Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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Chapter 2

Conjecture about zeta-values

The regulator morphism is introduced in §2.2, using the constructions from [KLMS2006]. It is more naturally defined with its target in Deligne homology, and all the necessary preliminaries about it are included in §2.1. Then in everything is put together to formulate the conjectural relation of Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ to the special values $\zeta_X^*(n)$. Finally, it is verified in §2.4 that the conjecture is compatible with disjoint unions, open-closed decompositions and taking the affine bundle $\mathbb{A}_X^r \to X$.

2.1 Deligne cohomology and homology

Now we are going review the definitions of Deligne cohomology and homology. These were introduced in Beĭlinson's seminal paper [Bei1984], so they are also known in the literature as "Deligne–Beĭlinson (co)homology", but I will use the term "Deligne (co)homology" for brevity. For the technical details, the reader may consult [EV1988] and [Jan1988].

For this section, $\mathcal{X}$ denotes a smooth complex algebraic variety over $\mathbb{C}$, and $\mathbb{Z} \subset A \subseteq \mathbb{R}$ denotes a subring of the ring of real numbers (eventually we will be interested in $A = \mathbb{Z}$ and $\mathbb{R}$). For a parameter $k \in \mathbb{Z}$ one can define (co)homology groups

$$H^i_{\mathcal{D}}(\mathcal{X}, A(k)), \quad H_i^{\mathcal{D}}(\mathcal{X}, A(k)).$$

Here k is a "twist" that may be any integer. In fact, for certain values of k the above groups have simpler description, and it will be our case.

We are going to assume that $\mathcal{X}$ is connected, of dimension $d_{\mathbb{C}}$. A **good**

compactification of X is given by

$$X \overset{j}{\hookrightarrow} \overline{X} \hookleftarrow D \tag{2.1.1}$$

where $j\colon X \hookrightarrow \overline{X}$ is an embedding into a proper smooth algebraic variety $\overline{X}$, and the complement $D := \overline{X} \setminus X$ is a normal crossing divisor (meaning that locally in the analytic topology, D has smooth components intersecting transversally). Such a good compactification always exists (this follows from Hironaka's resolution of singularities), and we fix one.

Deligne cohomology

We denote by $\Omega^\bullet_{X(\mathbb{C})}$ the de Rham complex of holomorphic differential forms on $X(\mathbb{C})$:

$$0 \to \mathcal{O}_{X(\mathbb{C})} \to \Omega^1_{X(\mathbb{C})} \to \Omega^2_{X(\mathbb{C})} \to \cdots \to \Omega^{d_\mathbb{C}}_{X(\mathbb{C})} \to 0$$

Further, let $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D)$ be the de Rham complex of meromorphic differential forms on $\overline{X}(\mathbb{C})$, holomorphic on $X(\mathbb{C})$, with at most logarithmic poles along $D(\mathbb{C})$. We consider the descending filtration of $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D)$ by subcomplexes

$$\begin{aligned}\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})}(\log D)\colon \quad 0 \to \cdots \to 0 \to \Omega^k_{\overline{X}(\mathbb{C})}(\log D) \to \Omega^{k+1}_{\overline{X}(\mathbb{C})}(\log D) \\ \to \cdots \to \Omega^{d_\mathbb{C}}_{\overline{X}(\mathbb{C})}(\log D) \to 0\end{aligned}$$

Let us fix some conventions related to the cones of complexes. If $u\colon A^\bullet \to B^\bullet$ is a morphism of complexes, the corresponding cone complex is given by

$$\operatorname{Cone}(u) := A^\bullet[1] \oplus B^\bullet,$$

together with the differentials

$$\begin{aligned} d^i\colon A^{i+1} \oplus B^i &\to A^{i+2} \oplus B^{i+1}, \\ (a,b) &\mapsto (-d_A^{i+1}(a), u(a) + d_B^i(b)). \end{aligned}$$

This gives us a short exact sequence of complexes

$$B^\bullet \rightarrowtail \operatorname{Cone}(u) \twoheadrightarrow A^\bullet[1]$$

and the corresponding distinguished triangle in the derived category

$$A^\bullet \to B^\bullet \to \operatorname{Cone}(u) \to A^\bullet[1]$$

2.1.1. Definition. Let A be a subring of $\mathbb{R}$. For $k \in \mathbb{Z}$ we denote

$$A(k) := (2\pi i)^k A \subset \mathbb{C}.$$

This is a $G_{\mathbb{R}}$-module, and we will also denote by $A(k)$ the corresponding ($G_{\mathbb{R}}$-equivariant) sheaf on $\mathcal{X}(\mathbb{C})$. For a fixed good compactification (2.1.1), the corresponding **Deligne–Beĭlinson complex** is the complex of sheaves on $\overline{\mathcal{X}}(\mathbb{C})$ given by

$$A(k)_{\mathcal{D}\text{-Б},(\overline{\mathcal{X}},\mathcal{X})} := \operatorname{Cone}\Bigl(Rj_* A(k) \oplus \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \xrightarrow{\epsilon-\iota} Rj_* \Omega^{\bullet}_{\mathcal{X}(\mathbb{C})}\Bigr)\,[-1],$$

where

$$\epsilon\colon Rj_* A(k) \to Rj_* \Omega^{\bullet}_{\mathcal{X}(\mathbb{C})}$$

is induced by the canonical morphism of sheaves $A(k) \to \mathcal{O}_{\mathcal{X}(\mathbb{C})}$ and

$$\iota\colon \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \to Rj_* \Omega^{\bullet}_{\mathcal{X}(\mathbb{C})}$$

is induced by a natural inclusion

$$\Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \xrightarrow{\simeq} j_* \Omega^{\bullet}_{\mathcal{X}(\mathbb{C})} = Rj_* \Omega^{\bullet}_{\mathcal{X}(\mathbb{C})},$$

which is a quasi-isomorphism of filtered complexes (see [Del1971, §3.1]). The corresponding **Deligne cohomology** groups are given by the hypercohomology of $A(k)_{\mathcal{D}\text{-Б},(\overline{\mathcal{X}},\mathcal{X})}$:

$$H^i_{\mathcal{D}}(\mathcal{X}, A(k)) := H^i(R\Gamma(\overline{\mathcal{X}}(\mathbb{C}), A(k)_{\mathcal{D}\text{-Б},(\overline{\mathcal{X}},\mathcal{X})})).$$

The distinguished triangle of sheaves on $\overline{\mathcal{X}}(\mathbb{C})$

$$A(k)_{\mathcal{D}\text{-Б},(\overline{\mathcal{X}},\mathcal{X})} \to Rj_* A(k) \oplus \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \xrightarrow{\epsilon-\iota} Rj_* \Omega^{\bullet}_{\mathcal{X}(\mathbb{C})} \to A(k)_{\mathcal{D}\text{-Б},(\overline{\mathcal{X}},\mathcal{X})}[1]$$

induces the (hyper)cohomology long exact sequence

(2.1.2)
$$\cdots \to H^i_{\mathcal{D}}(\mathcal{X}, A(k)) \to H^i(\mathcal{X}(\mathbb{C}), A(k)) \oplus F^k H^i_{dR}(\mathcal{X}(\mathbb{C})) \xrightarrow{\epsilon-\iota} H^i_{dR}(\mathcal{X}(\mathbb{C})) \to H^{i+1}_{\mathcal{D}}(\mathcal{X}, A(k)) \to \cdots$$

where

$$F^k H^i_{dR}(\mathcal{X}(\mathbb{C})) := \operatorname{im}\Bigl(\mathbb{H}^i(\overline{\mathcal{X}}(\mathbb{C}), \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D)) \hookrightarrow \mathbb{H}^i(\overline{\mathcal{X}}(\mathbb{C}), \Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D)) \cong H^i_{dR}(\mathcal{X}(\mathbb{C}))\Bigr)$$

denotes the Hodge filtration on the de Rham cohomology of $\mathcal{X}(\mathbb{C})$ (for details about this, see [Del1971] and [Voi2002, Chapter 8]). Using the above distinguished triangle / long exact sequence, one may show that the groups $H^i_{\mathcal{D}}(\mathcal{X}, A(k))$ in fact do not depend on the choice of a good compactification $j\colon \mathcal{X} \hookrightarrow \overline{\mathcal{X}}$ (see [EV1988, Lemma 2.8]). For this we will write simply "$A(k)_{\mathcal{D}\text{-}\mathrm{B}}$" instead of "$A(k)_{\mathcal{D}\text{-}\mathrm{B},(\overline{\mathcal{X}},\mathcal{X})}$" if $\mathcal{X}$ is clear from the context and a specific good compactification does not matter.

