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Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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Chapter 0

Preliminaries

In this chapter we are going to fix some notation and collect several basic results which we will use later on.

Unless specified otherwise, X will denote an **arithmetic scheme**, i.e. separated, of finite type over $\text{Spec } \mathbb{Z}$. Its small Zariski and étale sites will be denoted by X_{Zar} and $X_{\text{ét}}$ respectively. By $X(\mathbb{C})$ we denote the space of complex points of X equipped with the usual analytic topology. It comes with a natural action of the Galois group $G_{\mathbb{R}} := \text{Gal}(\mathbb{C}/\mathbb{R})$.

I start with some definitions and facts related to abelian groups in §0.1. Then in §0.2 I fix some conventions about complexes. In our constructions there will appear complexes of abelian groups of a very special kind: their cohomology is conjecturally \mathbb{Q}/\mathbb{Z} -dual of finitely generated abelian groups, so in §0.3 I collect some properties that are enjoyed by such complexes. We will also make use of sheaves of roots of unity, and §0.5 is dedicated to some observations about $\mu_m(\mathbb{C})$ viewed as $G_{\mathbb{R}}$ -modules. We are also going to use the equivariant cohomology of sheaves on $X(\mathbb{C})$ with an action of $G_{\mathbb{R}}$. I review the basic definitions in §0.6. Then in §0.7 I recall how a sheaf on $X_{\text{ét}}$ gives rise to a $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$. In §0.8 I recall the definitions of cohomology with compact support for sheaves on $X_{\text{ét}}$ and $X(\mathbb{C})$, and in §0.9 I review a slight modification of cohomology with compact support on $X_{\text{ét}}$ needed for arithmetic duality theorems, which will show up in §1.3. Then in §0.10 I sketch a proof that for any arithmetic scheme X , the cohomology groups $H_c^i(X(\mathbb{C}), \mathbb{Z})$ are finitely generated (this seems to be very standard, but I could not find a reference). Finally, §0.11 is dedicated to an overview of Bloch's cycle complexes.

0.1 Abelian groups

Let A be an abelian group. Then A_{tor} denotes the maximal torsion subgroup of A and A_{cotor} denotes the group A/A_{tor} . Similarly, A_{div} denotes the maximal divisible subgroup of A and A_{codiv} denotes the group A/A_{div} , and we have short exact sequences

$$\begin{aligned} 0 &\rightarrow A_{tor} \rightarrow A \rightarrow A_{cotor} \rightarrow 0, \\ 0 &\rightarrow A_{div} \rightarrow A \rightarrow A_{codiv} \rightarrow 0. \end{aligned}$$

Note that the image of a divisible group is divisible, so that a group homomorphism $f: A \rightarrow B$ induces functorially a homomorphism of divisible groups $f_{div}: A_{div} \rightarrow B_{div}$. If A is a divisible group, then

$$\mathrm{Hom}_{\mathbf{Ab}}(A, B) \cong \mathrm{Hom}_{\mathbf{DivAb}}(A, B_{div}),$$

so that taking the maximal divisible subgroup $(-)_div: \mathbf{Ab} \rightarrow \mathbf{DivAb}$ is right adjoint to the inclusion $\mathbf{DivAb} \hookrightarrow \mathbf{Ab}$.

For the group of homomorphisms $A \rightarrow B$ between two abelian groups, we will write simply $\mathrm{Hom}(A, B)$. For $m = 1, 2, 3, \dots$ we denote by

$${}_m A := \ker(A \xrightarrow{m} A) \cong \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, A)$$

the m -torsion subgroup of A , and dually,

$$A_m := \mathrm{coker}(A \xrightarrow{m} A) = A/mA.$$

We have thus an exact sequence

$$0 \rightarrow {}_m A \rightarrow A \xrightarrow{\times m} A \rightarrow A_m \rightarrow 0$$

The abelian group \mathbb{Q}/\mathbb{Z} is divisible, hence injective, meaning that the contravariant functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact. For the infinite cyclic group we have trivially

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z},$$

and for finite cyclic groups

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) &\cong {}_m(\mathbb{Q}/\mathbb{Z}) \\ &= \{[0/m], [1/m], [2/m], \dots, [m-1/m]\} \cong \mathbb{Z}/m\mathbb{Z}. \end{aligned}$$

It follows that if A is a finitely generated abelian group, then $A \cong \mathbb{Z}^{\oplus r} \oplus A_{tor}$, where A_{tor} is the finite maximal torsion subgroup in A , and

$$\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus A_{tor}.$$

Of course, this isomorphism is not canonical, as it requires a choice of generators.

0.1.1. Definition. If $B \cong \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ for a finitely generated abelian group A , we say that B is **of cofinite type**.

0.1.2. Observation. If A is a finitely generated abelian group, then there is a canonical isomorphism

$$\varinjlim_m \text{Hom}(A/mA, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \text{Hom}(A, \mathbb{Q}/\mathbb{Z}).$$

Proof. This isomorphism is induced by $A \rightarrow A/mA$, and then applying the functor $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and \varinjlim_m . It comes from the following easy observation: as \mathbb{Q}/\mathbb{Z} is a torsion group, if A is finitely generated, any homomorphism $A \rightarrow \mathbb{Q}/\mathbb{Z}$ is killed by some m , hence factors through $A/mA \rightarrow \mathbb{Q}/\mathbb{Z}$. ■

0.1.3. Lemma. Denote $(-)^D := \text{Hom}(-, \mathbb{Q}/\mathbb{Z})$. Let A and B be finitely generated abelian groups and let A^D and B^D be the corresponding groups of cofinite type. Then every extension of B^D by A^D is again a group of cofinite type. Namely, any such extension is equivalent to

$$(0.1.1) \quad 0 \rightarrow A^D \rightarrow C^D \rightarrow B^D \rightarrow 0$$

where

$$(0.1.2) \quad 0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$$

is an extension of A by B .

The statement seems trivial, especially because $\text{Ext}(A, B)$ and $\text{Ext}(B^D, A^D)$ are easily seen to be isomorphic finite groups. However, there is one subtle issue: it is not obvious why nonequivalent extensions (0.1.2) cannot for some reason give equivalent extensions (0.1.1). Indeed, between groups of cofinite type, there are many homomorphisms that are not induced from the corresponding finitely generated groups; for example,

$$(0.1.3) \quad \text{Hom}_{\mathbf{Ab}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \quad \text{while} \quad \text{Hom}_{\mathbf{Ab}}(\mathbb{Z}^D, \mathbb{Z}^D) \cong \widehat{\mathbb{Z}}.$$

A priori, these extra homomorphisms could give weird equivalences of extensions. This is not the case, but we need to be a little bit more careful to justify that.

