f_2$ *almost everywhere, then* $\int_X^{\mathrm{o}} f_1 \, \mathrm{d}\mu \leq \int_X^{\mathrm{o}} f_2 \, \mathrm{d}\mu$.

PROOF. It is clear from the above that $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ is a sublattice of $\mathscr{M}(X,\Omega;\mathbb{R})$. That it is actually also a solid subset follows from Lemma 9.3. The σ-Dedekind completeness of $\mathscr{M}(X,\Omega;\mathbb{R})$ is then inherited by its order ideal $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$.

In part (2), we have observed earlier that $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is an order ideal in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$. Suppose that $\{f_n\}_{n=1}^\infty \subset \mathscr{N}(X,\Omega,\mu;\mathbb{R})$ has a supremum f in the enveloping lattice. Replacing the sequence with the sequence $f_1, f_1 \vee f_2, f_1 \vee f_2 \vee f_3, \dots$, we may suppose that the sequence is increasing pointwise. Set $g(x) := \sup_{n \geq 1} f_n(x)$ for $x \in X$. Then $f \geq g \geq 0$ pointwise and $g \in \mathscr{M}(X,\Omega;\mathbb{R})$. On the other hand, the monotone convergence theorem (Theorem 9.9) will show that $\int_X^{\mathrm{o}} g \, \mathrm{d}\mu = 0$. Since $g \geq 0$, Lemma 9.5 implies that $g \in \mathscr{N}(X,\Omega,\mu;\mathbb{R}) \subseteq \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$. Since f is the least upper bound of the f_n in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$, we must have $f \leq g$ pointwise. We see that $f - g$ is an element of $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$, and conclude that $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is a σ-order ideal in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$.

The parts (3) and (4) have already been observed.

In part (5), we may, by redefining f_1 and f_2 to be zero at a set of measure zero, suppose that $f_1 \leq f_2$ pointwise. In that case, if $f_1 \leq f_2$, then $f_2 = f_1 + (f_2 - f_1)$ with $f_2 - f_1 \geq 0$. The additivity of the integral, which is positive on $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$, then shows that $\int_X^{\mathrm{o}} f_1 \, \mathrm{d}\mu \leq \int_X^{\mathrm{o}} f_2 \, \mathrm{d}\mu$. $\square$

We shall now introduce the generalisation of the classical L^1-space to the vector-valued case.

Since $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is an order ideal in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$, the quotient space

$$\mathrm{L}^1(X,\Omega,\mu;\mathbb{R}) := \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}) / \mathscr{N}(X,\Omega,\mu;\mathbb{R})$$

is again a Riesz space when the quotient is supplied with the partial ordering that is defined by the image of the positive cone $\mathscr{M}(X,\Omega;\mathbb{R}^+)$ under the quotient map. Furthermore, the quotient map is then a Riesz homomorphism. We shall write $[f]$ for the image of $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ under the quotient map.

In view of part (3) of Proposition 9.7, we can define an (order) integral on $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ by setting $\int_X^{\mathrm{o}} [f] \, \mathrm{d}\mu := \int_X^{\mathrm{o}} f \, \mathrm{d}\mu$ for $[f] \in \mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$. This integral is then linear, positive, and it is faithful on the positive cone $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ of $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$.

Recall that a Riesz space is said to be *order separable*, or to *have the countable sup-property*, if every non-empty subset that has a supremum contains an at most countably infinite subset that has the same supremum.

Theorem 9.8. *Let* (X,Ω,μ,E) *be a measure space, where* Ω *is a* σ*-algebra.*

Then the quotient map from $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ *onto* $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ *is a* σ*-order continuous Riesz homomorphism, and the map that sends* $[f] \in \mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ *to* $\int_X^{\mathrm{o}} f \, \mathrm{d}\mu \in E$ *is well-defined, linear, and positive on* $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$*. It is faithful on the positive cone* $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ *of* $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$*.*

If E *is order separable and monotone complete, then* $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ *is a Dedekind complete Riesz space. In that case, if* S *is a subset of* $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ *that is bounded from above and that is closed under taking finite suprema, and if* $\widetilde{f} \in \mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ *is such that* $[\widetilde{f}] = \bigvee \{ [f] : [f] \in S \}$*,* *then*

$$\int_X^{\mathrm{o}} \widetilde{f}\,\mathrm{d}\mu = \bigvee \left\{ \int_X^{\mathrm{o}} f\,\mathrm{d}\mu : f \in \mathrm{L}^1(X,\Omega,\mu;\mathbb{R}) \text{ and } [f] \in S \right\}. \tag{9.2}$$

PROOF. Since we know from part (2) of Proposition 9.7 that $\mathscr{N}(X,\Omega,\mu;\mathbb{R})$ is a σ-order ideal in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$, it follows from [**19**, Theorem 18.11] that the quotient map is σ-order continuous.

The basic properties of the integral have already been observed.

We shall now suppose that E is order separable and monotone complete, and proceed to show that $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ is then Dedekind complete. Equation (9.2) will be established during the proof.

Let S be a non-empty subset of $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ that is bounded from above by $[g]$ for some $g \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$. In order to show that its supremum exists, we may suppose that S is closed under taking finite suprema. Since the subset $\{\int_X^{\mathrm{o}} f\,\mathrm{d}\mu : f \in S\}$ of E is bounded from above by $\int_X^{\mathrm{o}} g\,\mathrm{d}\mu$, it has a supremum e in E. By the order separability of E, there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ such that $[f_n] \in S$ for all $n \geq 1$ and $e = \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu$. Using that the quotient map is a Riesz homomorphism and that S is closed under taking finite suprema, one easily sees that the sequence $f_1, f_1 \vee f_2, f_1 \vee f_2 \vee f_3, \ldots$ has the same properties. By passing to this new sequence we may and shall assume that also $f_n(x)\uparrow$ for every $x \in X$. Set $\widetilde{f}(x) := \bigvee_{n \geq 1} f_n(x)$ in $\overline{\mathbb{R}^+}$ for $x \in X$. The monotone convergence theorem (see Theorem 9.9) shows that $\int_X^{\mathrm{o}} \widetilde{f}\,\mathrm{d}\mu = \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu = e$. Since this is finite, Proposition 9.11 shows that $\widetilde{f}$ is almost everywhere finite. By redefining $\widetilde{f}$ and all f_n to be zero at the measurable subset of X where $\widetilde{f}$ equals ∞, we obtain a sequence $\{f_n\}_{n=1}^\infty \subseteq \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ and $\widetilde{f} \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$ such that $[f_n] \in S$ for all $n \geq 1$, $f_n \uparrow \widetilde{f}$ pointwise, and $\int_X^{\mathrm{o}} \widetilde{f}\,\mathrm{d}\mu = e$.

We claim that $[\widetilde{f}]$ is the supremum of S in $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$.

In order to show that $[\widetilde{f}]$ is an upper bound of S, let $[f] \in S$. Then $[f \vee f_n] = [f] \vee [f_n] \in S$ for all $n \geq 1$, so that $\int_X^{\mathrm{o}} f \vee f_n\,\mathrm{d}\mu \leq e$. Since $f \vee f_n \uparrow f \vee \widetilde{f}$ pointwise, we can use the monotone convergence theorem in the second step to see that

$$\begin{aligned} e &\geq \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} f \vee f_n\,\mathrm{d}\mu \\ &= \int_X^{\mathrm{o}} f \vee \widetilde{f}\,\mathrm{d}\mu \\ &\geq \int_X^{\mathrm{o}} \widetilde{f}\,\mathrm{d}\mu \\ &= e. \end{aligned}$$

We conclude that $\int_X^{\mathrm{o}} f \vee \widetilde{f} - \widetilde{f}\,\mathrm{d}\mu = 0$. Since $f \vee \widetilde{f} - \widetilde{f} \geq 0$, and since the integral is faithful on $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R}^+)$, this shows that $[f] \vee [\widetilde{f}] - [\widetilde{f}] = [f \vee \widetilde{f} - \widetilde{f}] = 0$. Hence $[\widetilde{f}] \geq [f]$.

In order to show that $[f]$ is the least upper bound of S, suppose that $[g]$ is any upper bound of S in $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$. Then certainly $[g] \geq [f_n]$ for all $n \geq 1$, i.e. $g(x) \geq f_n(x)$ for almost all $x \in X$. Collecting the countably infinitely many exceptional sets into one measurable subset of measure zero, we see that $g(x) \geq \bigvee_{n=1}^\infty f_n(x) = \widetilde{f}(x)$ for almost all $x \in X$. Hence $[g] \geq [\widetilde{f}]$.

This concludes the proof of the Dedekind completeness of $\mathrm{L}^1(X,\Omega,\mu;\mathbb{R})$ and equation (9.2).

□

9.2. Convergence theorems

We shall now establish the three basic convergence theorems in the partially ordered vector-valued context.

The monotone convergence theorem can be proved in the usual way.

Theorem 9.9 (Monotone convergence theorem). *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra. Let $\{f_n\}_{n=1}^\infty \subseteq \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ and $f \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ be such that $f_n(x)\uparrow f(x)$ in $\overline{\mathbb{R}^+}$ for almost all $x \in X$. Then*

$$\int_X^{\mathrm{o}} f_n \,\mathrm{d}\mu \uparrow \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$$

in $\overline{E}$.

Proof. In view of Lemma 9.5, we can, by redefining all f_n and f to be zero at a measurable subset of measure 0, suppose that $f_n(x)\uparrow f(x)$ in $\overline{\mathbb{R}^+}$ for all $x \in X$. For each $n \geq 1$, let $\{\varphi_i^n\}_{i=1}^\infty$ be a sequence in $\mathscr{E}(X,\Omega;\mathbb{R}^+)$ such that $\varphi_i^n \uparrow f_n$ in $\overline{\mathbb{R}^+}$ pointwise as $i \to \infty$. Set $\psi_n := \bigvee_{i=1}^n \varphi_n^i$ for $n \geq 1$. Then $\psi_n \in \mathscr{E}(X,\Omega;\mathbb{R}^+)$ for all $n \geq 1$ and $\psi_n \uparrow f$ in $\overline{E}$ pointwise. Hence $\int_X^{\mathrm{o}} f\,\mathrm{d}\mu = \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} \psi_n\,\mathrm{d}\mu$ by definition. On the other hand, since, for all $n \geq 1$, $\psi_n \leq f_n$ in $\overline{E}$ pointwise, we have $\int_X^{\mathrm{o}} \psi_n\,\mathrm{d}\mu \leq \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu$ for all $n \geq 1$. Hence $\int_X^{\mathrm{o}} f\,\mathrm{d}\mu \leq \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu$ in $\overline{E}$. Since the reverse inequality is clear, the proof is complete. □

Just as Proposition 5.5 implies Proposition 5.6, Theorem 9.9 implies our next result. It is a special case of the dominated convergence theorem; see part (4) of Theorem 9.13.

Corollary 9.10. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra. Let $\{f_n\}_{n=1}^\infty \subseteq \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ and $f \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ be such that $f_n(x)\downarrow f(x)$ in $\overline{\mathbb{R}^+}$ for almost all $x \in X$. If $\int_X^{\mathrm{o}} f_1\,\mathrm{d}\mu$ is finite, then*

$$\int_X^{\mathrm{o}} f_n \,\mathrm{d}\mu \downarrow \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$$

in E.

Proof. In view of 9.4 and Corollary 9.6, we can, after redefining all f_n and f to be zero at a suitable measurable subset of measure zero, suppose that the f_n have finite values and that $f_n \downarrow f$ pointwise. Then the functions $f_1 - f_n$ are well-defined elements of $\mathscr{M}(X,\Omega;\mathbb{R}^+)$. Since $\int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu$ is also finite for all n, we have $\int_X^{\mathrm{o}}(f_1-f_n)\,\mathrm{d}\mu = \int_X^{\mathrm{o}} f_1\,\mathrm{d}\mu - \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu$ for all $n \geq 1$. Similarly, $\int_X^{\mathrm{o}}(f_1-f)\,\mathrm{d}\mu = \int_X^{\mathrm{o}} f_1\,\mathrm{d}\mu - \int_X^{\mathrm{o}} f\,\mathrm{d}\mu$. Since $(f_1-f_n)\uparrow(f_1-f)$, an application of Theorem 9.9 shows that $\left(\int_X^{\mathrm{o}} f_1\,\mathrm{d}\mu - \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu\right) \uparrow \left(\int_X^{\mathrm{o}} f_1\,\mathrm{d}\mu - \int_X^{\mathrm{o}} f\,\mathrm{d}\mu\right)$ in E. We conclude that $\int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu \downarrow \int_X^{\mathrm{o}} f\,\mathrm{d}\mu$. □

A combination of Lemma 9.4 and Theorem 9.9 yields the following.

Proposition 9.11. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra. Let $\{f_n\}_{n=1}^\infty \subseteq \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ and $f \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$ be such that $f_n(x)\uparrow f(x)$ in $\overline{\mathbb{R}^+}$ for almost all $x \in X$, and suppose that $\bigvee_{n=1}^\infty \int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu$ is finite. Then $\{\, x \in X : \bigvee_{n=1}^\infty f_n(x) = \infty \,\}$ is a measurable subset of X of measure zero.*

As in the case where $E = \mathbb{R}$, Fatou's lemma is a consequence of the monotone convergence theorem.

Theorem 9.12 (Fatou's lemma). *Let (X, Ω, μ, E) be a measure space, where Ω is a σ-algebra and E is σ-Dedekind complete.*

If $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$, then

$$\int_X^{\mathrm{o}} \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \leq \bigvee_{n=1}^\infty \bigwedge_{k=n}^\infty \int_X^{\mathrm{o}} f_k \,\mathrm{d}\mu$$

in $\overline{E}$.

The σ-Dedekind completeness is necessary to guarantee that $\bigwedge_{k=n}^\infty \int_X^{\mathrm{o}} f_k \,\mathrm{d}\mu$ exists for all $n \geq 1$.

PROOF. For $n \geq 1$, set $g_n = \inf_{k\geq n} f_k$. Then $0 \leq g_n \leq f_k$ all $k \geq n$, so that Lemma 9.3 implies that

$$\int_X^{\mathrm{o}} g_n \,\mathrm{d}\mu \leq \bigwedge_{k\geq n} \int_X^{\mathrm{o}} f_k \,\mathrm{d}\mu \tag{9.3}$$

in $\overline{E}$ for all $n \geq 1$.

Applying Theorem 9.9 to the increasing sequence $g_n \uparrow \liminf_{n\geq 1} f_n$, and using equation (9.3), we then see that

$$\begin{aligned}\int_X^{\mathrm{o}} \liminf_{n\to\infty} f_n \,\mathrm{d}\mu &= \bigvee_{n=1}^\infty \int_X^{\mathrm{o}} g_n \,\mathrm{d}\mu \\ &\leq \bigvee_{n=1}^\infty \bigwedge_{k=n}^\infty \int_X^{\mathrm{o}} f_k \,\mathrm{d}\mu\end{aligned}$$

in $\overline{E}$.

□

As in the real case, the dominated convergence theorem follows from Fatou's lemma.

Theorem 9.13 (Dominated convergence theorem). *Let (X, Ω, μ, E) be a measure space, where Ω is a σ-algebra and E is σ-Dedekind complete.*

Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathscr{M}(X, \Omega; \mathbb{R})$, and let $f \in \mathscr{M}(X, \Omega; \mathbb{R})$ be such that $f_n(x) \to f(x)$ for almost all x in X.

If there exists $g \in \mathscr{M}(X, \Omega; \overline{\mathbb{R}^+})$ such that $\int_X^{\mathrm{o}} g \,\mathrm{d}\mu$ is finite, and such that, for all $n \geq 1$, $|f_n(x)| \leq g(x)$ in $\overline{\mathbb{R}^+}$ for almost all x in X, then:

(1) $f_n \in \mathscr{L}^1(X, \Omega, \mu; \mathbb{R})$ *for all* $n \geq 1$*;*
(2) $f \in \mathscr{L}^1(X, \Omega, \mu; \mathbb{R})$*;*
(3) $\bigwedge_{n=1}^\infty \bigvee_{k=n}^\infty \int_X^{\mathrm{o}} |f_k - f| \,\mathrm{d}\mu = 0$*;*
(4)
$$\int_X^{\mathrm{o}} f \,\mathrm{d}\mu = \bigvee_{n=1}^\infty \bigwedge_{k=n}^\infty \int_X^{\mathrm{o}} f_k \,\mathrm{d}\mu = \bigwedge_{n=1}^\infty \bigvee_{k=n}^\infty \int_X^{\mathrm{o}} f_k \,\mathrm{d}\mu.$$

PROOF. In view of Lemma 9.5, we may, by redefining all f_n, f, and g to be zero at a measurable subset of measure zero, suppose that $g \in \mathscr{M}(X,\Omega;\mathbb{R}^+)$, that $|f_n(x)| \leq g(x)$ for all $x \in X$, and that $f_n(x) \to f(x)$ for all $x \in X$. Since then also $|f|(x) \leq g(x)$ for all $x \in X$, part (1) of Proposition 9.7 shows that the f_n and f are in $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$.

We turn to part (3). Since $2g - |f_n - f| \geq 0$ pointwise, Theorem 9.12 shows that

$$\begin{aligned}\int_X^{\mathrm{o}} 2g \,\mathrm{d}\mu &= \int_X^{\mathrm{o}} \liminf_{n\geq 1}(2g - |f_n - f|)\,\mathrm{d}\mu \\ &\leq \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\int_X^{\mathrm{o}}(2g - |f_n - f|)\,\mathrm{d}\mu \\ &= \int_X^{\mathrm{o}} 2g\,\mathrm{d}\mu - \bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\int_X^{\mathrm{o}}|f_n - f|\,\mathrm{d}\mu,\end{aligned}$$

where the final equality is valid since $\int_X^{\mathrm{o}}(g - |f_n - f|)\,\mathrm{d}\mu$, $\int_X^{\mathrm{o}} g\,\mathrm{d}\mu$, and $\int_X^{\mathrm{o}}|f_n - f|\,\mathrm{d}\mu$ all lie in the finite order interval $[0, 2\int_X^{\mathrm{o}} g\,\mathrm{d}\mu]$ of E. Cancelling the finite element $\int_X^{\mathrm{o}} 2g\,\mathrm{d}\mu$, we see that $\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\int_X^{\mathrm{o}}|f_n - f|\,\mathrm{d}\mu \leq 0$. Since the reverse inequality is obvious, the proof of part (3) is complete.

