Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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Chapter 1

Weil-étale complexes

For an arithmetic scheme X (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$) and a strictly negative integer n, we are going to construct certain complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, following Flach and Morin [Mor2014, FM2016]. Here "W" stays for "Weil-étale" and "c" stays for "compact support".

The constructions are based on complexes of sheaves $\mathbb{Z}^c(n)$ on $X_{ét}$, discussed in §0.11. The basic properties of motivic cohomology for arithmetic schemes are still conjectural, and in order to make sense of all our constructions, we will need to assume in 1.1.1 that the groups $H^i(X_{ét},\mathbb{Z}^c(n))$ are finitely generated.

It is worth mentioning that the constructions in [FM2016] use other cycle complexes $\mathbb{Z}(n)$, mentioned in §0.11. If X has pure dimension d, then all this amounts to the renumbering

$$\mathbb{Z}^c(n) = \mathbb{Z}(d-n)[2d], \tag{1.0.1}$$

which should be taken into account when comparing formulas that will appear below with the formulas from [FM2016]. We use $\mathbb{Z}^c(n)$ instead of $\mathbb{Z}(n)$ precisely to avoid any references to the dimension of X (which is not assumed anymore to be equidimensional). Indeed, the dimensions of cohomology groups in many formulas in [FM2016] have terms "$2d$", and if one rewrites everything using (1.0.1), they magically disappear. This suggests that $\mathbb{Z}^c(n)$ is a more natural object than $\mathbb{Z}(n)$ in our situation.

In fact, §1.2 introduces a special definition of $\mathbb{Z}(n)$, motivated by [FM2016], which is unrelated to the cycle complexes. In our setting $n < 0$, the complex $\mathbb{Z}(n)$ will consist of a single étale sheaf, rather easy to define and understand.

Both $\mathbb{Z}^c(n)$ and $\mathbb{Z}(n)$ will appear in a certain arithmetic duality theorem in §1.3, which is stated as a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

In §1.4 I take a look at $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))$ and related complexes. Then using the duality theorem, I define in §1.5 a morphism in the derived category $\mathbf{D}(\mathbf{Ab})$

$$\alpha_{X,n} \colon R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n))$$

and declare $R\Gamma_{fg}(X, \mathbb{Z}(n))$ to be its cone:

$$R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is **almost perfect** in the sense of 0.3.3 (i.e. a perfect complex modulo possible 2-torsion in arbitrarily high degrees), canonical and functorial (despite being defined as a cone in the derived category).

Then §1.6 is dedicated to the definition of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. For this we will need a morphism

$$i_\infty^* \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}),$$

where $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$ stays for the $G_\mathbb{R}$-equivariant cohomology with compact support on $X(\mathbb{C})$. Then $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ will be given (sadly, up to a non-unique isomorphism in $\mathbf{D}(\mathbf{Ab})$) by the distinguished triangle

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\ \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

The sheaf $(2\pi i)^n\, \mathbb{Z}$ is the constant $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$, which is the image of $\mathbb{Z}(n)$ under the morphism α^* from §0.7 (see 1.6.2). The existence of i_∞^* relies on a rather nontrivial argument (theorem 1.6.4).

I show in §1.7 that there is a (non-canonical) splitting

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \cong \\ R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q})[-1].$$

Finally, §1.8 is dedicated to verifying that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is well-behaved with respect to open-closed decompositions of schemes $U \hookrightarrow X \leftarrow Z$. With the present definition, this cannot be shown for the complex itself, but we are going to establish a canonical isomorphism of the determinants

$$\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong \det_\mathbb{Z} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \otimes_\mathbb{Z} \det_\mathbb{Z} R\Gamma_{W,c}(Z, \mathbb{Z}(n)),$$

which will be enough for our purposes.

1.1 Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$

Practically all our constructions will make use of the following hypothesis for an arithmetic scheme X and a strictly negative integer $n < 0$.

1.1.1. Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$. The groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.

This is analogous to "$\mathbf{L}(X_{\acute{e}t}, n)$" (Conjecture 3.2) in [FM2016], but in our setting we need a statement for the dualizing cycle complexes $\mathbb{Z}^c(n)$. As we are going to see in 1.5.3, the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ actually implies that for any arithmetic scheme X the complex $\mathbb{Z}^c(n)$ is bounded from below and has some finite 2-torsion in higher degrees. This is related to the **Beilinson–Soulé vanishing conjecture**, which has not been proved yet.

1.2 Complexes of étale sheaves $\mathbb{Z}(n)$ for $n < 0$

For our construction, we need to make sense of "cycle complexes" $\mathbb{Z}(n)$ for $n < 0$. Here we recall a good definition of such an object, coming from [FM2016, §6.2].

First of all, if $\mathbb{Z}(n)$ is defined, then for any abelian group A and $n \geq 0$, one can define the corresponding complex with coefficients in A by

$$A(n) := \mathbb{Z}(n) \otimes^{\mathbf{L}}_{\mathbb{Z}} A.$$

The usual distinguished triangle

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$$

should give after tensoring with $\mathbb{Z}(n)$ a distinguished triangle of complexes of sheaves

$$\mathbb{Q}/\mathbb{Z}(n)[-1] \to \mathbb{Z}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}(n)$$

and we can use this to define the cycle complex $\mathbb{Z}(n)$ for $n < 0$. In this case we should have $\mathbb{Q}(n) = 0$, so the triangle above suggests that we should put

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] \quad \text{for } n < 0.$$

The complex $\mathbb{Q}/\mathbb{Z}(n)$ still does not make sense for $n < 0$, but we should have something like

$$\mathbb{Q}/\mathbb{Z}(n) = \bigoplus_p \mathbb{Z}/p^\infty\mathbb{Z}(n) = \bigoplus_p \varinjlim_r \mathbb{Z}/p^r\mathbb{Z}(n),$$

and we define for $n < 0$

$$\mathbb{Z}/p^r\mathbb{Z}(n) := j_{p!}\mu_{p^r}^{\otimes n},$$

where

1) j_p is the open immersion $X[1/p] \to X$, and $j_{p!}\colon \mathbf{Sh}(X[1/p]_{ét}) \to \mathbf{Sh}(X_{ét})$ denotes the extension by zero functor;

2) μ_{p^r} is the sheaf of roots of unity on $X[1/p]_{ét}$ represented by the commutative group scheme

$$X[1/p] \underset{\operatorname{Spec}\mathbb{Z}[1/p]}{\times} \operatorname{Spec}\mathbb{Z}[1/p][t]/(t^{p^r}-1) \to X[1/p];$$

3) $\mu_{p^r}^{\otimes n}$ is the sheaf on $X[1/p]_{ét}$ defined by

$$\mu_{p^r}^{\otimes n} := \underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r).$$

Therefore we are going to use the following definition.

1.2.1. Definition. For each $n < 0$ we consider the complex of sheaves on $X_{ét}$

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1].$$

1.3 An Artin–Verdier-like duality

At the heart of our constructions is a certain arithmetic duality theorem for cycle complexes obtained by Thomas Geisser in [Gei2010]. It generalizes the classical Artin–Verdier duality (originating from one of the Woods Hole seminars [AV1964]; one of the few thorough discussions in the literature is the second chapter of Milne's book [Mil2006]).

1.3.1. Proposition ("Artin–Verdier duality"). *For any $n < 0$ we have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \cong \varinjlim_m R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Proof. We unwind our definition of $\mathbb{Z}(n)$ for $n < 0$ and reduce everything to the results from [Gei2010]. It is worth remarking that Geisser uses notation "$R\Gamma_c$" for our "$R\widehat{\Gamma}_c$" (see §0.9).

As we have $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, it will be enough to show that for every prime p and $r = 1, 2, 3, \ldots$ there is a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]) \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]),$$

and then pass to the corresponding filtered colimits.

As in §1.2, the morphism $j_p\colon X[1/p] \hookrightarrow X$ denotes the canonical open immersion. We further denote by $f\colon X \to \operatorname{Spec}\mathbb{Z}$ the structure morphism of X and by f_p the morphism $X[1/p] \to \operatorname{Spec}\mathbb{Z}[1/p]$:

$$\begin{array}{ccc} X[1/p] & \xhookrightarrow{j_p} & X \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}\mathbb{Z}[1/p] & \hookrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

As we are going to change the base scheme, let us write "$\operatorname{Hom}_X(-,-)$" for the Hom between sheaves on $X_{ét}$ (and "$\underline{\operatorname{Hom}}_X(-,-)$" for the internal Hom). Instead of "$\operatorname{Hom}_{\operatorname{Spec} R}$", we will simply write "Hom_R".

By [Gei2010, Proposition 7.10, (c)], we have the following "exchange formulas". If we work with complexes of constructible sheaves on the étale site of schemes over the spectrum of a number ring $\operatorname{Spec} O_K$, then for a morphism ϕ of such schemes we have

$$R\phi_* \mathcal{D}(\mathcal{F}) \cong \mathcal{D}(R\phi_! \mathcal{F}), \tag{1.3.1}$$

$$R\phi^! \mathcal{D}(\mathcal{G}) \cong \mathcal{D}(\phi^* \mathcal{G}), \tag{1.3.2}$$

where the dualization is given by

$$\mathcal{D}(\mathcal{F}^\bullet) := R\underline{\operatorname{Hom}}_X(\mathcal{F}^\bullet, \mathbb{Z}^c(0)).$$

Applying the exchange formula (1.3.1) to our situation, we get

$$R\underline{\operatorname{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong Rj_{p*} R\underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)). \tag{1.3.3}$$

Using the other exchange formula (1.3.2), we may identify the sheaf $R\underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0))$:

$$R\underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)) \cong R\underline{\operatorname{Hom}}_{X[1/p]}(f_p^*\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)) \tag{1.3.4}$$

$$\cong Rf_p^! R\underline{\operatorname{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{\mathbb{Z}[1/p]}(0)) \tag{1.3.5}$$

$$\cong Rf_p^! R\underline{\operatorname{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{G}_m[1]) \tag{1.3.6}$$

$$\cong Rf_p^! R\underline{\operatorname{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}, \mathbb{G}_m)[2] \tag{1.3.7}$$

$$\cong Rf_p^! \mu_{p^r}^{\otimes(1-n)}[2] \tag{1.3.8}$$

Here (1.3.4) simply means that the sheaf $\mu_{p^r}^{\otimes n}$ on $X[1/p]$ is the same as the inverse image of the corresponding sheaf on $\operatorname{Spec}\mathbb{Z}[1/p]$. The quasi-isomorphism (1.3.5) is the first exchange formula. Then, (1.3.6) is the fact that the complex $\mathbb{Z}^c_{\mathbb{Z}[1/p]}(0)$ is quasi-isomorphic to $\mathbb{G}_{\mathrm{m}}[1]$ according to [Gei2010, Lemma 7.4]. Thanks to [Gei2004, Theorem 1.2], we may identify the sheaf $\mu_{p^r}^{\otimes(1-n)}$:

$$\mu_{p^r}^{\otimes(1-n)} \cong \mathbb{Z}_{\mathbb{Z}[1/p]}/p^r(1-n) = \mathbb{Z}^c_{\mathbb{Z}[1/p]}/p^r(n)[-2]. \tag{1.3.9}$$

Then [Gei2010, Corollary 7.9] tells us that

$$Rf_p^!\mathbb{Z}^c_{\mathbb{Z}[1/p]}/p^r(n) \cong \mathbb{Z}^c_{X[1/p]}/p^r(n). \tag{1.3.10}$$