Eventually we will be interested in a very special case when Deligne cohomology is particularly easy to describe.

2.1.2. Lemma. *For $k > d_{\mathbb{C}}$ and $A = \mathbb{R}$ we have a quasi-isomorphism of complexes*

$$R\Gamma(\overline{\mathcal{X}}(\mathbb{C}), \mathbb{R}(k)_{\mathcal{D}\text{-}\mathrm{B}}) \simeq R\Gamma(\mathcal{X}(\mathbb{C}), (2\pi i)^{k-1}\,\mathbb{R})[-1].$$

Proof. We have $\Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) = 0$ for $k > d_{\mathbb{C}}$, so that in this case

$$A(k)_{\mathcal{D}\text{-}\mathrm{B}} = \mathrm{Cone}(Rj_* A(k) \xrightarrow{\epsilon} Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})})[-1] \cong Rj_*\,\mathrm{Cone}(A(k) \xrightarrow{\epsilon} \Omega^\bullet_{\mathcal{X}(\mathbb{C})})[-1],$$

and we easily see that the complex of sheaves $\mathrm{Cone}(A(k) \xrightarrow{\epsilon} \Omega^\bullet_{\mathcal{X}(\mathbb{C})})[-1]$ on $\mathcal{X}(\mathbb{C})$ is given by

$$[A(k) \to \Omega^\bullet_{\mathcal{X}(\mathbb{C})}[-1]] := \left[0 \to \underset{0}{A(k)} \to \underset{1}{\mathcal{O}_{\mathcal{X}(\mathbb{C})}} \to \underset{2}{\Omega^1_{\mathcal{X}(\mathbb{C})}} \to \underset{3}{\Omega^2_{\mathcal{X}(\mathbb{C})}} \to \cdots \to \underset{d_{\mathbb{C}}+1}{\Omega^{d_{\mathbb{C}}}_{\mathcal{X}(\mathbb{C})}} \to 0 \right]$$

—that is, we have the constant sheaf $A(k) := (2\pi i)^k A$ in degree 0, followed by the whole holomorphic de Rham complex on $\mathcal{X}(\mathbb{C})$, shifted by one position. By the Poincaré lemma, we have a quasi-isomorphism of complexes of sheaves on $\mathcal{X}(\mathbb{C})$

$$\mathbb{C} \xrightarrow{\simeq} \Omega^\bullet_{\mathcal{X}(\mathbb{C})}, \tag{2.1.3}$$

and we also have a short exact sequence of $G_{\mathbb{R}}$-modules

$$(2\pi i)^k\,\mathbb{R} \rightarrowtail \mathbb{C} \twoheadrightarrow (2\pi i)^{k-1}\,\mathbb{R}. \tag{2.1.4}$$

Now (2.1.3) and (2.1.4) give us a quasi-isomorphism of complexes of sheaves on $\mathcal{X}(\mathbb{C})$

$$[\mathbb{R}(k) \to \Omega^\bullet_{\mathcal{X}(\mathbb{C})}[-1]] \simeq (2\pi i)^{k-1}\,\mathbb{R}[-1].$$

Putting all this together, we have

$$\begin{aligned} R\Gamma(\overline{X}(\mathbb{C}), \mathbb{R}(k)_{\mathcal{D}\text{-Б}}) &\simeq R\Gamma(\overline{X}(\mathbb{C}), Rj_*[(2\pi i)^k\,\mathbb{R} \to \Omega^\bullet_{X(\mathbb{C})}[-1]]) \\ &\simeq R\Gamma(\overline{X}(\mathbb{C}), Rj_*(2\pi i)^{k-1}\,\mathbb{R})[-1] \\ &\simeq R\Gamma(X(\mathbb{C}), (2\pi i)^{k-1}\,\mathbb{R})[-1]. \end{aligned}$$

■

Deligne homology

Deligne homology $H^{\mathcal{D}}_\bullet(X, A(k))$ is constructed in such a way that there is an isomorphism with Deligne cohomology

$$H^i_{\mathcal{D}}(X, A(k)) \xrightarrow{\cong} H^{\mathcal{D}}_{2d_{\mathbb{C}}-i}(X, A(d_{\mathbb{C}} - k)).$$

To do this, Jannsen in his article [Jan1988] replaces the singular cohomology $H^\bullet(X(\mathbb{C}), A(k))$ with Borel–Moore homology $H^{BM}_\bullet(X(\mathbb{C}), A(k))$, and de Rham cohomology $H^\bullet_{dR}(X(\mathbb{C}))$ with the corresponding object, which he calls "de Rham homology". It would be probably more correct to call $H^{\mathcal{D}}_\bullet(X, A(k))$ "Deligne–Borel–Moore homology".

We would like to compare homological and cohomological complexes, and for this the following convention will be used. To pass from a homological complex $C_\bullet$ to a cohomological complex $'C^\bullet$, we set

$$'C^i := C_{-i},$$

and the differentials are given by

$$('C^i \xrightarrow{d^i} {}'C^{i+1}) := (C_{-i} \xrightarrow{(-1)^{i+1}\,d} C_{-i-1})$$

(note the alternating signs).

As before, we fix a good compactification (2.1.1). Here are the ingredients for the definition of Deligne homology (we refer to [Jan1988] for details).

1. We consider the quotient complex

$$'C^\bullet(\overline{X}, D, A(k)) := {}'C^\bullet(\overline{X}(\mathbb{C}), A(k)) / {}'C^\bullet_D(\overline{X}(\mathbb{C}), A(k)),$$

   where $C_\bullet(\overline{X}(\mathbb{C}), A(k))$ denotes the complex of singular C^∞-chains on $\overline{X}(\mathbb{C})$ with coefficients in $A(k) := (2\pi i)^k A$, and $C^D_\bullet(\overline{X}(\mathbb{C}), A(k))$ is the subcomplex of chains with support on $D(\mathbb{C})$. We put $'C^\bullet$ instead of $C_\bullet$ to pass to cohomological complexes.

2. We denote by $\Omega^{p,q}_{\mathcal{X}(\mathbb{C})^\infty}$ the sheaf of C^∞-(p,q)-forms on $\mathcal{X}(\mathbb{C})$ (sometimes also denoted by $\mathcal{A}^{p,q}_{\mathcal{X}(\mathbb{C})}$).