We turn to part (4).

Since $g + f_n \geq 0$ for all $n \geq 1$, Fatou's lemma shows that

$$\begin{aligned}\int_X^{\mathrm{o}}(g + f)\,\mathrm{d}\mu &= \int_X^{\mathrm{o}}\liminf_{n\geq 1}(g + f_n)\,\mathrm{d}\mu \\ &\leq \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\int_X^{\mathrm{o}}(g + f_n)\,\mathrm{d}\mu \\ &= \int_X^{\mathrm{o}} g\,\mathrm{d}\mu + \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu,\end{aligned}$$

from which we see that

$$\int_X^{\mathrm{o}} f\,\mathrm{d}\mu \leq \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu. \tag{9.4}$$

Since $g - f_n \geq 0$ for all $n \geq 1$, Fatou's lemma shows that

$$\begin{aligned}\int_X^{\mathrm{o}}(g - f)\,\mathrm{d}\mu &= \int_X^{\mathrm{o}}\liminf_{n\geq 1}(g - f_n)\,\mathrm{d}\mu \\ &\leq \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\int_X^{\mathrm{o}}(g - f_n)\,\mathrm{d}\mu \\ &= \int_X^{\mathrm{o}} g\,\mathrm{d}\mu - \bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu,\end{aligned}$$

from which we see that

$$\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n\,\mathrm{d}\mu \leq \int_X^{\mathrm{o}} f\,\mathrm{d}\mu. \tag{9.5}$$

Combining equations (9.4) and (9.5) with Lemma 2.2, we have

$$\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n \,\mathrm{d}\mu \le \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \le \bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n \,\mathrm{d}\mu \le \bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}\int_X^{\mathrm{o}} f_n \,\mathrm{d}\mu,$$

which completes the proof of part (4). □

§10. Positive $\overline{E}$-valued measures arising from densities

As in the real case, given a measure μ and $f \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$, one can canonically construct a new measure from these data. In this section, we investigate the preservation of regularity of measures on locally compact Hausdorff spaces under the process. This will be an important ingredient when showing that finite measures representing positive algebra homomorphism from $\mathrm{C_c}(X)$ into partially ordered algebras are actually spectral measures; the latter notion will be introduced in Definition 11.3.

The first part of this section will still be in the context of an arbitrary measure space.

Proposition 10.1. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra. Let $f \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$. For $A \in \Omega$, set*

$$\mu^f(A) := \int_X^{\mathrm{o}} \chi_A f \,\mathrm{d}\mu.$$

Then $\mu^f : \Omega \to \overline{E^+}$ is a positive $\overline{E}$-valued measure, which is finite if and only if $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$.

If $A \subseteq X$ is a μ-null set, then A is a μ^f-null set.

If $g \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$, then

$$\int_X^{\mathrm{o}} g \,\mathrm{d}\mu^f = \int_X^{\mathrm{o}} gf \,\mathrm{d}\mu \tag{10.1}$$

in $\overline{E}$.

Proof. It is clear that μ^f is a positive $\overline{E}$-valued measure, since the monotone convergence theorem (Theorem 9.9) shows that it is σ-additive. It is likewise clear that it is a finite measure if and only if $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$, and that μ-null sets are μ^f-null sets.

For equation (10.1), consider $\varphi \in \mathscr{E}(X,\Omega;\mathbb{R}^+)$. Choose a representation $\varphi = \sum_{i=1}^n r_i\chi_{A_i}$ with $r_1,\dots,r_n \in \mathbb{R}^+$ and $A_1,\dots,A_n \in \Omega$. Then

$$\begin{aligned}\int_X^{\mathrm{o}} \varphi \,\mathrm{d}\mu^f &= \sum_{i=1}^n r_i\mu^f(A_i)\\ &= \sum_{i=1}^n r_i \int_X^{\mathrm{o}} \chi_{A_i} f \,\mathrm{d}\mu\\ &= \int_X^{\mathrm{o}} \left(\sum_{i=1}^n r_i\chi_{A_i}\right) f \,\mathrm{d}\mu\\ &= \int_X^{\mathrm{o}} \varphi f \,\mathrm{d}\mu.\end{aligned}$$

This establishes equation (10.1) for $g \in \mathscr{E}(X,\Omega;\mathbb{R}^+)$. For general $g \in \mathscr{M}(X,\Omega;\overline{\mathbb{R}^+})$, we choose a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathscr{E}(X,\Omega;\mathbb{R}^+)$ such that $\varphi_n(x) \uparrow g(x)$ for every x in X. Then $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu^f \uparrow \int_X^{\mathrm{o}} g \,\mathrm{d}\mu^f$ in $\overline{E}$ by definition, whereas $\int_X^{\mathrm{o}} \varphi_n f \,\mathrm{d}\mu \uparrow \int_X^{\mathrm{o}} g f \,\mathrm{d}\mu$ by Theorem 9.9. Hence equation (10.1) holds for g. ☐

We shall use the following two results in the context of locally compact Hausdorff spaces. The density f is now supposed to be bounded.

Lemma 10.2. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra. Let $f \in \mathscr{B}(X,\Omega;\mathbb{R}^+)$ and let $\{ S_i : i \in I \}$ be a subset of Ω such that $\bigwedge \{ \mu(S_i) : i \in I \} = 0$ in $\overline{E}$. Then $\bigwedge \{ \mu^f(S_i) : i \in I \} = 0$ in $\overline{E}$.*

Proof. It is clear that $\mu^f(S_i) \leq \|f\| \mu(S_i)$ in $\overline{E}$ for all $i \in I$. The statement then follows from part (7) of Lemma 2.3. ☐

Lemma 10.3. *Let (X,Ω,μ,E) be a measure space, where Ω is a σ-algebra and E is a monotone complete partially ordered vector space. Let $\mu \in \mathrm{M}(X,\Omega,\overline{E^+})$, and let $f \in \mathscr{B}(X,\Omega;\mathbb{R}^+)$. Suppose that $A \in \Omega$ is such that $\mu(A)$ is finite.*

(1) *If $\{ A_i : i \in I \}$ is a subset of Ω that is upward directed and such that $A_i \subseteq A$ for all $i \in I$ and*
$$\mu(A) = \bigvee \{ \mu(A_i) : i \in I \}$$
in $\overline{E}$, then
$$\mu^f(A) = \bigvee \{ \mu^f(A_i) : i \in I \}.$$
in $\overline{E}$.

(2) *If $\{ A_i : i \in I \}$ is a subset of Ω that is downward directed and such that $A \subseteq A_i$ for all $i \in I$ and*
$$\mu(A) = \bigwedge \{ \mu(A_i) : i \in I \}$$
in $\overline{E}$, then
$$\mu^f(A) = \bigwedge \{ \mu^f(A_i) : i \in I \}$$
in $\overline{E}$.

Proof. We prove part (1). We have
$$\mu(A) = \bigvee \{ \mu(A_i) : i \in I \}.$$
in $\overline{E}$. Since $\mu(A)$ and then also all $\mu(A_i)$ are finite, part (5) of Lemma 2.3 and part (4) of Lemma 5.4 show that
$$\bigwedge \{ \mu(A \setminus A_i) : i \in I \} = 0.$$
in $\overline{E}$. Applying Lemma 10.2, we find that
$$\bigwedge \{ \mu^f(A \setminus A_i) : i \in I \} = 0$$
in $\overline{E}$. Since $\mu^f(A)$ and all $\mu^f(A_i)$ are finite, part (6) of Lemma 2.3 and part (4) of Lemma 5.4 show that
$$\mu^f(A) = \bigvee \{ \mu^f(A_i) : i \in I \}$$
in $\overline{E}$.

We prove part (2). We have
$$\mu(A) = \bigwedge \{ \mu(A_i) : i \in I \}$$

in $\overline{E}$. Since $\mu(A)$ is finite, part (5) of Lemma 2.3 and part (4) of Lemma 5.4 show that

$$\bigwedge\{\,\mu(A_i\setminus A): i\in I\,\}=0$$

in $\overline{E}$. Applying Lemma 10.2, we find that

$$\bigwedge\{\,\mu^f(A_i\setminus A): i\in I\,\}=0$$

in $\overline{E}$. Since $\mu^f(A)$ is finite, part (6) of Lemma 2.3 and part (4) of Lemma 5.4 show that

$$\mu^f(A)=\bigwedge\{\,\mu^f(A_i): i\in I\,\}$$

in $\overline{E}$. □

We shall now apply the transition from μ to μ^f in the context of a locally compact Hausdorff space X with Borel σ-algebra $\mathscr{B}$, a monotone complete partially ordered vector space E, and a positive $\overline{E}$-valued measure $\mu:\mathscr{B}\to\overline{E^+}$. For $E=\mathbb{R}$, if $\mu:\mathscr{B}\to\overline{\mathbb{R}^+}$ is a regular Borel measure and if $f:X\to\overline{\mathbb{R}^+}$ is integrable, then $\mu^f:\mathscr{B}\to\overline{\mathbb{R}^+}$ is again a regular Borel measure; see [**14**, p. 220, Exercise 8]. The rest of this section is concerned with the vector-valued analogue of this phenomenon. These results are a key ingredient later on when establishing that a (sufficiently regular) finite measure representing a positive algebra homomorphism from $\mathrm{C_c}(X)$ into partially ordered algebras is actually a spectral measure; see Theorem 15.1.

The following result for finite measures is obvious from Lemma 10.3.

Corollary 10.4. *Let X be a locally compact Hausdorff space, let E be a monotone complete partially ordered vector space, let $\mu\in\mathrm{M}(X,\mathscr{B},E^+)$, and let $f\in\mathscr{B}(X,\mathscr{B};\mathbb{R}^+)$.*

(1) *If μ is a finite regular Borel measure on X, then so is μ^f.*
(2) *If μ is a finite quasi-regular Borel measure on X, then so is μ^f.*

We turn to the case where μ need not be finite.

Lemma 10.5. *Let X be a locally compact Hausdorff space, let E be a monotone complete partially ordered vector space, let $\mu\in\mathrm{M}(X,\mathscr{B},\overline{E^+})$, let $f\in\mathscr{B}(X,\mathscr{B};\mathbb{R}^+)$, and let $A\in\mathscr{B}$. Set $S:=\{\,x\in X: f(x)>0\,\}$.*

(1) *Suppose that at least one of the following is satisfied:*
   (a) *$\mu(A)$ is finite and μ is inner regular at A;*
   (b) *$\mu(S)$ is finite and μ is inner regular at $A\cap S$.*
   Then μ^f is inner regular at A.
(2) *If $\mu(A)$ is finite and μ is outer regular at A, then μ^f is outer regular at A.*
(3) *Suppose that at least one of the following is satisfied:*
   (a) *$\mu(A)$ is finite and μ is weakly inner regular at A;*
   (b) *$\mu(S)$ is finite and μ is weakly inner regular at $A\cap S$.*
   Then μ^f is weakly inner regular at A.

Proof. We prove part (1). The case where $\mu(A)$ is finite is immediate from part (1) of Lemma 10.3. In the second case, we note that obviously $\mu^f(A)=\mu^f(A\cap S)$. Since $A\cap S\subseteq S$ has finite μ-measure, we see, using what we have just established for $A\cap S$, that

$$\begin{aligned}\mu^f(A)&=\mu^f(A\cap S)\\&=\bigvee\{\,\mu^f(K): K\text{ is compact and }K\subseteq A\cap S\,\}\\&\leq\bigvee\{\,\mu^f(K): K\text{ is compact and }K\subseteq A\}\end{aligned}$$

$$\leq \mu^f(A).$$

Hence μ^f is inner regular at A.

Part (2) is immediate from part (2) of Lemma 10.3.

We prove part (3). As with part (1), the case where $\mu^f(A)$ is finite is immediate from part (1) of Lemma 10.3. In the second case, we note again that $\mu^f(A) = \mu^f(A \cap S)$, and that $A \cap S \subseteq S$ has finite μ-measure. Hence

$$\begin{aligned}\mu^f(A) &= \mu^f(A \cap S) \\ &= \bigvee \{\, \mu^f((A \cap S) \cap U) : U \in \Upsilon \,\} \\ &\leq \bigvee \{\, \mu^f(A \cap U) : U \in \Upsilon \,\} \\ &\leq \mu^f(A).\end{aligned}$$

Hence μ^f is weakly inner regular at A.

□

Our next result, again for possibly infinite measures, involves the restriction of measures to open subsets. If U is an open subset of X and if $f \in \mathscr{M}(X, \mathscr{B}; \mathbb{R}^+)$, then it is easy to see that, as in the real case,

$$\int_U^{\mathrm{o}} f|_U \,\mathrm{d}\mu_U = \int_X^{\mathrm{o}} \chi_U f \,\mathrm{d}\mu. \tag{10.2}$$

Indeed, this holds for characteristic functions, then for elementary functions, and then for f by the definition of the integral. Replacing f with $\chi_A f$ for $A \in \mathscr{B}_U$, this yields that $(\mu_U)^{f|_U}(A) = \mu^f(A)$, so that $(\mu_U)^{f|_U} = (\mu^f)_U$. We shall write μ_U^f for this common resulting measure on $\mathscr{B}_U$.

Theorem 10.6. *Suppose that E is a normal and monotone complete partially ordered vector space. Let $\mu \in \mathrm{M}(X, \mathscr{B}, \overline{E^+})$ be such that:*

1. *μ is a quasi-regular Borel measure on X;*
2. *μ_U is a regular Borel measure on U for all $U \in \Upsilon$.*

Let $f \in \mathscr{B}(X, \mathscr{B}; \mathbb{R}^+)$ be such that the subset $\{\, x \in X : f(x) > 0 \,\}$ of X has finite μ-measure. Then:

1. *μ^f is a quasi-regular Borel measure on X that is inner regular at all Borel sets;*
2. *μ_U^f is a finite regular Borel measure on U for all $U \in \Upsilon$.*

It follows from Corollary 7.8 that μ itself is also inner regular at all Borel sets, and from Lemma 7.4 that the μ_U are also finite measures for $U \in \Upsilon$. These are the properties that are inherited by μ^f.

Proof. We start by proving that μ^f is a quasi-regular Borel measure.

If K is compact, then K is contained in a relatively compact open subset U of X. Since μ_U is a finite measure, $\mu(K)$ is finite. Since f is bounded, it is then clear that $\mu^f(K)$ is finite. Hence μ^f is a Borel measure.

Since μ is weakly inner regular at all Borel sets, we know from Corollary 7.8 that μ is inner regular at all Borel sets. In particular, if $A \in \mathscr{B}$, then μ is inner regular at $A \cap \{\, x \in X : f(x) > 0 \,\}$. Lemma 10.5 then shows that μ^f is inner regular at A. Hence μ^f is inner regular at all Borel sets; in particular, μ^f is inner regular at all open subsets of X.

Let $A \in \mathscr{B}$. Since μ is weakly inner regular at all Borel sets, it is, in particular, weakly inner regular at $A \cap \{ x \in X : f(x) > 0 \}$. Lemma 10.5 then shows that μ^f is weakly inner regular at A. Hence μ^f is weakly inner regular at all Borel sets.

This concludes the proof of part (1).

Since the μ_U for $U \in \Upsilon$ are all finite regular Borel measures, Corollary 10.4 shows that the $(\mu_U)^{f|_U}$ are finite regular Borel measures. Since $(\mu_U)^{f|_U} = \mu_U^f$, this establishes part (2). □

Remark 10.7. The subtle point in Theorem 10.6 is the inner regularity of μ^f. This ultimately rests on Proposition 7.6, which uses the normality of E. Hence it is for the inner regularity that the normality of E is used.

§11. Positive measures with values in partially ordered algebras

In this section, we consider measures with values in partially ordered algebras. We study how these behave under left and right multiplication with a fixed positive element, and, in the context of a measure on a locally compact Hausdorff space, how regularity properties are inherited by the multiplied measures. The notion of a spectral measure with values in a partially ordered algebra is also introduced, and it is shown that the positive linear maps that these represent are positive algebra homomorphisms.

Let $\mathscr{A}$ be a σ-monotone complete partially ordered algebra, and let $\mu : 2^X \to \mathscr{A}$ be a set map. For $a \in \mathscr{A}^+$, we define $\mu_{a\cdot}, \mu_{\cdot a} : \Omega \to \mathscr{A}^+$ by

$$\mu_{a\cdot}(A) = a \cdot \mu(A),$$
$$\mu_{\cdot a}(A) = \mu(A) \cdot a.$$

Lemma 11.1. *Let (X, Ω) be a measurable space, where Ω is a σ-algebra. Suppose that $\mathscr{A}$ is a σ-monotone complete partially ordered algebra and that multiplication in $\mathscr{A}$ is σ-monotone continuous. Let $\mu \in \mathrm{M}(X, \Omega, \mathscr{A}^+)$, and let $a \in \mathscr{A}^+$.*

(1) *The set maps $\mu_{a\cdot}$ and $\mu_{\cdot a}$ are positive $\mathscr{A}$-valued measures. If $A \in \Omega$ is a μ-nullset, then it is a $\mu_{a\cdot}$-nullset and a $\mu_{\cdot a}$-nullset.*

(2) *If $f \in \mathscr{L}^1(X, \Omega, \mu; \mathbb{R})$, then $f \in \mathscr{L}^1(X, \Omega, \mu_{a\cdot}; \mathbb{R})$ and $f \in \mathscr{L}^1(X, \Omega, \mu_{\cdot a}; \mathbb{R})$, and*

$$\int_X^{\mathrm{o}} f \, \mathrm{d}\mu_{a\cdot} = a \cdot \left(\int_X^{\mathrm{o}} f \, \mathrm{d}\mu \right),$$
$$\int_X^{\mathrm{o}} f \, \mathrm{d}\mu_{\cdot a} = \left(\int_X^{\mathrm{o}} f \, \mathrm{d}\mu \right) \cdot a \tag{11.1}$$

in $\mathscr{A}$.