Finally, thanks to [Gei2010, Theorem 7.2 (a)] and [Gei2010, Proposition 2.3], we have $\mathbb{Z}^c_{X[1/p]}/p^r(n) \cong j_p^*\mathbb{Z}^c_X/p^r(n)$, and all the above gives

$$R\underline{\mathrm{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong Rj_{p*}j_p^*\mathbb{Z}^c_X/p^r(n) \cong \mathbb{Z}^c_X/p^r(n). \tag{1.3.11}$$

After applying $R\Gamma(X_{ét}, -)$, we get a quasi-isomorphism of complexes of abelian groups

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong R\Gamma(X_{ét}, \mathbb{Z}^c_X/p^r(n)). \tag{1.3.12}$$

Now according to the generalization of Artin–Verdier duality by Geisser [Gei2010, Theorem 7.8], we have

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c(0)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]). \tag{1.3.13}$$

So what we obtain at the end is a quasi-isomorphism

$$R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

This is almost what we need: if we apply $R\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}[-2])$, then, as $\widehat{H}^i_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1])$ are finite groups (because the sheaves $j_{p!}\mu_{p^r}^{\otimes n}$ are constructible), we have

$$\begin{aligned} &R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong \\ &\quad R\mathrm{Hom}(R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]), \mathbb{Q}/\mathbb{Z}[-2]) \\ &\qquad \cong R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]). \end{aligned}$$

■

The quasi-isomorphism

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \cong \varinjlim_m R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2])$$

that we just saw means that on the level of cohomology, we get

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \varinjlim_m \operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z})$$

(note that the group $\mathbb{Q}/\mathbb{Z}$ is divisible, so $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor, and the filtered colimit $\varinjlim_m$ is exact as well).

1.3.2. Proposition. *Assuming the conjecture* $\mathbf{L}^c(X_{\acute{e}t}, n)$ *(see 1.1.1), there is a quasi-isomorphism of complexes*

$$\varinjlim_m R\operatorname{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong R\operatorname{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Proof. As $\mathbb{Z}^c(n)$ is a complex of flat sheaves, the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

induces a short exact sequence of sheaves

$$0 \to \mathbb{Z}^c(n) \xrightarrow{\times m} \mathbb{Z}^c(n) \to \mathbb{Z}/m\mathbb{Z}^c(n) \to 0 \qquad (1.3.14)$$

The morphism $\mathbb{Z}^c(n) \to \mathbb{Z}/m\mathbb{Z}^c(n)$ induces some morphisms in cohomology

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^i(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)).$$

We claim that if we pass to the duals $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and then to the filtered colimits $\varinjlim_m$, then we obtain an isomorphism. (Note that both $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and $\varinjlim_m$ are exact.)

The short exact sequence (1.3.14) induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \longrightarrow H^i(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)) \xrightarrow{\delta^i} H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^{i+1}(X_{\acute{e}t}, \mathbb{Z}/m\mathbb{Z}^c(n)) \to \cdots$$

We further have exact sequences

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \longrightarrow \underbrace{H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))_m}_{=\ker \delta^i} \longrightarrow 0$$

$$0 \to \underbrace{{}_mH^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n))}_{=\operatorname{im} \delta^i} \to H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\times m} H^{i+1}(X_{\acute{e}t}, \mathbb{Z}^c(n))$$

that give us

$$0 \to H^i(X_{ét}, \mathbb{Z}^c(n))_m \to H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \to {}_m H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \to 0$$

Now if we take $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and filtered colimits $\varinjlim_m$, we get

$$(1.3.15) \quad 0 \to \varinjlim_m \operatorname{Hom}({}_m H^{i+1}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to$$
$$\varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \to 0$$

By the conjecture $\mathbf{L}^c(X_{ét}, n)$, the group $H^{i+1}(X_{ét}, \mathbb{Z}^c(n))$ is finitely generated, and therefore

$${}_m H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) = 0 \quad \text{for } m \gg 0,$$

which means that the first $\varinjlim_m$ in the short exact sequence (1.3.15) vanishes, and we obtain isomorphisms

$$\varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

It remains to note that the first $\varinjlim_m$ above is canonically isomorphic to

$$\operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}),$$

as we observed in 0.1.2 (again, thanks to finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$). ■

Let us summarize the results of this section.

1.3.3. Theorem. *Assuming the conjecture $\mathbf{L}^c(X_{ét}, n)$, there is a quasi-isomorphism*

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\cong} R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

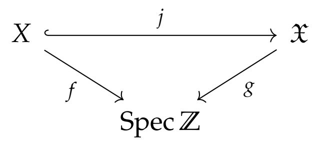
In particular, the conjecture $\mathbf{L}^c(X_{ét}, n)$ implies that the cohomology of $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))$ is of cofinite type.

1.4 Complexes $R\widehat{\Gamma}(G_{\mathbb{R}}, (Rf_! \mathbb{Z}(n))_{\mathbb{C}})$

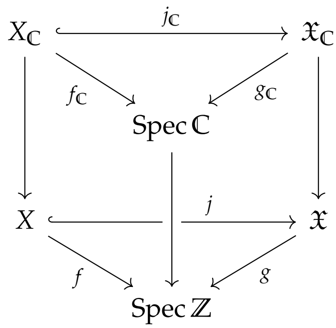
The duality theorem 1.3.3 deals with the complex $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))$, so let us make a little digression to understand it. By the definition from §0.9, it sits in the distinguished triangle

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\widehat{\Gamma}(G_{\mathbb{R}}, (Rf_! \mathbb{Z}(n))_{\mathbb{C}}) \to R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))[1]$$

To define cohomology with compact support, we pick a Nagata compactification



where j is an open immersion and g is proper. Then by definition, $Rf_!\mathbb{Z}(n) := Rg_*j_!\mathbb{Z}(n)$. As we are interested in the stalk of $Rf_!\mathbb{Z}(n)$ at $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Z}$, let us consider the base change to $\mathbb{C}$. The schemes $f\colon X \to \operatorname{Spec}\mathbb{Z}$ and $g\colon \mathfrak{X} \to \operatorname{Spec}\mathbb{Z}$ give us $f_{\mathbb{C}}\colon X_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$ and $g_{\mathbb{C}}\colon \mathfrak{X}_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$, and the open immersion $j\colon X \hookrightarrow \mathfrak{X}$ induces an open immersion $j_{\mathbb{C}}\colon X_{\mathbb{C}} \hookrightarrow \mathfrak{X}_{\mathbb{C}}$. We have the following commutative prism:



Note that the back face is also a pullback. The proper base change theorem [SGA 4, Exposé XII, Theéorème 5.1] applied to the right face of the prism (recall that the morphism g is proper) and the abelian torsion sheaf $j_!\mathbb{Z}(n)$ on $\mathfrak{X}_{ét}$, gives us an isomorphism

$$Rg_{\mathbb{C},*}(j_!\mathbb{Z}(n))_{\mathbb{C}} \cong (Rg_*j_!\mathbb{Z}(n))_{\mathbb{C}}. \tag{1.4.1}$$

Here $(j_!\mathbb{Z}(n))_{\mathbb{C}}$ denotes the inverse image of $j_!\mathbb{Z}(n)$ with respect to $\mathfrak{X}_{\mathbb{C}} \to \mathfrak{X}$, and $(Rg_*j_!\mathbb{Z}(n))_{\mathbb{C}}$ denotes the inverse image of $Rg_*j_!\mathbb{Z}(n)$ with respect to $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Z}$. Extension by zero commutes with base change, so we have

$$(j_!\mathbb{Z}(n))_{\mathbb{C}} \cong j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}),$$

and we may rewrite (1.4.1) as

$$\underbrace{Rg_{\mathbb{C},*}j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}})}_{=:Rf_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}})} \cong \underbrace{(Rg_*j_!\mathbb{Z}(n))}_{=:Rf_!\mathbb{Z}(n)}{}_{\mathbb{C}}. \tag{1.4.2}$$

Now we would like to apply Artin's comparison theorem [SGA 4, Exposé XVI, Théorème 4.1]. We have the following commutative square of sites:

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{C},ét} & \xleftarrow{\epsilon_{\mathfrak{X}}} & \mathfrak{X}_{\mathbb{C},cl} \\ {\scriptstyle g_{\mathbb{C},ét}}\downarrow & & \downarrow{\scriptstyle g_{\mathbb{C},cl}} \\ (\operatorname{Spec}\mathbb{C})_{ét} & \xleftarrow[\epsilon_{\mathbb{C}}]{} & (\operatorname{Spec}\mathbb{C})_{cl} \end{array}$$

and for the sheaf $j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}})$, Artin's theorem gives

$$Rg_{\mathbb{C},cl,*}\epsilon_{\mathfrak{X}}^* j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}) \cong \epsilon_{\mathbb{C}}^* Rg_{\mathbb{C},ét,*} j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}).$$

Note that we have

$$\epsilon_{\mathfrak{X}}^* j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}) \cong j_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}}),$$

where ϵ_X denotes the corresponding morphism of sites $X_{\mathbb{C},cl} \to X_{\mathbb{C},ét}$. Now

$$\begin{aligned} Rf_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}}) &:= Rg_{\mathbb{C},cl,*} j_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}}) \\ &\cong \epsilon_{\mathbb{C}}^* Rg_{\mathbb{C},ét,*} j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}) \\ &\overset{(1.4.2)}{\cong} \epsilon_{\mathbb{C}}^* (Rg_* j_! \mathbb{Z}(n))_{\mathbb{C}} \\ &=: \epsilon_{\mathbb{C}}^* (Rf_! \mathbb{Z}(n))_{\mathbb{C}}. \end{aligned}$$

Note that $\epsilon_{\mathbb{C}}^*$ is just an equivalence of categories, and both $Rf_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}})$ and $\epsilon_{\mathbb{C}}^*(Rf_!\mathbb{Z}(n))_{\mathbb{C}}$ may be viewed as complexes of abelian groups or, more precisely, of $G_{\mathbb{R}}$-modules.

Let us calculate the sheaf $\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}})$ on $X_{\mathbb{C},cl}$. Recall that by definition,

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1],$$

where

$$\mu_{p^r}^{\otimes n} := \underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

Base change to $X_{\mathbb{C}}$ and the inverse image ϵ_X^* commute with colimits. The sheaves $\mu_{p^r}^{\otimes n}$ become constant sheaves $\mu_{p^r}^{\otimes n}(\mathbb{C})$ on $X(\mathbb{C})$, and their colimit is given by 0.5.5.