3. We denote by ${}'\Omega^{p,q}_{\mathcal{X}(\mathbb{C})^\infty}$ the sheaf of distributions over $\Omega^{-p,-q}_{\mathcal{X}(\mathbb{C})^\infty}$. That is, for an open subset $U \subseteq X$ we have

$$ {}'\Omega^{p,q}_{\mathcal{X}(\mathbb{C})^\infty}(U) := \{\text{continuous linear functionals on } \Gamma_c(U, \Omega^{-p,-q}_{\mathcal{X}(\mathbb{C})^\infty})\}. $$

4. Both $\Omega^{\bullet,\bullet}_{\mathcal{X}(\mathbb{C})^\infty}$ and ${}'\Omega^{\bullet,\bullet}_{\mathcal{X}(\mathbb{C})^\infty}$ naturally form double complexes. We denote by $\Omega^{\bullet}_{\mathcal{X}(\mathbb{C})^\infty}$ and ${}'\Omega^{\bullet}_{\mathcal{X}(\mathbb{C})^\infty}$ the total complexes associated to $\Omega^{\bullet,\bullet}_{\mathcal{X}(\mathbb{C})^\infty}$ and ${}'\Omega^{\bullet,\bullet}_{\mathcal{X}(\mathbb{C})^\infty}$ respectively:

$$ \Omega^{n}_{\mathcal{X}(\mathbb{C})^\infty} := \bigoplus_{p+q=n} \Omega^{p,q}_{\mathcal{X}(\mathbb{C})^\infty}, \quad {}'\Omega^{n}_{\mathcal{X}(\mathbb{C})^\infty} := \bigoplus_{p+q=n} {}'\Omega^{p,q}_{\mathcal{X}(\mathbb{C})^\infty}. $$

5. As before, we consider the corresponding logarithmic de Rham complexes and their filtrations:

$$ \Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D) := \Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \otimes_{\Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})}} \Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}, $$
$$ \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D) := \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \otimes_{\Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})}} \Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}, $$

and similarly for ${}'\Omega$ instead of Ω.

2.1.3. Definition. In the above setting, for a fixed good compactification (2.1.1), consider the complex of abelian groups

$$ {}'C^{\bullet}_{\mathcal{D}}(\overline{\mathcal{X}}, D, A(k)) := $$
$$ \operatorname{Cone}\left(\begin{matrix} {}'C^{\bullet}(\overline{\mathcal{X}}, D, A(k)) \\ \oplus \\ \Gamma(\overline{\mathcal{X}}(\mathbb{C}), {}'\Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D)) \end{matrix} \xrightarrow{\epsilon - \iota} \Gamma(\overline{\mathcal{X}}(\mathbb{C}), {}'\Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D)) \right)[-1], $$

where ι is induced by the inclusion ${}'\Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D) \subset {}'\Omega^{\bullet}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D)$, and ϵ is given by the integration over chains (see [Jan1988] for details). The corresponding **Deligne homology** groups are given by

$$ {}'H^{i}_{\mathcal{D}}(\mathcal{X}, A(k)) := H^{i}({}'C^{\bullet}_{\mathcal{D}}(\overline{\mathcal{X}}, D, A(k))). $$

To understand the above definition, we should examine what each complex computes.

1. According to [Jan1988, Lemma 1.11], the complex ${}'C^\bullet(\overline{X}, D, A(k))$ calculates Borel–Moore homology of $X(\mathbb{C})$ with coefficients in $A(k)$: there are canonical isomorphisms

$$H^i({}'C^\bullet(\overline{X}, D, A(k))) \cong {}'H^i_{BM}(X(\mathbb{C}), A(k)) = H^{BM}_{-i}(X(\mathbb{C}), A(-k))$$

(see loc. cit. and [Ver1976, §1] for details on Borel–Moore homology).

2. According to [Jan1988, Corollary 1.13], there are quasi-isomorphisms of fine sheaves

$$Rj_*\, {}'\Omega^\bullet_{X(\mathbb{C})^\infty} = j_*\, {}'\Omega^\bullet_{X(\mathbb{C})^\infty} \xleftarrow{\simeq} j_* \Omega^\bullet_{X(\mathbb{C})^\infty}[2d_\mathbb{C}] \xleftarrow{\simeq} \Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D)[2d_\mathbb{C}] \xrightarrow{\simeq} {}'\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D)$$

and then Jannsen defines

$${}'H^i_{dR}(X(\mathbb{C})) := H^i(\Gamma(X(\mathbb{C}), {}'\Omega^\bullet_{X(\mathbb{C})^\infty})) \cong H^i(\Gamma(\overline{X}(\mathbb{C}), {}'\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D)))$$

to be the **de Rham homology** of $X(\mathbb{C})$ (this is by no means standard terminology).

3. De Rham homology carries a Hodge filtration defined by

$$F^k\, {}'H^i_{dR}(X(\mathbb{C})) := \operatorname{im}\Big(H^i(\Gamma(\overline{X}(\mathbb{C}), {}'\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})^\infty}(\log D))) \hookrightarrow H^i(\Gamma(\overline{X}(\mathbb{C}), {}'\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D))) \cong {}'H^i_{dR}(X(\mathbb{C}))\Big)$$

(the fact that this map is injective is in a sense dual to the corresponding fact for the Hodge filtration on de Rham cohomology).

The above considerations and the definition of Deligne homology give us the long exact sequence

$$(2.1.5)\quad \cdots \to {}'H^i_\mathcal{D}(X, A(k)) \to {}'H^i_{BM}(X(\mathbb{C}), A(k)) \oplus F^k\, {}'H^i_{dR}(X(\mathbb{C})) \xrightarrow{\epsilon - \iota} {}'H^i_{dR}(X(\mathbb{C})) \to {}'H^{i+1}_\mathcal{D}(X, A(k)) \to \cdots$$

from which one may see that the groups ${}'H^i_\mathcal{D}(X, A(k))$ do not depend on the choice of a good compactification $X \hookrightarrow \overline{X}$ (again, see [Jan1988, Corollary 1.13]).

Twisted Poincaré duality

According to [Jan1988, Theorem 1.15], Deligne cohomology and homology are related through the “twisted Poincaré duality”[*]

$$H^{2d_{\mathbb{C}}+i}_{\mathcal{D}}(X, A(d_{\mathbb{C}}+k)) \xrightarrow{\cong} {}'H^i_{\mathcal{D}}(X, A(k)). \tag{2.1.6}$$

In fact, Jannsen establishes a quasi-isomorphism of complexes of abelian groups

$$R\Gamma(\overline{X}(\mathbb{C}), A(k+d_{\mathbb{C}})_{\mathcal{D}\text{-B},(\overline{X},X)}[2d_{\mathbb{C}}]) \simeq {}'C^\bullet_{\mathcal{D}}(\overline{X}, D, A(k)), \tag{2.1.7}$$

where the left hand side computes $H^{2d_{\mathbb{C}}+i}_{\mathcal{D}}(X, A(d_{\mathbb{C}}+k))$ (definition 2.1.1) and the right hand side computes ${}'H^i_{\mathcal{D}}(X, A(k))$ (definition 2.1.3). The duality is best understood if we use the homological numbering

$$H_i^{\mathcal{D}}(X, A(k)) := {}'H^{-i}_{\mathcal{D}}(X, A(-k))$$

(sic! the sign of the twist gets flipped as well), and also look at the isomorphism of the long exact sequences (2.1.2) and (2.1.5) (see [Jan1988, Remark 1.16 b)]). The duality takes the familiar form

$$H^i_{\mathcal{D}}(X, A(k)) \xrightarrow{\cong} H^{\mathcal{D}}_{2d_{\mathbb{C}}-i}(X, A(d_{\mathbb{C}}-k)).$$

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^i_{\mathcal{D}}(X, A(k)) & \xrightarrow{\cong} & H^{\mathcal{D}}_{2d_{\mathbb{C}}-i}(X, A(d_{\mathbb{C}}-k)) \\
\downarrow & & \downarrow \\
H^i(X(\mathbb{C}), A(k)) \oplus F^k H^i_{dR}(X(\mathbb{C})) & \xrightarrow{\cong} & H^{BM}_{2d_{\mathbb{C}}-i}(X(\mathbb{C}), A(d_{\mathbb{C}}-k)) \oplus F_{d_{\mathbb{C}}-k} H^{dR}_{2d_{\mathbb{C}}-i}(X(\mathbb{C})) \\
\downarrow{\scriptstyle \epsilon-\iota} & & \downarrow{\scriptstyle \epsilon-\iota} \\
H^i_{dR}(X(\mathbb{C})) & \xrightarrow{\cong} & H^{dR}_{2d_{\mathbb{C}}-i}(X(\mathbb{C})) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

As in 2.1.2, eventually we will be interested in a very special case where the Hodge filtration part does not enter.

