Positive representations of algebras of continuous functions

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This thesis consists of two chapters. Chapter I focuses on positive representations of algebras of continuous functions on partially ordered vector spaces. Chapter II focuses on simultaneous power factorization in modules over Banach algebras.

Preface

One of the main questions in representation theory is to determine if arbitrary representations are built from irreducible ones. For unitary representations of locally compact groups and for representations of C*-algebras on Hilbert spaces, the answer is affirmative and a rather satisfactory theory is available. There may be several reasons why representation theory in Hilbert spaces is so well developed. For analysts, these are the easiest of Banach spaces to work with. Furthermore, their central position in quantum theory has put these spaces into the mathematical spotlight in the first half of the 20th century, and the story of representations in these spaces and of operator algebras has unfolded naturally since then.

Outside quantum theory, however, Hilbert spaces are often not a canonical type of function spaces to work with in the natural sciences. Integrability of functions (reflecting finiteness of quantities) or boundedness are often more natural conditions than the square integrability that is imposed on wave functions. Functions in these spaces are usually real-valued, and, more often than not, there is a meaningful notion of positivity in these spaces. Symmetry groups of the system tend to act in a natural way on these spaces as positive linear maps. Positive representations of groups and algebras on partially ordered vector space are, therefore, quite natural. Yet there is no theory available of, say, positive representations on Banach lattices, that even remotely resembles that of representations on Hilbert spaces.

This thesis covers a part of such a theory of positive representations.

An important class of C*-algebras that is of interest in its own right, and that is particularly helpful when understanding more complicated situations in representation theory in Hilbert spaces, are the commutative C*-algebras. These are the algebras of continuous complex-valued functions on a locally compact Hausdorff space that vanish at infinity. One of the tools to describe their representations on Hilbert spaces are the spectral measures that correspond to such representations.

When developing representation theory in an ordered context, it is desirable to have a similar tool available for positive representations of such algebras of continuous real-valued functions on partially ordered vector spaces, and, in particular, on Banach lattices. The principal aim of Chapter I of this thesis is, therefore, to establish the existence of spectral measures corresponding to such positive representations. We refer to the Summary for more details on this part of the work. It has lead, among others, to an abstract spectral theorem for positive homomorphisms from algebras of continuous functions into partially ordered algebras that has the existence of projection-valued spectral measures in both the ordered context and in the Hilbert space context as immediate consequences.

Chapter II of this thesis is inspired by the classical Cohen-Hewitt factorization theorem. In a Banach algebra with a bounded left approximate identity, this states that every element can be factored. There is a more general statement for non-degenerate Banach modules over such algebras. In our context, supposing that the Banach algebra and the module are ordered, and that the module is a positive module over the algebra, it is natural to ask if such factorizations of a positive element of the module are possible with both factors positive. This is not generally possible, but for our algebras

of continuous functions the answer is affirmative. Apart from this, Chapter II contains a systematic investigation of simultaneous power factorization in modules over Banach algebras, generalising and unifying several results in the literature.

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