1.4.1. Proposition. *There is an isomorphism of constant $G_{\mathbb{R}}$-equivariant sheaves on $X_{\mathbb{C},cl}$*

$$\epsilon_X^*(\mathbb{Z}(n))_{\mathbb{C}} \cong (2\pi i)^n\, \mathbb{Q}/\mathbb{Z}[-1].$$

This implies that the complex $Rf_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}})$ may be identified with $R\Gamma_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])$, and in particular, we have a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1]) \cong R\widehat{\Gamma}(G_{\mathbb{R}},(Rf_!\mathbb{Z}(n))_{\mathbb{C}}), \tag{1.4.3}$$

where

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1]) := R\widehat{\Gamma}(G_{\mathbb{R}},R\Gamma_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])).$$

1.4.2. Proposition. *We have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1]) \cong R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}).$$

Proof. Consider the short exact sequence of $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$

$$0 \to (2\pi i)^n\,\mathbb{Z} \to (2\pi i)^n\,\mathbb{Q} \to (2\pi i)^n\,\mathbb{Q}/\mathbb{Z} \to 0$$

which gives us a distinguished triangle

$$\begin{aligned} &R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \\ &\qquad \to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}) \to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})[1] \end{aligned}$$

and the corresponding long exact sequence in cohomology

$$\begin{aligned} \cdots \to \widehat{H}_c^{i-1}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \to \widehat{H}_c^{i-1}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}) \to \\ \widehat{H}_c^{i}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \to \widehat{H}_c^{i}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \to \cdots \end{aligned}$$

Now in the spectral sequence

$$E_2^{pq} = \widehat{H}^p(G_{\mathbb{R}},H_c^q(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})) \Longrightarrow \widehat{H}_c^{p+q}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}),$$

the groups $\widehat{H}^p(G_{\mathbb{R}},H_c^q(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}))$ are $\mathbb{Q}$-vector spaces, and they are 2-torsion for all $p \in \mathbb{Z}$ (keep in mind that we are working with Tate cohomology). This means that $E_2^{pq} = 0$ for all $p,q \in \mathbb{Z}$, and

$$\widehat{H}_c^i(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) = 0.$$

We conclude that the morphism

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1]) \to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

induces isomorphisms on cohomology. ■

Combining the last proposition with (1.4.3), we obtain the following result.

1.4.3. Theorem. *There is a quasi-isomorphism of complexes*

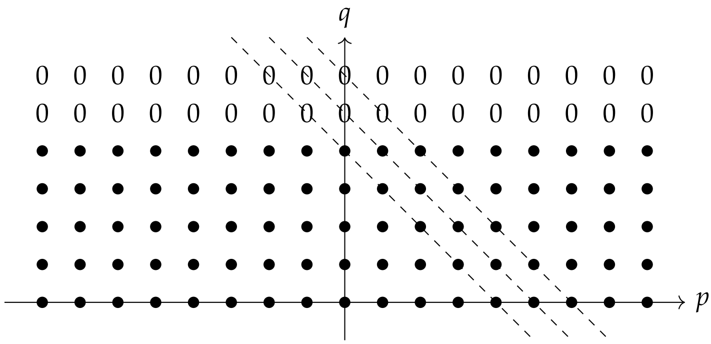
$$R\widehat{\Gamma}(G_{\mathbb{R}}, (Rf_!\mathbb{Z}(n))_{\mathbb{C}}) \cong R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}).$$

The cohomology of these complexes is given by finite 2*-torsion groups.*

Proof. Tate (hyper)cohomology groups of $G_{\mathbb{R}}$ are always killed by $\#G_{\mathbb{R}} = 2$ (see 0.9.1). To see that in our case these 2-torsion groups are finite, we may consider the spectral sequence

$$E_2^{pq} = \widehat{H}^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})) \Longrightarrow \widehat{H}_c^{p+q}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}).$$

According to 0.10.1, the groups $H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ are finitely generated for all q, and they vanish for $q \gg 0$ and $q < 0$. This means that the second page of the spectral sequence looks like

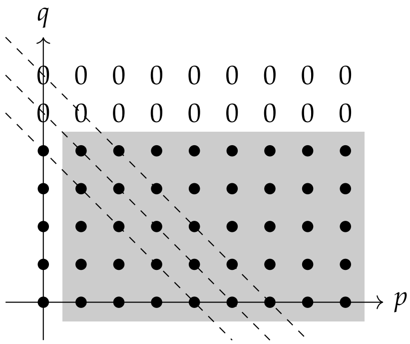


where all objects are *finite* 2-torsion. ■

For the sake of completeness and for further reference, let us look at spectral sequences similar to the one in the last proof, but with the usual group cohomology instead of Tate cohomology. If we replace $\widehat{H}$ with H, then $H^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}))$ is not necessarily 2-torsion for $p = 0$, and the second page of the spectral sequence

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})) \Longrightarrow H_c^{p+q}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$$

looks like



where the shaded part E_2^{pq}, $p > 0$ consists of finitely generated 2-torsion groups, the line E_2^{0q} consists of finitely generated groups, and the objects E_2^{pq} are zero for $q \gg 0$. It follows that the groups $H^i(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ are all finitely generated as well, and they are torsion for $i \gg 0$. This is in fact 2-torsion, and we may see this as follows. If $P_\bullet \twoheadrightarrow \mathbb{Z}$ is the bar-resolution of $\mathbb{Z}$ by free $\mathbb{Z}G_{\mathbb{R}}$-modules, then the morphism of complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2-N} & & \\ \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \end{array}$$

$$\begin{aligned} \text{“2”}\colon P_\bullet &\to P_\bullet, \\ (2-N)\colon P_0 &\to P_0, \\ 2\colon P_i &\to P_i \quad \text{for } i > 1, \end{aligned}$$

which induces multiplication by 2 on $H^i(G, -)$ for $i > 0$ is null-homotopic [Wei1994, Theorem 6.5.8]. It is not multiplication by 2 in degree 0, but as the complex $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ is bounded, we see that it induces multiplication by 2 on $H^i(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ for $i \gg 0$. So we just proved the following.

1.4.4. Lemma. *The complex*

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) = R\Gamma(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}))$$

is almost perfect in the sense of 0.3.3.

As for $\mathbb{Q}/\mathbb{Z}$-coefficients, we may analyze a similar spectral sequence

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})) \Longrightarrow H^{p+q}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z}).$$

The second page will have groups of cofinite type on the line E_2^{0q} (see 0.10.1) and finite 2-torsion groups E_2^{pq} for $p > 0$. We have filtrations

$$H^{p+q} = F^0(H^{p+q}) \supseteq F^1(H^{p+q}) \supseteq F^2(H^{p+q}) \supseteq \cdots \supseteq F^{p+q}(H^{p+q}) \supset F^{p+q+1}(H^{p+q}) = 0 \tag{1.4.4}$$

where

$$0 \to F^{p+1}(H^{p+q}) \to F^p(H^{p+q}) \to E_\infty^{pq} \to 0$$

Note that E_∞^{0q} will be groups of cofinite type, and E_∞^{pq} will be finite 2-torsion groups for $p > 0$, as we are going to have

$$0 \to E_{r+1}^{0q} \to E_r^{0q} \to T \to 0$$

where T is finite 2-torsion, and similarly,

$$E_{r+1}^{pq} \cong \ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1}$$

$$E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} E_r^{p+r,q-r+1}$$

where E_r^{pq} is finite 2-torsion for $p > 0$. It follows by induction that all the members of the filtration (1.4.4) are finite groups, except for $F^0(H^{p+q}) = H^{p+q}$ itself, which is of cofinite type, being an extension of a group of cofinite type E_∞^{0q} by a finite group $F^1(H^{p+q})$ (see 0.1.3). We also see that H^{p+q} is 2-torsion for $p + q \gg 0$. This gives us the following result.

1.4.5. Lemma. *The complex*

$$R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z}) = R\Gamma(G_\mathbb{R}, R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z}))$$

is almost of cofinite type in the sense of 0.3.7.

1.5 Complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$

1.5.1. Definition. The morphism $\alpha_{X,n}$ in $\mathbf{D}(\mathbf{Ab})$ is given by the composition of morphisms

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \xleftarrow{\cong} R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

Here the first arrow is induced by $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), -)$ and the canonical projection $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$. The second arrow is a quasi-isomorphism given by theorem 1.3.3. The third arrow is the morphism (0.9.8) from cohomology with compact support à la Milne to the usual cohomology with compact support.

Then the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined as a cone of $\alpha_{X,n}$ in $\mathbf{D}(\mathbf{Ab})$:

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

1.5.2. Remark. If $X(\mathbb{R}) = \emptyset$, then $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the same as $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ (see 0.9.2), so that in this case we have an isomorphism of distinguished triangles

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) & \xrightarrow{\simeq} & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) & \dashrightarrow^{\simeq} & R\Gamma_{fg}(X, \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \xrightarrow{\mathrm{id}} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])
\end{array}$$

where the left column is the result of application of $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), -)$ to an appropriate rotation of the triangle

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$$

We conclude that

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \simeq R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]).$$

However, this holds only if $X(\mathbb{R}) = \emptyset$. In what follows, we are not going to make such an assumption on X, even though it would save quite some technical work. It is still helpful to keep in mind the special case $X(\mathbb{R}) = \emptyset$.

The complex of sheaves $\mathbb{Z}^c(n)$ is bounded from below, under the assumption that their cohomology groups are finitely generated (which is our conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, stated in 1.1.1).

1.5.3. Lemma. *Assuming the conjecture* $\mathbf{L}^c(X_{\acute{e}t}, n)$*, we have*

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0 \quad \textit{for } i < -2 \dim X.$$

Proof. The complex of sheaves $\mathbb{Z}^c(n)$ is flat, so the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

gives us a short exact sequence of étale sheaves

$$0 \to \mathbb{Z}^c(n) \to \mathbb{Q}^c(n) \to \mathbb{Q}/\mathbb{Z}^c(n) \to 0$$

and then applying $R\Gamma(X_{ét},-)$, we obtain a distinguished triangle in $\mathbf{D}(\mathbf{Ab})$

$$R\Gamma(X_{ét},\mathbb{Z}^c(n)) \to R\Gamma(X_{ét},\mathbb{Q}^c(n)) \to R\Gamma(X_{ét},\mathbb{Q}/\mathbb{Z}^c(n)) \to R\Gamma(X_{ét},\mathbb{Z}^c(n))[1]$$

Now according to [Mor2014, Lemma 5.12] (note that the proof there also uses Geisser's duality), we have

$$H^i(X_{ét},\mathbb{Q}/\mathbb{Z}^c(n)) = 0 \quad \text{for } i < -2\dim X,$$

and the above triangle implies that

$$H^i(X_{ét},\mathbb{Q}^c(n)) \cong H^i(X_{ét},\mathbb{Z}^c(n)) \quad \text{for } i < -2\dim X.$$

However, $H^i(X_{ét},\mathbb{Q}^c(n))$ is a $\mathbb{Q}$-vector space, and according to the conjecture $\mathbf{L}^c(X_{ét},n)$, the groups $H^i(X_{ét},\mathbb{Z}^c(n))$ are finitely generated over $\mathbb{Z}$. This means that for $i < -2\dim X$ these groups are trivial. ■

1.5.4. Proposition. *The complex $R\Gamma_{fg}(X,\mathbb{Z}(n))$ is almost perfect in the sense of 0.3.3, i.e. its cohomology groups $H^i_{fg}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X,\mathbb{Z}(n)))$ are finitely generated, trivial for $i \ll 0$, and only have 2-torsion for $i \gg 0$.*

Proof. By the definition of $R\Gamma_{fg}(X,\mathbb{Z}(n))$, we have a long exact sequence in cohomology

$$\cdots \to \operatorname{Hom}(H^{2-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) \xrightarrow{H^i(\alpha_{X,n})} H^i_c(X_{ét},\mathbb{Z}(n)) \longrightarrow H^i_{fg}(X,\mathbb{Z}(n)) \xrightarrow{\delta^i}$$
$$\operatorname{Hom}(H^{1-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}) \xrightarrow{H^{i+1}(\alpha_{X,n})} H^{i+1}_c(X_{ét},\mathbb{Z}(n)) \longrightarrow \cdots$$

We consider short exact sequences

$$0 \longrightarrow \ker\delta^i \longrightarrow H^i_{fg}(X,\mathbb{Z}(n)) \longrightarrow \operatorname{im}\delta^i \longrightarrow 0$$

where $\ker\delta^i = \operatorname{coker} H^i(\alpha_{X,n})$ and $\operatorname{im}\delta^i = \ker H^{i+1}(\alpha_{X,n})$.