[*] The word “twisted” means that the isomorphism takes into account the twist $k \in \mathbb{Z}$. However, this duality is also twisted in the sense that, unlike the usual Poincaré duality, it does not come from some nondegenerate pairing.

2.1.4. Lemma. *For $k > 0$ and $A = \mathbb{R}$ we have a quasi-isomorphism of complexes*

$$'C^\bullet_{\mathcal{D}}(\overline{\mathcal{X}}, D, A(k)) \simeq R\mathrm{Hom}(R\Gamma_c(\mathcal{X}(\mathbb{C}), (2\pi i)^{1-k}\,\mathbb{R}), \mathbb{R})[-1] \\ =: R\Gamma_{BM}(\mathcal{X}(\mathbb{C}), (2\pi i)^{1-k}\,\mathbb{R})[-1].$$

Proof. The right hand side calculates Bore–Moore homology, which is by definition dual to cohomology with compact support. In case $k > 0$ we have $'\Omega^{\geq k}_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D) = 0$, and the Deligne homology complex is defined by

$$'C^\bullet_{\mathcal{D}}(\overline{\mathcal{X}}, D, \mathbb{R}(k)) \\ := \mathrm{Cone}\left('C^\bullet(\overline{\mathcal{X}}, D, (2\pi i)^k\,\mathbb{R}) \xrightarrow{\epsilon} \Gamma(\overline{\mathcal{X}}(\mathbb{C}), '\Omega^\bullet_{\overline{\mathcal{X}}(\mathbb{C})^\infty}(\log D))\right)[-1]$$

Probably the correct way to obtain the result would be to analyze this directly and argue as in 2.1.2. There the map ϵ was essentially the comparison between singular cohomology and de Rham cohomology of $\mathcal{X}(\mathbb{C})$, and in the present situation there should be a similar comparison between Borel–Moore homology and cohomology of $'\Omega^\bullet$, which is dual to the de Rham cohomology with compact support.

As a shortcut, let us assume that $\mathcal{X}(\mathbb{C})$ is connected of dimension $2d_{\mathbb{C}}$. The quasi-isomorphism (2.1.7) together with the quasi-isomorphism from 2.1.2 and the Poincaré duality (in the correct version that takes into account the twists) give us

$$'C^\bullet_{\mathcal{D}}(\overline{\mathcal{X}}, D, \mathbb{R}(k)) \simeq R\Gamma(\mathcal{X}(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}-(1-k)}\,\mathbb{R})[2d_{\mathbb{C}} - 1] \\ \simeq R\mathrm{Hom}(R\Gamma_c(\mathcal{X}(\mathbb{C}), (2\pi i)^{1-k}\,\mathbb{R}), \mathbb{R})[-1].$$

If X is not connected, the above may be done separately for the connected components. ∎

I still note that the above argument does the trick and uses only the arguments from Jannsen's paper, but it is *morally wrong*: Jannsen derives (2.1.7) from the Poincaré duality, and in the above proof we applied the duality again.

2.2 The regulator morphism

Now as always in this text, X denotes a scheme over $\mathrm{Spec}\,\mathbb{Z}$; separated of finite type. At this point we also assume that $X_{\mathbb{C}}$ is smooth, quasi-projective. Let us also assume for the moment that X is of pure dimension d, so that

$$d_{\mathbb{C}} := \dim_{\mathbb{C}} X_{\mathbb{C}} = d - 1.$$

However, later on we will see that this assumption is superficial. We fix a good compactification

$$X_{\mathbb{C}} \overset{j}{\hookrightarrow} \overline{X_{\mathbb{C}}} \hookleftarrow D$$

The regulators for higher Chow groups $CH^n(X,p) := H^{2n-p}(X_{\acute{e}t}, \mathbb{Z}(n))$ were introduced by Bloch in [Blo1986b] as morphisms

$$H^\bullet(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^\bullet(X_{\mathbb{C}}, \mathbb{Z}(n)) \to H^\bullet_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(n)).$$

Here we are going to use the construction from [KLMS2006] which is given on the level of complexes, not merely separate cohomology groups. The reader is advised to review §0.11 for the definitions of different cycle complexes $z_r(-,-\bullet)$, $z^r(-,-\bullet)$, $z^r_\square(-,-\bullet)$, which will all be used now.

The construction from [KLMS2006, §5.9] gives us a morphism of complexes

$$z^r_{\square,\mathbb{R}}(\overline{X_{\mathbb{C}}}, -\bullet)/z^{r-1}_{\square,\mathbb{R}}(D, -\bullet) \to {}'C^{2r-2d_{\mathbb{C}}+\bullet}_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, X, \mathbb{Z}(r-d_{\mathbb{C}})).$$

Here $z^r_{\square,\mathbb{R}}(-,-\bullet)$ are certain subcomplexes of the cubical cycle complexes $z^r_\square(-,-\bullet)$; I refer to [KLMS2006, §5.4] for the precise definition. According to [KLMS2006, §5.9], there are quasi-isomorphisms

$$z^r_{\square,\mathbb{R}}(\overline{X_{\mathbb{C}}}, -\bullet)/z^{r-1}_{\square,\mathbb{R}}(D, -\bullet) \xrightarrow{\simeq} z^r_\square(\overline{X_{\mathbb{C}}}, -\bullet)/z^{r-1}_\square(D, -\bullet) \xrightarrow{\simeq} z^r_\square(X_{\mathbb{C}}, -\bullet),$$

and finally, we have an isomorphism in the derived category

$$z^r_\square(X_{\mathbb{C}}, -\bullet) \cong z^r(X_{\mathbb{C}}, -\bullet).$$

All this means that in the derived category, we may treat the morphism of Kerr, Lewis, and Müller-Stach as

$$z^r(X_{\mathbb{C}}, -\bullet) \to {}'C^{2r-2d_{\mathbb{C}}+\bullet}_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(r-d_{\mathbb{C}})). \tag{2.2.1}$$

It gives a "regulator" in the following sense. Taking the corresponding $(-i)$-th cohomology groups and using the duality (2.1.6), we obtain

$$AJ\colon CH^r(X_{\mathbb{C}}, i) \to {}'H^{2r-2d_{\mathbb{C}}-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(r-d_{\mathbb{C}})) \xleftarrow{\cong} H^{2r-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(r)).$$

According to [KLMS2006, §5.5], if $X_{\mathbb{C}}$ is projective, then the composition

$$CH^r(X_{\mathbb{C}}, i) \xrightarrow{AJ} H^{2r-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(r)) \xrightarrow{\pi_{\mathbb{R}}} H^{2r-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r))$$

coincides with the regulator defined by Goncharov in [Gon1995].