Proof. Part (1) is clear from the definitions; the σ-monotone continuity of the multiplication in E guarantees that $\mu_{a\cdot}$ and $\mu_{\cdot a}$ are again σ-additive. In part (2), we cover the case for $\mu_{a\cdot}$; the argument for $\mu_{\cdot a}$ is similar. It is clear from the definitions that equation (11.1) holds for characteristic functions of measurable subsets, and then it holds for elementary functions. If $f \in \mathscr{L}^1(X, \Omega, \mu; \mathbb{R}^+)$, we choose a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathscr{E}(X, \Omega; \mathbb{R}^+)$ such that $\varphi_n \uparrow f$ pointwise. Then the definition of the integral implies that

$$\int_X^{\mathrm{o}} f \, \mathrm{d}\mu_{a\cdot} = \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \varphi_n \, \mathrm{d}\mu_{a\cdot} = \bigvee_{n=1}^{\infty} a \cdot \left(\int_X^{\mathrm{o}} \varphi_n \, \mathrm{d}\mu \right) = a \cdot \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \varphi_n \, \mathrm{d}\mu = a \cdot \left(\int_X^{\mathrm{o}} f \, \mathrm{d}\mu \right),$$

where the penultimate inequality is validated by the fact that left multiplication in E is σ-monotone continuous. This shows that $\mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+) \subseteqq \mathscr{L}^1(X,\Omega,\mu_{a\cdot};\mathbb{R}^+)$, and that equation (11.1) holds for $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R}^+)$. For the general case, the proof follows by splitting $f \in \mathscr{L}^1(X,\Omega,\mu;\mathbb{R})$ into its positive and negative parts. ☐

For later use, we record the following obvious facts in the context of locally compact Hausdorff spaces.

Lemma 11.2. *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a monotone complete partially ordered algebra with monotone continuous multiplication, let $\mu \in \mathrm{M}(X,\mathscr{B},\mathscr{A}^+)$, let $a \in \mathscr{A}^+$, and let $A \in \mathscr{B}$.*

1. *If μ is inner regular at A, then so are $\mu_{a\cdot}$ and $\mu_{\cdot a}$.*
2. *If μ is outer regular at A, then so are $\mu_{a\cdot}$ and $\mu_{\cdot a}$.*
3. *If μ is weakly inner regular at A, then so are $\mu_{a\cdot}$ and $\mu_{\cdot a}$.*
4. *If μ is a regular Borel measure, then so are $\mu_{a\cdot}$ and $\mu_{\cdot a}$;*
5. *If μ is a quasi-regular Borel measure, then so are $\mu_{a\cdot}$ and $\mu_{\cdot a}$.*

In later sections, our interest will be in the existence of algebra-valued measures that have a special property. As a prelude, we already introduce the pertinent notion in the present section, and give a few elementary properties.

Definition 11.3. Let (X,Ω) be a measure space, where Ω is a σ-algebra, and Let $\mathscr{A}$ be a σ-monotone complete partially ordered algebra. A positive $\mathscr{A}$-valued measure $P : \Omega \to \mathscr{A}^+$ is called a *spectral measure* if

$$P(A_1 \cap A_2) = P(A_1)P(A_2) \tag{11.2}$$

for all $A_1, A_2 \in \Omega$.

If $\mathscr{A}$ has a unit element e, it is not required that $P(X) = e$.

By definition, a spectral measure is always finite. The reason for this is the following. Any reasonable multiplicative structure on $\overline{\mathscr{A}^+}$ would be such that $0\cdot\infty = 0$ and $a\cdot\infty = \infty$ for all $a > 0$. If P is not finite, then $P(X) = \infty$. In that case, if $A \in \Omega$, then $P(A) = P(A\cap X) = P(A)P(X) = P(A)\cdot\infty$. It follows that either $P(A) = 0$ or $P(A) = \infty$. Such degenerate measures are hardly of interest.

The following is clear.

Lemma 11.4. *Let (X,Ω) be a measurable space, where Ω is a σ-algebra, and let $\mathscr{A}$ be a σ-monotone complete partially ordered algebra. If $P : \Omega \to \mathscr{A}^+$ is a spectral measure, then $\{ P(A) : A \in \Omega \} \subseteq \mathscr{A}^+$ is an ordered commutative multiplicative semigroup that consists of positive idempotents in $\mathscr{A}$. It has an identity element $P(X)$, which is also its largest element; 0 is its smallest element. If $A_1 \cap A_2 = \emptyset$, then $P(A_1)P(A_2) = 0$.*

Lemma 11.5. *Let (X,Ω) be a measurable space, where Ω is a σ-algebra, let $\mathscr{A}$ be a σ-monotone complete partially ordered algebra such that multiplication is σ-order continuous, and let $P : \Omega \to \mathscr{A}^+$ be a spectral measure.*

If $f, g \in \mathscr{L}^1(X,\Omega,P;\mathbb{R}^+)$, then $fg \in \mathscr{L}^1(X,\Omega,P;\mathbb{R}^+)$, and

$$\int_X^{\mathrm{o}} fg\,\mathrm{d}P = \int_X^{\mathrm{o}} f\,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g\,\mathrm{d}P. \tag{11.3}$$

PROOF. It is an immediate consequence of the definition of the integral of elementary functions and equation (11.2) that equation (11.3) holds for $f, g \in \mathscr{E}(X, \Omega; \mathbb{R}^+)$. For general $f, g \in \mathscr{L}^1(X, \Omega, P; \mathbb{R}^+)$, we choose sequences $\{\varphi_n\}_{n=1}^\infty$, $\{\psi_n\}_{n=1}^\infty \subseteqq \mathscr{E}(X, \Omega; \mathbb{R}^+)$ such that $\varphi_n \uparrow f$ and $\psi_n \uparrow g$. Then $\phi_n \psi_n \uparrow f g$. Using Lemma 4.2 and the definition of the order integral, we see that

$$\int_X^{\mathrm{o}} f g \,\mathrm{d}P \uparrow \int_X^{\mathrm{o}} \varphi_n \psi_n \,\mathrm{d}P = \int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}P \cdot \int_X^{\mathrm{o}} \psi_n \,\mathrm{d}P \uparrow \int_X^{\mathrm{o}} f \,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g \,\mathrm{d}P.$$

□

The following is now clear from Lemma 11.5, Proposition 9.7, and Theorem 9.8.

Theorem 11.6. *Let (X, Ω) be a measurable space, where Ω is a σ-algebra, let $\mathscr{A}$ be a σ-monotone complete partially ordered algebra such that multiplication is σ-order continuous, and let $P : \Omega \to \mathscr{A}^+$ be a spectral measure.*

(1) *The space $\mathscr{L}^1(X, \Omega, P; \mathbb{R})$ is a σ-Dedekind complete Riesz algebra, and*

$$\int_X^{\mathrm{o}} f g \,\mathrm{d}P = \int_X^{\mathrm{o}} f \,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g \,\mathrm{d}P. \tag{11.4}$$

for $f, g \in \mathscr{L}^1(X, \Omega, P; \mathbb{R})$.

The σ-order ideal $\mathscr{N}(X, \Omega, P; \mathbb{R})$ in $\mathscr{L}^1(X, \Omega, P; \mathbb{R})$ is also an algebra ideal.

(2) *The space $\mathrm{L}^1(X, \Omega, P; \mathbb{R})$ is a Riesz algebra, and the quotient map from the Riesz algebra $\mathscr{L}^1(X, \Omega, P; \mathbb{R})$ to $\mathrm{L}^1(X, \Omega, P; \mathbb{R})$ is a σ-order continuous Riesz algebra homomorphism.*

(3) *The map that sends $[f] \in \mathrm{L}^1(X, \Omega, P; \mathbb{R})$ to $\int_X^{\mathrm{o}} f \,\mathrm{d}P \in \mathscr{A}$ is a well-defined positive algebra homomorphism from $\mathrm{L}^1(X, \Omega, P; \mathbb{R})$ into $\mathscr{A}$. It is faithful on the positive cone $\mathrm{L}^1(X, \Omega, \mu; \mathbb{R}^+)$ of $\mathrm{L}^1(X, \Omega, \mu; \mathbb{R})$.*

(4) *If $\mathscr{A}$ is order separable and monotone complete, then $\mathrm{L}^1(X, \Omega, P; \mathbb{R})$ is Dedekind complete.*

§12. Riesz representation theorem for $\mathrm{C_c}(X)$: normed case

In this section, we present our first generalisation of the classical Riesz representation theorem for $\mathrm{C_c}(X)$, where X is a locally compact Hausdorff space. Let us recall the theorem for convenience.

Theorem 12.1 (Riesz representation theorem for $\mathrm{C_c}(X)$; real case). *Let X be a locally compact Hausdorff space, and let $T : \mathrm{C_c}(X) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique regular Borel measure μ on X such that*

$$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$$

for all f in $\mathrm{C_c}(X)$.

The measure μ is finite if and only if T is continuous with respect to the supremum norm topology on $\mathrm{C_c}(X)$.

Theorem 12.3 shows that, apart from the equivalence statement about continuity and finiteness of the measure, this can be generalised to a larger class of normed partially ordered vector spaces.

We introduce the following usual notation.

Definition 12.2. If X is a locally compact Hausdorff space and S is an subset of X, then we shall write $f \prec S$ to denote that $f \in \mathrm{C_c}(X)$, that $0 \le f(x) \le 1$ for all $x \in X$, and that $\mathrm{supp}(f) \subseteq S$. We shall write $S \prec f$ to denote that $f \in \mathrm{C_c}(X)$, that $0 \le f(x) \le 1$ for all $x \in X$, and that $f(x) = 1$ for all $s \in S$.

Theorem 12.3 (Riesz representation theorem for $\mathrm{C_c}(X)$; normed case)**.** *Let E be a monotone complete partially ordered vector space that is also a normed space. Suppose that E has the following properties:*

(1) *If $x_\lambda \downarrow 0$ in E, then $\|x_\lambda\| \to 0$;*
(2) *If $\{x_n\}_{n=1}^\infty \subset E^+$ is such that $\sum_{n=1}^\infty \|x_n\| < \infty$, then $\bigvee_{N=1}^\infty \sum_{n=1}^N x_n$, which exists in $\overline{E}$, is actually finite.*
(3) *If $x \in E$ is such that $(x, x') \ge 0$ for all norm bounded positive σ-order continuous linear functionals on E, then $x \in E^+$.*

Let X be a non-empty locally compact Hausdorff space, and let $T : \mathrm{C_c}(X) \to E$ be a positive linear map.

Then there exists a unique regular Borel measure $\mu : \mathscr{B} \to \overline{E^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu$$

for all $f \in \mathrm{C_c}(X)$.

If V is a non-empty open subset of X, then

$$\mu(V) = \bigvee \{\, T(f) : f \prec V \,\} \tag{12.1}$$

in $\overline{E}$.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{\, T(f) : K \prec f \,\} \tag{12.2}$$

in $\overline{E}$.

Consequently, μ is a finite measure if and only if $\{\, T(f) : f \in \mathrm{C_c}(X)^+, \|f\| \le 1 \,\}$ is order bounded in E; this is automatically the case if X is compact.

If μ is a finite measure and if the norm of E is monotone on E^+, then T is continuous with respect to the norm topologies on $\mathrm{C_c}(X)$ (supremum norm) and E.

Let us mention explicitly that under (1) it is not required that $\|x_\lambda\| \downarrow 0$.

In the context of a normed Riesz space E, the requirement under (2) is the so-called Riesz-Fischer property. It characterises the Banach lattices among the normed Riesz spaces; see [**29**, Theorem 16.2].

Before we embark on the lengthy proof, let us note that the real numbers evidently satisfy all requirements in the theorem, so that it contains the classical theorem as a special case (one has to use the fact that a norm bounded subset of the real numbers is order bounded to obtain the full equivalence between continuity of T and the finiteness of the measure in Theorem 12.1).

More generally, *every Banach lattice with order continuous norm satisfies the requirements in the theorem*. Indeed, the first requirement is satisfied by definition, and the second one is a consequence of the well-known fact that the sum of a convergent series with positive terms in a normed Riesz space is also the supremum of the sequence of its partial sums; see e.g. [**29**, Theorem 15.3]. The third requirement follows from Proposition 3.10 and its proof, where it was also observed that a Banach lattice with order continuous norm is Dedekind complete.

The global layout of the following proof of Theorem 12.3 is modelled after that of [**2**, Theorem 28.3] for the real case; the part where it is established that the measure as constructed actually represents the positive linear map is essentially that of [**8**, Proposition 7.2.11] for the real case. Although our statement is a little stronger, the proof of equation (12.2) is essentially that of the corresponding statement in the real case in [**13**, Theorem 7.2].

The arguments in the proof rather strongly resemble those for the real case, but there are certainly modifications. The proof that the set map μ^* in the proof below is an outer measure requires a special argument in the vector-valued case. Furthermore, we shall use the fact that the space is Archimedean a few times; in the real case it is not always so obvious in the various proofs in the literature that this is a property of the real numbers that is crucial for the proof. For example, the part of the proof of [**13**, Theorem 7.2] where it is shown that the measure represents the integral concludes with an argument invoking the lattice structure of $\mathbb{R}$. That does not apply in the more general context, and one could, therefore, argue that it is the 'wrong' argument and that the Archimedean property of $\mathbb{R}$ should have been used instead. Even though this is adaptation is easily implemented, this shows once more that it is easy to go astray when thinking that a proof for the real case 'can obviously be generalised'. One has to go over each detail again.

Proof of Theorem 12.3. We define a set map $\mu^* : 2^X \to \overline{E}$ as follows. Set $\mu(\emptyset) := 0$, and set

$$\mu(V) := \bigvee \{\, T(f) : f \prec V \,\} \tag{12.3}$$

in $\overline{E}$ for all non-empty open subsets of X. Note that the set in the right hand side of this equation is upward directed as a consequence of the positivity of T, so that the supremum indeed exists in $\overline{E}$. Indeed, if $f_1, f_2 \prec V$, then $f_1 \vee f_2 \prec V$, and $T(f_1 \vee f_2) \geq T(f_1)$ and $T(f_1 \vee f_2) \geq T(f_2)$.

Evidently, $\mu(V) \geq 0$ for all open subsets V of X, and $\mu(V_1) \leq \mu(V_2)$ if V_1 and V_2 are open subsets of X such that $V_1 \subseteq V_2$.

For an arbitrary subset A of X, we set

$$\mu^*(A) := \bigwedge \{\, \mu(V) : V \text{ is open and } A \subseteq V \,\}. \tag{12.4}$$

in $\overline{E}$. Note that the set in the right hand side of this equation is downward directed, so that the infimum indeed exists in $\overline{E}$. Indeed, if V_1, V_2 are open subsets of X such that $A \subseteq V_1, V_2$, then $A \subseteq V_1 \cap V_2$, and $\mu^*(V_1 \cap V_2) \leq \mu^*(V_1)$ and $\mu^*(V_1 \cap V_2) \leq \mu^*(V_2)$ as a consequence of the monotonicity of μ on the open subsets of X that we have already observed. This monotonicity on the open subsets of X also shows that $\mu(V) = \mu^*(V)$ for all open subsets V of X. If V is an open subset of X, we shall write $\mu(V)$ for this common element of $\overline{E^+}$.

Evidently, $\mu^*(A) \geq 0$ for all subsets A of X, and $\mu^*(A_1) \leq \mu^*(A_2)$ if A_1 and A_2 are subsets of X such that $A_1 \subseteq A_2$.

The first main step in the proof consists of showing that μ^* is an outer measure.

We shall now proceed to show this.

Since $\mu^*(\emptyset) = 0$ and since the monotonicity of μ^* has already been established, we are left with the σ-sub-additivity.

We first establish the σ-sub-additivity of μ^* for open subsets of X, on which it coincides with μ. Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open subsets of X. Then $\bigcup_{n=1}^{\infty} V_n$ is open, and we are to show that $\mu\left(\bigcup_{n=1}^{\infty} V_n\right) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(V_n)$ in $\overline{E}$. In doing so, we may and shall suppose that all V_n are non-empty.

Suppose that $f \prec \bigcup_{n=1}^{\infty} V_n$. Since $\operatorname{supp}(f)$ is compact, there exists $m \geq 1$ such that $\operatorname{supp}(f) \subseteq \bigcup_{n=1}^{m} V_n$. Then [**21**, Theorem 2.13] furnishes $f_1, \ldots, f_m$ in $\mathrm{C_c}(X)$ such that $f_n \prec V_n$ for $n = 1, \ldots, m$ and $\sum_{n=1}^{m} f_n(x) = 1$ for all $x \in \operatorname{supp}(f)$. Then clearly $f \leq \sum_{n=1}^{m} f_n$, so

$$T(f) \leq \sum_{n=1}^{m} T(f_n) \leq \sum_{n=1}^{m} \mu(V_n) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(V_n).$$

in $\overline{E}$. This is true for all $f \prec \bigcup_{n=1}^{\infty} V_n$, so we have

$$\mu\left(\bigcup_{n=1}^{\infty} V_n\right) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(V_n)$$

in $\overline{E}$. Hence μ^* is σ-sub-additive on the open subsets of X.

We shall now establish the σ-sub-additivity of μ^* for arbitrary subsets of X.

Let $\{A_n\}_{n=1}^{\infty}$ be sequence of subsets of X. We are going to show that $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu^*(A_n)$ in $\overline{E}$. In doing so, we may and shall suppose that

$$x := \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n)$$

is actually an element of E. Then $\mu^*(A_n)$ is also finite for $n \geq 1$, so that, by part (3) of Lemma 2.3,

$$\mu^*(A_n) = \bigwedge \{\mu(V) : V \text{ is open}, A_n \subseteq V, \text{ and } \mu(V) \in E\}.$$

That is, $(\mu(V) - \mu^*(A_n)) \downarrow 0$ as V runs over the open neighbourhoods of A_n. By requirement (1) in the theorem, we also have $\|\mu(V) - \mu^*(A_n)\| \to 0$.

After these preparations, we fix a norm bounded positive σ-order continuous linear functional x' on E and $\varepsilon > 0$.

For each $n \geq 1$, we can then find an open neigbourhood V_n of A_n such that $\mu(V_n)$ is finite and $\|\mu(V_n) - \mu^*(A_n)\| < \varepsilon/2^n$. As a consequence of requirement (2), we can set

$$y := \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} (\mu(V_n) - \mu^*(A_n))$$

as an element of E.