By the definition of $\alpha_{X,n}$, the morphism $H^i(\alpha_{X,n})$ factors as

$$\operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \to \operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

Here the morphism $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is identity, except for some *finite* 2-torsion. Indeed, this morphism sits in the long exact sequence (0.9.9):

$$\cdots \to \widehat{H}^{i-1}(G_\mathbb{R}, (Rf_!\mathbb{Z}(n))_\mathbb{C}) \to \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to \widehat{H}^i(G_\mathbb{R}, (Rf_!\mathbb{Z}(n))_\mathbb{C}) \to \cdots$$

and $\widehat{H}^i(G_\mathbb{R}, (Rf_!\mathbb{Z}(n))_\mathbb{C})$ is finite 2-torsion according to 1.4.3.

The group $H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finitely generated according to the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ (see 1.1.1). If this group is of the form $\mathbb{Z}^{\oplus r} \oplus T$, the morphism $H^i(\alpha_{X,n})$ is given by

$$\mathbb{Q}^{\oplus r} \twoheadrightarrow (\mathbb{Q}/\mathbb{Z})^{\oplus r} \rightarrowtail \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

where $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \rightarrowtail \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the inclusion of the maximal divisible subgroup in the group of cofinite type

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

Both kernel and cokernel of the above map are finitely generated, hence $H^i_{fg}(X, \mathbb{Z}(n))$ is finitely generated.

As we observed in 1.5.3, again assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, we may deduce that the complex $\mathbb{Z}^c(n)$ is bounded from below. This means that for $i \ll 0$ we have

$$\ker H^{i+1}(\alpha_{X,n}) = 0, \quad H^i_{fg}(X, \mathbb{Z}(n)) \cong \operatorname{coker} H^i(\alpha_{X,n}) = H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)).$$

For $i < 1$ we have $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, and for $i \gg 0$ we know that

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) = 0,$$

again by boundedness of $\mathbb{Z}^c(n)$ from below. The only difference between $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ and $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is some finite 2-torsion. ∎

1.5.5. Observation. *$R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in $\mathbf{D}(\mathbf{Ab})$.*

Proof. The complex $R\operatorname{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ consists of $\mathbb{Q}$-vector spaces, and $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect, so we are in the situation of 0.3.6. ∎

1.5.6. Observation. *Fix a distinguished triangle defining $R\Gamma_{fg}(X,\mathbb{Z}(n))$:*

$$RHom(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét},\mathbb{Z}(n)) \xrightarrow{f} R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{g} RHom(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1])$$

1) *For each $m = 1,2,3,\ldots$ the morphism*

$$f\otimes\mathbb{Z}/m\mathbb{Z}\colon R\Gamma_c(X_{ét},\mathbb{Z}(n))\otimes^{\mathbf{L}}_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} R\Gamma_{fg}(X,\mathbb{Z}(n))\otimes^{\mathbf{L}}_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z}$$

is iso. Further, we have

$$R\Gamma_c(X_{ét},\mathbb{Z}(n))\otimes^{\mathbf{L}}_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z} \cong R\Gamma_c(X_{ét},\mathbb{Z}/m\mathbb{Z}(n)) := R\Gamma_c(X_{ét},\mathbb{Z}(n)\otimes^{\mathbf{L}}\mathbb{Z}/m\mathbb{Z}).$$

2) *The morphism*

$$g\otimes\mathbb{Q}\colon R\Gamma_{fg}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\cong} RHom(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1])$$

is iso.

Proof. The statement 1) follows from the fact that the complexes

$$RHom(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[\ldots])$$

consist of $\mathbb{Q}$-vector spaces, and thus

$$RHom(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[\ldots])\otimes^{\mathbf{L}}_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z} \simeq RHom(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[\ldots])\otimes_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z}\simeq 0.$$

Next, 2) follows from the fact that the cohomology of the étale sheaf $\mathbb{Z}(n)$ is torsion, and therefore

$$H^i(R\Gamma_c(X_{ét},\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q}) \cong H^i_c(X_{ét},\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q} = 0,$$
$$R\Gamma_c(X_{ét},\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q}\simeq 0.$$

■

1.6 Complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$

To define complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, we first construct a morphism

$$i_\infty^*\colon R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}).$$

By definition, it sits in the morphism of distinguished triangles

$$
\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_{X,n}} & & \downarrow \\
R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \xrightarrow{u_\infty^*} & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
R\Gamma_{fg}(X,\mathbb{Z}(n)) & \overset{i_\infty^*}{\dashrightarrow} & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-1]) & \longrightarrow & 0
\end{array}
\tag{1.6.1}
$$

Here

$$u_\infty^*\colon R\Gamma(X_{\acute{e}t},\mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

is some morphism, to be defined below, such that the composition $u_\infty^* \circ \alpha_{X,n}$ is zero. Then by the axiom (TR3) there exists some morphism i_∞^*. The fact that $u_\infty^* \circ \alpha_{X,n} = 0$ will be a delicate issue, which is the main goal of this section. However, once we know that, i_∞^* is automatically unique.

1.6.1. Observation. *If i_∞^* exists, then it is unique.*

Proof of 1.6.1. We may apply 0.3.6, because $R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2])$ is a complex of $\mathbb{Q}$-vector spaces and both

$$R\Gamma_{fg}(X,\mathbb{Z}(n)) \quad \text{and} \quad R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

are almost perfect complexes by 1.5.4 and 1.4.4. ∎

1.6.2. Proposition. *Consider the morphism*

$$\alpha^*\colon \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(G_{\mathbb{R}},X(\mathbb{C})),$$

as described in §0.7. For the sheaf

$$\mathbb{Q}/\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}$$

defined in §1.2 we have an isomorphism of $G_{\mathbb{R}}$-equivariant constant sheaves on $X(\mathbb{C})$

$$\alpha^*\mathbb{Q}/\mathbb{Z}(n) \cong \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}} =: (2\pi i)^n\,\mathbb{Q}/\mathbb{Z}.$$

Proof. First of all, since α^* is the composition of certain inverse image functors γ^* and ϵ^* (which are left adjoint) and an equivalence of categories δ_*, the functor α^* preserves colimits, and in particular

$$\alpha^* \mathbb{Q}/\mathbb{Z}(n) \cong \bigoplus_p \varinjlim_r \alpha^* j_{p!} \mu_{p^r}^{\otimes n}. \tag{1.6.2}$$

Another formal observation is that the base change from $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec}\mathbb{C}$ factors through the base change to $\operatorname{Spec}\mathbb{Z}[1/p]$, and then $j_p^* \circ j_{p!} = \mathrm{id}_{\mathbf{Sh}(X[1/p]_{ét})}$:

$$\begin{array}{ccccc}
\mathbf{Sh}(X[1/p]_{ét}) & \xrightarrow{j_{p!}} & \mathbf{Sh}(X_{ét}) & \xrightarrow{\gamma^*} & \mathbf{Sh}(X_{\mathbb{C},ét}) \\
 & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle j_p^*} & \nearrow & \\
 & & \mathbf{Sh}(X[1/p]_{ét}) & &
\end{array}$$

which means that we may safely erase "$j_{p!}$" in (1.6.2), and everything boils down to calculating the sheaves

$$\alpha^* \mu_{p^r}^{\otimes n} = \alpha^* \underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

As we base change to $\operatorname{Spec}\mathbb{C}$, the étale sheaf μ_{p^r} simply becomes the constant sheaf $\mu_{p^r}(\mathbb{C})$ on $X(\mathbb{C})$, and

$$\alpha^* \mu_{p^r}^{\otimes n} = \underline{\mathrm{Hom}}_{X(\mathbb{C})}(\mu_{p^r}^{\otimes(-n)}(\mathbb{C}), \mathbb{Z}/p^r\mathbb{Z}).$$

In 0.5.5 we calculated the colimit of such things to be $(2\pi i)^n\, \mathbb{Q}/\mathbb{Z}$. ■

1.6.3. Definition. The morphism

$$u_\infty^* \colon R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$$

is given by the composition

$$\begin{aligned}
& R\Gamma_c(X_{ét}, \mathbb{Z}(n)) := R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))[-1] \\
& \xrightarrow{v_\infty^*[-1]} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z})[-1] \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})
\end{aligned}$$

Here the last arrow is induced by $(2\pi i)^n\, \mathbb{Q}/\mathbb{Z}[-1] \to (2\pi i)^n\, \mathbb{Z}$, which comes from the distinguished triangle of constant $G_\mathbb{R}$-equivariant sheaves

$$(2\pi i)^n\, \mathbb{Z} \to (2\pi i)^n\, \mathbb{Q} \to (2\pi i)^n\, \mathbb{Q}/\mathbb{Z} \to (2\pi i)^n\, \mathbb{Z}[1]$$

and the arrow

$$v_\infty^* \colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z})$$

is induced by the morphism

$$\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathbb{Q}/\mathbb{Z}(n)) \cong \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z})$$

(see 0.8.3 and 1.6.2).

1.6.4. Theorem. *For any arithmetic scheme X one has $u_\infty^* \circ \alpha_{X,n} = 0$ in the derived category.*

This seems to be rather nontrivial; our proof will be based on the following result about ℓ-adic cohomology.

1.6.5. Proposition. *Let $f\colon X \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic scheme (that is, with f separated, of finite type). Let $n < 0$. Then for any prime ℓ we have*

$$(H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0.$$

Proof. Let us recall some facts about ℓ-adic cohomology. We refer to [SGA 5, Exposé VI] for details. Let us first consider the sheaf $\mathbb{Z}_\ell(n)$. It is a **constructible $\mathbb{Z}_\ell$-sheaf*** on X in the sense of [SGA 5, Exposé VI, 1.1.1]. We would like to compare the cohomology of $\mathbb{Z}_\ell(n)$ on $X_{\overline{\mathbb{Q}},ét}$ and $X_{\overline{\mathbb{F}_p},ét}$, where p is some prime different from ℓ, to be determined later. For this we fix some algebraic closures $\overline{\mathbb{Q}}/\mathbb{Q}$ and $\overline{\mathbb{F}_p}/\mathbb{F}_p$ and consider the corresponding morphisms

$$\overline{\eta}\colon \operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\mathbb{Z}, \quad \overline{x}\colon \operatorname{Spec}\overline{\mathbb{F}_p} \to \operatorname{Spec}\mathbb{Z}.$$

Let $X_{\overline{\mathbb{Q}},ét}$ and $X_{\overline{\mathbb{F}_p},ét}$ be the pullbacks of X along the above morphisms:

$$\begin{array}{ccccc} X_{\overline{\mathbb{Q}}} & \longrightarrow & X & \longleftarrow & X_{\overline{\mathbb{F}_p}} \\ {\scriptstyle f_{\overline{\mathbb{Q}}}}\downarrow & \lrcorner & \downarrow{\scriptstyle f} & \llcorner & \downarrow{\scriptstyle f_{\overline{\mathbb{F}_p}}} \\ \operatorname{Spec}\overline{\mathbb{Q}} & \xrightarrow[\overline{\eta}]{} & \operatorname{Spec}\mathbb{Z} & \xleftarrow[\overline{x}]{} & \operatorname{Spec}\overline{\mathbb{F}_p} \end{array}$$

According to [SGA 5, Exposé VI, 2.2.3], the proper base change theorem holds for constructible $\mathbb{Z}_\ell$-sheaves. It gives us isomorphisms

$$H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Z}_\ell(n)) \cong (R^i f_! \mathbb{Z}_\ell(n))_{\overline{\eta}}, \quad H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n)) \cong (R^i f_! \mathbb{Z}_\ell(n))_{\overline{x}},$$

where $R^i f_! \mathbb{Z}_\ell(n)$ is the same sheaf on $\operatorname{Spec}\mathbb{Z}$, and we take its different stalks to get cohomology with compact support on different fibers. The construction of higher direct images with proper support $R^i f_! \mathcal{F}$ for ℓ-adic sheaves is given in [SGA 5, Exposé VI, §2.2]. The key nontrivial fact that we need

*Or simply $\mathbb{Z}_\ell$-**sheaf** in the terminology of [SGA $4\frac{1}{2}$, Rapport].

is that for every morphism (of locally noetherian schemes) $f\colon X \to Y$, separated of finite type, if $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf on X, then $R^i f_! \mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf on Y.