We consider (2.2.1) for $r = d - n$, where d is the dimension of X and $n < 0$ as always denotes a strictly negative integer. We obtain

$$z^{d-n}(X_{\mathbb{C}}, -\bullet) \to {}'C_{\mathcal{D}}^{2-2n+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n)),$$

which we may also write as

$$R\Gamma(X_{\mathbb{C},Zar}, \mathcal{Z}_{X_{\mathbb{C}}}^{d-n}[2n]) \cong z^{d-n}(X_{\mathbb{C}}, -\bullet)[2n] \to {}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n))$$

(the first isomorphism is 0.11.9). We consider now the composition

$$\begin{aligned} R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) = R\Gamma(X_{\acute{e}t}, \mathcal{Z}_X^{d-n}[2n]) &\to R\Gamma(X_{Zar}, \mathcal{Z}_X^{d-n}[2n]) \\ \to R\Gamma(X_{\mathbb{C},Zar}, \mathcal{Z}_{X_{\mathbb{C}}}^{d-n}[2n]) &\to {}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n)) \\ &\xrightarrow{\pi_{\mathbb{R}}} {}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(1-n)) \end{aligned}$$

As $n < 0$, the target complex may be simplified thanks to 2.1.4:

$$'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n)) \simeq R\Gamma_{BM}(\mathcal{X}(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1]$$

Taking $G_{\mathbb{R}}$-invariants (all the complexes involved in the definitions of Deligne (co)homology and all statements about them are $G_{\mathbb{R}}$-equivariant) we obtain a morphism

$$\text{(2.2.2)} \qquad Reg\colon R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1].$$

2.2.1. Remark. This suggests that in our situation $n < 0$ the regulator probably has en easier definition which could work under weaker assumptions on $X_{\mathbb{C}}$.

In what follows, we are going to use the $\mathbb{R}$-dual to (2.2.2):

$$\text{(2.2.3)} \quad Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}).$$

Compatibility of the regulator with basic operations on schemes

2.2.2. Lemma (Compatibility of the regulator with open-closed decompositions). *Suppose that we have an open-closed decomposition of arithmetic schemes $U \hookrightarrow X \leftarrow Z$ such that $U_{\mathbb{C}}$, $X_{\mathbb{C}}$, $Z_{\mathbb{C}}$ are smooth, quasi-projective varieties. Then* *the corresponding regulator morphisms yield a morphism of distinguished triangles*

$$\begin{array}{ccc}
R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) & \xrightarrow{Reg_Z} & R\Gamma_{BM}(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1] \\
\downarrow & & \downarrow \\
R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) & \xrightarrow{Reg_X} & R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1] \\
\downarrow & & \downarrow \\
R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) & \xrightarrow{Reg_U} & R\Gamma_{BM}(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1] \\
\downarrow & & \downarrow \\
R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1] & \xrightarrow{Reg_Z[1]} & R\Gamma_{BM}(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[2]
\end{array}$$

Proof. This follows from the functoriality of the construction of Kerr, Lewis, and Müller-Stach with respect to proper and flat morphisms, as discussed in [Wei2017, §3]. ■

2.2.3. Lemma (Compatibility of the regulator with affine bundles). *For an arithmetic scheme X such that $X_\mathbb{C}$ is smooth and quasi-projective, consider the affine space of dimension r over X and the corresponding set of complex points:*

$$\begin{array}{ccc}
\mathbb{A}^r_X & \longrightarrow & \mathbb{A}^r \\
{\scriptstyle p}\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & \operatorname{Spec}\mathbb{Z}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{A}^r_X(\mathbb{C}) & \longrightarrow & \mathbb{A}^r(\mathbb{C}) \\
{\scriptstyle p}\downarrow & \lrcorner & \downarrow \\
X(\mathbb{C}) & \longrightarrow & *
\end{array}$$

There is a commutative diagram

$$\begin{array}{ccc}
R\Gamma_c(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] & \xrightarrow{\cong} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})[-2r-1] \\
{\scriptstyle Reg^\vee_{\mathbb{A}^r_X,n}}\downarrow & & \downarrow{\scriptstyle Reg^\vee_{X,n-r}[-2r]} \\
R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r)), \mathbb{R})[-2r]
\end{array}$$

Proof. The diagram is the $\mathbb{R}$-dual to

$$\begin{array}{ccc}
R\Gamma_{BM}(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})[1] & \xleftarrow{\cong} & R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{Z})[2r+1] \\
{\scriptstyle Reg_{\mathbb{A}^r_X,n}}\uparrow & & \uparrow{\scriptstyle Reg_{X,n-r}[2r]} \\
R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)) & \xleftarrow{\cong} & R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r]
\end{array}$$

so it will be enough to check that the latter tensored with $\mathbb{R}$ commutes, which amounts to the commutativity of the following diagrams of $\mathbb{R}$-vector spaces:

$$\begin{array}{ccc} H^{\bullet+1}_{BM}(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) & \xleftarrow{\cong} & H^{\bullet+2r+1}_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R}) \\ \uparrow{\scriptstyle Reg_{\mathbb{A}^r_X,n}} & & \uparrow{\scriptstyle Reg_{X,n-r}[2r]} \\ H^\bullet(\mathbb{A}^r_X, \mathbb{R}^c(n)) & \xleftarrow{\cong} & H^{\bullet+2r}(X, \mathbb{R}^c(n-r)) \end{array}$$

Now on the level of separate cohomology groups, we may use Bloch's construction from [Blo1986b]. Namely, after unwinding our definitions, everything amounts to checking that Bloch's regulator is compatible with the "homotopy isomorphisms" for the cycle complex cohomology and Deligne cohomology:

$$\begin{array}{ccc} H^\bullet_{\mathcal{D}}(\mathbb{A}^1 \times X_{\mathbb{C}}, \mathbb{R}(n)) & \xleftarrow{\cong} & H^\bullet_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(n)) \\ \uparrow{\scriptstyle \text{Bloch's reg.}} & & \uparrow{\scriptstyle \text{Bloch's reg.}} \\ H^\bullet(\mathbb{A}^1 \times X_{\mathbb{C}}, \mathbb{R}(n)) & \xleftarrow{\cong} & H^\bullet(X_{\mathbb{C}}, \mathbb{R}(n)) \end{array}$$

■

The regulator conjecture

In order to relate the regulator to our machinery, we need make the following assumption.

2.2.4. Conjecture B(X, n). For an arithmetic scheme X and $n < 0$, the morphism $Reg^\vee$ (the $\mathbb{R}$-dual of the regulator) is an isomorphism in the derived category.

2.2.5. Remark. This is a standard but very strong assumption. For instance, if X is defined over a finite field, then $X(\mathbb{C}) = \emptyset$, and the conjecture implies that the cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are torsion.

2.2.6. Theorem. *Let X be an arithmetic scheme such that $X_{\mathbb{C}}$ is a smooth quasi-projective variety. Let $n < 0$ be a strictly negative integer for which the conjecture* **B**(X, n) *holds. Then there exists a morphism*

$$\smile\theta\colon R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}[1]$$

giving a long exact sequence

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+2}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

i.e. turning $H^\bullet_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$ into an acyclic complex of finite dimensional vector spaces.