Let $N \geq 1$. Then

$$\begin{aligned}\left[\sum_{n=1}^{N} \mu(V_n)\right] - x &\leq \sum_{n=1}^{N} \mu(V_n) - \sum_{n=1}^{N} \mu^*(A_n) \\ &= \sum_{n=1}^{N} (\mu(V_n) - \mu^*(A_n)) \\ &\leq y,\end{aligned}$$

so that

$$\sum_{n=1}^{N} \mu(V_n) \leq x + y.$$

We conclude that $\bigvee_{N=1}^{\infty}\sum_{n=1}^{N}\mu(V_n)$ is finite, and that

$$\bigvee_{N=1}^{\infty}\sum_{n=1}^{N}\mu(V_n)\leq x+y.$$

Since $\bigcup_{n=1}^{\infty}A_n\subseteq\bigcup_{n=1}^{\infty}V_n$, we then see, using the monotonicity of μ^* and the σ-sub-additivity of μ on the open subsets of X, that

$$\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right)\leq\mu^*\left(\bigcup_{n=1}^{\infty}V_n\right)=\mu\left(\bigcup_{n=1}^{\infty}V_n\right)\leq\bigvee_{N=1}^{\infty}\sum_{n=1}^{N}\mu(V_n)\leq x+y.$$

In particular, $\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right)$ is finite. Hence we see, using the σ-order continuity of x' in the second step, that

$$\begin{aligned}\left(\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right),x'\right)&\leq(x+y,x')\\&=(x,x')+\sum_{n=1}^{\infty}(\mu(V_n)-\mu^*(A_n),x')\\&\leq(x,x')+\sum_{n=1}^{\infty}\|\mu(V_n)-\mu^*(A_n)\|\|x'\|\\&\leq(x,x')+\varepsilon\|x'\|.\end{aligned}$$

Letting $\varepsilon\downarrow 0$, we conclude that

$$\left(\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right),x'\right)\leq(x,x').$$

Since this is true for all norm bounded positive σ-order continuous linear functionals x', requirement (3) implies that

$$\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right)\leq x=\bigvee_{N=1}^{\infty}\sum_{n=1}^{N}\mu(A_n).$$

We have now shown that μ^* is an outer measure.

The next step is to show that every Borel set is μ^*-measurable. The fact that the restriction of μ^* to the Borel σ-algebra is a regular Borel measure will simultaneously be established during the process.

This step is itself broken up into a number of intermediate results.

We claim that $\mu^*(K)$ is finite for every compact subset K of X.

To see this, suppose that K is a compact subset X. Using [**20**, Theorem 2.7], we can choose a relatively open subset V of X such that $K\subseteq V$. By Urysohn's Lemma (see [**20**, Theorem 2.12]), there exists $g\in\mathrm{C_c}(X)$ such that $\overline{V}\prec g$. From this, we see that

$$\mu^*(K)\leq\mu^*(V)=\mu(V)=\bigvee\{\,T(f):f\prec V\,\}\leq T(g),$$

so that $\mu^*(K)$ is finite.

We claim that μ^* is finitely additive on the compact subsets of X.

To see this, it is sufficient to show that $\mu^*(K_1\cup K_2)\geq\mu^*(K_1)+\mu^*(K_2)$ for any two disjoint compact subsets K_1 and K_2 of X. Since X is a Hausdorff space, there exist open subsets U_1 and U_2 of X such that $K_1\subseteq U_1$, $K_2\subseteq U_2$, and $U_1\cap U_2=\emptyset$. Let V be an arbitrary open neighbourhood of $K_1 \cup K_2$. Then $K_1 \subseteq V \cap U_1$, $K_2 \subseteq V \cap U_2$, and $(V \cap U_1) \cap (V \cap U_2) = \emptyset$. The latter disjointness implies that $f_1 + f_2 \prec V$ whenever $f_1 \prec V \cap U_1$ and $f \prec V \cap U_2$. Using this, we see that

$$\begin{aligned}
\mu^*(K_1) + \mu^*(K_2) &\leq \mu(V \cap V_1) + \mu(V \cap V_2) \\
&= \sum_{i=1}^{2} \bigvee \{\, T(f_i) : f_i \prec V \cap V_i \,\} \\
&= \bigvee \{\, T(f_1) + T(f_2) : f_1 \prec V \cap U_1 \text{ and } f_2 \prec V \cap U_2 \,\} \\
&= \bigvee \{\, T(f_1 + f_2) : f_1 \prec V \cap U_1 \text{ and } f_2 \prec V \cap U_2 \,\} \\
&\leq \bigvee \{\, T(f) : f \prec V \,\} \\
&= \mu(V).
\end{aligned}$$

Recalling the defining equation (12.4), we see that $\mu^*(K_1) + \mu(K_2) \leq \mu^*(K_1 \cup K_2)$, as required.

We claim that $\mu^*(\operatorname{supp}(f)) \geq T(f)$ *whenever* $f \prec X$.

Indeed, if V is any open neighbourhood of $\operatorname{supp}(f)$, then $f \prec V$, so that $T(f) \leq \mu(V)$. Recalling the defining equation (12.4), we can see that $\mu^*(\operatorname{supp}(f)) \geq T(f)$, as required.

We claim that

$$\mu(V) = \bigvee \{\, \mu^*(K) : K \textit{ is compact and } K \subseteq V \,\} \tag{12.5}$$

in $\overline{E}$ *for every open subset* V *of* X.

To see this, we may suppose that $V \neq \emptyset$. In that case, using the previous result and the monotonicity of μ^*, we have

$$\begin{aligned}
\mu(V) &= \bigvee \{\, T(f) : f \prec V \,\} \\
&\leq \bigvee \{\, \mu^*(\operatorname{supp} f) : f \prec V \,\} \\
&\leq \bigvee \{\, \mu^*(K) : K \text{ is compact and } K \subseteq V \,\} \\
&\leq \mu^*(V) \\
&= \mu(V).
\end{aligned}$$

This establishes our claim.

We claim that $\mu^*(K \cup V) = \mu^*(K) + \mu(V)$ *whenever* K *is a compact subset of* X*,* V *is an open subset of* X*, and* $K \cap V = \emptyset$.

To see this, we use equation (12.5) and the finite additivity of μ^* on the compact subsets of X to justify that

$$\begin{aligned}
\mu^*(K \cup V) &\leq \mu^*(K) + \mu(V) \\
&= \mu^*(K) + \bigvee \{\, \mu^*(K_V) : K_V \text{ is compact and } K_V \subseteq V \,\} \\
&= \bigvee \{\, \mu^*(K) + \mu^*(K_V) : K_V \text{ is compact and } K_V \subseteq V \,\} \\
&= \bigvee \{\, \mu^*(K \cup K_V) : K_V \text{ is compact and } K_V \subseteq V \,\} \\
&\leq \mu^*(K \cup V).
\end{aligned}$$

This establishes our claim.

We claim that every Borel set is μ^**-measurable.*

To see this, it is sufficient to show that every open subset of X is μ^*-measurable. Let V be an open subset of X. In order to show that V is μ^*-measurable, it is sufficient to show that $\mu^*(A) \geq \mu^*(A \cap V) + \mu^*(A \cap V^{\mathrm{c}})$ whenever A is a subset of X such that $\mu^*(A)$ is finite.

We first show this if A is an open subset of X. In that case, let K be a compact subset of $A \cap V$. Then $A = K \cup (A \setminus K)$ as a disjoint union. Since $A \setminus K$ is an open subset of X, our previous result shows that

$$\mu(A) = \mu^*(K) + \mu(A \setminus K).$$

Since $A \cap V^{\mathrm{c}} \subseteq A \cap K^{\mathrm{c}}$, this implies that

$$\mu^*(K) + \mu^*(A \cap V^{\mathrm{c}}) \leq \mu^*(K) + \mu(A \setminus K) = \mu(A).$$

Using equation (12.5) for the open subset $A \cap V$ of X, we see that

$$\mu(A \cap V) + \mu^*(A \cap V^{\mathrm{c}}) \leq \mu(A)$$

for all open subsets A of X, as desired.

Suppose now that A is an arbitrary subset of X such that $\mu^*(A)$ is finite. If U is an open neighbourhood of A such that $\mu(U)$ is finite, then, using what we have just established in the first step, we see that

$$\mu(U) \geq \mu(U \cap V) + \mu^*(U \cap V^{\mathrm{c}}) \geq \mu^*(A \cap V) + \mu^*(A \cap V^{\mathrm{c}}).$$

Since this is evidently also true if $\mu(U) = \infty$, the defining equation (12.4), shows that

$$\mu^*(A) \geq \mu^*(A \cap V) + \mu^*(A \cap V^{\mathrm{c}}).$$

This concludes the proof that all Borel sets are μ^*-measurable.

In the remainder of the proof, we shall write μ for the restriction of μ^* to the Borel σ-algebra of X.

We have already seen that μ is finite on the compact subsets of X. Furthermore, inner regularity at all open subsets of X was established in equation (12.5), and outer regularity at all Borel sets was built in by the defining equation (12.4).

Hence μ is a regular Borel measure on X.

Equation (12.1) *is simply the defining equation* (12.3).

We shall now establish equation (12.2).

For this, we need a preparatory result that we shall also use in the sequel of the proof.

We claim that $\mu(K) \leq T(f)$ whenever K is a compact subset of X and $f \in \mathrm{C_c}(X)$ is such that $\chi_K \leq f$.

To see this, fix ε such that $0 < \varepsilon < 1$, and set $U_\varepsilon = \{ x \in X : f(x) > 1 - \varepsilon \}$. Then U_ε is open and $K \subseteq U_\varepsilon$. If $g \prec U_\varepsilon$, then $g \leq (1-\varepsilon)^{-1} f$, so that

$$\mu(K) \leq \mu(U_\varepsilon) = \bigvee \{ T(g) : g \prec U_\varepsilon \} \leq (1-\varepsilon)^{-1} T(f).$$

Hence $\mu(K) \leq T(f) + \varepsilon\mu(K)$ for all $\epsilon > 0$. Since $\mu(K)$ is finite and E is Archimedean, this implies that $\mu(K) \leq T(f)$.

Continuing with the proof of equation (12.2), we see from what we have just demonstrated that $\mu(K) \leq \bigwedge \{ T(f) : K \prec f \}$. In order to establish the reverse inequality, let V be an open neighbourhood of K. By Urysohn's Lemma (see [**21**, Theorem 2.7]), there exists a function f such that $K \prec f \prec V$. Then $\mu(V) \geq T(f)$. Equation (12.4) then implies that

$$\mu(K) = \bigwedge \{ \mu(V) : V \text{ is open and } K \subseteq V \} \geq \bigwedge \{ T(f) : K \prec f \}.$$

This concludes the proof of equation (12.2).

We shall now show that $T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)$.

For this, we need a final preparatory result.

We claim that $\mu(K) \geq T(f)$ whenever K is a compact subset of X and $f \in \mathrm{C_c}(X)$ is such that $0 \leq f \leq \chi_K$.

To see this, let V be an open neighbourhood of K. Then $f \prec V$, so that $\mu(V) \geq T(f)$ by the defining equation (12.3), and then $\mu(K) \geq T(f)$ by the defining equation (12.4).

Continuing with the proof that $T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)$, we note that we may clearly suppose that $f \in \mathrm{C_c}(X)^+$. Fix $\varepsilon > 0$. For all $n \geq 1$, define $f_n : X \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 0 & \text{if } f(x) \leq (n-1)\varepsilon; \\ f(x) - (n-1)\varepsilon & \text{if } (n-1)\varepsilon < f(x) \leq n\varepsilon; \\ \varepsilon & \text{if } n\varepsilon < f(x). \end{cases}$$

A moment's thought shows that $f_n \in \mathrm{C_c}(X)^+$ and $f_n(X) \subseteq [0, \varepsilon]$ for all $n \geq 1$, and that $\sum_{n=1}^{\infty} f_n(x) = f(x)$, where the series is a finite sum for every $x \in X$. There exists $N \geq 1$ such that $f_n = 0$ for all $n > N$, so that actually $f = \sum_{n=1}^{N} f_n$ is a finite sum. We set $K_0 := \mathrm{supp}(f)$ and $K_n := \{ x \in X : f(x) \geq n\varepsilon \}$ for $n \geq 1$. Then $\varepsilon \chi_{K_n} \leq f_n \leq \varepsilon \chi_{K_{n-1}}$ for all $n \geq 1$. The first of these inequalities, written as $\chi_{K_n} \leq f_n/\varepsilon$, implies that $\mu(K_n) \leq T(f_n)/\varepsilon$ for $n \geq 1$ by one of the claims established above. The second of these inequalities, written as $f_n/\varepsilon \leq \chi_{K_{n-1}}$, implies that $\mu(K_{n-1}) \geq T(f_n)/\varepsilon$ for all $n \geq 1$ by our most recent auxiliary result. Thus we have $\varepsilon\mu(K_n) \leq T(f_n) \leq \varepsilon\mu(K_{n-1})$ for $n \geq 1$. It is evident from the monotonicity of the order integral that $\varepsilon\mu(K_n) \leq \int_X^{\mathrm{o}} f_n \, \mathrm{d}\mu \leq \varepsilon\mu(K_{n-1})$ for $n \geq 1$. Since $f = \sum_{n=1}^{N} f_n$, a summation of the inequalities shows that

$$\varepsilon \sum_{i=1}^{N} \mu(K_n) \leq T(f) \leq \varepsilon \sum_{i=1}^{N} \mu(K_{n-1})$$

and that

$$\varepsilon \sum_{i=1}^{N} \mu(K_n) \leq \int_X^{\mathrm{o}} f \, \mathrm{d}\mu \leq \varepsilon \sum_{i=1}^{N} \mu(K_{n-1}).$$

This implies that $T(f) - \int_X^{\mathrm{o}} f \, \mathrm{d}\mu \leq \varepsilon(\mu(K_0) - \mu(K_N)) \leq \varepsilon\mu(K_0) = \varepsilon\mu(\mathrm{supp}(f))$. Since $\varepsilon > 0$ is arbitrary and E is Archimedean, we see that $T(f) - \int_X^{\mathrm{o}} f \, \mathrm{d}\mu \leq 0$. The reverse inequality is likewise true by a similar argument, and we conclude that $T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)$, as required.

We turn to the uniqueness of μ.

Suppose that μ' is another regular Borel measure on X such that $T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu'$ for all $f \in \mathrm{C_c}(X)$. By regularity, it is sufficient to show that $\mu(K) = \mu'(K)$ for all compact subsets K of X.

Let V be a open neighbourhood of K. The Urysohn Lemma furnishes $f \in \mathrm{C_c}(X)$ with $K \prec f \prec V$ (see [**21**, Theorem 2.13]), and then

$$\mu'(K) = \int_X^{\mathrm{o}} \chi_K \, \mathrm{d}\mu' \leq \int_X^{\mathrm{o}} f \, \mathrm{d}\mu' = T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu \leq \int_X^{\mathrm{o}} \chi_V \, \mathrm{d}\mu = \mu(V).$$

Using the outer regularity of μ at K, we see that $\mu'(K) \leq \mu(K)$. The reverse inequality likewise holds, and we conclude that μ is the unique regular Borel measure on X that represents T.

If the space X is compact, then $\{ T(f) : f \in \mathrm{C_c}(X)^+, \|f\| \leq 1 \}$ is obviously order bounded in E, since it has $T(\mathbf{1})$ as an upper bound.

The statement concerning the continuity of T is obvious. $\square$

We have the following consequence.

Theorem 12.4. *Let X be a locally compact Hausdorff space, and let E be a monotone complete partially ordered vector spaces with the properties (1), (2), and (3) in Theorem 12.3.*

If $\mu : \mathscr{B} \to \overline{E^+}$ is a positive $\overline{E}$-valued regular Borel measure, define $T_\mu : \mathrm{C_c}(X) \to E$ by $T_\mu(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for $f \in \mathrm{C_c}(X)$.

Then the map $\mu \mapsto T_\mu$ is an order preserving cone isomorphism between the cone $\mathrm{M_{rB}}(X, \mathscr{B}, \overline{E^+})$ of positive $\overline{E}$-valued regular Borel measures on X and $\mathrm{L_r}(\mathrm{C_c}(X), E)^+$ of positive linear operators from $\mathrm{C_c}(X)$ into E.

Proof. The combination of Theorems 8.10 and 12.3 shows that the map is a cone isomorphism, and clearly $T_\mu \le T_\nu$ if $\mu \le \nu$. Conversely, if $T_\mu \le T_\nu$, then equation (12.1) shows that $\mu(V) \le \nu(V)$ for all open subsets V of X. The outer regularity of μ and ν at all Borel sets then implies that $\mu \le \nu$. $\square$

It is easy to give examples of positive linear maps $T : \mathrm{C_c}(X) \to E$ where Theorem 12.3 applies and where the representing measure μ is infinite. Indeed, let X be a locally compact Hausdorff space and let E be a Banach lattice with order continuous norm. Suppose that (ρ, e) is a pair where ρ is a regular Borel measure $\rho : \mathscr{B} \to \overline{\mathbb{R}^+}$, and where $e \in E^+ \setminus \{0\}$. Set $T_{\rho,e}(f) := \int_X f \,\mathrm{d}\rho \cdot e$ for $f \in \mathrm{C_c}(X)$. Then $T_{\rho,e} : \mathrm{C_c}(X) \to E$ is positive, and its representing measure $\mu_{\rho,e}$ is given by $\mu_{\rho,e}(A) = \rho(A) \cdot e$ for $A \in \mathscr{B}$. Hence $\mu_{\rho,e}$ is infinite precisely if ρ is infinite. Although this yields examples where the representing measures are infinite, these are hardly interesting. However, one can use such operators as building blocks for others. Indeed, if $(\rho_1, e_1), \dots, (\rho_n, e_n)$ are n such pairs and ρ_1 is infinite, then $\bigvee_{i=1}^n T_{\rho_i,e_i} \in \mathrm{L_r}(\mathrm{C_c}(X), E)^+$, and its representing measure is again infinite. This is immediate from the fact that $\bigvee_{i=1}^n T_{\rho_i,e_i} \ge T_{\rho_1,e_1}$ and equation (12.1); one does not need Theorem 12.4 for this. Subsequently, one can take positive linear combinations of such finite suprema to obtain a non-zero cone of positive operators from $\mathrm{C_c}(X)$ into E such that every non-zero element has an infinite representing measure.