Proof. Consider the category \mathbf{Ab}_{ft} of finitely generated abelian groups. It is a full abelian subcategory of the category \mathbf{Ab} . The contravariant functor

$$(-)^D := \text{Hom}(-, \mathbb{Q}/\mathbb{Z}) : \mathbf{Ab}_{ft}^{\circ} \rightarrow \mathbf{Ab}.$$

is exact and faithful, but it is very far from being full, as we observed in (0.1.3). Let us denote the image of the functor $(-)^D$ by \mathbf{Ab}_{cft} . It is the category whose objects are groups of cofinite type A^D for some finitely generated A , and morphisms $B^D \rightarrow A^D$ in \mathbf{Ab}_{cft} are induced by morphisms $A \rightarrow B$ of finitely generated groups. This means that $(-)^D$ restricts to an (anti)equivalence of abelian categories

$$(0.1.4) \quad (-)^D := \text{Hom}(-, \mathbb{Q}/\mathbb{Z}): \mathbf{Ab}_{ft}^{\circ} \xrightarrow{\cong} \mathbf{Ab}_{cft}.$$

The category \mathbf{Ab}_{ft} has enough projective objects (and no nontrivial injective objects). Dually, \mathbf{Ab}_{cft} has enough injective objects: they are \mathbb{Q}/\mathbb{Z} -dual to the projective objects in \mathbf{Ab}_{ft} :

$$\begin{array}{ccc} & P & \\ \tilde{f} \swarrow & \downarrow f & \\ A & \twoheadrightarrow B & \end{array} \rightsquigarrow \begin{array}{ccc} & P^D & \\ \tilde{f}^D \swarrow & \uparrow f^D & \\ A^D & \longleftarrow B^D & \end{array}$$

Now assume that for some finitely generated groups A and B we want to calculate

$$\text{Ext}_{\mathbf{Ab}}^1(A, B) \cong R^1 \text{Hom}_{\mathbf{Ab}}(-, B)(A) = R^1 \text{Hom}_{\mathbf{Ab}_{ft}}(-, B)(A) \cong \text{Ext}_{\mathbf{Ab}_{ft}}^1(A, B).$$

To do this, we may pick a projective resolution $P_{\bullet} \twoheadrightarrow A$, and then calculate the cohomology group $H^1 \text{Hom}(P_{\bullet}, B)$. Note that we may build this projective resolution from finitely generated groups, i.e. inside the category \mathbf{Ab}_{ft} . Then thanks to the (anti)equivalence of categories (0.1.4), we have

$$(0.1.5) \quad \text{Ext}_{\mathbf{Ab}}^1(A, B) \cong \text{Ext}_{\mathbf{Ab}_{ft}}^1(A, B) \cong \text{Ext}_{\mathbf{Ab}_{cft}}^1(B^D, A^D).$$

The group

$$\text{Ext}_{\mathbf{Ab}_{cft}}^1(B^D, A^D) \cong R^1 \text{Hom}_{\mathbf{Ab}_{cft}}(B^D, -)(A^D)$$

may be calculated by taking the same resolution $P_{\bullet} \twoheadrightarrow A$, dualizing it to obtain an injective resolution $A^D \hookrightarrow P_{\bullet}^D$ by groups of cofinite type, and then calculating $H^1 \text{Hom}_{\mathbf{Ab}_{cft}}(B^D, P_{\bullet}^D)$. Note that $\text{Hom}_{\mathbf{Ab}_{cft}}(B^D, P_{\bullet}^D)$ is a subcomplex in $\text{Hom}_{\mathbf{Ab}}(B^D, P_{\bullet}^D)$, and we have the corresponding homomorphism on H^1

$$(0.1.6) \quad \text{Ext}_{\mathbf{Ab}_{cft}}^1(B^D, A^D) \rightarrow \text{Ext}_{\mathbf{Ab}}^1(B^D, A^D).$$

I claim that it is an isomorphism. Indeed, by additivity of $\text{Ext}_{\mathcal{A}}^1(-, -)$, it is enough to see this for the only interesting case $A = \mathbb{Z}/m\mathbb{Z}$ and $B = \mathbb{Z}$. The projective resolution

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{1 \mapsto [1]} \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

gives us the corresponding injective resolution of $\mathbb{Z}/m\mathbb{Z}^D \cong \mathbb{Z}/m\mathbb{Z}$:

$$0 \rightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{[1] \mapsto [1/m]} \mathbb{Q}/\mathbb{Z} \xrightarrow{\times m} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

After applying $\text{Hom}_{\mathcal{A}}(\mathbb{Z}^D, -)$ for $\mathcal{A} = \mathbf{Ab}_{\text{cft}}, \mathbf{Ab}$, we get two complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{\mathbb{Z}} & \xrightarrow{\times m} & \widehat{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

On H^1 this indeed induces an isomorphism $\mathbb{Z}/m\mathbb{Z} \rightarrow \widehat{\mathbb{Z}}/m\widehat{\mathbb{Z}} \cong \mathbb{Z}/m\mathbb{Z}$. Combining the isomorphism (0.1.6) with (0.1.5), we obtain an isomorphism

$$\text{Ext}_{\mathbf{Ab}}^1(A, B) \cong \text{Ext}_{\mathbf{Ab}}^1(B^D, A^D).$$

It remains to pass to the Yoneda Ext, which I suggest to denote for the moment by $\text{YExt}_{\mathcal{A}}^1(A, B)$, and which corresponds to the equivalence classes of extensions

$$0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$$

with respect to the Baer sum. If we have enough projectives or injectives in \mathcal{A} , so that $\text{Ext}_{\mathcal{A}}^1(A, B)$ exists, then we have an isomorphism of abelian groups

$$\text{YExt}_{\mathcal{A}}^1(A, B) \cong \text{Ext}_{\mathcal{A}}^1(A, B)$$

—see e.g. [Wei1994, §3.4]. In our situation, this gives an isomorphism

$$\begin{aligned} \text{YExt}_{\mathbf{Ab}}^1(A, B) &\xrightarrow{\cong} \text{YExt}_{\mathbf{Ab}}^1(B^D, A^D), \\ [B \twoheadrightarrow C \twoheadrightarrow A] &\mapsto [A^D \twoheadrightarrow C^D \twoheadrightarrow B^D] \end{aligned}$$

■

0.1.4. Example. If T is a finite abelian group, then

$$\text{Ext}(\mathbb{Q}/\mathbb{Z}, T) \cong \text{Ext}(T, \mathbb{Z}) \cong T.$$

Indeed, by additivity of $\text{Ext}(-, -)$, it is enough to check this for cyclic groups $T \cong \mathbb{Z}/m\mathbb{Z}$, and in this case, after applying $\text{Hom}(-, \mathbb{Z})$ to the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

we obtain

$$\begin{aligned}
 0 \rightarrow \underbrace{\text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})}_{=0} &\rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \\
 &\rightarrow \text{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \rightarrow \underbrace{\text{Ext}(\mathbb{Z}, \mathbb{Z})}_{=0} \rightarrow \underbrace{\text{Ext}(\mathbb{Z}, \mathbb{Z})}_{=0} \rightarrow 0
 \end{aligned}$$

In particular, for prime p , the corresponding p nonequivalent extensions of \mathbb{Q}/\mathbb{Z} by $\mathbb{Z}/p\mathbb{Z}$ arise as follows. First, there is the split extension