Proof. Recall that according to 1.7.1, we have isomorphisms

$$R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{R}\cong RHom(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1]\oplus R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]. \tag{2.2.4}$$

Using this and the morphism $Reg^\vee$, we may define θ in the obvious way:

$$\begin{array}{c}
R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\\
\downarrow\cong\\
RHom(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1]\ \oplus\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\\
\twoheaddownarrow\\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\\
\downarrow Reg^\vee\\
RHom(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})\\
\downarrow\\
RHom(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})\ \oplus\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})\\
\downarrow\cong\\
R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]
\end{array}$$

On the level of cohomology, these morphisms give us

$$\smile\theta\colon H^i_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\to H^{i+1}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}.$$

If $Reg^\vee$ is a quasi-isomorphism, we obtain an exact sequence

$$\cdots\to H^i_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\xrightarrow{\smile\theta}H^{i+1}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\xrightarrow{\smile\theta}H^{i+2}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\to\cdots$$

Indeed, let us denote for brevity

$$A^\bullet := RHom(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1],$$
$$B^\bullet := R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1].$$

Then $Reg^\vee$ conjecturally gives isomorphisms $H^i(B^\bullet)\xrightarrow{\cong}H^{i+1}(A^\bullet)$, and the above sequence looks like

$$
\begin{array}{ccccccccc}
 & H^i(A^\bullet) & & H^{i+1}(A^\bullet) & & H^{i+2}(A^\bullet) & \\
\cdots & \oplus & \overset{\cong}{\nearrow} & \oplus & \overset{\cong}{\nearrow} & \oplus & \cdots \\
 & H^i(B^\bullet) & & H^{i+1}(B^\bullet) & & H^{i+2}(B^\bullet) &
\end{array}
$$

which is clearly exact. ■

2.3 The conjecture $\mathbf{C}(X,n)$

In the previous section we built a morphism

$$\smile \theta\colon R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\to R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]$$

that produces an acyclic complex of finitely generated $\mathbb{R}$-vector spaces

$$H^\bullet_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}.$$

This means that there is a canonical trivialization isomorphism

$$
\begin{aligned}
(2.3.1)\quad \lambda\colon \mathbb{R} &\xrightarrow{\cong} \det\nolimits_{\mathbb{R}} H^\bullet_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\cong} \det\nolimits_{\mathbb{R}} R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\\
&\xrightarrow{\cong} (\det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)))\otimes\mathbb{R}.
\end{aligned}
$$

Another way to get the same morphism is to go back to the definition of $\smile \theta$ and recall that it uses the splitting

$$
\begin{aligned}
(2.3.2)\quad & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1])\oplus R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\\
&\xrightarrow{\cong} R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}
\end{aligned}
$$

and the quasi-isomorphism

$$Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}).$$

These two give us an isomorphism
(2.3.3)

$$
\begin{array}{ccc}
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-2] & & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1])\\
\oplus & \xrightarrow[\cong]{Reg^\vee[-1]\oplus\mathrm{id}} & \oplus\\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] & & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\\
 & \searrow & \cong\big\downarrow(2.3.2)\\
 & & R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{R}
\end{array}
$$

which after taking the determinants gives us a canonical isomorphism

(2.3.4)

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} (\det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})) \otimes_{\mathbb{R}} (\det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}))^{-1} \xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Now in terms of the trivialization morphism λ, we are ready to formulate our main conjecture, which is similar to [Mor2014, Conjecture 4.2] and [FM2016, Conjecture 5.12, 5.13].

2.3.1. Conjecture $\mathbf{C}(X,n)$. For an arithmetic scheme X and $n < 0$

a) assume that the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds;

b) assume that $X_{\mathbb{C}}$ is smooth, quasi-projective, so that the regulator morphism $Reg^\vee$ exists; assume that the conjecture $\mathbf{B}(X,n)$ holds;

c) assume that the zeta-function of X

$$\zeta(X,s) := \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}}$$

has a meromorphic continuation near $s = n$.

Then

1) the leading coefficient $\zeta^*(X,n)$ of the Taylor expansion of $\zeta(X,s)$ at $s = n$ is given up to sign by

$$\lambda(\zeta^*(X,n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)),$$

where λ is the trivialization morphism defined in (2.3.1);

2) the vanishing order of $\zeta(X,n)$ at $s = n$ is given by the weighted alternating sum of ranks of $H^i_{W,c}(X, \mathbb{Z}(n))$:

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)). \tag{2.3.5}$$

2.3.2. Remark. The sum in (2.3.5) is finite, because as we saw in 1.6.8, the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ implies that the complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is perfect.

Since the conjectures $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X,n)$ imply that the groups

$$H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$$

form an acyclic complex, the usual Euler characteristic of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ vanishes:

$$\begin{aligned}\chi(R\Gamma_{W,c}(X, \mathbb{Z}(n))) &= \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} = 0.\end{aligned}$$

The sum in (2.3.5) is known as the **secondary Euler characteristic**:

$$\chi'(C^\bullet) := \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk} H^i(C^\bullet).$$

For a distinguished triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

usually

$$\chi'(B^\bullet) \neq \chi'(A^\bullet) + \chi'(C^\bullet),$$

unless the triangle is split, but the secondary Euler characteristic is still a natural invariant for acyclic complexes and it arises in various natural contexts; see [Ram2016].

2.3.3. Remark. The parts 1) and 2) of the conjecture $\mathbf{C}(X, n)$ are equivalent to Conjecture 5.12 and Conjecture 5.13 from [FM2016] if X is proper and regular. This is rather straightforward to see by going through the constructions of Flach and Morin and comparing them to our constructions. Then it is showed in [FM2016, §5.6] that their conjecture 5.12 is compatible with the Tamagawa number conjecture.

2.3.4. Proposition. *Assuming the conjectures* $\mathbf{L}^c(X_{\acute{e}t}, n)$ *and* $\mathbf{B}(X, n)$*, the weighted sum of ranks of* $H^i_{W,c}(X, \mathbb{Z}(n))$ *equals the Euler characteristic of*

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R});$$

that is,

$$\begin{aligned}\sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)) &= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \\ &=: \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})).\end{aligned}$$

Proof. Thanks to the splitting

$$\begin{aligned}&R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} \cong \\ &\qquad R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1]\end{aligned}$$

and the quasi-isomorphism

$$Reg^\vee\colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}),$$

we have

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong \\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-2],$$

so that

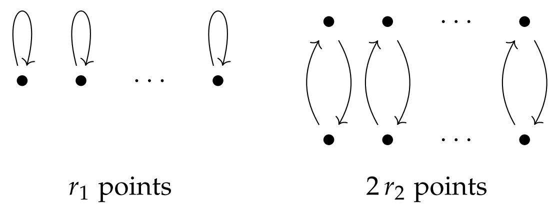
$$H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \oplus H^{i-2}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}).$$

Now

$$\begin{aligned} \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \\ = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} (H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ + \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} (H^{i-2}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ - \sum_{i\in\mathbb{Z}} (-1)^i \cdot (i+1) \cdot \dim_\mathbb{R} H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ = - \sum_{i\in\mathbb{Z}} (-1)^i \dim_\mathbb{R} H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ = \sum_{i\in\mathbb{Z}} (-1)^i \dim_\mathbb{R} H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}). \end{aligned}$$

■

2.3.5. Elementary example. Here is one easy illustration for 2.3.4. If $X = \operatorname{Spec} O_K$ is a number ring, then the space $X(\mathbb{C})$ consists of $r_1 + 2\,r_2$ points, corresponding to the real places of K and complex places coming in conjugate pairs:



Now $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$ in this case may be identified with the complex having just a single $G_\mathbb{R}$-module in degree 0, namely

$$((2\pi i)^n\,\mathbb{R})^{\oplus r_1} \oplus ((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2},$$

where $G_\mathbb{R}$ acts on $((2\pi i)^n\,\mathbb{R})^{\oplus r_1}$ by complex conjugation, while the action on $((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2}$ is given by $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$ on each summand $(2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R}$. If n is odd, then the action of $G_\mathbb{R}$ on $((2\pi i)^n\,\mathbb{R})^{\oplus r_1}$ has no fixed points, and if n is even, this action is trivial. As for the other part $((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2}$, we see that the space of $G_\mathbb{R}$-fixed points has real dimension r_2, regardless of the parity of n. Thus in this case

$$H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) = \begin{cases} r_2, & n \text{ odd},\ i = 0; \\ r_1 + r_2, & n \text{ even},\ i = 0; \\ 0 & i \neq 0. \end{cases}$$

Therefore

$$\chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}. \end{cases}$$

This agrees with the vanishing order of the Dedekind zeta function $\zeta(\operatorname{Spec} O_K, s)$ at strictly negative integers.