§13. Riesz representation theorems for $\mathrm{C_c}(X)$: normal case

In this section, we present two representation theorems for positive linear maps $T : \mathrm{C_c}(X) \to E$ in the case where E is a normal and monotone complete partially ordered vector space that need not be a normed space.

The idea is to use the classical Riesz representation theorem (which is now a special case of Theorem 12.3) as a stepping stone by invoking the positive order continuous linear functionals on E—of which there are sufficiently many by the normality of E.

Of course, linear functionals can act only on finite elements of $\overline{E}$, and this the reason that we first prove a representation theorem where the set $\{\, T(f) : f \prec X \,\}$ is supposed to be order bounded. This will serve to guarantee that everything in sight is a finite element of $\overline{E}$. It is thus that Theorem 13.4 for the finite case is established. The second representation theorem for the general case is then obtained from the finite case; see Theorem 13.5.

We shall have additional use for Theorem 13.4 for the finite case later on; see e.g. the proofs of Theorems 14.7 and 14.10.

We start by introducing a notation.

Suppose that (X,Ω) is a measurable space, that E is a σ-monotone complete partially ordered vector space, and that $\mu:\Omega\to E^+$ is a (finite-valued) set map. If $x':E\to\mathbb{R}$ is a positive linear functional, we define $\mu_{x'}:\Omega\to\mathbb{R}^+$ by $\mu_{x'}(A):=(\mu(A),x')$ for $A\in\Omega$.

The following is immediate from the definitions and Proposition 3.9.

Lemma 13.1. *Let (X,Ω) be a measurable space, let E be a σ-monotone complete partially ordered vector space, and let $\mu:\Omega\to E^+$ be a set map.*

(1) *If $\mu:\Omega\to E^+$ is a measure, then $\mu_{x'}:\Omega\to\mathbb{R}^+$ is a finite measure for all $x'\in(E_{\mathrm{c}}^{\sim})^+$.*
(2) *If E is σ-normal, then the following are equivalent:*
   (a) *$\mu:\Omega\to E^+$ is a finite measure;*
   (b) *$\mu_{x'}:\Omega\to\mathbb{R}^+$ is a finite measure for all $x'\in(E_{\mathrm{c}}^{\sim})^+$.*

Likewise, the following follows directly from the definitions and Proposition 3.8.

Lemma 13.2. *Let X be a locally compact Hausdorff space with Borel σ-algebra $\mathscr{B}$, let E be normal and monotone complete partially ordered vector space, and let $\mu:\Omega\to E^+$ be a set map.*

Then the following are equivalent:

(1) *$\mu:\Omega\to E^+$ is a finite measure;*
(2) *$\mu_{x'}:\Omega\to\mathbb{R}^+$ is a finite measure for all $x'\in(E_{\mathrm{n}}^{\sim})^+$.*

If this is the case, then the following are equivalent for a Borel set A:

(1) *$\mu:\Omega\to E^+$ is inner regular (resp. outer regular, resp. weakly inner regular) at A;*
(2) *$\mu_{x'}:\Omega\to\mathbb{R}^+$ is inner regular (resp. outer regular, resp. weakly inner regular) at A for all $x'\in E_{\mathrm{n}}^{\sim}$.*

We shall have use for the following immediate consequence of the above two results. Of course, finite measures are always Borel measures, but we have still included the redundant adjective for reasons of uniformity.

Corollary 13.3. *Let X be a locally compact Hausdorff space with Borel σ-algebra $\mathscr{B}$, let E be normal and monotone complete partially ordered vector space, and let $\mu:\Omega\to E^+$ be a set map.*

Then the following are equivalent:

(1) *$\mu:\Omega\to E^+$ is a finite regular (resp. quasi-regular) Borel measure on X;*
(2) *$\mu_{x'}:\Omega\to\mathbb{R}^+$ is a finite regular (resp. quasi-regular) Borel measure on X for all $x'\in(E_{\mathrm{n}}^{\sim})^+$.*

After these preparations, we can now establish our first representation theorem for normal and monotone complete partially ordered vector spaces.

Theorem 13.4 (Riesz representation theorem for $\mathrm{C_c}(X)$; finite normal case)**.** *Let X be a locally compact Hausdorff space, let E be a normal and monotone complete partially ordered vector space. Let $T:\mathrm{C_c}(X)\to E$ be a positive linear map such that $\{\,T(f):f\in\mathrm{C_c}(X)^+,\ \|f\|\le 1\,\}$ is order bounded in E; this is automatically the case if X is compact.*

Then there exists a unique regular Borel measure $\mu:\mathscr{B}\to\overline{E^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f)=\int_X^{\mathrm{o}} f\,\mathrm{d}\mu$$

for all $f\in\mathrm{C_c}(X)$.

The measure μ is finite.

If V is a non-empty open subset of X, then

$$\mu(V)=\bigvee\{\,T(f):f\prec V\,\}\tag{13.1}$$

in E.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{\, T(f) : K \prec f \,\} \tag{13.2}$$

in E.

PROOF. If the space X is compact, then $\{\, T(f) : f \in \mathrm{C_c}(X)^+, \|f\| \leq 1 \,\}$ is obviously order bounded in E, since it has $T(\mathbf{1})$ as an upper bound.

For a general locally compact space, suppose that $\{\, T(f) : f \in \mathrm{C_c}(X)^+, \|f\| \leq 1 \,\}$ is order bounded in E.

Motivated by the proof of Theorem 12.3, we define the set map $\mu : \mathscr{B} \to E^+$ by setting $\mu(\emptyset) := 0$,

$$\mu(V) := \bigvee \{\, T(f) : f \prec V \,\} \tag{13.3}$$

in $\overline{E}$ for every non-empty open subset V of X, and

$$\mu(A) := \bigwedge \{\, \mu(V) : V \text{ is open and } A \subseteq V \,\} \tag{13.4}$$

(13.5)

in $\overline{E}$ for every Borel set A. The remarks that were made in the beginning of the proof of Theorem 12.3 apply here as well, showing that both $\mu(V)$ and $\mu(A)$ are well-defined, and that the two definitions agree on the open subsets of X. It is clear from the hypotheses that $\mu(A)$ is, in fact, a finite element of $\overline{E}$ for every Borel set A, so that we can let linear functionals on E act on the range of μ.

We shall show that the set map μ has all required properties.

To this end, we define, for every $x' \in (E_{\mathrm{n}}^{\sim})^+$, the map $T_{x'} : \mathrm{C_c}(X) \to \mathbb{R}$ by setting

$$T_{x'}(f) := (T(f), x')$$

for $f \in \mathrm{C_c}(X)$. Then $T_{x'}$ is a positive linear functional on $\mathrm{C_c}(X)$. The classical Riesz representation theorem (the case where $E = \mathbb{R}$ in Theorem 13.4) informs us that, for all $x' \in (E_{\mathrm{n}}^{\sim})^+$, there exists a unique regular Borel measure $\nu_{x'} : \mathscr{B} \to \overline{\mathbb{R}^+}$ such that

$$T_{x'}(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\nu_{x'} \tag{13.6}$$

for all $f \in \mathrm{C_c}(X)$. Since $\{\, T_{x'}(f) : f \prec X \,\}$ is evidently order bounded in $\mathbb{R}$, Theorem 12.3 shows that $\nu_{x'}$ is, in fact, a finite regular Borel measure on X for all $x' \in (E_{\mathrm{n}}^{\sim})^+$.

For each $x' \in (E_{\mathrm{n}}^{\sim})^+$, we know from equation (12.1) that

$$\nu_{x'}(V) = \sup \{\, T_{x'}(f) : f \prec V \,\}$$

for all non-empty open subsets V of X. On the other hand, we see from the defining equation (13.3) that

$$\mu_{x'}(V) = \sup \{\, T_{x'}(f) : f \prec V \,\}$$

for all $x' \in (E_{\mathrm{n}}^{\sim})^+$ and all non-empty open subsets V of X. Hence $\mu_{x'}(V) = \nu_{x'}(V)$ for all non-empty open subsets V of X and all $x' \in (E_{\mathrm{n}}^{\sim})^+$; this is trivially true for the empty subset.

Furthermore, the outer regularity of $\nu_{x'}$ means that

$$\nu_{x'}(A) = \inf \{\, \nu_{x'}(V) : V \text{ is open and } A \subseteq V \,\}.$$

Using the defining equation (13.3) and what we have just observed for the open subsets of X, we therefore see that

$$\begin{aligned}\mu_{x'}(A) &= \inf\{\,\mu_{x'}(V) : V \text{ is open and } A \subseteq V\,\}\\ &= \inf\{\,\nu_{x'}(V) : V \text{ is open and } A \subseteq V\,\}\\ &= \nu_{x'}(A).\end{aligned}$$

We conclude that $\mu_{x'} = \nu_{x'}$ for all $x' \in (E_{\mathrm{n}}^{\sim})^{+}$. Since we know that $\nu_{x'}$ is a regular Borel measure on X for all x', it now follows from Corollary 13.3 that μ is a regular Borel measure on X.

It is clear from equation (13.3) that μ is a finite measure.

Equation (13.1) holds by construction.

If K is a compact subset of X, then equation (12.2) shows that

$$\mu_{x'}(K) = \nu_{x'}(K) = \bigwedge\{\,T_{x'}(f) : K \prec f\,\}.$$

It then follows from Proposition 3.8 that

$$\mu(K) = \bigwedge\{\,T(f) : K \prec f\,\},$$

which is equation (13.2).

Let $f \in \mathrm{C_c}(X)^{+}$, and choose a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $\mathscr{E}(X, \mathscr{B}; \mathbb{R}^{+})$ such that $\varphi_n \uparrow f$ in E pointwise. Fix $x' \in (E_{\mathrm{n}}^{\sim})^{+}$. Using the definition and equation (13.6), where we already know that $\mu_{x'} = \nu_{x'}$, we see that

$$\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu_{x'} \uparrow \int_X^{\mathrm{o}} f \,\mathrm{d}\mu_{x'} = T_{x'}(f).$$

On the other hand, it is clear from the definition of the integral of elementary functions that

$$\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu_{x'} = \left(\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu, x'\right)$$

for all $n \geq 1$. Since $\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu \uparrow \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ by definition, the σ-order continuity of x' therefore implies that

$$\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu_{x'} = \left(\int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu, x'\right) \uparrow \left(\int_X^{\mathrm{o}} f \,\mathrm{d}\mu, x'\right).$$

We conclude that $T_{x'}(f) = \left(\int_X^{\mathrm{o}} f \,\mathrm{d}\mu, x'\right)$. Since this holds for all $x' \in (E_{\mathrm{n}}^{\sim})^{+}$ and E is normal, it follows that $T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)^{+}$. By linearity, this is then also true for general $f \in \mathrm{C_c}(X)$.

The uniqueness of μ as an a priori possibly infinite regular Borel measure representing T follows as in the conclusion of the proof of Theorem 12.3. □

We can now also obtain a representation theorem when $\{\,T(f) : f \prec X\,\}$ need not be order bounded. The idea is to start by using Theorem 13.4 locally. If $U \in \Upsilon$ (that is: if U is an open relatively compact subset of X), then there exists $f \in \mathrm{C_c}(X)$ such that $\overline{U} \prec \widetilde{f}$. This implies that $f \leq \widetilde{f}$ whenever $f \prec U$, so that $T(f) \leq T(\widetilde{f})$ whenever $f \prec U$. We are now in the finite case that is covered by Theorem 13.4, so that there exists a unique regular Borel measure μ_U on U such that $T(f) = \int_U^{\mathrm{o}} f \,\mathrm{d}\mu_U$ for all $f \in \mathrm{C_c}(X)$ that are supported in U. These local measures μ_U on U can then seen to be the restrictions of a global measure on X using a technique that is also employed in [**26**, proof of Theorem 1].

The details are contained in the proof of the following theorem.

Theorem 13.5 (Riesz representation theorem for $\mathrm{C_c}(X)$; normal case). *Let X be a locally compact Hausdorff space, let E be a normal and monotone complete partially ordered vector space, and let $T : \mathrm{C_c}(X) \to E$ be a positive linear map.*

Then there exists a unique measure $\mu : \mathscr{B} \to \overline{E^+}$ such that:

(1) *For every $U \in \Upsilon$, the restriction μ_U of μ to $\mathscr{B}_U$ is a regular Borel measure on U;*
(2) *μ is weakly inner regular at all Borel sets;*
(3) *$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)$.*

This measure μ is a quasi-regular Borel measure.

If $\widetilde{\mu}$ is a regular Borel measure on X such that $T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)$, then $\widetilde{\mu} \geq \mu$, and $\widetilde{\mu}$ and μ agree at all open subsets of X, at all compact subsets of X, at all Borel sets with finite $\widetilde{\mu}$-measure, and at all Borel sets with infinite μ-measure.

As a consequence of the last part: if A is a Borel set such that $\widetilde{\mu}(A) \neq \mu(A)$, then A is neither open nor compact, $\widetilde{\mu}(A) = \infty$, and $\mu(A)$ is finite. It also follows that, if T is represented by a finite regular Borel measure $\widetilde{\mu}$, then it must be the case that $\widetilde{\mu} = \mu$.

Proof. The uniqueness is easily taken care of. For this, we note that, for a non-empty open subset U of X, we may canonically view $\mathrm{C_c}(U)$ as a subset of $\mathrm{C_c}(X)$. Indeed, if $f \in \mathrm{C_c}(U)$, then we can extend f canonically to a continuous compactly supported function on X by setting $f(x) := 0$ for all $x \in X$ such that $x \notin U$. Then equation (10.2) shows that $T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu = \int_X^{\mathrm{o}} \chi_U f \,\mathrm{d}\mu = \int_U^{\mathrm{o}} f \,\mathrm{d}\mu_U$ for all $f \in \mathrm{C_c}(U)$. Since μ_U is supposed to be a regular Borel measure on U, the uniqueness statement in Theorem 13.4 implies that μ_U is uniquely determined. Since μ is weakly inner regular at all Borel sets, i.e. since

$$\mu(A) = \bigvee \{\, \mu(A \cap U) : U \in \Upsilon \,\} = \bigvee \{\, \mu_U(A \cap U) : U \in \Upsilon \,\},$$

for every Borel set A, we see that μ itself is unique.

This proof of the uniqueness also shows how to find μ, as follows. If $U \in \Upsilon$, then, as explained preceding the theorem, there exists a unique regular Borel measure μ_U on U such that $T(f) = \int_U^{\mathrm{o}} f \,\mathrm{d}\mu_U$ for all $f \in \mathrm{C_c}(U)$.

We shall now patch these μ_U together to obtain a measure on X.

As a preparation for this, we note the following.

If $U, V \in \Upsilon$ and $U \cap V \neq \emptyset$, then the restriction of μ_U to $U \cap V$ is a regular Borel measure on $U \cap V$ that represents the restriction of T to $\mathrm{C_c}(U \cap V)$. Since the same holds for μ_V, the uniqueness statement of Theorem 13.4 shows that the restrictions of μ_U and μ_V to $U \cap V$ coincide.

We can now define a set function on $\mathscr{B}$ that will turn out to have the desired properties. If A is Borel set, we set

$$\mu(A) := \bigvee \{\, \mu_U(A \cap U) : U \in \Upsilon \,\} \tag{13.7}$$

in $\overline{E}$. The supremum in the right hand side of this equation exists in $\overline{E}$, since the set is upward directed. Indeed, if $U, V \in \Upsilon$, then $\mu_U(A \cap U) = \mu_{U \cup V}(A \cap U) \leq \mu_{U \cup V}(A \cap (U \cup V))$, and likewise $\mu_V(A \cap V) \leq \mu_{U \cup V}(A \cap (U \cup V))$.

We claim that $\mu(A) = \mu_{U_0}(A)$ whenever A is a Borel set and $U_0 \in \Upsilon$ is such that $A \subseteq U_0$.

To see this, note that certainly $\mu(A) \geq \mu_{U_0}(A \cap U_0) = \mu_{U_0}(A)$. Furthermore, if $U \in \Upsilon$, then $\mu_U(A \cap U) = \mu_U(A \cap U_0 \cap U) = \mu_{U_0}(A \cap U_0 \cap U) \leq \mu_{U_0}(A)$. This implies that $\mu(A) \leq \mu_{U_0}(A)$. We conclude that $\mu(A) = \mu_{U_0}(A)$, as required.

We shall now show that μ is a measure.

It is clear that $\mu(\emptyset) = 0$. The first step towards σ-additivity is to show that μ is finitely additive. Let A and B be disjoint Borel sets. Then

$$\begin{aligned}
\mu(A\cup B) &= \bigvee \{\, \mu_U((A\cup B)\cap U) : U \in \Upsilon \,\} \\
&= \bigvee \{\, \mu_U((A\cap U)\cup(B\cap U)) : U \in \Upsilon \,\} \\
&= \bigvee \{\, \mu_U(A\cap U) + \mu_U(B\cap U) : U \in \Upsilon \,\} \\
&\le \mu(A) + \mu(B) \\
&= \bigvee \{\, \mu_U(A\cap U) : U \in \Upsilon \,\} + \bigvee \{\, \mu_V(B\cap V) : V \in \Upsilon \,\} \\
&= \bigvee \{\, \mu_U(A\cap U) + \mu_V(B\cap V) : U,V \in \Upsilon \,\} \\
&= \bigvee \{\, \mu_{U\cup V}(A\cap U) + \mu_{U\cup V}(B\cap V) : U,V \in \Upsilon \,\} \\
&= \bigvee \{\, \mu_{U\cup V}((A\cap U)\cup(B\cap V)) : U,V \in \Upsilon \,\} \\
&\le \bigvee \{\, \mu_{U\cup V}((A\cup B)\cap(U\cup V) : U,V \in \Upsilon \,\} \\
&= \bigvee \{\, \mu_U((A\cup B)\cap U) : U \in \Upsilon \,\} \\
&= \mu(A\cup B).
\end{aligned}$$

It follows that μ is finitely additive.