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

which is dual to the extension

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$$

Then the remaining $p - 1$ extensions are of the form

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{[1] \mapsto [m/p]} \mathbb{Q}/\mathbb{Z} \xrightarrow{\times p} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

where $m = 1, 2, \dots, p - 1$. Here we identify $\mathbb{Z}/p\mathbb{Z}$ with the cyclic subgroup $\{0, \frac{1}{p}, \frac{2}{p}, \dots, \frac{p-1}{p}\} \subset \mathbb{Q}/\mathbb{Z}$. These extensions are dual to

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{1 \mapsto [m]} \mathbb{Z}/p\mathbb{Z} \rightarrow 0$$

They are not equivalent for different m , because if we have a commutative diagram

$$\begin{array}{ccccccc}
 & & & \mathbb{Q}/\mathbb{Z} & & & \\
 & & [1] \mapsto [m_1/p] & \nearrow & \searrow \times p & & \\
 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & & & \mathbb{Q}/\mathbb{Z} & \longrightarrow 0 \\
 & & [1] \mapsto [m_2/p] & \searrow & \nearrow \times p & & \\
 & & & \mathbb{Q}/\mathbb{Z} & & &
 \end{array}$$

then $m_1 = m_2$. ▲

0.2 Complexes

Let us recall a couple of constructions from homological algebra. For an abelian category \mathcal{A} we denote by $\mathbf{C}(\mathcal{A})$ the category of cohomological complexes in \mathcal{A} , by $\mathbf{K}(\mathcal{A})$ the corresponding homotopy category, and by $\mathbf{D}(\mathcal{A})$ the derived category.

For a complex C^\bullet and $p \in \mathbb{Z}$, the shifted complex $C^\bullet[p]$ is defined by

$$(C^\bullet[p])^i := C^{i+p}, \quad d_{C^\bullet[p]}^i := (-1)^p d^{i+p}.$$

With this convention, $H^i(C^\bullet[p]) = H^{i+p}(C^\bullet)$. (Note that some sources, e.g. [Wei1994, 1.2.4], use another renumbering $(C^\bullet[p])^i := C^{i-p}$.)

0.2.1. Definition. A (cohomological) **double complex** $(C^{\bullet\bullet}, d_h^{\bullet\bullet}, d_v^{\bullet\bullet})$ is given by objects $C^{i,j} \in \mathcal{A}$ for $i, j \in \mathbb{Z}$, **horizontal differentials**

$$d_h^{i,j}: C^{i,j} \rightarrow C^{i+1,j},$$

and **vertical differentials**

$$d_v^{i,j}: C^{i,j} \rightarrow C^{i,j+1},$$

such that for all $i, j \in \mathbb{Z}$

$$(0.2.1) \quad d_v^{i+1,j} \circ d_h^{i,j} + d_h^{i,j+1} \circ d_v^{i,j} = 0;$$

that is, we have a diagram with anti-commutative squares

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & C^{i-1,j+1} & \longrightarrow & C^{i,j+1} & \xrightarrow{d_h^{i,j+1}} & C^{i+1,j+1} & \longrightarrow & \dots \\ & & \uparrow & & d_v^{i,j} \uparrow & & d_v^{i+1,j} \uparrow & & \\ \dots & \longrightarrow & C^{i-1,j} & \longrightarrow & C^{i,j} & \xrightarrow{d_h^{i,j}} & C^{i+1,j} & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & C^{i-1,j-1} & \longrightarrow & C^{i,j-1} & \longrightarrow & C^{i+1,j-1} & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Assume that in \mathcal{A} exist arbitrary products $\prod_i A_i$ and coproducts $\bigoplus_i A_i$. Then the corresponding **total complexes** (with respect to direct sum and product) are given by

$$(\text{Tot}^\oplus C^{\bullet\bullet})^m := \bigoplus_{i+j=m} C^{i,j}, \quad (\text{Tot}^\Pi C^{\bullet\bullet})^m := \prod_{i+j=m} C^{i,j},$$

together with the obvious differentials $d^m: (\text{Tot} C^{\bullet\bullet})^m \rightarrow (\text{Tot} C^{\bullet\bullet})^{m+1}$ defined via $d_h^{\bullet\bullet}$ and $d_v^{\bullet\bullet}$. The identity $d^{m+1} \circ d^m = 0$ is satisfied thanks to the condition (0.2.1).

Note that if $C^{i,j} = 0$ for $i \ll 0$ and for $j \ll 0$, then for each m there are only finitely many nonzero objects $C^{i,j}$ such that $i + j = m$, and in this case $\text{Tot}^{\oplus} C^{\bullet\bullet} = \text{Tot}^{\Pi} C^{\bullet\bullet}$.

0.2.2. Definition. Let $(A_{\bullet}, d_A^{\bullet})$ be a homological complex and $(B^{\bullet}, d_B^{\bullet})$ a cohomological complex. Then the corresponding **Hom double complex** $\text{Hom}^{\bullet\bullet}(A_{\bullet}, B^{\bullet})$ is the double complex of abelian groups given by

$$\text{Hom}^{i,j}(A_{\bullet}, B^{\bullet}) := \text{Hom}_{\mathcal{A}}(A_i, B^j),$$

together with the differentials for $f \in \text{Hom}_{\mathcal{A}}(A_i, B^j)$

$$(0.2.2) \quad \begin{aligned} d_h^{i,j} f &:= f \circ d_{i+1}^A, \\ d_v^{i,j} f &:= (-1)^{i+j+1} d_B^j \circ f. \end{aligned}$$

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \rightarrow & \text{Hom}(A_{i-1}, B^{j+1}) & \rightarrow & \text{Hom}(A_i, B^{j+1}) & \rightarrow & \text{Hom}(A_{i+1}, B^{j+1}) & \rightarrow \cdots \\ & & \uparrow & & d_v^{i,j} \uparrow & & \uparrow & \\ \cdots & \rightarrow & \text{Hom}(A_{i-1}, B^j) & \rightarrow & \text{Hom}(A_i, B^j) & \xrightarrow{d_h^{i,j}} & \text{Hom}(A_{i+1}, B^j) & \rightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow & \\ \cdots & \rightarrow & \text{Hom}(A_{i-1}, B^{j-1}) & \rightarrow & \text{Hom}(A_i, B^{j-1}) & \rightarrow & \text{Hom}(A_{i+1}, B^{j-1}) & \rightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow & \\ & & \vdots & & \vdots & & \vdots & \end{array}$$

The sign in (0.2.2) is introduced to make the squares anti-commute, turning $\text{Hom}^{\bullet\bullet}(A_{\bullet}, B^{\bullet})$ into a double complex in the sense of 0.2.1.

0.2.3. Definition. Let $(A^{\bullet}, d_A^{\bullet})$ and $(B^{\bullet}, d_B^{\bullet})$ be two cohomological complexes. Then we may turn A^{\bullet} into a homological complex A_{\bullet} by setting $A_i := A^{-i}$ and $d_i^A := d_A^{-i}: A_i \rightarrow A_{i-1}$. The complex

$$\text{Hom}^{\bullet}(A^{\bullet}, B^{\bullet}) := \text{Tot}^{\Pi} \text{Hom}^{\bullet\bullet}(A_{\bullet}, B^{\bullet})$$

is called the **Hom complex**.