2.3.6. Trivial example. If X is a variety over $\mathbb{F}_q$, then

$$\zeta(X, s) = Z(X, q^{-s}),$$

where

$$Z(X, t) := \exp\left(\sum_{k \geq 1} \frac{\#X(\mathbb{F}_{q^k})}{k}\, t^k\right)$$

is Weil zeta function. Now if $\zeta(X, s)$ has a zero or pole at s, we have necessarily

$$\operatorname{Re} s = i/2, \quad 0 \leq i \leq 2 \dim X$$

—this may be seen from Weil's conjectures (see e.g. [Kat1994, p. 26–27]). In particular, there are no zeros nor poles for $s < 0$, and the identity (2.3.4) is trivially correct in this case:

$$\operatorname{ord}_{s=n} \zeta(X, s) = 0 = \chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})).$$

2.4 Stability of the conjecture under some operations on schemes

The following properties are clear from the definition of the zeta function of an arithmetic scheme:

1) If $U \hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition, then

$$\zeta(X,s) = \zeta(U,s) \cdot \zeta(Z,s). \tag{2.4.1}$$

2) For $r \geq 0$, consider the affine space $\mathbb{A}^r_X := \mathbb{A}^r_{\mathbb{Z}} \times X$. Then

$$\zeta(\mathbb{A}^r_X, s) = \zeta(X, s-r). \tag{2.4.2}$$

This suggests that our conjecture $\mathbf{C}(X,n)$ should also be compatible with open-closed decompositions and considering the affine space over X. Our goal is to verify that. We need to establish several lemmas.

2.4.1. Lemma. *The morphism λ is compatible with open-closed decompositions $U \hookrightarrow X \hookleftarrow Z$. Such a decomposition gives a commutative diagram*

$$\begin{array}{ccc}
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} & \xrightarrow[\cong]{x\otimes y \mapsto xy} & \mathbb{R} \\
{\scriptstyle \lambda_U \otimes \lambda_Z} \downarrow \cong & & \cong \downarrow {\scriptstyle \lambda_X} \\
\begin{array}{c} (\det_{\mathbb{Z}} R\Gamma_{W,c}(U,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \\ \otimes_{\mathbb{R}} \\ (\det_{\mathbb{Z}} R\Gamma_{W,c}(Z,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \end{array} & \xrightarrow{\cong} & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}
\end{array}$$

Where the bottom row is induced by the canonical isomorphism from 1.8.1.

Proof. This follows from the compatibility of the regulator with open-closed decompositions (see 2.2.2) and the ad-hoc isomorphism

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(U,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \cong \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$$

constructed in 1.8.1. ■

2.4.2. Lemma. *The morphism λ is compatible with affine bundles. We have a commutative diagram*

$$\begin{array}{ccc}
 & \mathbb{R} & \\
{\scriptstyle \lambda_{\mathbb{A}^r_X}} \swarrow & & \searrow {\scriptstyle \lambda_X} \\
(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathbb{A}^r_X,\mathbb{Z}(n)) \otimes \mathbb{R} & \xrightarrow{\cong} & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n-r)) \otimes \mathbb{R}
\end{array}$$

Proof. Follows from 2.2.3. ■

2.4.3. Lemma. *There is a quasi-isomorphism*

$$R\Gamma_{BM}(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \simeq R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\, \mathbb{R})[2r]$$

or dually,

$$R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \simeq R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\, \mathbb{R})[-2r]. \tag{2.4.3}$$

Proof. We already assumed that $X_{\mathbb{C}}$ is smooth to formulate the conjecture. Further, let us assume for simplicity that $X(\mathbb{C})$ is connected of dimension $d_{\mathbb{C}}$. Then Poincaré duality tells us that

$$R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \simeq \\ R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}+r-n}\, \mathbb{R}), \mathbb{R}[-2d_{\mathbb{C}} - 2r])$$

and

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\, \mathbb{R}) \\ \simeq R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}+r-n}\, \mathbb{R}), \mathbb{R}[-2d_{\mathbb{C}}]) \\ \simeq R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}+r-n}\, \mathbb{R}), \mathbb{R}[-2d_{\mathbb{C}}]).$$

If $X(\mathbb{C})$ is not connected, we may apply the same argument to each connected component separately. This gives us (2.4.3). ■

2.4.4. Proposition.

0) If $X = \coprod_{0 \le i \le r} X_i$ is a finite disjoint union of arithmetic schemes, then

- *0a) the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ is equivalent to the conjunction of conjectures $\mathbf{L}^c(X_{i,\acute{e}t}, n)$ for $i = 0, \ldots, r$;*
- *0b) the conjecture $\mathbf{B}(X, n)$ is equivalent to the conjunction of conjectures $\mathbf{B}(X_i, n)$ for $i = 0, \ldots, r$.*

1) If $U \hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition, then

- *1a) if two out of three conjectures $\mathbf{L}^c(U_{\acute{e}t}, n)$, $\mathbf{L}^c(Z_{\acute{e}t}, n)$, $\mathbf{L}^c(X_{\acute{e}t}, n)$ hold, then the other one holds as well;*
- *1b) if two out of three conjectures $\mathbf{B}(U, n)$, $\mathbf{B}(Z, n)$, $\mathbf{B}(X, n)$ hold, then the other one holds as well.*

2) For $r \geq 0$, consider the affine space $\mathbb{A}^r_X$:

$$\begin{array}{ccc} \mathbb{A}^r_X & \longrightarrow & \mathbb{A}^r_{\mathbb{Z}} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

2a) the conjectures $\mathbf{L}^c(\mathbb{A}^r_{X,ét}, n)$ and $\mathbf{L}^c(X_{ét}, n-r)$ are equivalent;

2b) the conjectures $\mathbf{B}(\mathbb{A}^r_X, n)$ and $\mathbf{B}(X, n-r)$ are equivalent.

Proof. Part 0) really deserved to be numbered by 0, because it is quite obvious: for finite disjoint unions $X := \coprod_{0\leq i\leq r} X_i$ we have

$$R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \cong \bigoplus_{0\leq i\leq r} R\Gamma(X_{i,ét}, \mathbb{Z}^c(n)),$$

which implies 0a). Similarly, for 0b), we note that the regulator morphism and its dual decompose as

$$Reg_X \cong \bigoplus_{0\leq i\leq r} Reg_{X_i} \quad \text{and} \quad Reg^\vee_X \cong \bigoplus_{0\leq i\leq r} Reg^\vee_{X_i}.$$

As for open-closed decompositions, recall that in this situation we have a distinguished triangle (see 0.11.1)

$$R\Gamma(Z_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(U_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{ét}, \mathbb{Z}^c(n))[1]$$

The associated long exact sequence in cohomology

$$\cdots \to H^i(Z_{ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^i(U_{ét}, \mathbb{Z}^c(n)) \\ \to H^{i+1}(Z_{ét}, \mathbb{Z}^c(n)) \to \cdots$$

implies 1a). For 1b), we apply $R\mathrm{Hom}(-, \mathbb{R})$ to the morphism of triangles from 2.2.2:

$$\begin{array}{ccc} R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] & \xrightarrow{Reg^\vee_U} & R\mathrm{Hom}(R\Gamma(U_{ét}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \downarrow & & \downarrow \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] & \xrightarrow{Reg^\vee_X} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \downarrow & & \downarrow \\ R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] & \xrightarrow{Reg^\vee_Z} & R\mathrm{Hom}(R\Gamma(Z_{ét}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \downarrow & & \downarrow \\ R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) & \xrightarrow{Reg^\vee_U[1]} & R\mathrm{Hom}(R\Gamma(U_{ét}, \mathbb{Z}^c(n)), \mathbb{R})[1] \end{array}$$

Here if two of the arrows $Reg_U^\vee$, $Reg_X^\vee$, $Reg_Z^\vee$ is a quasi-isomorphism, the third one is also a quasi-isomorphism by the triangulated 5-lemma.