Now that we know that μ is finitely additive, the σ-additivity is easily seen to be consequence of the fact that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} \mu(A_n)$ for every increasing sequence $\{A_n\}_{n=1}^{\infty}$ of Borel sets; we shall now show this. Using Proposition 5.5, we see that

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \bigvee \left\{ \mu_U\left(\left(\bigcup_{n=1}^{\infty} A_n\right)\cap U\right) : U \in \Upsilon \right\} \\
&= \bigvee \left\{ \bigvee_{n=1}^{\infty} \mu_U(A_n\cap U) : U \in \Upsilon \right\} \\
&= \bigvee_{n=1}^{\infty} \left(\bigvee \{\, \mu_U(A_n\cap U) : U \in \Upsilon \,\}\right) \\
&= \bigvee_{n=1}^{\infty} \mu(A_n),
\end{aligned}$$

as required.

This concludes the proof that μ is a measure.

The measure μ is weakly inner regular at all Borel sets.

Since we know that the restrictions of μ to the $U \in \Upsilon$ are the μ_U, this is clear from equation (13.7).

It is now easy to show that $T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for all $f \in \mathrm{C_c}(X)$.

Indeed, one can choose $U \in \Upsilon$ such that $f \in \mathrm{C_c}(U)$. Since we have already observed that the restriction of μ to U is μ_U, we see that $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu = \int_U^{\mathrm{o}} f \,\mathrm{d}\mu_U$, and $\int_U^{\mathrm{o}} f \,\mathrm{d}\mu_U = T(f)$ by construction.

The measure μ is a Borel measure.

Since every compact subset of X is contained in a relatively open compact subset of X, and since μ_U is a finite measures for all $U \in \Upsilon$, this is clear.

The measure μ is inner regular at all open sets.

Let V be an open subset of X. Using that, for each $U \in \Upsilon$, μ_U is inner regular at all open subsets of U, we see that

$$\begin{aligned}
\mu(V) &= \bigvee \{\, \mu_U(V \cap U) : U \in \Upsilon \,\} \\
&= \bigvee_{U \in \Upsilon} \left\{ \bigvee \{\, \mu_U(K) : K \text{ is compact and } K \subseteq V \cap U \,\} \right\} \\
&= \bigvee \{\, \mu_U(K) : K \text{ is compact and } K \subseteq V \cap U \text{ for some } U \in \Upsilon \,\} \\
&= \bigvee \{\, \mu(K) : K \text{ is compact and } K \subseteq V \cap U \ \text{ for some } U \in \Upsilon \,\} \\
&\leq \bigvee \{\, \mu(K) : K \text{ is compact and } K \subseteq V \,\} \\
&\leq \mu(V).
\end{aligned}$$

Since μ is a Borel measure that is inner regular at all open subsets of X and weakly inner regular at all Borel sets, it is a quasi-regular Borel measure.

We turn to the remaining statements.

Suppose that $\widetilde{\mu}$ is a regular Borel measure that also represents T. Then, for all $u \in \Upsilon$, $\widetilde{\mu}_U$ is a regular Borel measure that represents the restriction of T to $C_c(U)$. By uniqueness, we see that $\widetilde{\mu}_U = \mu_U$ for all $U \in \Upsilon$. This implies that $\widetilde{\mu}$ and μ agree at all compact subsets of X and then, by inner regularity, also at all open subsets of X. Knowing this, and using the outer regularity of $\widetilde{\mu}$, we see that, for a Borel set A,

$$\begin{aligned}
\widetilde{\mu}(A) &= \bigwedge \{\, \widetilde{\mu}(V) : V \text{ is open and } A \subseteq V \,\} \\
&= \bigwedge \{\, \mu(V) : V \text{ is open and } A \subseteq V \,\} \\
&\geq \mu(A),
\end{aligned}$$

so that $\widetilde{\mu} \geq \mu$.

Clearly, if $\mu(B) = \infty$, then $\widetilde{\mu}(B) = \infty$ as well.

Finally, suppose A is a Borel set such that $\widetilde{\mu}(A)$ is finite. Using Proposition 7.6 in the second step, we see that

$$\begin{aligned}
\mu(A) &\leq \widetilde{\mu}(A) \\
&= \bigvee \{\, \widetilde{\mu}(K) : K \text{ is compact and } K \subseteq A \} \\
&= \bigvee \{\, \mu(K) : K \text{ is compact and } K \subseteq A \} \\
&\leq \mu(A).
\end{aligned}$$

Hence $\mu(A) = \widetilde{\mu}(A)$. □

Remark 13.6. The representation theorem for the normed case (Theorem 12.3) and the representation theorem (Theorem 13.5) for the normal case both have their virtues. For example, the theorem for the normed case does not apply to ℓ^∞, since it does not have order continuous norm. On the other hand, the normal representation theorem *does* apply to ℓ^∞, since the space is Dedekind complete and the coordinate functionals are order continuous linear functionals that separate the points. If both theorems apply, as is e.g. the case for Banach lattices with order continuous norms, then the normed representation theorem (asserting that there is a representing regular Borel measure) gives a stronger conclusion than the normal representation theorem (asserting that there is a representing quasi-regular Borel measure).

§14. Riesz representation theorems for $\mathrm{C}_0(X)$

In this section, we consider representation theorems for positive linear maps $T : \mathrm{C}_0(X) \to E$ that are defined on $\mathrm{C}_0(X)$ rather than on $\mathrm{C}_c(X)$.

As a motivation for our approach, let us recall the classical theorem for $E = \mathbb{R}$ on the existence of a representing measure, formulated a little sharper than is usually done. We do not include the customary statement about the norm, since such investigations are outside the current body of work.

Theorem 14.1 (Riesz representation theorem for $\mathrm{C}_0(X)$; real case)**.** *Let X be a locally compact Hausdorff space, and let $T : \mathrm{C}_0(X) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique regular Borel measure μ on X such that*

$$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \tag{14.1}$$

for all f in $\mathrm{C}_c(X)$. This measure is finite, and equation (14.1) *also holds for all $f \in \mathrm{C}_0(X)$.*

This is easily derived from Theorem 12.1, the classical representation theorem for $\mathrm{C}_c(X)$. Indeed, since $\mathrm{C}_0(X)$ is a Banach lattice and T is positive, T is continuous. Then so is its restriction to $\mathrm{C}_c(X)$, and in that case Theorem 12.1 asserts that the unique representing regular Borel measure for $\mathrm{C}_c(X)$ is finite. This implies that $\mathrm{C}_0(X) \subseteq \mathscr{L}^1(X, \mathscr{B}, \mu; \mathbb{R})$, so that we can define a map $f \mapsto \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ from $\mathrm{C}_0(X)$ into $\mathbb{R}$. This positive linear map is continuous, and it agrees with the continuous map $f \mapsto T(f)$ on the dense subspace $\mathrm{C}_c(X)$ of $\mathrm{C}_0(X)$. Hence they coincide, and we conclude that equation (14.1) holds for all $f \in \mathrm{C}_0(X)$.

The important ingredient in the above argument is the automatic continuity of positive linear maps from Banach lattices into normed Riesz spaces. We can also use this in a more general context. Here is a first example.

Lemma 14.2. *Let X be a locally compact Hausdorff space, let E be a normed Riesz space, and let $T : \mathrm{C}_0(X) \to E$ be a positive linear map. Suppose that there exists a measure $\mu : \mathscr{B} \to \overline{E^+}$ such that*

$$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \tag{14.2}$$

for all f in $\mathrm{C}_c(X)$ and such that $\mathrm{C}_0(X) \subseteq \mathscr{L}^1(X, \mathscr{B}, \mu; \mathbb{R})$.

Then equation (14.2) *also holds for all $f \in \mathrm{C}_0(X)$.*

Proof. The map $f \to \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ is a positive linear map from $\mathrm{C}_c(X)$ into E. Hence it is continuous. Since $T : \mathrm{C}_c(X) \to E$ is likewise continuous, and since the two maps agree on $\mathrm{C}_c(X)$, they are equal. □

Looking at our representation theorems, there is one in which normed spaces (possibly normed Riesz spaces) figure, namely, Theorem 12.3. Lemma 14.2 makes clear that it is relevant to point out cases in which we know a priori that the representing measure for $\mathrm{C}_c(X)$ from Theorem 12.3 is finite, i.e. that $\{\, T(f) : f \prec X \,\}$ is order bounded, once we have the extra information that T is the restriction of a positive linear map defined on $\mathrm{C}_0(X)$. This is indeed possible, as is shown by the following result. It includes Theorem 14.1 as a special case.

We recall (see [**3**, p. 232] that a Banach lattice is a *KB-space* if every increasing norm bounded net in the positive cone is norm convergent. As is well known, the norm limit is then also the supremum of the net. In particular, every increasing norm bounded net in the positive cone of a KB-space is order bounded.

Furthermore, a KB-space satisfies all requirements in Theorem 13.5. Indeed, it has order continuous norm (see e.g. [**23**, Theorem 7.1]), so that it is Dedekind complete (see e.g. [**3**, Corollary 4.10]). The fact that is a Banach space, together with the fact that a norm limit of an increasing sequence in a Banach lattice is also its supremum, shows that the second requirement is fulfilled. The third follows immediately from the normality of every Banach lattice with order continuous norm (see Proposition 3.10).

After these observations, the next result is easily proved.

Theorem 14.3. *Let X be a locally compact Hausdorff space, let E be a KB-space, and let $T : \mathrm{C}_0(X) \to E$ be a positive linear map.*

Then there exists a unique regular Borel measure $\mu : \mathscr{B} \to \overline{E^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f) = {}^{\mathrm{o}}\!\!\int_X f \,\mathrm{d}\mu \tag{14.3}$$

for all $f \in \mathrm{C_c}(X)$.

The measure μ is finite, and equation (14.3) *also holds for all $f \in \mathrm{C}_0(X)$.*

If V is a non-empty open subset of X, then

$$\mu(V) = \bigvee \{\, T(f) : f \prec V \,\}$$

in E.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{\, T(f) : K \prec f \,\}$$

in E.

Proof. As observed preceding the theorem, Theorem 12.3 can be applied. The measure μ that is obtained satisfies $\mu(X) = \{\, T(f) : f \in \mathrm{C_c}(X)^+,\ \|f\| \leq 1 \,\}$. Since T is positive, it is continuous, so that the set $\{\, T(f) : f \in \mathrm{C_c}(X)^+,\ \|f\| \leq 1 \,\}$ is norm bounded. Since it is also upward directed, as a consequence of the positivity of T, the remarks preceding the theorem show that it is order bounded. We conclude that μ is a finite measure, and then Lemma 14.2 applies to show that equation (14.3) is valid for all $f \in \mathrm{C}_0(X)$. □

We turn to the normal spaces, where we have Theorem 13.4 to start from. As we shall see, we shall actually obtain a generalisation of Theorem 14.3. For the normal spaces, we need the following companion result of Lemma 14.2.

Lemma 14.4. *Let X be a locally compact Hausdorff space, let E be a σ-monotone complete partially ordered vector space such that the positive linear functionals separate the points of E, and let $T : \mathrm{C}_0(X) \to E$ be a positive linear map. Suppose that there exists a measure $\mu : \mathscr{B} \to \overline{E^+}$ such that*

$$T(f) = {}^{\mathrm{o}}\!\!\int_X f \,\mathrm{d}\mu \tag{14.4}$$

for all f in $\mathrm{C_c}(X)$ and such that $\mathrm{C}_0(X) \subseteq \mathscr{L}^1(X, \mathscr{B}, \mu; \mathbb{R})$.

Then equation (14.4) *also holds for all $f \in \mathrm{C}_0(X)$.*

Proof. Fix a positive linear functional x' on E. Then the maps $f \mapsto (T(f), x')$ and $f \mapsto (\int_X^{\mathrm{o}} f \, \mathrm{d}\mu, x')$ are both positive linear functionals on the Banach lattice $\mathrm{C}_0(X)$. Hence they are continuous. Since they agree on the dense subspace $\mathrm{C_c}(X)$ of $\mathrm{C}_0(X)$, they are equal. We conclude that $(T(f), x') = (\int_X^{\mathrm{o}} f \, \mathrm{d}\mu, x')$ for all $f \in \mathrm{C}_0(X)$ and all positive linear functionals on E. Since the latter separate the points of E, it follows that $T(f) = \int_X^{\mathrm{o}} f \, \mathrm{d}\mu$ for all $f \in \mathrm{C}_0(X)$. □

Completely analogous to the above line of reasoning for normed spaces, it is, in view of Lemma 14.4, relevant to point out cases where we know a priori that the representing measure for $\mathrm{C_c}(X)$ in Theorem 13.4 is finite, i.e. that $\{ T(f) : f \prec X \}$ is order bounded, once we have the extra information that T is the restriction of a positive linear map defined on $\mathrm{C}_0(X)$.

This is indeed possible again. For this, we recall two definitions.

Firstly, the norm on a Banach lattice is said to be a *Levi norm* if every norm bounded increasing net in the positive cone has a supremum.

Secondly, a Riesz space is called *perfect* if the natural Riesz homomorphism from E into $(E_{\mathrm{n}}^{\sim})_{\mathrm{n}}^{\sim}$ is a surjective isomorphism. The relevance of these spaces for us lies in the following alternate characterisation due to Nakano; see [**3**, Theorem 1.71].

Theorem 14.5. *A Riesz space E is a perfect Riesz space if and only if the following two conditions hold:*

(1) *E is normal;*
(2) *whenever an increasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ in E^+ is such that $\sup(x_\lambda, x') < \infty$ for each $x' \in (E_{\mathrm{n}}^{\sim})^+$, then this net has a supremum in E.*

The class of spaces for which we have a priori boundedness is a variation on Nakano's characterisation of the perfect Riesz spaces. Let us say that a Riesz space is *quasi-perfect* if the two following conditions are satisfied:

(1) E is normal;
(2) whenever an increasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ in E^+ is such that $\sup(x_\lambda, x') < \infty$ for each $x' \in (E^{\sim})^+$, then this net has a supremum in E.

Clearly, a quasi-perfect Riesz space is Dedekind complete.

Lemma 14.6. *The following inclusions between classes of Riesz spaces hold.*

(1) *Every KB-space is a normal Banach lattice space with a Levi norm;*
(2) *Every normal Banach lattice with a Levi norm is a quasi-perfect Riesz space;*
(3) *Every KB-space is a perfect Riesz space.*
(4) *Every perfect Riesz space is a quasi-perfect Riesz space.*

Consequently, all KB-spaces, all normal Banach lattices with a Levi norm, and all perfect Riesz spaces are quasi-perfect Riesz spaces.

Proof. We prove part (1). As noted earlier, a KB-space has order continuous norm, so that it is normal. If $\{x_\lambda\}_{\lambda\in\Lambda}$ is a norm bounded increasing net in the positive cone of a KB-space, then it is norm convergent. As mentioned earlier, the norm limit of the net is then also the supremum of the net.

We prove part (2). Suppose that $\{x_\lambda\}_{\lambda\in\Lambda}$ is an increasing net in the positive cone of a normal Banach lattice E with a Levi norm such that $\sup(x_\lambda, x') < \infty$ for each $x' \in (E^{\sim})^+$. Then $\{ (x_\lambda, x') : \lambda \in \Lambda \}$ is bounded in $\mathbb{R}$ for every $x' \in E^{\sim} = E^*$. We conclude that $\{x_\lambda\}_{\lambda\in\Lambda}$ is norm bounded. Since the norm is supposed to be a Levi norm, the increasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ has a supremum in E.

We prove part (3). As noted above, a KB-space is normal. Suppose that $\{x_\lambda\}_{\lambda\in\Lambda}$ is an increasing net in the positive cone of a KB-space E such that $\sup(x_\lambda, x') < \infty$ for each $x' \in (E_n^\sim)^+$. This implies that the set $\{ (x_\lambda, x') : \lambda \in \Lambda \}$ is bounded in $\mathbb{R}$ for every $x' \in E_n^\sim = E^*$, where the latter equality holds because E has order continuous norm. We conclude that $\{x_\lambda\}_{\lambda\in\Lambda}$ is norm bounded. Since the space is a KB-space, the increasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ has a norm limit in E, and this norm limit is then also the supremum of the net.

Part (4) is clear. □

We can now establish the analogue of Theorem 14.3 for normal spaces.

Theorem 14.7. *Let X be a locally compact Hausdorff space, let E be a quasi-perfect Riesz space (such as a KB-space, a normal Banach lattice with a Levi norm, or a perfect Riesz space), and let $T : C_0(X) \to E$ be a positive linear map.*

Then there exists a unique regular Borel measure $\mu : \mathscr{B} \to \overline{E^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \tag{14.5}$$

for all $f \in C_c(X)$.

The measure μ is finite, and equation (14.5) *also holds for all $f \in C_0(X)$.*

If V is a non-empty open subset of X, then

$$\mu(V) = \bigvee \{ T(f) : f \prec V \}$$

in E.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{ T(f) : K \prec f \}$$

in E.

Proof. Since E is normal and monotone complete, Theorem 13.4 applies, and then Lemma 14.4 implies that it is sufficient to show that the set $\{ T(f) : f \prec X \}$ is order bounded. To see this, we consider, for each $x' \in (E^\sim)^+$, the map $f \mapsto (T(f), x')$ from $C_0(X)$ to $\mathbb{R}$. Since $C_0(X)$ is a Banach lattice, the positivity of the map implies that it is norm bounded. Since the set $\{ T(f) : f \prec V \}$ is upward directed, the fact that E is quasi-perfect now implies that this set has a supremum in E. Consequently, it is order bounded. □

Theorem 14.7 has a consequence that will become important later on when considering representations of $C_0(X)$ on KB-spaces. We need a preparatory result that seems worth recording explicitly.

Proposition 14.8. *Let E be a KB-space. Then the regular norm on $L_r(E)$ is a Levi norm.*

Proof. Suppose that $0 \leq \{S_\lambda\}_{\lambda\in\Lambda} \uparrow$ in $L_r(E)$ is a net that is bounded in the regular norm, which coincides with the operator norm on these positive operators. Choose $M \geq 0$ such that $\|S_\lambda\| \leq M$ for all $\lambda \in \Lambda$. If $x \in E^+$, then $\{S_\lambda x\}_{\lambda\in\Lambda}$ is an increasing net in E^+. Furthermore, we have $\|S_\lambda x\| \leq M\|x\|$ for all $\lambda \in \Lambda$. Since E is a KB-space, the supremum of the increasing norm bounded positive net $\{S_\lambda x\}_{\lambda\in\Lambda}$ exists in E. Then Proposition 3.1 shows that the net $\{S_\lambda\}_{\lambda\in\Lambda}$ has a supremum in $L_r(E)$. □

If E is a KB-space, then Theorem 3.13 shows that the order continuity of the norm on E implies that $L_r(E)$ is a normal partially ordered vector space. Since we have established in

Proposition 14.8 that the regular norm on $\mathrm{L_r}(E)$ is a Levi norm, Theorem 14.7 applies to the partially ordered vector space $\mathrm{L_r}(E)$ and yields the following.