According to [SGA 5, Exposé VI, 1.2.6], for a projective system of abelian sheaves $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ on $X_{\acute{e}t}$, the following are equivalent:

1) $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf,

2) every open subscheme $U \subset X$ is a finite union of locally closed pieces Z_i where $\mathcal{F}|_{Z_i}$ is a **twisted constant constructible $\mathbb{Z}_\ell$-sheaf**[*].

Being "twisted constant" means that each sheaf $\mathcal{F}_n$ in the projective system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is locally constant. The importance of twisted constant sheaves is explained by the following property [SGA 5, Exposé VI, 1.2.4, 1.2.5]: for a connected locally noetherian scheme X, the category of twisted constant $\mathbb{Z}_\ell$-constructible sheaves on X is equivalent to the category of finitely generated $\mathbb{Z}_\ell$-modules with a continuous action of the étale fundamental group $\pi_1^{\acute{e}t}(X)$.

In our setting, all this means that there exists an open subscheme

$$U = \operatorname{Spec} \mathbb{Z}_S \subset \operatorname{Spec} \mathbb{Z},$$

where $\mathbb{Z}_S$ denotes the localization of $\mathbb{Z}$ at a finite set of primes S, such that the sheaves $R^i f_! \mathbb{Z}_\ell(n)$ are twisted constant on U. By removing all the necessary bad primes, we can make sure this holds for all i.

Now according to [Elements, Book IX, Proposition 20], there exists some prime $p \notin S$ (that is, $(p) \in U$), for which we may consider the following picture:

$$\begin{array}{ccccc}
X_{\overline{\mathbb{Q}}} & \longrightarrow & X_U & \longleftarrow & X_{\overline{\mathbb{F}_p}} \\
{\scriptstyle f_{\overline{\mathbb{Q}}}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f_U} & \llcorner & \big\downarrow{\scriptstyle f_{\overline{\mathbb{F}_p}}} \\
\operatorname{Spec} \overline{\mathbb{Q}} & \xrightarrow[\overline{\eta}]{} & U & \xleftarrow[\overline{x}]{} & \operatorname{Spec} \overline{\mathbb{F}_p}
\end{array}$$

It follows that we have isomorphisms

(1.6.3)
$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong (R^i f_{U,!} \mathbb{Z}_\ell(n))_{\overline{\eta}} \cong (R^i f_{U,!} \mathbb{Z}_\ell(n))_{\overline{x}} \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)),$$

of finitely generated $\mathbb{Z}_\ell$-modules with continuous action of

$$\pi_1^{\acute{e}t}(U) \cong \operatorname{Gal}(\mathbb{Q}_S/\mathbb{Q}),$$

where $\mathbb{Q}_S/\mathbb{Q}$ denotes a maximal extension of $\mathbb{Q}$ unramified outside of S. We note that $(R^i f_{U,!} \mathbb{Z}_\ell(n))_{\overline{\eta}}$ naturally carries an action of $\pi_1^{\acute{e}t}(U, \overline{\eta})$, while

[*] A **faisceau lisse** in the terminology of [SGA $4\frac{1}{2}$, Rapport].

$(R^i f_{U,!}\mathbb{Z}_\ell(n))_{\overline{x}}$ carries an action of $\pi_1^{ét}(U,\overline{x})$, and the isomorphism in the middle of (1.6.3) sweeps under the rug an identification of $\pi_1^{ét}(U,\overline{\eta})$ with $\pi_1^{ét}(U,\overline{x})$.

To state this more accurately, note that the $\mathbb{Z}_\ell$-module $H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Z}_\ell(n))$ carries a natural action of $G_{\mathbb{Q}}$ while $H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))$ carries a natural action of $G_{\mathbb{F}_p}$. After making the necessary choices, we have $G_{\mathbb{Q}_p}\subset G_{\mathbb{Q}}$ and a short exact sequence

$$1\to I_p\to G_{\mathbb{Q}_p}\to G_{\mathbb{F}_p}\to 1$$

where I_p is the inertia subgroup, acting trivially on $H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Z}_\ell(n))$. We have thus isomorphisms of finitely generated $\mathbb{Z}_\ell$-modules

$$H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Z}_\ell(n))\cong H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n)),$$

equivariant under the action of $G_{\mathbb{Q}_p}/I_p$ on the left hand side and of $G_{\mathbb{F}_p}$ on the right hand side. To relate all this to $\mathbb{Q}_\ell(n)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$-coefficients, note that we have the following isomorphic long exact sequences in cohomology

(1.6.4)
$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^{i-1}_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) & \xrightarrow{\cong} & H^{i-1}_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\
H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Z}_\ell(n)) & \xrightarrow{\cong} & H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n)) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell(n)) & \xrightarrow{\cong} & H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell(n)) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) & \longrightarrow & H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \\
\downarrow{\scriptstyle \cong} & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Here

$$H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell(n)) = H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Z}_\ell(n))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,$$
$$H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell(n)) = H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,$$

and the arrows ϕ above are canonical localization morphisms. The horizontal arrows are equivariant isomorphisms in the above sense. Note that we have

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}} \rightarrowtail H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}_p}/I_p} \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}},$$

so in order to prove that

$$(H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0,$$

it will be enough to show that

$$(H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = 0.$$

From now on we move to the characteristic p and consider the fixed points of $G_{\mathbb{F}_p}$ acting on the $\mathbb{Z}_\ell$-module $H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. In the long exact sequence (1.6.4), we have (keeping in mind that ϕ is merely the localization morphism):

$$\begin{aligned}
\ker\phi &= H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))_{tor},\\
\ker\psi &= \operatorname{im}\phi \cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))/\ker\phi\\
&= \frac{H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))}{H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))_{tor}} =: H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))_{cotor},\\
\operatorname{im}\psi &= H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div}.
\end{aligned}$$

This gives us a short exact sequence

$$0 \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))_{cotor} \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n)) \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div} \to 0$$

After taking the $G_{\mathbb{F}_p}$-invariants, we obtain a long exact sequence of cohomology groups

$$\begin{aligned}
(1.6.5)\quad & 0 \to (H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))_{cotor})^{G_{\mathbb{F}_p}} \to H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}}\\
& \to (H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \to H^1(G_{\mathbb{F}_p},H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))_{cotor}) \to \cdots
\end{aligned}$$

We claim that

$$(1.6.6)\qquad H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} = 0.$$

Indeed, according to [SGA 7, Exposé XXI, 5.5.3], the eigenvalues of the geometric Frobenius acting on $H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell)$ are algebraic integers. We are twisting $\mathbb{Q}_\ell$ by n, so the eigenvalues of Frobenius lie in $p^{-n}\,\overline{\mathbb{Z}}$. Since $n < 0$ by our assumption, this implies that 1 does not occur as an eigenvalue.

Now (1.6.6) and the long exact sequence (1.6.5) imply that there is a monomorphism

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor}),$$

which restricts to a monomorphism between the maximal divisible subgroups

$$((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})_{div}.$$

However, $H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})$ is a finitely generated $\mathbb{Z}_\ell$-module, and therefore its maximal divisible subgroup is trivial. We have therefore

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = ((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} = 0.$$

(For the first equality, note that for any G-module A one has $((A_{div})^G)_{div} = (A^G)_{div}$.) ■

Now we are ready to prove 1.6.4. The morphism $\alpha_{X,n}$ is defined on

$$R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]),$$

which is a complex of $\mathbb{Q}$-vector spaces, so it will be enough to show that v_∞^* is a torsion element in the abelian group

$$\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)),\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})).$$

The complexes $R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$ are almost of cofinite type in the sense of 0.3.7. Indeed, we observed it in 1.4.5 for $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$, and for $R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$, by the duality theorem 1.3 we have

$$\begin{aligned} H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) = H^{i-1}_c(X_{ét}, \mathbb{Z}(n)) &\overset{\text{up to 2-torsion}}{\approx} \widehat{H}^{i-1}_c(X_{ét}, \mathbb{Z}(n)) \\ &\cong \mathrm{Hom}(H^{3-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}(n)) \end{aligned}$$

and the groups $H^{3-i}(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated by our conjecture $\mathbf{L}^c(X_{ét}, n)$ (see 1.1.1), trivial for $i \ll 0$ by 1.5.3 (again, assuming $\mathbf{L}^c(X_{ét}, n)$) and finite 2-torsion for $i \gg 0$. Therefore, according to 0.3.8, to show that $v_\infty^*\colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$ is torsion in $\mathbf{D}(\mathbf{Ab})$, it is enough to show that the corresponding morphisms on the maximal divisible subgroups

$$H^i_c(v^*_\infty)_{div}\colon H^i_c(X_{ét},\mathbb{Q}/\mathbb{Z}(n))_{div} \to H^i_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z})_{div}$$

are all trivial.

The morphism $H^i_c(v^*_\infty)$ factors through $H^i_c(X_{\overline{\mathbb{Q}},ét},\mu^{\otimes n})^{G_\mathbb{Q}}$, where $\mu^{\otimes n}$ is the sheaf of all roots of unity on $X_{\overline{\mathbb{Q}},ét}$ twisted by n. We have therefore

$$\begin{array}{ccc} H^i_c(X_{ét},\mathbb{Q}/\mathbb{Z}(n))_{div} & \xrightarrow{H^i_c(v^*_\infty)_{div}} & H^i_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z})_{div} \\ & \searrow \qquad \nearrow & \\ & \left(H^i_c(X_{\overline{\mathbb{Q}},ét},\mu^{\otimes n})^{G_\mathbb{Q}}\right)_{div} & \end{array}$$

Now

$$\left(H^i_c(X_{\overline{\mathbb{Q}},ét},\mu^{\otimes n})^{G_\mathbb{Q}}\right)_{div} \cong \left(\bigoplus_\ell H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}\right)_{div} \cong \bigoplus_\ell \left(H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}\right)_{div},$$

where all summands are trivial according to 1.6.5. ∎

1.6.6. Corollary. *The morphism i^*_∞ is torsion in the derived category, i.e. $i^*_\infty\otimes\mathbb{Q} = 0$.*

Proof. Let us examine the morphism of distinguished triangles (1.6.1) that defines i^*_∞; in particular, the commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{ét},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) \\ {\scriptstyle u^*_\infty}\downarrow & \swarrow {\scriptstyle i^*_\infty} & \\ R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) & & \end{array}$$

According to 0.3.6, the morphism

$$\operatorname{Hom}_{\mathbf{D(Ab)}}(R\Gamma_{fg}(X,\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})) \to \operatorname{Hom}_{\mathbf{D(Ab)}}(R\Gamma_c(X_{ét},\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))$$

induced by the composition with $R\Gamma_c(X_{ét},\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n))$, is mono, and therefore

$$\operatorname{Hom}_{\mathbf{D(Ab)}}(R\Gamma_{fg}(X,\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))\otimes_\mathbb{Z}\mathbb{Q} \to \operatorname{Hom}_{\mathbf{D(Ab)}}(R\Gamma_c(X_{ét},\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))\otimes_\mathbb{Z}\mathbb{Q}$$

is mono as well. However, we just saw in the proof of 1.6.4 that $u_\infty^* \otimes \mathbb{Q} = 0$, and this implies that $i_\infty^* \otimes \mathbb{Q} = 0$. ∎

Now that we know that i_∞^* exists (and is unique), we are ready to define Weil-étale complexes.