0.3 Derived category of abelian groups

Most of the time we are going to work in the derived category $\mathbf{D}(\mathbf{Ab})$ of complexes of abelian groups, and occasionally the derived category of $\mathbf{D}(\mathbb{R}\text{-Vect})$

of complexes of real vector spaces. The canonical reference for derived categories is Verdier's thesis [Verdier-thèse], and in particular I am going to use Verdier's original axioms (TR1)–(TR4).

It is rather easy to describe how objects and morphisms in the category $\mathbf{D}(\mathbf{Ab})$ look like, thanks to the fact that $\text{Ext}_{\mathbb{Z}}^i(-, -) = 0$ for $i > 1$. Let us recall the general (well-known) result.

0.3.1. Lemma. *Let \mathcal{A} be a hereditary abelian category, i.e. an abelian category such that*

$$\text{Ext}_{\mathcal{A}}^i(A, B) = 0 \quad \text{for all } A, B \in \mathcal{A}, i > 1$$

(when $\mathcal{A} = \mathbf{R}\text{-Mod}$, this condition is equivalent to R being a hereditary ring; in particular, \mathbb{Z} and any principal ideal domain is hereditary).

1) In the derived category $\mathbf{D}(\mathcal{A})$ every complex A^\bullet is isomorphic to the complex

$$\cdots \rightarrow H^{i-1}(A^\bullet) \xrightarrow{0} H^i(A^\bullet) \xrightarrow{0} H^{i+1}(A^\bullet) \rightarrow \cdots$$

that is,

$$A^\bullet \cong \bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i] \cong \prod_{i \in \mathbb{Z}} H^i(A^\bullet)[-i].$$

2) The morphisms in $\mathbf{D}(\mathcal{A})$ are given by

$$\begin{aligned} \text{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet) &\cong \\ &\prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i \in \mathbb{Z}} \text{Ext}_{\mathcal{A}}^1(H^i(A^\bullet), H^{i-1}(B^\bullet)). \end{aligned}$$

Proof. For the first part, for each $i \in \mathbb{Z}$ let us consider the short exact sequence

$$0 \rightarrow \ker d^{i-1} \rightarrow A^{i-1} \xrightarrow{p} \text{im } d^{i-1} \rightarrow 0$$

Applying the functor $\text{Hom}_{\mathcal{A}}(H^i(A^\bullet), -)$ gives us a long exact sequence of Yoneda Exts

$$\begin{aligned} \cdots \rightarrow \text{Ext}_{\mathcal{A}}^1(H^i(A^\bullet), \ker d^{i-1}) &\rightarrow \text{Ext}_{\mathcal{A}}^1(H^i(A^\bullet), A^{i-1}) \\ &\xrightarrow{p_*} \text{Ext}_{\mathcal{A}}^1(H^i(A^\bullet), \text{im } d^{i-1}) \rightarrow \text{Ext}_{\mathcal{A}}^2(H^i(A^\bullet), \ker d^{i-1}) \rightarrow \cdots \end{aligned}$$

where the last Ext vanishes by our assumption on \mathcal{A} , and therefore p_* is surjective, which in particular means that the class of the short exact sequence

$$0 \rightarrow \text{im } d^{i-1} \rightarrow \ker d^i \rightarrow H^i(A^\bullet) \rightarrow 0$$

lies in the image in p_{**} , so that there exists an object B^i sitting in the following morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{i-1} & \longrightarrow & B^i & \longrightarrow & H^i(A^\bullet) & \longrightarrow & 0 \\ & & \downarrow p & & \lrcorner & & \downarrow \text{id} & & \\ 0 & \longrightarrow & \text{im } d^{i-1} & \longrightarrow & \ker d^i & \longrightarrow & H^i(A^\bullet) & \longrightarrow & 0 \end{array}$$

This gives us morphisms of complexes

$$\begin{array}{ccc} & [A_{i-1}^{i-1} \rightarrow B_i^i] & \\ & \swarrow & \searrow \\ A^\bullet & & H^i(A^\bullet)[-i] \end{array}$$

that induce isomorphisms in cohomology in degree i :

$$\begin{array}{ccccccccc} \dots & \longrightarrow & A^{i-2} & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} & \longrightarrow & \dots \\ & & \uparrow & & \text{id} \uparrow & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & A^{i-1} & \longrightarrow & B^i & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i(A^\bullet) & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

Passing to direct sums of the complexes $H^i(A^\bullet)[-i]$ and $[A_{i-1}^{i-1} \rightarrow B_i^i]$ gives us quasi-isomorphisms that form the desired isomorphism in $\mathbf{D}(\mathcal{A})$:

$$\begin{array}{ccc} & C^\bullet & \\ & \swarrow \cong & \searrow \cong \\ A^\bullet & & \bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i] \end{array}$$

We note that $\bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i]$ has the universal property of both product and coproduct in the category of complexes.

Now for the second part, we note that since by our assumptions on \mathcal{A} ,

$$\text{Hom}_{\mathbf{D}(\mathcal{A})}(A, B[i]) = \begin{cases} \text{Hom}_{\mathcal{A}}(A, B), & i = 0, \\ \text{Ext}_{\mathcal{A}}^1(A, B), & i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

we have by the calculation in 1),

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet) &\cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}\left(\bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i], \prod_{j \in \mathbb{Z}} H^j(B^\bullet)[-j]\right) \\ &\cong \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(H^i(A^\bullet), H^j(B^\bullet)[i-j]) \\ &\cong \prod_{i \in \mathbb{Z}} \left(\mathrm{Hom}_{\mathcal{A}}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \mathrm{Ext}_{\mathcal{A}}^1(H^i(A^\bullet), H^{i-1}(B^\bullet)) \right). \end{aligned}$$

■

0.3.2. Remark. One can also obtain information about $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ using the following **hyperext spectral sequence**:

$$E_2^{pq} = \prod_{i \in \mathbb{Z}} \mathrm{Ext}_{\mathcal{A}}^p(H^i(A^\bullet), H^{q+i}(B^\bullet)) \implies \mathrm{Ext}_{\mathbf{D}(\mathcal{A})}^{p+q}(A^\bullet, B^\bullet)$$

(see e.g. [Verdier-thèse, Chapitre III, §4.6.10] and [Wei1994, §5.7.9]). For a hereditary category $\mathrm{Ext}_{\mathcal{A}}^p = 0$, unless $p = 0, 1$, and this spectral sequence consists of two columns and therefore gives us short exact sequences

$$\begin{aligned} 0 \rightarrow \prod_{i \in \mathbb{Z}} \mathrm{Ext}_{\mathcal{A}}^1(H^i(A^\bullet), H^{i-1}(B^\bullet)) &\rightarrow \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet) \\ &\rightarrow \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(H^i(A^\bullet), H^i(B^\bullet)) \rightarrow 0 \end{aligned}$$

However, one should be careful with boundedness of A^\bullet and B^\bullet to make sure that the spectral sequence exists.