Theorem 14.9. *Let X be a locally compact Hausdorff space, let E be a KB-space, and let $T : \mathrm{C}_0(X) \to \mathrm{L_r}(E)$ be a positive linear map.*

Then there exists a unique regular Borel measure $\mu : \mathscr{B} \to \overline{\mathrm{L_r}(E)_+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \tag{14.6}$$

for all $f \in \mathrm{C_c}(X)$.

The measure μ is finite, and equation (14.6) *also holds for all $f \in \mathrm{C}_0(X)$.*

If V is a non-empty open subset of X, then

$$\mu(V) = \bigvee \{\, T(f) : f \prec V \,\}$$

in E.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{\, T(f) : K \prec f \,\}$$

in E.

We conclude this section with a result for partially ordered vector spaces of operators on a Hilbert space. It has some interest of its own, and will become important later on when considering representations of (the complexification of) $\mathrm{C}_0(X)$ on Hilbert spaces.

Theorem 14.10. *Let X be a locally compact Hausdorff space, let $\mathscr{H}$ be a complex Hilbert space, and let L be a weakly closed complex linear subspace of $\mathscr{B}(\mathscr{H})$ that contains the identity operator I on $\mathscr{H}$. Let L_{sa} be the real vector space that consists of the self-adjoint elements of L, supplied with the partial ordering that is inherited from the usual partial ordering on $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$. Let $T : \mathrm{C}_0(X) \to L_{\mathrm{sa}}$ be a positive linear map.*

Then there exists a unique regular Borel measure $\mu : \mathscr{B} \to \overline{L^+_{\mathrm{sa}}}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$T(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \tag{14.7}$$

for all $f \in \mathrm{C_c}(X)$.

The measure μ is finite, and equation (14.7) *also holds for all $f \in \mathrm{C}_0(X)$.*

If V is a non-empty open subset of X, then

$$\mu(V) = \bigvee \{\, T(f) : f \prec V \,\}$$

in E.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{\, T(f) : K \prec f \,\}$$

in E.

Proof. We claim that T is continuous. To see this, let $\mathrm{C}_0(X, \mathbb{C})$ denote the complex-valued continuous functions on X that vanish at infinity. Then T extends to a linear map from $\mathrm{C}_0(X, \mathbb{C})$ to $\mathscr{B}(\mathscr{H})$ that is positive again. Since positive linear maps between complex C*-algebras are continuous (see [**22**, p. 194]), our claim follows.

If A is a positive operator on a Hilbert space, then $A \leq \|A\| I$. Hence $T(f) \leq \|T\| I$ whenever $f \prec X$. Since $I \in L_{\mathrm{sa}}$, this shows that $\{ T(f) : f \prec X \}$ is order bounded in L_{sa}. Theorem 13.4 applies and shows that a measure μ exists that has all the properties in the theorem, except that we still need to show that equation (14.7) holds for all $f \in \mathrm{C_0}(X)$.

For this, we consider, for $x \in \mathscr{H}$, the linear maps $f \to \langle T(f)x, x\rangle$ and $f \to \left\langle \left(\int_X^{\mathrm{o}} f \,\mathrm{d}\mu \right) \cdot x, x, \right\rangle$ from $\mathrm{C_0}(X)$ into $\mathbb{R}$. Being positive, they are continuous. Since they agree on $\mathrm{C_c}(X)$, they are equal. Hence $\langle T(f)x, x\rangle = \left\langle \left(\int_X^{\mathrm{o}} f \,\mathrm{d}\mu \right) \cdot x, x \right\rangle$ for all $f \in \mathrm{C_0}(X)$ and $x \in \mathscr{H}$. This implies that equation (14.7) holds for all $f \in \mathrm{C_0}(X)$. □

§15. Positive representations of $\mathrm{C_c}(X)$ and $\mathrm{C_0}(X)$

In this section, we are concerned with a positive algebra homomorphism π from $\mathrm{C_c}(X)$ or $\mathrm{C_0}(X)$ into a partially ordered algebra. We know from earlier work that, under conditions, π has a representing measure since it a positive linear map. The fact that π preserves multiplication is then enough to guarantee that, if it is finite, the representing measure is actually a spectral measure.

Specialising to the case where $\mathscr{A}$ is an algebra of operators, we shall see how two results in the literature that look similar, but that do no appear to be very strongly related, are actually both consequences of the following abstract result.

Conforming to the usual notation for spectral measures in the literature, we shall now denote our representing measures with P.

Theorem 15.1 (Spectral measures for positive homomorphisms of $\mathrm{C_c}(X)$ and $\mathrm{C_0}(X)$ into algebras)**.** *Let X be a locally compact Hausdorff space and let $\mathscr{A}$ be a normal and monotone complete partially ordered algebra with monotone continuous multiplication.*

Let $\pi : \mathrm{C_c}(X) \to \mathscr{A}$ be a positive algebra homomorphism, and suppose that the subset $\{ \pi(f) \colon f \in \mathrm{C_c}(X)^+, \|f\| \leq 1 \}$ of $\mathscr{A}$ is order bounded; this is automatically the case if X is compact.

Then there exists a unique regular Borel measure $P : \mathscr{B} \to \overline{\mathscr{A}^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$\pi(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}P \tag{15.1}$$

for all $f \in \mathrm{C_c}(X)$.

The measure P is a spectral measure.

If π is the restriction of a positive algebra homomorphism $\pi : \mathrm{C_0}(X) \to \mathscr{A}$, then equation (15.1) *also holds for $f \in \mathrm{C_0}(X)$.*

If V is a non-empty open subset of X, then

$$P(V) = \bigvee \{ \pi(f) : f \prec V \} \tag{15.2}$$

in $\mathscr{A}$.

If K is a compact subset of X, then

$$P(K) = \bigwedge \{ \pi(f) : K \prec f \} \tag{15.3}$$

in $\mathscr{A}$.

The map π extends to a positive algebra homomorphism from $\mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$ into $\mathscr{A}$, denoted by π again, by setting $\pi(f) := \int_X^{\mathrm{o}} f \,\mathrm{d}P$ for $f \in \mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$.

For $a \in \mathscr{A}^+$, the following are equivalent:

(1) $a\pi(f) = \pi(f)a$ *for all* $f \in \mathrm{C_c}(X)$;
(2) $aP(A) = P(A)a$ *for all Borel sets* A;
(3) $a\pi(f) = \pi(f)a$ *for all* $f \in \mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$.

PROOF. An application of Theorem 13.4 gives all statements in the theorem, except that the finite regular Borel measure $P : \mathscr{B} \to \mathscr{A}^+$ supplied by it is a spectral measure, that it will also represent π on the larger space $\mathrm{C_0}(X)$, that the extended map to $\mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$ is a positive homomorphism, and that the commutativity statements are valid. We shall now show these four extra components.

We start with the fact that P is a spectral measure. For this, we need a few preparations.

Firstly, if $\varphi \in \mathscr{B}(X, \mathscr{B}; \mathbb{R}^+)$, then we define

$$P^{\varphi}(A) := \int_X^{\mathrm{o}} \chi_A \varphi \, \mathrm{d}P$$

for $A \in \mathscr{B}$. Then Proposition 10.1 implies that $P^{\varphi} : \mathscr{B} \to \mathscr{A}^+$ is a finite positive measure and that

$$\int_X^{\mathrm{o}} \psi \, \mathrm{d}P^{\varphi} = \int_X^{\mathrm{o}} \varphi\psi \, \mathrm{d}P \tag{15.4}$$

for all $\psi \in \mathscr{B}(X, \mathscr{B}; \mathbb{R})$. Corollary 10.4 shows that P^{φ} is a finite regular Borel measure again.

Secondly, if $\varphi \in \mathscr{B}(X, \mathscr{B}; \mathbb{R}^+)$, then we define

$$P_{\int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \cdot}(A) := \left(\int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \right) \cdot P(A)$$

for $A \in \mathscr{B}$. Then Lemma 11.1 implies that $P_{\int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \cdot} : \mathscr{B} \to \mathscr{A}$ is a finite positive measure, and that

$$\int_X^{\mathrm{o}} \psi \, \mathrm{d}P_{\int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \cdot} = \int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \cdot \int_X^{\mathrm{o}} \psi \, \mathrm{d}P \tag{15.5}$$

for all $\psi \in \mathscr{B}(X, \mathscr{B}; \mathbb{R})$. Furthermore, Lemma 11.2 shows that $P_{\int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \cdot}$ is a finite regular Borel measure again.

Thirdly, we claim that $P(A)P(B) = P(B)P(A)$ for all $A, B \in \mathscr{B}$.

To see this, let $g \in \mathrm{C_c}(X)$, and let V be a non-empty open subset of X. Then equation (15.2), together with the monotone continuity of the left and right multiplications, implies that

$$\begin{aligned}
\pi(g)P(V) &= \pi(g) \bigvee \{ \pi(f) : f \prec V \} = \bigvee \{ \pi(g)\pi(f) : f \prec V \} \\
&= \bigvee \{ \pi(gf) : f \prec V \} = \bigvee \{ \pi(fg) : f \prec V \} \\
&= \bigvee \{ \pi(f)\pi(g) : f \prec V \} = \bigvee \{ \pi(f) : f \prec V \} \pi(g) \\
&= P(V)\pi(g).
\end{aligned}$$

A similar argument then shows that $P(U)P(V) = P(V)P(U)$ for all open subsets U and V of X. The outer regularity of P can subsequently be used to conclude that $P(V)P(B) = P(B)P(V)$ for all $B \in \mathscr{B}$ and all open subsets V of X, and then once more to see that $P(A)P(B) = P(B)P(A)$ for all $A, B \in \mathscr{B}$.

Fourthly, we claim that $\int_X^{\mathrm{o}} \varphi \, \mathrm{d}P \cdot \int_X^{\mathrm{o}} \psi \, \mathrm{d}P = \int_X^{\mathrm{o}} \psi \, \mathrm{d}P \cdot \int_X^{\mathrm{o}} \varphi \, \mathrm{d}P$ for all φ and ψ in $\mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$.

Indeed, for $\varphi, \psi \in \mathscr{E}(X, \mathscr{B}; \mathbb{R}^+)$ this follows from what we have just noted. This, together with the definition of the order integral and Lemma 4.2, implies the validity for $\varphi, \psi \in \mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R}^+)$; the general case then follows.

After these preparations, we can now start with the actual argument. It relies heavily on the uniqueness statement in Theorem 13.4.

We claim that $P_{\int_X^{\mathrm{o}} f\,\mathrm{d}P\cdot} = P^f$ for all $f \in \mathrm{C_c}(X)^+$.

To see this, let $g \in \mathrm{C_c}(X)$. Using equation (15.5) in the first step, the multiplicativity of π in the third (this is actually the only place where it is used when showing that P is spectral), and equation (15.4) in the fifth, we see that

$$
\begin{aligned}
\int_X^{\mathrm{o}} g\,\mathrm{d}P_{\int_X^{\mathrm{o}} f\,\mathrm{d}P\cdot} &= \int_X^{\mathrm{o}} f\,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g\,\mathrm{d}P \\
&= \pi(f)\pi(g) \\
&= \pi(fg) \\
&= \int_X^{\mathrm{o}} fg\,\mathrm{d}P \\
&= \int_X^{\mathrm{o}} g\,\mathrm{d}P^f.
\end{aligned}
$$

Since $P_{\int_X^{\mathrm{o}} f\,\mathrm{d}P\cdot}$ and P^f are both regular Borel measures, an appeal to the uniqueness part of Theorem 13.4 establishes our claim.

We claim that $\int_X^{\mathrm{o}} fg\,\mathrm{d}P = \int_X^{\mathrm{o}} f\,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g\,\mathrm{d}P$ for all $f \in \mathrm{C_c}(X)$ and $g \in \mathscr{B}(X, \mathscr{B}; \mathbb{R})$.

To see this, we may suppose that $f \in \mathrm{C_c}(X)^+$. Using equation (15.4) in the first step, our previous claim in the second, and equation (15.5) in the third, we see that

$$
\begin{aligned}
\int_X^{\mathrm{o}} fg\,\mathrm{d}P &= \int_X^{\mathrm{o}} g\,\mathrm{d}P^f \\
&= \int_X^{\mathrm{o}} g\,\mathrm{d}P_{\int_X^{\mathrm{o}} f\,\mathrm{d}P\cdot} \\
&= \int_X^{\mathrm{o}} f\,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g\,\mathrm{d}P.
\end{aligned}
$$

We claim that $P_{\int_X^{\mathrm{o}} f\,\mathrm{d}P\cdot} = P^f$ for all $f \in \mathscr{B}(X, \mathscr{B}; \mathbb{R}^+)$.

To see this, let $g \in \mathrm{C_c}(X)$. Using equation (15.5) in the first step, the fact that $\int_X^{\mathrm{o}} f\,\mathrm{d}P$ and $\int_X^{\mathrm{o}} g\,\mathrm{d}P$ commute in the second, and our previous claim in the third, see that

$$
\begin{aligned}
\int_X^{\mathrm{o}} g\,\mathrm{d}P_{\int_X^{\mathrm{o}} f\,\mathrm{d}P\cdot} &= \int_X^{\mathrm{o}} f\,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g\,\mathrm{d}P \\
&= \int_X^{\mathrm{o}} g\,\mathrm{d}P \cdot \int_X^{\mathrm{o}} f\,\mathrm{d}P \\
&= \int_X^{\mathrm{o}} fg\,\mathrm{d}P \\
&= \int_X^{\mathrm{o}} g\,\mathrm{d}P^f.
\end{aligned}
$$

Since $P_{\int_X^{\mathrm{o}} f \,\mathrm{d}P\cdot}$ and P^f are both regular Borel measures, an appeal to the uniqueness part of Theorem 13.4 establishes our claim.

We claim that $\int_X^{\mathrm{o}} f\,g \,\mathrm{d}P = \int_X^{\mathrm{o}} f \,\mathrm{d}P \int_X^{\mathrm{o}} g \,\mathrm{d}P$ *for all* $f,g \in \mathscr{B}(X,\mathscr{B};\mathbb{R}^+)$.

To see this, we may suppose that $f \in \mathscr{B}(X,\mathscr{B};\mathbb{R}^+)$. Using equation (15.4) in the first step, our previous claim in the second, and equation (15.5) in the third, we see that

$$\begin{aligned}\int_X^{\mathrm{o}} f\,g \,\mathrm{d}P &= \int_X^{\mathrm{o}} g \,\mathrm{d}P^f \\ &= \int_X^{\mathrm{o}} g \,\mathrm{d}P_{\int_X^{\mathrm{o}} f \,\mathrm{d}P\cdot} \\ &= \int_X^{\mathrm{o}} f \,\mathrm{d}P \cdot \int_X^{\mathrm{o}} g \,\mathrm{d}P.\end{aligned}$$

We conclude that P is a spectral measure.

This follows from our previous claim, when applied to the characteristic functions of two Borel sets A and B.

If $\pi : \mathrm{C_c}(X) \to \mathscr{A}$ *is the restriction of a positive algebra homomorphism* $\pi : \mathrm{C_0}(X) \to \mathscr{A}$*, then equation* (15.1) *also holds for* $f \in \mathrm{C_0}(X)$.

This is immediate from Lemma 14.4.

The extended map $\pi : \mathscr{L}^1(X,\mathscr{B},P;\mathbb{R}) \to \mathscr{A}$ *is a positive homomorphism.*

This is clear from Theorem 11.6.

We turn to the commutativity statements.

These are easily taken care of, and do not rely on the fact that P is a spectral measure.

We show that part (1) implies part (2). Since both left and right multiplication in $\mathscr{A}$ are monotone continuous, it follows from the explicit expression for $P(V)$ that $aP(V) = P(V)a$ for all open subsets V of X. The outer regularity of P and the monotone order continuity of left and right multiplication in $\mathscr{A}$ then imply that $aP(A) = P(V)a$ for all Borel sets A.

We show that part (2) implies part (3). It is immediate that $a \cdot \int_X^{\mathrm{o}} \varphi \,\mathrm{d}P = \int_X^{\mathrm{o}} \varphi \,\mathrm{d}P \cdot a$ for all $\varphi \in \mathscr{E}(X,\mathscr{B};\mathbb{R}^+)$, and then the definition of the order integral, combined with the σ-monotone continuity of the left and right multiplication in $\mathscr{A}$, implies that $a \cdot \int_X^{\mathrm{o}} f \,\mathrm{d}P = \int_X^{\mathrm{o}} f \,\mathrm{d}P \cdot a$ for all $f \in \mathscr{L}^1(X,\mathscr{B},P;\mathbb{R}^+)$. The case for general $f \in \mathscr{L}^1(X,\mathscr{B},P;\mathbb{R})$ follows.

It is clear that part (3) implies part (1).

$\square$

We shall now specialise to cases where $\mathscr{A}$ is an algebra of operators.

As a preparation, let us recall the following. Suppose that A is a normed algebra, and that π is a bounded representation of A on a normed space E. Then the representation is said to be *non-degenerate* if E is the closed linear span of the elements $\pi(a)x$ for $a \in A$ and $x \in E$. If A has a bounded left approximate identity, then it is not difficult to see that there is a largest invariant subspace of E such that the restricted representation of A on this subspace is non-degenerate. This subspace is closed; in fact, it is the closed linear span of the elements $\pi(a)x$ for $a \in A$ and $x \in E$. We shall call this subspace the *non-degenerate component of* E, and denote it by E_{nd}. If, moreover, A is a Banach algebra and E is a Banach space, then the Cohen-Hewitt factorisation theorem [**7**, Theorem 10, p. 61], shows that, actually, $E_{\mathrm{nd}} = \{\,\pi(a)x : a \in A, x \in E\,\}$.

