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Citation

Beshenov, A. (2018, December 10). *Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology*. Retrieved from <https://hdl.handle.net/1887/68224>

Version: Not Applicable (or Unknown)

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Cover Page



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Issue Date: 2018-12-10

Stellingen

behorende bij het proefschrift

Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

van Alexey Beshenov

In everything what follows, X is an arithmetic scheme (separated, of finite type over $\text{Spec } \mathbb{Z}$) and n is a *strictly negative integer*.

We denote by $\mathbb{Z}^c(n)$ the dualizing Bloch's cycle complex of sheaves on $X_{\text{ét}}$, and by $\mathbb{Z}(n)$ the complex of sheaves $\bigoplus_p \varinjlim_r j_{p!} \mu_{p^r}^{\otimes n}[-1]$, where $j_p: X[1/p] \hookrightarrow X$ is the canonical open immersion for each prime p and $\mu_{p^r}^{\otimes n}$ is the sheaf of p^r -th roots of unity on $X[1/p]_{\text{ét}}$ twisted by n .

We denote by $R\Gamma_c(X_{\text{ét}}, \mathcal{F}^\bullet)$ the étale cohomology with compact support and by $R\widehat{\Gamma}_c(X_{\text{ét}}, \mathcal{F}^\bullet)$ the modified étale cohomology with compact support, as defined e.g. in Milne's book "Arithmetic Duality theorems".

For brevity, we write $[A^\bullet, B^\bullet]$ instead of $\text{RHom}(A^\bullet, B^\bullet)$.

All the main constructions are done assuming the **conjecture** $\mathbf{L}^c(X_{\text{ét}}, n)$: *the cohomology groups $H^i(X_{\text{ét}}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.*

I. Assuming $\mathbf{L}^c(X_{\text{ét}}, n)$, there is a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{\text{ét}}, \mathbb{Z}(n)) \xrightarrow{\cong} [R\Gamma(X_{\text{ét}}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]].$$

II. Assume $\mathbf{L}^c(X_{\text{ét}}, n)$ and let $\alpha_{X,n}$ be the composition of morphisms of complexes

$$[R\Gamma(X_{\text{ét}}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] \rightarrow [R\Gamma(X_{\text{ét}}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]] \xleftarrow{\cong} R\widehat{\Gamma}_c(X_{\text{ét}}, \mathbb{Z}(n)) \rightarrow R\Gamma_c(X_{\text{ét}}, \mathbb{Z}(n))$$

Let $R\Gamma_{fg}(X, \mathbb{Z}(n))$ be a cone of $\alpha_{X,n}$:

$$[R\Gamma(X_{\text{ét}}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\text{ét}}, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \rightarrow [R\Gamma(X_{\text{ét}}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]]$$

Then the cohomology groups $H^i(R\Gamma_{fg}(X, \mathbb{Z}(n)))$ are finitely generated, trivial for $i \ll 0$, and only have 2-torsion for $i \gg 0$.

III. For any prime ℓ the group $H_c^i(X_{\overline{\mathbb{Q}}_{\text{ét}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{\text{G}_\mathbb{Q}}$ has no nontrivial divisible elements.

IV. Assume $\mathbf{L}^c(X_{\text{ét}}, n)$ and let $\alpha_{X,n}$ be as above. Let

$$u_\infty^*: R\Gamma_c(X_{\text{ét}}, \mathbb{Z}(n)) \rightarrow R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{Z})$$

be the canonical comparison morphism, discussed in §§0.7–0.8 of the thesis. Then $u_\infty^* \circ \alpha_{X,n} = 0$. Let i_∞^* be a morphism of complexes defined via

$$\begin{array}{ccccccc} [R\Gamma(X, \mathbb{Z}^c(n)), \mathbb{Q}[-2]] & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{\text{ét}}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & \cdots \\ \downarrow & & \downarrow u_\infty^* & & \downarrow i_\infty^* & & \downarrow \\ 0 & \longrightarrow & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{Z}) & \xrightarrow{\text{id}} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

and let $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ be a mapping fiber of i_∞^* :

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{Z}) \rightarrow R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

Then $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is a perfect complex.

* * * * *

To formulate the next result, we denote by $\mathbf{C}(X, n)$ the following conjecture.

a) assume that the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds;

b) assume that $X_{\mathbf{C}}$ is smooth, quasi-projective, so that the regulator morphism

$$\text{Reg}: R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{R})[1]$$

exists and assume the **regulator conjecture**: the \mathbb{R} -dual is a quasi-isomorphism

$$\text{Reg}^{\vee}: R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{R})[-1] \xrightarrow{\cong} [R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}].$$

c) assume that the zeta-function $\zeta(X, s)$ has a meromorphic continuation near $s = n$.

Then

1) the leading coefficient $\zeta^*(X, n)$ of the Taylor expansion of $\zeta(X, s)$ at $s = n$ is given up to sign by

$$\lambda(\zeta^*(X, n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)),$$

where λ is the trivialization morphism defined using the regulator in §2.3 of the thesis;

2) the vanishing order of $\zeta(X, n)$ at $s = n$ is given by the weighted alternating sum of ranks of $H_{W,c}^i(X, \mathbb{Z}(n))$:

$$\text{ord}_{s=n} \zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \text{rk}_{\mathbb{Z}} H_{W,c}^i(X, \mathbb{Z}(n)).$$

* * * * *

V. The conjecture $\mathbf{C}(X, n)$ is compatible with disjoint unions, open-closed decompositions and taking affine bundles in the following sense.

- If $X = \coprod_{0 \leq i \leq r} X_i$ is a disjoint union of arithmetic schemes, then the conjectures $\mathbf{C}(X_i, n)$ for $i = 0, \dots, r$ together imply $\mathbf{C}(X, n)$.
- If $U \hookrightarrow X \hookrightarrow Z$ is an open-closed decomposition of an arithmetic scheme, then if two out of three conjectures $\mathbf{C}(U, n)$, $\mathbf{C}(Z, n)$, $\mathbf{C}(X, n)$ hold, the other one holds as well.
- The conjecture $\mathbf{C}(\mathbb{A}_{X,r}^r, n)$ is equivalent to $\mathbf{C}(X, n - r)$.

VI. Sometimes it is possible to talk about unique cones in the derived category. For a distinguished triangle $A^{\bullet} \xrightarrow{u} B^{\bullet} \xrightarrow{v} C^{\bullet} \xrightarrow{w} A^{\bullet}[1]$ assume that A^{\bullet} is a complex such that $H^i(A^{\bullet})$ are finite dimensional \mathbb{Q} -vector spaces and C^{\bullet} is "almost perfect", meaning that $H^i(C^{\bullet})$ are finitely generated groups, zero for $i \ll 0$ and have only 2-torsion for $i \gg 0$. Then the cone of u is unique up to a unique isomorphism in the derived category.

- VII. If A and B are finitely generated abelian groups, then up to equivalence, every extension of $\text{Hom}(B, \mathbb{Q}/\mathbb{Z})$ by $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is \mathbb{Q}/\mathbb{Z} -dual to an extension of A by B .
- VIII. The order of zero of the Dedekind zeta function of a number field K at $n < 0$ may be interpreted via the equivariant cohomology of $X = \text{Spec } O_K$ as $\text{ord}_{s=n} \zeta_K(s) = \dim_{\mathbb{R}} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{R})$.