1.6.7. Definition. $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is an object in the derived category $\mathbf{D}(\mathbf{Ab})$ which is a mapping fiber of i_∞^*:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

The **Weil-étale cohomology with compact support** is given by

$$H^i_{W,c}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n))).$$

Note that this defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ up to a non-unique isomorphism in $\mathbf{D}(\mathbf{Ab})$, and the groups $H^i_{W,c}(X, \mathbb{Z}(n))$ are also defined up to a non-unique isomorphism.

1.6.8. Proposition. *The conjecture $\mathbf{L}^c(X_{ét}, n)$ implies that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is a perfect complex.*

Proof. By definition, we have a long exact sequence in cohomology

$$\cdots \to H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \to H^i_{W,c}(X, \mathbb{Z}(n)) \to H^i_{fg}(X, \mathbb{Z}(n)) \xrightarrow{H^i(i_\infty^*)} H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \to \cdots$$

The groups $H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$ and $H^i_{fg}(X, \mathbb{Z}(n))$ are finitely generated by 1.4.4 and 1.5.4. They vanish for $i \ll 0$, but they are finite 2-torsion for $i \gg 0$. I claim that $H^i(i_\infty^*)$ is an isomorphism for $i \gg 0$, meaning that this 2-torsion in higher degrees does not appear in $H^i_{W,c}(X, \mathbb{Z}(n))$. We have a commutative diagram

$$\begin{array}{ccc} H^i_c(X_{ét}, \mathbb{Z}(n)) & \longrightarrow & H^i_{fg}(X, \mathbb{Z}(n)) \\ {\scriptstyle H^i(u_\infty^*)}\downarrow & \swarrow {\scriptstyle H^i(i_\infty^*)} & \\ H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) & & \end{array}$$

The morphism $H^i(u_\infty^*)$ is iso for $i \gg 0$, hence $H^i(i_\infty^*)$ is surjective for $i \gg 0$. However, $H^i_{fg}(X, \mathbb{Z}(n))$ and $H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$ have the same 2-torsion for $i \gg 0$,and $H^i(i_\infty^*)$ is iso for $i \gg 0$.

∎

1.6.9. Proposition. *The determinant* $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ *is well-defined up to a canonical isomorphism.*

Proof. For two different choices of a mapping fiber of i_∞^*, we obtain an isomorphism of distinguished triangles

$$\begin{array}{ccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \overset{\cong}{\dashrightarrow} & R\Gamma_{W,c}(X,\mathbb{Z}(n))' \\
\downarrow & & \downarrow \\
R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{\text{id}} & R\Gamma_{fg}(X,\mathbb{Z}(n)) \\
\downarrow{\scriptstyle i_\infty^*} & & \downarrow{\scriptstyle i_\infty^*} \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) & \xrightarrow{\text{id}} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] & \overset{\cong}{\dashrightarrow} & R\Gamma_{W,c}(X,\mathbb{Z}(n))'[1]
\end{array}$$

Here the dashed arrows are not canonical, but this does not affect the determinants, because these are functorial with respect to isomorphisms of triangles (see 0.4.1). The only technical issue is that the complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$ and $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ may have unbounded 2-torsion, unless $X(\mathbb{R}) = \emptyset$. However, we know that the arrow

$$H^i(i_\infty^*)\colon H^i_{fg}(X,\mathbb{Z}(n)) \to H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$$

is an isomorphism for $i \gg 0$. Therefore, taking the truncations $\tau_{\le m}$ for m big enough, we obtain a commutative diagram where the columns still induce long exact sequences in cohomology:

$$\begin{array}{ccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \overset{\cong}{\dashrightarrow} & R\Gamma_{W,c}(X,\mathbb{Z}(n))' \\
\downarrow & & \downarrow \\
\tau_{\le m} R\Gamma_{fg}(X,\mathbb{Z}(n)) & \xrightarrow{\text{id}} & \tau_{\le m} R\Gamma_{fg}(X,\mathbb{Z}(n)) \\
\downarrow{\scriptstyle i_\infty^*} & & \downarrow{\scriptstyle i_\infty^*} \\
\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) & \xrightarrow{\text{id}} & \tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] & \overset{\cong}{\dashrightarrow} & R\Gamma_{W,c}(X,\mathbb{Z}(n))'[1]
\end{array}$$

which gives us the desired canonical isomorphism

$$\begin{aligned}
&\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \\
&\quad \det_{\mathbb{Z}} \tau_{\le m} R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} (\det_{\mathbb{Z}} \tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}))^{-1} \\
&\qquad \cong \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))'.
\end{aligned}$$

■

1.6.10. Remark. Our methods establish existence of i_∞^* only as a morphism in the derived category and $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is defined only up to a non-canonical quasi-isomorphism. It is probably possible to construct i_∞^* as a canonical morphism in the category of complexes. This would give us a canonical construction of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ as a complex. Another possibility to make things canonical is to work with the derived ∞-category [Lur2006].

The reader will note that the non-canonicity of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ in the present construction is not only aesthetically unpleasant, but will also give us some technical troubles later on, for instance in §1.8.

1.7 Splitting of $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}$

The following result will be crucial in the next chapter.

1.7.1. Proposition. *There is a direct sum decomposition*

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1].$$

This isomorphism is not canonical, but induces a canonical isomorphism

$$(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \det_{\mathbb{Q}}(R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ \cong \det_{\mathbb{Q}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1].$$

Proof. Everything has to do with the cohomology of $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ and the morphism i_∞^* being torsion. In fact we already noted in 1.5.6 that the distinguished triangle defining $R\Gamma_{fg}(X, \mathbb{Z}(n))$

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ \xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

after tensoring with $\mathbb{Q}$ gives us an isomorphism

$$g \otimes \mathbb{Q}\colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]).$$

Now examine the triangle that defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \xrightarrow{h} R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}) \\ \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

According to 1.6.6, the morphism i_∞^* is torsion, so that $i_\infty^* \otimes \mathbb{Q} = 0$ and tensoring with $\mathbb{Q}$ gives a distinguished triangle

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{h\otimes\mathbb{Q}} R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \xrightarrow{0} R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}[1]$$

To shorten the notation, let us write $[-,-]$ instead of $R\mathrm{Hom}(-,-)$ and $(-)_{\mathbb{Q}}$ instead of $-\otimes_{\mathbb{Z}}\mathbb{Q}$. We have an isomorphism of distinguished triangles

(1.7.1)

$$\begin{array}{ccccc}
R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}} & \xrightarrow{\mathrm{id}} & R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}} & \overset{\cong}{\dashrightarrow} & \begin{array}{c}[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})[-1]\end{array} \\
\downarrow{\scriptstyle h\otimes\mathbb{Q}} & & \downarrow{\scriptstyle (h\circ g)\otimes\mathbb{Q}} & & \downarrow \\
R\Gamma_{fg}(X,\mathbb{Z}(n))_{\mathbb{Q}} & \underset{\cong}{\xrightarrow{g\otimes\mathbb{Q}}} & [R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] & \xrightarrow{\mathrm{id}} & [R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow \\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) & \xrightarrow{\mathrm{id}} & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) & \xrightarrow{\mathrm{id}} & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}}[1] & \xrightarrow{\mathrm{id}} & R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}}[1] & \overset{\cong}{\dashrightarrow} & \begin{array}{c}[R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})\end{array}
\end{array}$$

Here the right triangle is distinguished, being the direct sum of the distinguished triangles

$$R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \xrightarrow{\mathrm{id}} R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \\ \to 0 \to R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q})$$

and

$$R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})[-1] \to 0 \to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \\ \xrightarrow{\mathrm{id}} R\Gamma_c(G_{\mathbb{Q}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})$$

The two dashed arrows in (1.7.1) exist thanks to the axiom (TR3), and they are isomorphisms by the triangulated 5-lemma. We note that these arrows are by no means unique[*]. To see that the obtained splitting is canonical on

[*]It is a well-known lemma that in a triangulated category, a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ splits whenever one of the morphisms u, v, w is zero—[Verdier-thèse, Chapitre II, Corollaire 1.2.6]. I basically recalled the proof for our case to stress that such a splitting is not canonical.

the level of determinants, we argue as in 1.6.9. The isomorphism of triangles

$$\begin{array}{ccc}
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] & \xrightarrow{\text{id}} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \\
\downarrow & & \downarrow \\
R\Gamma_{W,c}(X, \mathbb{Z}(n))_{\mathbb{Q}} & \xrightarrow[\cong]{f} & \begin{array}{c}[R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1]\end{array} \\
\downarrow & & \downarrow \\
R\Gamma_{fg}(X, \mathbb{Z}(n))_{\mathbb{Q}} & \xrightarrow[\cong]{g\otimes\mathbb{Q}} & [R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]] \\
\downarrow{\scriptstyle 0} & & \downarrow \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}) & \xrightarrow{\text{id}} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})
\end{array}$$

induces by 0.4.1 a commutative diagram

$$\begin{array}{ccc}
\begin{array}{c}\det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \\ \otimes_{\mathbb{Q}} \\ \det_{\mathbb{Q}} R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}\end{array} & \xrightarrow{\cong} & \det_{\mathbb{Q}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \\
{\scriptstyle \text{id}\otimes\det(g\otimes\mathbb{Q})}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \det(f)} \\
\begin{array}{c}\det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \\ \otimes_{\mathbb{Q}} \\ \det_{\mathbb{Q}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1]\end{array} & \xrightarrow{\cong} & \det_{\mathbb{Q}} \begin{pmatrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1]\end{pmatrix}
\end{array}$$

Here the top arrow is canonical, and the left arrow as well; composing them, we obtain a canonical isomorphism

$$\det_{\mathbb{Q}}(R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong$$
$$\det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{Q}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1].$$

■

1.7.2. Remark. This means that for the Weil-étale cohomology with rational coefficients, we could take as the definition

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1],$$

which would simplify things a lot. However, it is crucial for us to work with $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. In the next chapter, this will mean that we will state conjectures about special values of $\zeta(X, s)$ up to a sign ± 1 and not merely up to a multiplier $x \in \mathbb{Q}^\times$. Of course the latter would be much easier.

1.8 Compatibilities with open-closed decompositions

We say that we have an **open-closed decomposition** of a scheme X if there are given morphisms

$$U \hookrightarrow X \leftarrow Z$$

where $U \hookrightarrow X$ is an inclusion of an open subscheme of X and $Z \to X$ is a closed immersion where $Z = X \setminus U$. The goal of this section is to prove the following result.