15.1. Positive representations of $\mathrm{C_0}(X)$ on KB-spaces

In the case of positive representations of $\mathrm{C_0}(X)$ on KB-spaces, we have the following.

Theorem 15.2 (Existence of spectral measures for positive representations of $\mathrm{C_0}(X)$ on KB-spaces)**.** *Let X be a locally compact Hausdorff space, let E be a KB-space, and let $\pi : \mathrm{C_0}(X) \to \mathrm{L_r}(E)$ be a positive algebra homomorphism.*

Then there exists a unique regular Borel measure $P : \mathscr{B} \to \overline{\mathrm{L_r}(E)^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$\pi(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}P \tag{15.6}$$

for all $f \in \mathrm{C_c}(X)$.

The measure P is a spectral measure, and equation (15.6) *also holds for $f \in \mathrm{C_0}(X)$.*

If V is a non-empty open subset of X, then

$$P(V) = \bigvee \{\, \pi(f) : f \prec V \,\} \tag{15.7}$$

in $\mathrm{L_r}(E)$.

If K is a compact subset of X, then

$$P(K) = \bigwedge \{\, \pi(f) : K \prec f \,\} \tag{15.8}$$

in $\mathrm{L_r}(E)$.

The map π extends to a positive algebra homomorphism from $\mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$ into $\mathrm{L_r}(E)$, denoted by π again, by setting $\pi(f) := \int_X^{\mathrm{o}} f \,\mathrm{d}P$ for $f \in \mathscr{L}^1(X, \mathscr{B}, P; \mathbb{R})$.

For $S \in \mathrm{L_r}(E)$, the following are equivalent:

1. *$S\pi(f) = \pi(f)S$ for all $f \in \mathrm{C_0}(X)$;*
2. *$S\pi(f) = \pi(f)S$ for all $f \in \mathrm{C_c}(X)$;*
3. *$SP(A) = P(A)S$ for all Borel sets A;*
4. *$S\pi(f) = \pi(f)S$ for all $f \in \mathscr{B}(X, \mathscr{B}; \mathbb{R})$.*

The positive projection $P(X)$ projects E onto E_{nd}. Hence the representation $\pi : \mathrm{C_0}(X) \to \mathrm{L_r}(E)$ is non-degenerate if and only if $P(X) = I$.

Proof. This is immediate from the combination of Theorem 14.3, Proposition 4.3, and Theorem 15.1, except for the statement about commutants (which are no longer only for positive elements of the algebra) and the image of $P(X)$. We shall now establish these.

We start with the statements about commutants.

It is clear that part (1) implies part (2).

We show that part (2) implies part (3).

To see this, we employ an argument that relies on the uniqueness statement in the Riesz representation theorem again. It can also be found in e.g. [**12**]. We define, for $x \in E^+$ and $x' \in (E^*)^+$, the map $P_{x,x'} : \mathscr{B} \to \mathbb{R}$ by setting $P_{x,x'}(A) := (P(A)x, x')$ for $A \in \mathscr{B}$. Using Proposition 3.1 and the σ-order continuity of x', it is then easy to see that $P_{x,x'}$ is a real-valued positive measure. Using Proposition 3.1 and the order continuity of x', it is also easy to see that $P_{x,x'}$ is a real-valued positive regular Borel measure. Furthermore, it follows from Proposition 3.1, the definition of the order integral, equation (15.6), and the σ-order continuity of x' that

$$\bigl(\pi(f)x, x'\bigr) = \int_X f \,\mathrm{d}P_{x,x'} \tag{15.9}$$

for all $f \in \mathscr{B}(X,\mathscr{B};\mathbb{R})$.

After this, we extend our definition of $P_{x,x'} : \mathscr{B} \to \mathbb{R}$ to arbitrary $x \in E$ and $x' \in E^*$ by setting $P_{x,x'}(A) := (P(A)x, x')$ for $A \in \mathscr{B}$ again. Since E and E^* are generated by their positive cones, the previous remarks imply that $P_{x,x'}$ is a signed regular Borel measure, and that equation (15.9) also holds for all $x \in E$, $x' \in E^*$, and $f \in \mathscr{B}(X,\mathscr{B};\mathbb{R})$.

If $S \in \mathrm{L_r}(E)$ commutes with $\pi(f)$ for all $f \in \mathrm{C_c}(X)$, then

$$\big(S\pi(f)x, x'\big) = \big(\pi(f)x, S'x'\big) = \int_X f \,\mathrm{d}P_{x,S'x'}$$

for all $f \in \mathrm{C_c}(X)$, $x \in E$ and $x' \in E^*$. On the other hand,

$$\big(S\pi(f)x, x'\big) = \big(\pi(f)Sx, x'\big) = \int_X f \,\mathrm{d}P_{Sx,x'}$$

for all $f \in \mathrm{C_c}(X)$, $x \in E$ and $x' \in E^*$. The uniqueness part of the classical Riesz representation theorem then shows that $P_{x,S'x'} = P_{Sx,x'}$ for all $x \in E$ and $x' \in E^*$, i.e. that $(SP(A), x') = (P(A)Sx, x')$ for all $x \in E$, $x' \in E^*$ and $A \in \mathscr{B}$. This implies that $SP(A) = P(A)S$ for all $A \in \mathscr{B}$.

We show that part (3) implies part (4). For this, we note that $\pi : \mathscr{B}(X,\mathscr{B};\mathbb{R}) \to \mathrm{L_r}(E)$, being a positive linear map between Banach lattices, is automatically continuous when $\mathrm{L_r}(E)$ is supplied with the regular norm. Since $\mathscr{E}(X,\mathscr{B};\mathbb{R}^+) - \mathscr{E}(X,\mathscr{B};\mathbb{R}^+)$ is norm dense in $\mathscr{B}(X,\mathscr{B};\mathbb{R})$, $\pi((\mathscr{E}(X,\mathscr{B};\mathbb{R}^+) - \mathscr{E}(X,\mathscr{B};\mathbb{R}^+)))$ is dense in $\pi(\mathscr{B}(X,\mathscr{B};\mathbb{R}))$ in the regular norm. The continuity of multiplication in $\mathrm{L_r}(E)$ then shows that part (3) implies part (4).

Is is trivial that part (4) implies part (1).

We shall now establish that $P(X)$ projects E onto its non-degenerate component.

Suppose that $x \in E_{\mathrm{nd}}$. Then there exists $f \in \mathrm{C_0}(X)$ and $y \in E$ such that $x = \pi(f)y$. This implies that $P(X)x = \pi(\mathbf{1})\pi(f)y = \pi(\mathbf{1}\cdot f)y = \pi(f)y = x$. Hence $E_{\mathrm{nd}} \subseteq P(X)E$.

Conversely, let $x \in E$ and suppose that $x \in P(X)E$, i.e. suppose that there exists $y \in E$ such that $x = P(X)y$. Then the fact that $P(X)$ is a strong operator limit of operators of the form $\pi(f)$ for $f \prec X$, shows that x is the norm limit of elements of the form $\pi(f)y$ for $f \prec X$. Since these are all in E_{nd} and E_{nd} is closed, we see that $x \in E_{\mathrm{nd}}$. Hence $P(X)E \subseteq E_{\mathrm{nd}}$.

□

We shall now comment on the relation between the present Theorem 15.2 and [**12**, Theorem 5.6], where the existence of spectral measures representing positive representations of $\mathrm{C_0}(X)$ on a KB-space E is also asserted.

In [**12**], a spectral measure on a locally compact Hausdorff space is a map $Q : \mathscr{B} \to \mathrm{L_r}(E)^+$ such that:

(1) $Q(\emptyset) = 0$;
(2) $Q\left(\bigcup_{n=1}^{\infty} A_n\right)x = \sum_{n=1}^{\infty} Q(A_n)x$ for every $x \in E$ and every pairwise disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{B}$;
(3) $Q(A_1 \cap A_2) = Q(A_1)Q(A_2)$ for all $A_1, A_2 \in \mathscr{B}$.

For $x \in E$ and $x' \in E^*$, we can then define $Q_{x,x'} : \mathscr{B} \to \mathbb{R}$ by setting

$$Q_{x,x'}(A) := \big(Q(A)x, x'\big)$$

for $A \in \mathscr{B}$. Then $Q_{x,x'}$ is a signed measure.

Such a spectral measure Q defines a linear map

$$\pi_Q : \big(\mathscr{E}(X,\mathscr{B};\mathbb{R}^+) - \mathscr{E}(X,\mathscr{B};\mathbb{R}^+)\big) \to \mathrm{L_r}(E)$$

in the obvious way, and [**12**, Theorem 4.1] shows that π_Q extends by norm continuity to a positive representation $\pi : \mathscr{B}(X,\mathscr{B};\mathbb{R}) \to \mathrm{L_r}(E)$ of $\mathscr{B}(X,\mathscr{B};\mathbb{R})$ on E. For $\varphi \in \mathscr{B}(X,\mathscr{B};\mathbb{R})$, $\pi_Q(\varphi)$ is then the unique element of $\mathrm{L_r}(E)$ such that

$$\big(\pi_Q(\varphi)x, x'\big) = \int_X \varphi \,\mathrm{d}Q_{x,x'}$$

for all $x \in E^+$ and $x' \in (E^*)^+$.

Then [**12**, Theorem 5.6] asserts that, if $\pi : \mathrm{C_0}(X) \to \mathrm{L_r}(E)$ is a positive representation of $\mathrm{C_0}(X)$ on a KB-space E, there exists a unique spectral measure Q on X such that $\pi_Q = \pi$ and such that $Q_{x,x'}$ is a signed regular real-valued Borel measures on X for all $x \in E$ and $x' \in E^*$.

Since our measure P from the present Theorem 15.2 has the properties in the previous paragraph, the spectral measures P from Theorem 15.2 and Q from [**12**, Theorem 5.6] coincide.

As a consequence, we see that P is σ-additive in the strong operator topology. This, however, was also a priori clear from Lemma 5.2: for a Banach lattices E with order continuous norm, the σ-additivity of an $\mathrm{L_r}(E)$-valued measure in our ordered sense and that in the sense of the strong operator topology coincide.

It is instructive to compare the proofs of Theorem 15.2 and of [**12**, Theorem 2.1].

Our approach towards Theorem 15.2 is via an existence result for spectral measures corresponding to positive homomorphisms from $\mathrm{C_0}(X)$ into partially order algebras. This is an order-theoretical approach, and the spectral measure is obtained directly, without a detour via $\mathscr{B}(X,\mathscr{B};\mathbb{R})$ as in the proof of [**12**, Theorem 2.1].

The proof of [**12**, Theorem 2.1] follows more classical lines, and is as follows. Suppose that $\pi : \mathrm{C_0}(X) \to \mathrm{L_r}(E)$ is a positive representation of $\mathrm{C_0}(X)$ on a KB-space E. For $x \in E$ and $x' \in E^*$, the classical Riesz representation theorem supplies a regular signed Borel measure $\mu_{x,x'}$ such that

$$\big(\pi(f)x, x'\big) = \int_X f \,\mathrm{d}\mu_{x,x'}.$$

for all $f \in \mathrm{C_0}(X)$. For every $\varphi \in \mathscr{B}(X,\mathscr{B};\mathbb{R})$, the bilinear form $[\cdot,\cdot]_\varphi$ on $E \times E^*$, defined by

$$[x, x']_\varphi := \int_X \varphi \,\mathrm{d}\mu_{x,x'},$$

is then bounded. Therefore, there exist a bounded operator $\widetilde{\pi}(\varphi) : E \to E^{**}$ such that $(\widetilde{\pi}(\varphi)x, x') = [x, x']_\varphi$ for all $x \in E$ and $x' \in E^*$. The fact that E is a band in E^{**} (this characterises the KB-spaces among the Banach lattices) is then used to see that $\widetilde{\pi}(\varphi)$ actually maps E into itself (as embedded in E^{**}). Subsequently, one shows that the map $\widetilde{\pi} : \mathscr{B}(X,\mathscr{B};\mathbb{R}) \to \mathrm{L_r}(E)$ that is thus obtained is a positive algebra homomorphism; it extends π by construction. The spectral measure is then finally found by setting $P(A) := \widetilde{\pi}(\chi_A)$ for a Borel set A. This approach involves a detour via $\mathscr{B}(X,\mathscr{B};\mathbb{R})$. The order-theoretical component in this proof is much more limited; one could say that it is primarily a Banach space approach.

15.2. Representations of $\mathrm{C_0}(X,\mathbb{C})$ on Hilbert spaces

In this section, we let $\mathrm{C_0}(X,\mathbb{C})$ denote the continuous complex-valued functions on a locally compact Hausdorff space X that vanish at infinity. We shall consider representations of the (complex) C^*-algebra $\mathrm{C_0}(X,\mathbb{C})$ on complex Hilbert spaces. For these, we have the following result. Its formulation uses a few self-evident notations, and an implicit definition of the order integral of complex-valued functions by considering their real and imaginary parts.

Theorem 15.3 (Existence of spectral measures for representations of $\mathrm{C}_0(X,\mathbb{C})$ on Hilbert spaces)**.** *Let X be a locally compact Hausdorff space, let $\mathscr{H}$ be a complex Hilbert space, and let $\pi : \mathrm{C}_0(X,\mathbb{C}) \to \mathscr{B}(\mathscr{H})$ be a *-homomorphism.*

Let $\mathscr{A}$ be the commutative von Neumann algebra that is generated by $\pi(\mathrm{C}_0(X,\mathbb{C}))$, supplied with the partial ordering that is inherited from the usual partial ordering on $\mathscr{B}(\mathscr{H})_{\mathrm{sa}}$.

Then there exists a unique regular Borel measure $\mu : \mathscr{B} \to \overline{\mathscr{A}_{\mathrm{sa}}^+}$ on the Borel σ-algebra $\mathscr{B}$ of X such that

$$\pi(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu \tag{15.10}$$

for all $f \in \mathrm{C_c}(X)$.

The measure P is a spectral measure, and equation (15.10) *also holds for all $f \in \mathrm{C}_0(X,\mathbb{C})$.*

For $f \in \mathrm{C}_0(X,\mathbb{C})$, $\pi(f)$ is the unique element of $\mathscr{B}(\mathscr{H})$ such that

$$\langle \pi(f)x, y\rangle = \int_X f \,\mathrm{d}P_{x,y} \tag{15.11}$$

for all $x,y \in \mathscr{H}$, where the complex-valued regular Borel measure $P_{x,y} : \mathscr{B} \to \mathbb{C}$ is defined, for $x,y \in \mathscr{H}$, by setting $P_{x,y}(A) := \langle P(A)x, y\rangle$ for $A \in \mathscr{B}$.

If V is a non-empty open subset of X, then

$$\mu(V) = \bigvee \{\, T(f) : f \prec V \,\}$$

in $\mathscr{A}_{\mathrm{sa}}$.

If K is a compact subset of X, then

$$\mu(K) = \bigwedge \{\, T(f) : K \prec f \,\}$$

in $\mathscr{A}_{\mathrm{sa}}$.

The map π extends to an involutive algebra homomorphism from $\mathscr{L}^1(X,\mathscr{B},P;\mathbb{C})$ into $\mathscr{A}$, denoted by π again, by setting $\pi(f) := \int_X^{\mathrm{o}} f \,\mathrm{d}P$ for $f \in \mathscr{L}^1(X,\mathscr{B},P;\mathbb{C})$.

For $S \in \mathscr{B}(\mathscr{H})$, the following are equivalent:

(1) *$S\pi(f) = \pi(f)S$ for all $f \in \mathrm{C}_0(X,\mathbb{C})$;*
(2) *$S\pi(f) = \pi(f)S$ for all $f \in \mathrm{C_c}(X,\mathbb{C})$;*
(3) *$SP(A) = P(A)S$ for all Borel sets A;*
(4) *$S\pi(f) = \pi(f)S$ for all $f \in \mathscr{B}(X,\mathscr{B},\mathbb{C})$.*

The orthogonal projection $P(X)$ projects $\mathscr{H}$ onto $\mathscr{H}_{\mathrm{nd}}$. Hence the representation $\pi : \mathrm{C}_0(X,\mathbb{C}) \to \mathscr{B}(\mathscr{H})$ is non-degenerate if and only if $P(X) = I$.

PROOF. Most of the statements are immediate from a combination of Theorem 14.10, Proposition 4.4, and Theorem 15.1, combined with complex or sesquilinear extension . The remaining ones are established by routine verification, using the $P_{x,y}$ in a fashion analogous to the use of the $P_{x,x'}$ in the proof of Theorem 15.2. □

Remark 15.4. Let us make a few comments on the theorem.

(1) Using the $P_{x,y}$ for $x,y \in \mathscr{H}$, one easily shows, similarly as for Theorem 15.2, that the spectral measure P coincides with the one in the literature for non-degenerate representations (see e.g. [**14**, Corollary 1.55], where it is supposed that π is non-degenerate). Consequently, P is then σ-additive in the strong operator topology, but we also already knew this from Lemma 5.3.

(2) The familiar statement that the $P(A)$ for $A \in \mathscr{B}$ are in the bicommutant of $\pi(C_0(X,\mathbb{C}))$ is immediate from the theorem, since

$$P(\Omega) \subset P(\Omega)'' = \big(P(\Omega)'\big)' = \big(\pi(C_0(X,\mathbb{C}))'\big)' = \pi(C_0(X,\mathbb{C}))''.$$

(3) Via the present approach, it is immediate that the $P(A)$ for $A \in \mathscr{B}$ are in $\mathscr{A}$. In more classical approaches, it is shown that the $P(A)$ are in the bicommutant of $\pi(C_0(X,\mathbb{C}))$, and von Neumann's bicommutant theorem is then needed to show that they are in $\mathscr{A}$.

(4) We are not aware of a reference for the explicit expressions for the spectral measures of open and compact subsets of X as strong operator limits as they follow from the theorem.

(5) In the literature, spectral measures with values in the orthogonal projections on a Hilbert space are supposed to be such that the projection corresponding to the entire underlying point set is the identity map. One then needs the assumption that a representation of $C_0(X,\mathbb{C})$ be non-degenerate to ensure that the associated spectral measure has this property. Our treatment shows that one need not work with such 'unital' spectral measures or non-degenerate representations.

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