Recall that a complex of abelian groups C^\bullet is called **perfect** if it is quasi-isomorphic to a bounded complex of finitely generated free (= projective) abelian groups. This is the same as asking $H^i(C^\bullet)$ to be finitely generated abelian groups, and $H^i(C^\bullet) = 0$ for all but finitely many i . In §1.5 we are going to construct certain complexes $R\Gamma_{fg}(X, \mathbb{Z}(n))$ that are *almost* perfect, in the sense that their cohomology groups $H_{fg}^i(X, \mathbb{Z}(n))$ are finitely generated, vanish for $i \ll 0$, and for $i \gg 0$ they are finite 2-torsion (that is, killed by multiplication by 2). Let us introduce the following notion.

0.3.3. Definition. Let C^\bullet be an object in $\mathbf{D}(\mathbf{Ab})$. We say that C^\bullet is **almost perfect** if

- 1) $H^i(C^\bullet)$ are finitely generated groups,
- 2) $H^i(C^\bullet) = 0$ for $i \ll 0$,
- 3) $H^i(C^\bullet)$ is 2-torsion for $i \gg 0$.

I warn the reader that this terminology was invented by myself and serves only to simplify the exposition.

0.3.4. Lemma.

- 1) If C^\bullet and C'^\bullet are almost perfect, then the group $\mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups.
- 2) If A^\bullet is a complex such that $H^i(A^\bullet)$ are finite dimensional \mathbb{Q} -vector spaces and C^\bullet is a complex such that $H^i(C^\bullet)$ are finitely generated abelian groups, then the group $\mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(A^\bullet, C^\bullet)$ is divisible.

Proof. By 0.3.1 we have

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(C^\bullet, C'^\bullet) \cong \prod_{i \in \mathbb{Z}} \mathrm{Hom}(H^i(C^\bullet), H^i(C'^\bullet)) \oplus \prod_{i \in \mathbb{Z}} \mathrm{Ext}(H^i(C^\bullet), H^{i-1}(C'^\bullet)).$$

Note that by our assumptions, both groups $\prod_{i \in \mathbb{Z}} \mathrm{Hom}(H^i(C^\bullet), H^i(C'^\bullet))$ and $\prod_{i \in \mathbb{Z}} \mathrm{Ext}(H^i(C^\bullet), H^{i-1}(C'^\bullet))$ will be of the form $G \oplus T$, where G is a finitely generated abelian group and T is 2-torsion. Assume now that some element $x \in \mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(C^\bullet, C'^\bullet)$ is divisible by all powers of 2. If it lies in the finitely generated part, then $x = 0$; if it lies in the 2-torsion part, then again $x = 0$.

Similarly, in part 2), we have

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(A^\bullet, C^\bullet) \cong \prod_{i \in \mathbb{Z}} \mathrm{Hom}(H^i(A^\bullet), H^i(C^\bullet)) \oplus \prod_{i \in \mathbb{Z}} \mathrm{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet)).$$

Now by our assumptions $\mathrm{Hom}(H^i(A^\bullet), H^i(C^\bullet)) = 0$ for all i . Then each group $\mathrm{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet))$ is a direct sum of finitely many groups isomorphic to $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ and $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$, and $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is divisible while $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z}) = 0$. This means that the group $\mathrm{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet))$ is divisible for each i , and hence their direct product over i is divisible. ■

Recall that the axiom (TR1) tells us that every morphism $v: A^\bullet \rightarrow B^\bullet$ may be completed to a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$. Here C^\bullet is called the **cone** of u . The axiom (TR3) tells that for every commutative diagram with distinguished rows

$$(0.3.1) \quad \begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow f & & \downarrow g & & & & \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array}$$

there exists some morphism $h: C^\bullet \rightarrow C'^\bullet$ giving a morphism of distinguished triangles

$$(0.3.2) \quad \begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow f & & \downarrow g & & \downarrow \exists! h & & \downarrow f[1] \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array}$$

The cone C^\bullet in (TR1) and the morphism h in (TR3) are neither unique nor canonical. Two different cones of the same morphism are necessarily isomorphic, but the isomorphism between them is not unique, because it is provided by (TR3). This is a well-known issue with the derived category formalism, and in the present text we are going to encounter some problems related to it. For now, let us recall a useful standard argument which shows that at least in some special cases, things are uniquely defined.

0.3.5. Observation ((TR3) and (TR1) with uniqueness; \approx [BBD1982, Proposition 1.1.9, Corollaire 1.1.10]). Consider the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category \mathcal{A} .

- 1) For a commutative diagram (0.3.1), assume that the homomorphism of abelian groups

$$w^*: \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \rightarrow \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet)$$

induced by w is trivial. Then there exists a unique morphism $h: C^\bullet \rightarrow C'^\bullet$ giving a morphism of triangles (0.3.2).

- 2) For a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$, assume that for any other cone C'^\bullet of u the morphism w^* is trivial. Then in fact the cone of u is unique up to a unique isomorphism.

Proof. In 1), the existence of $C^\bullet \rightarrow C'^\bullet$ is the axiom (TR3), and the interesting part is uniqueness. Since $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(-, C'^\bullet)$ is a cohomological functor, applied to the first distinguished triangle, it gives us an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \xrightarrow{w^*} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet) \xrightarrow{v^*} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(B^\bullet, C'^\bullet).$$

If $w^* = 0$, we conclude that v^* is a monomorphism. This means that there is a unique morphism h such that $h \circ v = v' \circ g$. Now in 2), if C^\bullet and C'^\bullet are two different cones of u , we have a commutative diagram

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ A^\bullet & \xrightarrow{u'} & B^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A^\bullet[1] \end{array}$$

As always, by the “triangulated 5-lemma”, the dashed arrow is an isomorphism, and it is unique thanks to 1). ■

Here is a particular case that we are going to use.

0.3.6. Corollary. *Consider the derived category $\mathbf{D}(\mathbf{Ab})$.*

1) *Suppose we have a commutative diagram with distinguished rows*

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow f & & \downarrow g & & & & \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array}$$

where A^\bullet is a complex such that $H^i(A^\bullet)$ are finite dimensional \mathbb{Q} -vector spaces and C^\bullet and C'^\bullet are almost perfect complexes. Then there exists a unique (!) morphism $h: C^\bullet \rightarrow C'^\bullet$ giving a morphism of triangles

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\ \downarrow f & & \downarrow g & & \downarrow \exists! h & & \downarrow f[1] \\ A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1] \end{array}$$

2) *For a distinguished triangle*

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

assume that A^\bullet is a complex such that $H^i(A^\bullet)$ are finite dimensional \mathbb{Q} -vector spaces and C^\bullet is an almost perfect complex. Then the cone of u is unique up to a unique isomorphism.