1.8.1. Proposition. *An open-closed decomposition of arithmetic schemes*

$$U \hookrightarrow X \leftarrow Z$$

induces a canonical isomorphism

(1.8.1)
$$\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong \det_{\mathbb{Z}} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{W,c}(Z, \mathbb{Z}(n)).$$

Morally, an open-closed decomposition should induce a distinguished triangle of Weil-étale complexes

$$(1.8.2) \quad R\Gamma_{W,c}(U, \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{W,c}(Z, \mathbb{Z}(n)) \to R\Gamma_{W,c}(U, \mathbb{Z}(n))[1]$$

and the corresponding long exact sequence in cohomology

$$\cdots \to H^i_{W,c}(U, \mathbb{Z}(n)) \to H^i_{W,c}(X, \mathbb{Z}(n)) \to H^i_{W,c}(Z, \mathbb{Z}(n)) \to H^{i+1}_{W,c}(U, \mathbb{Z}(n)) \to \cdots$$

However, with the definition of $R\Gamma_{W,c}(-, \mathbb{Z}(n))$ that we have at the moment, obtaining such a distinguished triangle seems to be a nontrivial task, and even the complexes in (1.8.2) are defined only up to a non-unique isomorphism in the derived category.

1.8.2. Remark. The main technical issue is the following. Given a morphism of distinguished triangles

$$(1.8.3) \quad \begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

sometimes it is tempting to consider its "cone", i.e. complete the above diagram to a 3×3-diagram with distinguished rows and columns

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\
\downarrow & & \downarrow & & \downarrow & \text{(ac)} & \downarrow \\
X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2]
\end{array}$$

where all squares commute, except for the bottom right square, which anti-commutes*. Whenever it is possible, Neeman in [Nee1991] says that (1.8.3) is **middling good**. Unfortunately, not every morphism of triangles is middling good (see [Nee1991, Example 2.6]). It seems like the best result one can obtain in general is that given a diagram with distinguished rows

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

there exists *some* morphism $Z \to Z'$ making the above diagram into a middling good morphism of triangles (this is done using the axiom (TR4); see e.g. [BBD1982, Proposition 1.1.11] or [May2001, Lemma 2.6]).

The reader may consult [Nee1991] for a thorough discussion of this issue. The bottom line is that we should be careful and never expect an arbitrary morphism of distinguished triangles to be completed to a 3×3-diagram.

*The anti-commutativity comes from the following sign issue. The rotation axiom (TR2) says that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished. So for a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, its full rotation by 1 is not $X[1] \xrightarrow{u[1]} Y[1] \xrightarrow{v[1]} Z[1] \xrightarrow{w[1]} X[2]$ but rather $X[1] \xrightarrow{-u[1]} Y[1] \xrightarrow{-v[1]} Z[1] \xrightarrow{-w[1]} X[2]$. The latter is isomorphic to, say, $X[1] \xrightarrow{u[1]} Y[1] \xrightarrow{v[1]} Z[1] \xrightarrow{-w[1]} X[2]$, so we just have to put a minus sign somewhere. The usual convention is that in the 3×3-diagram, the bottom right square anti-commutes.

$R\Gamma_{fg}(X, \mathbb{Z}(n))$ and open-closed decompositions

For an open-closed decomposition $U \hookrightarrow X \leftarrow Z$, the cohomology of $\mathbb{Z}^c(n)$ gives a distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1]$$

(see 0.11.1). Applying to it $R\mathrm{Hom}(-, \mathbb{Q}[-2])$, we obtain a distinguished triangle

$$\begin{aligned}(1.8.4)\quad & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \\ & \to R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])\end{aligned}$$

Similarly, for étale cohomology with compact support, we have a distinguished triangle
(1.8.5)

$$R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(Z_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n))[1]$$

Then one can to check that $(\alpha_{U,n}, \alpha_{X,n}, \alpha_{Z,n})$ give a morphism of triangles (1.8.4) and (1.8.5).

1.8.3. Lemma. *We have the following commutative diagram in the derived category:*

(1.8.6)

$$\begin{array}{ccc}
R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\alpha_{U,n}} & R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\alpha_{Z,n}} & R\Gamma_c(Z_{\acute{e}t}, \mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \xrightarrow{\alpha_{U,n}[1]} & R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n))[1]
\end{array}$$

Now in the diagram (1.8.6) we may pick a cone of each arrow $\alpha_{U,n}$, $\alpha_{X,n}$, $\alpha_{Z,n}$, which is by definition $R\Gamma_{fg}(U, \mathbb{Z}(n))$, $R\Gamma_{fg}(X, \mathbb{Z}(n))$, $R\Gamma_{fg}(Z, \mathbb{Z}(n))$ respectively. According to 1.5.5, each of these is defined up to a unique isomorphism in the derived category.

$$
\begin{array}{ccccccc}
[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \overset{\alpha_{U,n}}{\to} & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) & \dashrightarrow & R\Gamma_{fg}(U,\mathbb{Z}(n)) & \dashrightarrow & [R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow & & \downarrow & & & & \downarrow \\
[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \overset{\alpha_{X,n}}{\to} & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \dashrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \dashrightarrow & [R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow & & \downarrow & & & & \downarrow \\
[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \overset{\alpha_{Z,n}}{\to} & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) & \dashrightarrow & R\Gamma_{fg}(Z,\mathbb{Z}(n)) & \dashrightarrow & [R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow & & \downarrow & & & & \downarrow \\
[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] & \dashrightarrow & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] & \dashrightarrow & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] & \dashrightarrow & [R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}]
\end{array}
$$

For the above diagram, by the axiom (TR3), there are morphisms

$$
\begin{aligned}
R\Gamma_{fg}(U,\mathbb{Z}(n)) &\to R\Gamma_{fg}(X,\mathbb{Z}(n)),\\
R\Gamma_{fg}(X,\mathbb{Z}(n)) &\to R\Gamma_{fg}(Z,\mathbb{Z}(n)),\\
R\Gamma_{fg}(Z,\mathbb{Z}(n)) &\to R\Gamma_{fg}(U,\mathbb{Z}(n))[1]
\end{aligned}
$$

making everything commute. According to 0.3.6, these arrows are uniquely defined.

(1.8.7)

$$
\begin{array}{ccccccc}
[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \overset{\alpha_{U,n}}{\to} & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) & \dashrightarrow & R\Gamma_{fg}(U,\mathbb{Z}(n)) & \dashrightarrow & [R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow & & \downarrow & \text{(a)} & \downarrow \exists! & & \downarrow \\
[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \overset{\alpha_{X,n}}{\to} & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) & \dashrightarrow & R\Gamma_{fg}(X,\mathbb{Z}(n)) & \dashrightarrow & [R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow & & \downarrow & \text{(b)} & \downarrow \exists! & & \downarrow \\
[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]] & \overset{\alpha_{Z,n}}{\to} & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) & \dashrightarrow & R\Gamma_{fg}(Z,\mathbb{Z}(n)) & \dashrightarrow & [R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] \\
\downarrow & & \downarrow & \text{(c)} & \downarrow \exists! & & \downarrow \\
[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]] & \dashrightarrow & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] & \dashrightarrow & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] & \dashrightarrow & [R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}]
\end{array}
$$

Obtained this way, the third column

(1.8.8)

$$R\Gamma_{fg}(U,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(Z,\mathbb{Z}(n)) \to R\Gamma_{fg}(U,\mathbb{Z}(n))[1]$$

is uniquely defined, but a priori it is not a distinguished triangle.

1.8.4. At least in the case $X(\mathbb{R}) = \emptyset$, as we already observed in 1.5.2,

$$
\begin{aligned}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]) &\simeq R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)),\\
R\Gamma_{fg}(X,\mathbb{Z}(n)) &\simeq R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Z}[-1]),
\end{aligned}
$$

and one easily sees that this actually gives us an isomorphism between (1.8.8) and the distinguished triangle

$$\begin{aligned} &R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \\ &\qquad \to R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \to R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}). \end{aligned}$$

In particular, (1.8.8) is distinguished.

1.8.5. In general, as we noted in 1.5.6, tensoring the diagram with $\mathbb{Q}$ and $\mathbb{Z}/m\mathbb{Z}$, gives us isomorphisms

$$\begin{array}{ccc} R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q} & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \\ \downarrow & & \downarrow \\ R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \\ \downarrow & & \downarrow \\ R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes \mathbb{Q} & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \\ \downarrow & & \downarrow \\ R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q}[1] & \xrightarrow{\cong} & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \end{array}$$

and

$$\begin{array}{ccc} R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}/m(n)) & \xrightarrow{\cong} & R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \\ \downarrow & & \downarrow \\ R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}/m(n)) & \xrightarrow{\cong} & R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \\ \downarrow & & \downarrow \\ R\Gamma_c(Z_{\acute{e}t}, \mathbb{Z}/m(n)) & \xrightarrow{\cong} & R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \\ \downarrow & & \downarrow \\ R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}/m(n))[1] & \xrightarrow{\cong} & R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m[1] \end{array}$$

This means that the triangles

$$\begin{aligned} &R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q} \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \\ &\qquad \to R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes \mathbb{Q} \to R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q}[1] \end{aligned} \tag{1.8.9}$$

and

$$\begin{aligned} &R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \\ &\qquad \to R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \to R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m[1] \end{aligned} \tag{1.8.10}$$

are distinguished.

1.8.6. Let us make use of (1.8.10). For each prime p we may consider the "derived p-adic completions"

$$R\Gamma_{fg}(-,\mathbb{Z}(n))^{\wedge}_{p} := R\varprojlim_{k}(R\Gamma_{fg}(-,\mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z}),$$

as discussed in [BS2013] and [Stacks, Tag 091N]. This will give us a distinguished triangle

$$R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_{p} \to R\Gamma_{fg}(X,\mathbb{Z}(n))^{\wedge}_{p} \to R\Gamma_{fg}(Z,\mathbb{Z}(n))^{\wedge}_{p} \to R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_{p}[1]$$

It induces a long exact sequence in cohomology, which thanks to [Stacks, 0A06] and flatness of $\mathbb{Z}_p$ may be identified with

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^i(R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_{p}) & \xrightarrow{\cong} & H^i_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \\
\downarrow & & \downarrow \\
H^i(R\Gamma_{fg}(X,\mathbb{Z}(n))^{\wedge}_{p}) & \xrightarrow{\cong} & H^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \\
\downarrow & & \downarrow \\
H^i(R\Gamma_{fg}(Z,\mathbb{Z}(n))^{\wedge}_{p}) & \xrightarrow{\cong} & H^i_{fg}(Z,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \\
\downarrow & & \downarrow \\
H^{i+1}(R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_{p}) & \xrightarrow{\cong} & H^{i+1}_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Now the exactness of

$$\cdots \to H^i_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^i_{fg}(Z,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^{i+1}_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to \cdots$$

for each prime p implies that the sequence

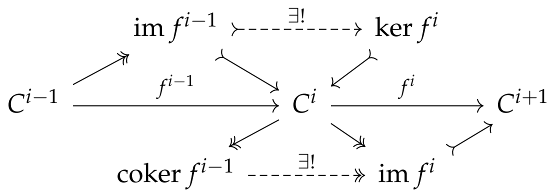
$$\cdots \to H^i_{fg}(U,\mathbb{Z}(n)) \to H^i_{fg}(X,\mathbb{Z}(n)) \to H^i_{fg}(Z,\mathbb{Z}(n)) \to H^{i+1}_{fg}(U,\mathbb{Z}(n)) \to \cdots$$

is exact as well. This uses the fact that the groups $H^i_{fg}(-,\mathbb{Z}(n))$ are finitely generated and $\mathbb{Z}_p$ is flat.