In 2), we have according to [Mor2014, Lemma 5.11] a quasi-isomorphism of complexes of sheaves on $X_{ét}$

$$Rp_* \mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-r)[2r],$$

so that there is a quasi-isomorphism

$$R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n)) \xrightarrow{\simeq} R\Gamma(X_{ét}, \mathbb{Z}^c(n-r))[2r]. \tag{2.4.4}$$

This establishes 2a). As for 2b), it follows from commutativity of the diagram from 2.2.3:

$$\begin{array}{ccc} R\Gamma_c(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] & \xrightarrow{\cong} & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})[-2r-1] \\ \downarrow{\scriptstyle Reg^\vee_{\mathbb{A}^r_X,n}} & & \downarrow{\scriptstyle Reg^\vee_{X,n-r}[-2r]} \\ R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n)), \mathbb{R}) & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n-r)), \mathbb{R})[-2r] \end{array}$$

Here the left vertical arrow is a quasi-isomorphism if and only if the right vertical arrow is a quasi-isomorphism. ■

2.4.5. Theorem.

0) *If $X = \coprod_{0 \le i \le r} X_i$ is a disjoint union of arithmetic schemes, then the conjectures $\mathbf{C}(X_i, n)$ for $i = 0, \ldots, r$ together imply $\mathbf{C}(X, n)$.*

1) *If $U \hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition of an arithmetic scheme, then if two out of three conjectures $\mathbf{C}(U, n)$, $\mathbf{C}(Z, n)$, $\mathbf{C}(X, n)$ hold, the other one holds as well.*

2) *The conjecture $\mathbf{C}(\mathbb{A}^r_X, n)$ is equivalent to $\mathbf{C}(X, n-r)$.*

Proof. The conjecture $\mathbf{C}(X, n)$ has two different parts: one about the special value $\zeta^*(X, n)$ and the other one about the vanishing order of $\zeta(X, s)$ at $s = n$. For the special value part of the conjecture, the claim holds thanks to 2.4.1 and 2.4.2. The vanishing order part is actually easier, because it is just about counting ranks of cohomology groups.

In the view of (2.4.1) and (2.4.2), we have

$$\mathrm{ord}_{s=n}\, \zeta(X, s) = \mathrm{ord}_{s=n}\, \zeta(U, s) + \mathrm{ord}_{s=n}\, \zeta(Z, s)$$

and

$$\mathrm{ord}_{s=n}\, \zeta(\mathbb{A}^r_X, s) = \mathrm{ord}_{s=n-r}\, \zeta(X, s).$$

This means that 0), 1), 2) would follow respectively from the identities

(2.4.5)
$$\sum_{j\in\mathbb{Z}}(-1)^j\cdot j\cdot \mathrm{rk}_{\mathbb{Z}}\, H^j_{W,c}(X,\mathbb{Z}(n)) \stackrel{?}{=} \sum_{0\le i\le r}\sum_{j\in\mathbb{Z}}(-1)^j\cdot j\cdot \mathrm{rk}_{\mathbb{Z}}\, H^j_{W,c}(X_i,\mathbb{Z}(n)),$$

(2.4.6)
$$\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n)) \stackrel{?}{=} \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(U,\mathbb{Z}(n)) + \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(Z,\mathbb{Z}(n)),$$

(2.4.7)
$$\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(\mathbb{A}^r_X,\mathbb{Z}(n)) \stackrel{?}{=} \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n-r)).$$

As for (2.4.5), it is enough to revise the construction of Weil-étale complexes and note that

$$R\Gamma_{W,c}(\coprod_{0\le i\le r} X_i,\mathbb{Z}(n)) \cong \bigoplus_{0\le i\le r} R\Gamma_{W,c}(X_i,\mathbb{Z}(n)).$$

Alternatively, thanks to 2.3.4, we may rewrite (2.4.5) as

$$\chi(R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) \stackrel{?}{=} \sum_{0\le i\le r}\chi(R\Gamma_c(G_\mathbb{R},X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R})),$$

which is evident, as Euler characteristic is additive with respect to direct sums of complexes:

$$\begin{aligned}\chi(R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) &= \chi(R\Gamma_c(G_\mathbb{R},\coprod_{0\le i\le r} X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R}))\\ &= \chi(\bigoplus_{0\le i\le r} R\Gamma_c(G_\mathbb{R},X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R}))\\ &= \sum_{0\le i\le r}\chi(R\Gamma_c(G_\mathbb{R},X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R})).\end{aligned}$$

Similarly, (2.4.6) is equivalent to

$$\chi(R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) \stackrel{?}{=} \chi(R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) + \chi(R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{R})),$$

which is now obviously true, being the additivity of the usual Euler characteristic for the distinguished triangle

$$\begin{aligned}R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{R}) &\to R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})\\ &\to R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[1]\end{aligned}$$

Similarly, the identity (2.4.7) is equivalent to

$$\chi(R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) \stackrel{?}{=} \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})).$$

The two complexes

$$R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \quad \text{and} \quad R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})$$

are quasi-isomorphic according to (2.4.3), modulo the shift by $2r$, which is an even number, so it does not affect the Euler characteristic. ■

Similarly to the relation (2.4.2), for projective spaces $\mathbb{P}^r_X := \mathbb{P}^r_{\mathbb{Z}} \times X$ we have

$$\zeta(\mathbb{P}^r_X, s) = \prod_{0 \le i \le r} \zeta(X, s-i).$$

Note that this follows by induction from (2.4.1) and (2.4.2). For $r = 0$, this is trivial. For the induction step, assume that the above formula holds for $\mathbb{P}^{r-1}_X$. Then for $\mathbb{P}^r_X$ we may consider the open-closed decomposition

$$\mathbb{A}^r_X \hookrightarrow \mathbb{P}^r_X \hookleftarrow \mathbb{P}^{r-1}_X$$

and then

$$\begin{aligned}\zeta(\mathbb{P}^r_X, s) &= \zeta(\mathbb{A}^r_X, s) \cdot \zeta(\mathbb{P}^{r-1}_X, s) \\ &= \zeta(X, s-r) \cdot \prod_{0 \le i \le r-1} \zeta(X, s-i) = \prod_{0 \le i \le r} \zeta(X, s-i).\end{aligned}$$

Applying the same inductive reasoning, we immediately deduce from 2.4.5 the compatibility of our main conjecture with taking the projective space.

2.4.6. Corollary. *For each arithmetic scheme X, assume $\mathbf{C}(X, n-i)$ holds for $i = 0, \ldots, r$. Then $\mathbf{C}(\mathbb{P}^r_X, n)$ holds.*

Conclusion

The conjecture $\mathbf{C}(X, n)$ is known for some special cases, e.g. thanks to its equivalence to the Tamagawa number conjecture in case when X is proper and regular (see the remark 2.3.3). It is now possible to take these cases as an input, and then formally deduce $\mathbf{C}(X, n)$ for new schemes constructed using the operations of disjoint unions, open-closed gluing and affine bundles. Note that these operations allow us to obtain non-smooth schemes.