Proof. In this situation, according to 0.3.4, the group $\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups and $\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet[1], C'^\bullet)$ is divisible. This means that there are no nontrivial homomorphisms

$$\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet[1], C'^\bullet) \rightarrow \mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet)$$

and we may apply 0.3.5. ■

We are going to encounter certain complexes whose cohomology groups are of cofinite type, i.e. \mathbb{Q}/\mathbb{Z} -dual of finitely generated abelian groups. Again, they will be bounded below, but may have 2-torsion in higher degrees. For this we introduce a definition similar to 0.3.3.

0.3.7. Definition. Let A^\bullet be an object in $\mathbf{D}(\mathbf{Ab})$. We say that A^\bullet is **almost of cofinite type** if

- 1) $H^i(A^\bullet)$ are groups of cofinite type for all i ,
- 2) $H^i(A^\bullet) = 0$ for $i \ll 0$,
- 3) $H^i(A^\bullet)$ is 2-torsion for $i \gg 0$ (in fact, finite 2-torsion according to 1)).

0.3.8. Observation. Suppose that A^\bullet and B^\bullet are almost of cofinite type. Then a morphism $f: A^\bullet \rightarrow B^\bullet$ is torsion in $\mathbf{D}(\mathbf{Ab})$ (i.e. a torsion element in the group $\text{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet)$), i.e. $f \otimes \mathbb{Q} = 0$ if and only if the morphisms

$$H^i(f): H^i(A^\bullet) \rightarrow H^i(B^\bullet)$$

are torsion, that is, they are trivial on the maximal divisible subgroups:

$$(H^i(f)_{\text{div}}: H^i(A^\bullet)_{\text{div}} \rightarrow H^i(B^\bullet)_{\text{div}}) = 0.$$

Proof. By 0.3.1 we have

$$\begin{aligned} \text{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet) &\cong \\ &\prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i \in \mathbb{Z}} \text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet)). \end{aligned}$$

As the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ are of the form $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T$, where T is finite, we have

$$\begin{aligned} \text{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'} \oplus T') &\cong \\ \underbrace{\text{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})}_{=0} \oplus \text{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, T') & \\ \oplus \underbrace{\text{Ext}(T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})}_{=0} \oplus \text{Ext}(T, T'), & \end{aligned}$$

where $\text{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})$ and $\text{Ext}(T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})$ are trivial because \mathbb{Q}/\mathbb{Z} is a divisible group; then $\text{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, T') \cong \text{Ext}(\mathbb{Q}/\mathbb{Z}, T')^{\oplus r} \cong T'^{\oplus r}$ by 0.1.4, and $\text{Ext}(T, T')$ is also finite, being a direct sum of

$$\text{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m, n)\mathbb{Z}.$$

For $i \gg 0$, the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ will be finite 2-torsion, and therefore $\text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ will be finite 2-torsion as well. It follows that the whole product $\prod_{i \in \mathbb{Z}} \text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ is of the form $G \oplus T$, where G is finite and T is possibly infinite 2-torsion. We have

$$(G \oplus T) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Similarly, the group $\prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet))$ will consist of some part of the form $\widehat{\mathbb{Z}}^{\oplus r} \oplus G$, where G is finite, and some 2-torsion part, which is killed by tensoring with \mathbb{Q} . It follows that there is an isomorphism

$$\text{Hom}_{\mathbf{D}(\text{Ab})}(A^\bullet, B^\bullet) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

After unwinding the proof of 0.3.1, one sees that this arrow is what it should be:

$$f \otimes \mathbb{Q} \mapsto (H^i(f) \otimes \mathbb{Q})_{i \in \mathbb{Z}}.$$

■

0.3.9. Observation. *If A^\bullet is a complex of \mathbb{Q} -vector spaces and B^\bullet is a complex almost of cofinite type, then there is an isomorphism of abelian groups*

$$\begin{aligned} \text{Hom}_{\mathbf{D}(\text{Ab})}(A^\bullet, B^\bullet) &\xrightarrow{\cong} \prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet)), \\ f &\mapsto (H^i(f))_{i \in \mathbb{Z}}. \end{aligned}$$

Proof. I claim that in the formula 0.3.1

$$\text{Hom}_{\mathbf{D}(\text{Ab})}(A^\bullet, B^\bullet) \cong \prod_{i \in \mathbb{Z}} \text{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i \in \mathbb{Z}} \text{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$$

the summand with Ext groups vanishes. Indeed, each group $H^{i-1}(B^\bullet)$ is of the form $\mathbb{Q}/\mathbb{Z}^{\oplus r} \oplus T$, where \mathbb{Q}/\mathbb{Z} is injective, hence $\text{Ext}(-, \mathbb{Q}/\mathbb{Z}) = 0$, and T is a finite torsion group, hence $\text{Ext}(V, T) = 0$ if V is a \mathbb{Q} -vector space. ■

0.4 Determinants of complexes

We are going use determinants of complexes defined by Knudsen and Mumford. The reader may consult [GKZ1994, Appendix A] for a nice introduction and the original paper [KM1976] for the technical details.

For a perfect complex of R -modules P^\bullet , or in general for a perfect complex in the derived category $\mathbf{D}(R\text{-Mod})$ one may define its **determinant**

$$\det_R P_\bullet := \bigotimes_{i \in \mathbb{Z}} \det_R H^i(P^\bullet)^{(-1)^i}.$$

0.4.1. Fact ([KM1976, p. 43, Corollary 2]). *For a distinguished triangle of perfect complexes in $\mathbf{D}(R\text{-Mod})$*

$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$$

we have a canonical isomorphism

$$\det_R A^\bullet \otimes_R \det_R C^\bullet \xrightarrow{\cong} \det_R B^\bullet.$$

It is functorial with respect to isomorphisms of distinguished triangles: for such an isomorphism

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A^\bullet[1] \\ \cong \downarrow f & & \cong \downarrow g & & \cong \downarrow h & & \cong \downarrow f[1] \\ A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow & A'^\bullet[1] \end{array}$$

we have a commutative diagram

$$\begin{array}{ccc} \det_R A^\bullet \otimes \det_R C^\bullet & \xrightarrow{\cong} & \det_R B^\bullet \\ \det(f) \otimes \det(h) \downarrow \cong & & \cong \downarrow \det(g) \\ \det_R A'^\bullet \otimes \det_R C'^\bullet & \xrightarrow{\cong} & \det_R B'^\bullet \end{array}$$

Note that in particular, if we consider the direct sum of distinguished triangles

$$A^\bullet \xrightarrow{\text{id}} A^\bullet \rightarrow 0 \rightarrow A^\bullet[1] \quad \text{and} \quad 0 \rightarrow B^\bullet \xrightarrow{\text{id}} B^\bullet \rightarrow 0$$

then we obtain a distinguished triangle

$$A^\bullet \rightarrow A^\bullet \oplus B^\bullet \rightarrow B^\bullet \rightarrow A^\bullet[1]$$

and 0.4.1 gives us a canonical isomorphism

$$\det_R A^\bullet \otimes_R \det_R B^\bullet \xrightarrow{\cong} \det_R (A^\bullet \oplus B^\bullet).$$

0.5 Roots of unity

The m -th complex roots of unity

$$\mu_m(\mathbb{C}) := \{z \in \mathbb{C}^\times \mid z^m = 1\} = \{e^{2\pi i k/m} \mid k = 0, \dots, m-1\}$$

form an abelian group with respect to multiplication. It also carries a natural action of the Galois group $G_{\mathbb{R}} := \text{Gal}(\mathbb{C}/\mathbb{R})$ by complex conjugation, making $\mu_m(\mathbb{C})$ into a $G_{\mathbb{R}}$ -module.