Indeed, given a morphism of finitely generated abelian groups $f\colon A \to B$, one sees that f is an isomorphism if and only if $f \otimes \mathbb{Z}_p\colon A \otimes \mathbb{Z}_p \to B \otimes \mathbb{Z}_p$ is an isomorphism for all p. Now a complex

$$(C^\bullet, f^\bullet)\colon \qquad \cdots \to C^{i-1} \xrightarrow{f^{i-1}} C^i \xrightarrow{f^i} C^{i+1} \to \cdots$$

is acyclic if and only if for each i in the diagram



$\operatorname{im} f^{i-1} \rightarrowtail \ker f^i$ is an isomorphism. Therefore, by the above and flatness of $\mathbb{Z}_p$, if $C^\bullet$ are finitely generated groups, $(C^\bullet, f^\bullet)$ is acyclic if and only if $(C^\bullet \otimes \mathbb{Z}_p, f^\bullet \otimes \mathbb{Z}_p)$ is acyclic for all p.

I suspect that the triangle (1.8.8) is actually distinguished, but the above argument at least settles that (1.8.8) induces a long exact sequence in cohomology, which will be enough for our purposes.

1.8.7. Remark. The argument from 1.8.6 may seem a bit too twisted, but there is a reason for that. We have to apply first $- \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z}$, and then take the limit $R \varprojlim_k$ because

$$R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z} \simeq 0,$$

while the complex

$$R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}_p[-2])$$

is not trivial. Intuitively, the whole argument comes from faithful flatness of $\widehat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$. We still looked at each p separately to make use of the derived completion $R \varprojlim_k (- \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z})$, which behaves nicely.

$R\Gamma_{W,c}(X, \mathbb{Z}(n))$ and open-closed decompositions

Recall now that the Weil-étale complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ was defined only up to a non-unique isomorphism in the derived category by the distinguished triangle

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

where the morphism i_∞^* is uniquely defined by the commutative triangle

$$
\begin{array}{ccccc}
 & & R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & & \\
 & \swarrow & & \searrow^{u_\infty^*} & \\
R\Gamma_{fg}(X, \mathbb{Z}(n)) & & \overset{i_\infty^*}{\underset{\exists!}{\dashrightarrow}} & & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})
\end{array}
$$

(see 1.6.1).

1.8.8. Lemma. *For an open-closed decomposition $U \hookrightarrow X \leftarrow Z$ the morphism u_∞^* gives a morphism between the corresponding distinguished triangles of cohomology with compact support:*

$$
\begin{array}{ccc}
R\Gamma_c(U_{ét}, \mathbb{Z}(n)) & \xrightarrow{u_{\infty,U}^*} & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & \text{(d)} & \downarrow \\
R\Gamma_c(X_{ét}, \mathbb{Z}(n)) & \xrightarrow{u_{\infty,X}^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & \text{(e)} & \downarrow \\
R\Gamma_c(Z_{ét}, \mathbb{Z}(n)) & \xrightarrow{u_{\infty,Z}^*} & R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & \text{(f)} & \downarrow \\
R\Gamma_c(U_{ét}, \mathbb{Z}(n))[1] & \xrightarrow{u_{\infty,U}^*[1]} & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})[1]
\end{array}
\tag{1.8.11}
$$

Proof. Follows from the definition of u_∞^* and the fact that α^* is compatible with the triangles associated to open-closed decompositions, as we verified in 0.8.4.

■

We now may assemble everything into the commutative prism displayed on the next page.

$$
\begin{tikzcd}
& R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) \arrow[dl] \arrow[dr, "u^*_{\infty,U}"] \arrow[ddd] & & \\
R\Gamma_{W,c}(U,\mathbb{Z}(n)) \arrow[r, dashed] & R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[r, dashed, "i^*_{\infty,U}", "\exists !"'] \arrow[ddd] & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\mathbb{Z}) \arrow[r, dashed] \arrow[ddd] & R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \\
& \text{(a)} & \text{(d)} & \\
& R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[dl] \arrow[dr, "u^*_{\infty,X}"] \arrow[ddd] & & \\
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r, dashed] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, dashed, "i^*_{\infty,X}", "\exists !"'] \arrow[ddd] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\mathbb{Z}) \arrow[r, dashed] \arrow[ddd] & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \\
& \text{(b)} & \text{(e)} & \\
& R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) \arrow[dl] \arrow[dr, "u^*_{\infty,Z}"] \arrow[ddd] & & \\
R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \arrow[r, dashed] & R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[r, dashed, "i^*_{\infty,Z}", "\exists !"'] \arrow[ddd] & R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),(2\pi i)^n\mathbb{Z}) \arrow[r, dashed] \arrow[ddd] & R\Gamma_{W,c}(Z,\mathbb{Z}(n))[1] \\
& \text{(c)} & \text{(f)} & \\
& R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] \arrow[dl] \arrow[dr, "u^*_{\infty,U}[1]"] & & \\
R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \arrow[r, dashed] & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[r, dashed, "i^*_{\infty,U}[1]", "\exists !"'] & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\mathbb{Z})[1] \arrow[r, dashed] & R\Gamma_{W,c}(U,\mathbb{Z}(n))[2]
\end{tikzcd}
\tag{1.8.12}
$$

Here the squares (a), (b), (c) are the ones that appear in the diagram (1.8.7); (d), (e), (f) are the ones that appear in (1.8.11); the arrows $i^*_{\infty,U}, i^*_{\infty,X}, i^*_{\infty,Z}$ are the unique morphisms in the derived category that make the triangles commute.

Morally, we should have a distinguished triangle of perfect complexes

$$R\Gamma_{W,c}(U,\mathbb{Z}(n)) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \to R\Gamma_{W,c}(U,\mathbb{Z}(n))[1]$$

which would give us then a canonical isomorphism of determinants (1.8.1). However, the diagram (1.8.12) a priori does not give a distinguished triangle for Weil-étale cohomology, it gives only a sequence of morphisms that is not necessarily distinguished (again, recall the discussion in 1.8.2). Let us instead give an ad hoc workaround on the level of determinants.

1.8.9. First let us assume for simplicity that $X(\mathbb{R}) = \emptyset$. Then the complexes

$$R\Gamma_{fg}(U,\mathbb{Z}(n)),\ R\Gamma_{fg}(X,\mathbb{Z}(n)),\ R\Gamma_{fg}(Z,\mathbb{Z}(n)),$$
$$R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}),\ R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}),\ R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

are perfect (they do not have 2-torsion in arbitrarily high degrees), and it makes sense to talk about their determinants. From the corresponding columns in (1.8.12) we obtain canonical isomorphisms

$$\det_\mathbb{Z} R\Gamma_{fg}(X,\mathbb{Z}(n)) \cong \det_\mathbb{Z} R\Gamma_{fg}(U,\mathbb{Z}(n)) \otimes_\mathbb{Z} \det_\mathbb{Z} R\Gamma_{fg}(Z,\mathbb{Z}(n)),$$

$$\det_\mathbb{Z} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \cong \det_\mathbb{Z} R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \otimes_\mathbb{Z} \det_\mathbb{Z} R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z});$$

and the rows give us isomorphisms

$$\det_\mathbb{Z} R\Gamma_{W,c}(U,\mathbb{Z}(n)) \cong \det_\mathbb{Z} R\Gamma_{fg}(U,\mathbb{Z}(n)) \otimes_\mathbb{Z} (\det_\mathbb{Z} R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1},$$

$$\det_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \det_\mathbb{Z} R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_\mathbb{Z} (\det_\mathbb{Z} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1},$$

$$\det_\mathbb{Z} R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \cong \det_\mathbb{Z} R\Gamma_{fg}(Z,\mathbb{Z}(n)) \otimes_\mathbb{Z} (\det_\mathbb{Z} R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1}.$$

Combining all the above, we obtain the desired isomorphism (1.8.1).

1.8.10. Now let us treat the general case, when possibly $X(\mathbb{R}) \neq \emptyset$. The above argument does not quite make sense, because the involved complexes are not bounded above. We consider the morphism of long exact sequences in cohomology given by $H^\bullet(i_\infty^*)$. We know that $H^i(i_\infty^*)$ is an isomorphism for $i \gg 0$, so truncating the long exact sequences at a sufficiently large degree $m \gg 0$, we obtain

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^i_{fg}(U, \mathbb{Z}(n)) & \longrightarrow & H^i_c(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^i_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^i_{fg}(Z, \mathbb{Z}(n)) & \longrightarrow & H^i_c(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow{\scriptstyle \delta^i_{fg}} & & \downarrow{\scriptstyle \delta^i_c} \\
H^{i+1}_{fg}(U, \mathbb{Z}(n)) & \longrightarrow & H^{i+1}_c(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^m_{fg}(U, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m_c(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^m_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^m_{fg}(Z, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m_c(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\
\downarrow{\scriptstyle \delta^m_{fg}} & & \downarrow{\scriptstyle \delta^m_c} \\
\operatorname{coker} \delta^m_{fg} & \xrightarrow{\cong} & \operatorname{coker} \delta^m_c \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array} \tag{1.8.13}$$

Note that the horizontal arrows that are isomorphisms induce canonical isomorphisms between the determinants. In particular, there is a canonical isomorphism

$$\det_\mathbb{Z} \operatorname{coker} \delta^m_{fg} \cong \det_\mathbb{Z} \operatorname{coker} \delta^m_c,$$

and hence a *canonical* isomorphism

$$\det_\mathbb{Z} \operatorname{coker} \delta^m_{fg} \otimes_\mathbb{Z} (\det_\mathbb{Z} \operatorname{coker} \delta^m_c)^{-1} \cong \mathbb{Z}. \tag{1.8.14}$$

Now the exact columns of (1.8.13) give us canonical isomorphisms

$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n))^{(-1)^i} \cong \tag{1.8.15}$$
$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(U,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(Z,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \operatorname{coker} \delta^m_{fg}$$

and

$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})^{(-1)^i} \cong \tag{1.8.16}$$
$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})^{(-1)^i} \otimes_{\mathbb{Z}}$$
$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})^{(-1)^i} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \operatorname{coker} \delta^m_c.$$

For fixed distinguished rows as in (1.8.12), we have canonical isomorphisms

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n))^{(-1)^i}$$
$$\cong \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})^{(-1)^{i+1}},$$

and similarly for U and Z in place of X. Combining these with (1.8.15), (1.8.16), and (1.8.14), gives us a canonical isomorphism (1.8.1).

1.8.11. Remark. A cheap way to get around the above technical problems is to consider the Weil-étale cohomology with coefficients in $\mathbb{Z}[1/2]$, i.e. tensor the distinguished triangle defining $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ with the flat $\mathbb{Z}$-module $\mathbb{Z}[1/2]$. Then the resulting distinguished triangle

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \to R\Gamma_{fg}(U,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$$
$$\to R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}[1/2]) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2][1]$$

consists of perfect complexes, as we just killed the 2-torsion. But then in the next chapter, we would be able to state the special value conjecture only up to some power of 2.