Let us fix some (standard) conventions for G -modules. We write the action of G on the left. If A and B are G -modules, then we denote by $A \otimes B$

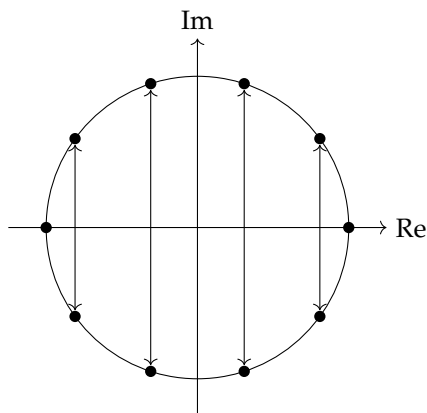
the tensor product of A and B over \mathbb{Z} together with the action of G defined by

$$g(a \otimes b) := g \cdot a \otimes g \cdot b.$$

This tensor product in the category of G -modules is left adjoint to the internal Hom, which we denote by $\underline{\text{Hom}}(A, B)$. The action of G on the latter is given by

$$(gf)(a) := g \cdot f(g^{-1} \cdot a)$$

for a group homomorphism $f: A \rightarrow B$.



The action of $G_{\mathbb{R}}$ on $\mu_{10}(\mathbb{C})$.

As an abelian group, $\mu_m(\mathbb{C})$ is non-canonically isomorphic to $\mathbb{Z}/m\mathbb{Z}$. Similarly, the group of all roots of unity $\text{colim}_m \mu_m(\mathbb{C}) = \bigoplus_p \varinjlim_r \mu_{p^r}(\mathbb{C})$ is isomorphic to $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$. Now we are going to write down such isomorphisms in a canonical way, without forgetting about the action of $G_{\mathbb{R}}$. Further, we introduce a twist by n . In the setting of this text, n is a negative integer, but for the sake of completeness, let us do that for any integer n .

0.5.1. Definition (Tate twists). Let $n \in \mathbb{Z}$.

- If $n = 0$, then

$$\mu_m(\mathbb{C})^{\otimes 0} := \mathbb{Z}/m\mathbb{Z},$$

where $\mathbb{Z}/m\mathbb{Z}$ is taken with the trivial action of $G_{\mathbb{R}}$.

- If $n > 0$, then

$$\mu_m(\mathbb{C})^{\otimes n} := \underbrace{\mu_m(\mathbb{C}) \otimes \cdots \otimes \mu_m(\mathbb{C})}_n$$

is the n -th tensor power of $\mu_m(\mathbb{C})$ with the canonical action of $G_{\mathbb{R}}$.

- If $n < 0$, then

$$\mu_m(\mathbb{C})^{\otimes n} := \underline{\text{Hom}}(\mu_m(\mathbb{C})^{\otimes(-n)}, \mathbb{Z}/m\mathbb{Z}),$$

where in this case the action of $G_{\mathbb{R}}$ is given by

$$\bar{f}(z) := f(\bar{z}).$$

0.5.2. Lemma. *There is a canonical isomorphism of $G_{\mathbb{R}}$ -modules*

$$\begin{aligned} \mu_m(\mathbb{C}) &\xrightarrow{\cong} \frac{2\pi i \mathbb{Z}}{m(2\pi i) \mathbb{Z}}, \\ e^{2\pi i k/m} &\mapsto 2\pi i k. \end{aligned}$$

Proof. The given explicit map is pretty self-explanatory, but the reader might appreciate the fact that this comes from the snake lemma. Let us consider the following morphism of short exact sequences of $G_{\mathbb{R}}$ -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 2\pi i \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{z \mapsto e^z} & \mathbb{C}^\times & \longrightarrow & 1 \\ & & \downarrow -\times m & & \downarrow -\times m & & \downarrow (-)^m & & \\ 0 & \longrightarrow & 2\pi i \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{z \mapsto e^z} & \mathbb{C}^\times & \longrightarrow & 1 \end{array}$$

Note that all the involved arrows are $G_{\mathbb{R}}$ -equivariant. The map in the middle has trivial kernel and cokernel, so by the snake lemma, there is a canonical isomorphism between the kernel of the last map, which is $\mu_m(\mathbb{C})$, and the cokernel of the first map, which is $\frac{2\pi i \mathbb{Z}}{m(2\pi i) \mathbb{Z}}$:

$$\mu_m(\mathbb{C}) \xrightarrow{\cong} \frac{2\pi i \mathbb{Z}}{m(2\pi i) \mathbb{Z}}.$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mu_m(\mathbb{C}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 2\pi i \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{z \mapsto e^z} & \mathbb{C}^\times & \longrightarrow & 1 \\ & & \downarrow -\times m & & \downarrow -\times m & & \downarrow (-)^m & & \\ 0 & \longrightarrow & 2\pi i \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{z \mapsto e^z} & \mathbb{C}^\times & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \frac{2\pi i \mathbb{Z}}{m(2\pi i) \mathbb{Z}} & \longrightarrow & 0 & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$



0.5.3. Lemma. For $n > 0$ we have a canonical isomorphism of $G_{\mathbb{R}}$ -modules

$$\mu_m(\mathbb{C})^{\otimes n} \cong \frac{(2\pi i)^n}{m(2\pi i)^n \mathbb{Z}}.$$

Proof. From the previous calculation and the canonical $G_{\mathbb{R}}$ -equivariant isomorphism

$$\underbrace{(2\pi i) \mathbb{Z} \otimes \cdots \otimes (2\pi i) \mathbb{Z}}_n \xrightarrow{\cong} (2\pi i)^n \mathbb{Z},$$

$$(2\pi i) a_1 \otimes \cdots \otimes (2\pi i) a_n \mapsto (2\pi i)^n a_1 \cdots a_n$$

we obtain

$$\underbrace{\mu_m(\mathbb{C}) \otimes \cdots \otimes \mu_m(\mathbb{C})}_n \xrightarrow{\cong} \underbrace{\frac{2\pi i \mathbb{Z}}{m(2\pi i) \mathbb{Z}} \otimes \cdots \otimes \frac{2\pi i \mathbb{Z}}{m(2\pi i) \mathbb{Z}}}_n \cong \frac{(2\pi i)^n \mathbb{Z}}{m(2\pi i)^n \mathbb{Z}}.$$

■

0.5.4. Lemma. For $n < 0$ we have a canonical isomorphism of $G_{\mathbb{R}}$ -modules

$$\mu_m(\mathbb{C})^{\otimes n} := \underline{\text{Hom}}(\mu_m^{\otimes(-n)}(\mathbb{C}), \mathbb{Z}/m\mathbb{Z}) \cong \frac{(2\pi i)^n \mathbb{Z}}{m(2\pi i)^n \mathbb{Z}}.$$

Proof. We claim that there is a $G_{\mathbb{R}}$ -equivariant isomorphism

(0.5.1)

$$\underline{\text{Hom}}\left(\frac{(2\pi i)^{-n} \mathbb{Z}}{m(2\pi i)^{-n} \mathbb{Z}}, \mathbb{Z}/m\mathbb{Z}\right) \cong \underline{\text{Hom}}((2\pi i)^{-n} \mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\cong} \frac{(2\pi i)^n \mathbb{Z}}{m(2\pi i)^n \mathbb{Z}}.$$

Note that $-n$ got replaced with n , for the reason which will be apparent in a second. A homomorphism $f: (2\pi i)^{-n} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is determined by the image of a generator $f((2\pi i)^{-n} \cdot 1)$, so we may define the second isomorphism in (0.5.1) by

$$(0.5.2) \quad \Phi: f \mapsto (2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1).$$

It is clearly an isomorphism of abelian groups, and it only remains to check that it is $G_{\mathbb{R}}$ -equivariant, i.e. that for all $f: (2\pi i)^{-n} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ holds

$$\Phi(\bar{f}) = \overline{\Phi(f)}.$$

We have indeed

$$\begin{aligned} \Phi(\bar{f}) &= (2\pi i)^n \cdot \bar{f}((2\pi i)^{-n} \cdot 1) = (2\pi i)^n \cdot \overline{f((2\pi i)^{-n} \cdot 1)} \\ &= (-1)^n (2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1) \end{aligned}$$

and

$$\overline{\Phi(f)} = \overline{(2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1)} = (-1)^n (2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1).$$

■

0.5.5. Lemma. *The $G_{\mathbb{R}}$ -module of all roots of unity twisted by n is canonically isomorphic to the $G_{\mathbb{R}}$ -module $\frac{(2\pi i)^n \mathbb{Q}}{(2\pi i)^n \mathbb{Z}}$.*

$$\operatorname{colim}_m \mu_m(\mathbb{C})^{\otimes n} := \bigoplus_p \lim_{\substack{\longrightarrow \\ r}} \mu_{p^r}(\mathbb{C})^{\otimes n} \cong \frac{(2\pi i)^n \mathbb{Q}}{(2\pi i)^n \mathbb{Z}}.$$

Proof. Using the previous calculations and observing that the transition morphisms in the colimit are $G_{\mathbb{R}}$ -equivariant,

$$\bigoplus_p \lim_{\substack{\longrightarrow \\ r}} \mu_{p^r}(\mathbb{C})^{\otimes n} \cong \bigoplus_p \lim_{\substack{\longrightarrow \\ r}} \frac{(2\pi i)^n \mathbb{Z}}{p^r (2\pi i)^n \mathbb{Z}} \cong \frac{(2\pi i)^n \mathbb{Q}}{(2\pi i)^n \mathbb{Z}}.$$

■

Somewhat abusively, from now on we will write simply “ $(2\pi i)^n \mathbb{Q}/\mathbb{Z}$ ” for $\frac{(2\pi i)^n \mathbb{Q}}{(2\pi i)^n \mathbb{Z}}$.

0.6 G -equivariant sheaves

G -equivariant sheaves on topological spaces are discussed in Grothendieck’s Tohoku paper [Tôhoku]:

Nous appellerons G -faisceau sur $X = X(G)$ un faisceau (d’ensembles) A sur X , dans lequel G opère de façon compatible avec ses opérations sur X . Pour donner un sens à cette définition, on pourra par exemple considérer A comme espace étalé dans X ; nous n’insisterons pas.

In this section I will give some explanation of the notion of a G -equivariant sheaf and collect certain relevant results. What follows is a rather straightforward generalization of the usual sheaf theory, so I omit some details. Probably the best way to motivate the definition is to recall the construction of the sheaf of sections of a continuous map.

0.6.1. Classical example. Let X be a topological space. Consider the category $\mathbf{Top}/_X$ of spaces over X where the objects are continuous maps of topological

spaces $p: E \rightarrow X$ and the morphisms are commutative diagrams

$$(0.6.1) \quad \begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \swarrow p' \\ & X & \end{array}$$

For a topological space over X given by $p: E \rightarrow X$, the corresponding **sheaf of sections** is the sheaf of sets defined by

$$\mathcal{F}(U) := \text{Hom}_{\mathbf{Top}/X}(U, E) = \left\{ \begin{array}{ccc} U & \xrightarrow{s} & E \\ & \searrow & \swarrow p \\ & X & \end{array} \right\}$$

for each open subset $U \subset X$. The restriction maps are obvious: an inclusion of open subsets $i: V \hookrightarrow U$ induces contravariantly

$$\text{res}_{VU} := \text{Hom}_{\mathbf{Top}/X}(i, E): \mathcal{F}(U) \rightarrow \mathcal{F}(V),$$

and the sheaf axiom is also easy to verify. A morphism over X of the form (0.6.1) gives rise to a morphism of the corresponding sheaves of sections: for each open subset $U \subset X$ we get a map

$$\begin{aligned} \phi_U: \text{Hom}_{\mathbf{Top}/X}(U, E) &\rightarrow \text{Hom}_{\mathbf{Top}/X}(U, E'), \\ \begin{array}{ccc} U & \xrightarrow{s} & E \\ & \searrow & \swarrow p \\ & X & \end{array} &\mapsto \begin{array}{ccc} U & \xrightarrow{s} & E \xrightarrow{f} E' \\ & \searrow & \downarrow p \swarrow p' \\ & X & \end{array} \end{aligned}$$

and for each $V \subset U$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) := \text{Hom}_{\mathbf{Top}/X}(U, E) & \xrightarrow{\phi_U} & \text{Hom}_{\mathbf{Top}/X}(U, E') =: \mathcal{F}'(U) \\ \text{res}_{VU} \downarrow & & \downarrow \text{res}'_{VU} \\ \mathcal{F}(V) := \text{Hom}_{\mathbf{Top}/X}(V, E) & \xrightarrow{\phi_V} & \text{Hom}_{\mathbf{Top}/X}(V, E') =: \mathcal{F}'(V) \end{array}$$

clearly commutes. So formation of the sheaf of sections is a functor

$$\Gamma: \mathbf{Top}/X \rightarrow \mathbf{Sh}(X).$$

0.6.2. G -equivariant example. For a discrete group G , consider the category of G -spaces $G\text{-Top}$ where the objects are topological spaces X with a specified action of G by homeomorphisms $\sigma_X: G \times X \rightarrow X$, and morphisms $f: X \rightarrow Y$ are continuous G -equivariant maps:

$$\begin{array}{ccc} G \times X & \xrightarrow{\text{id} \times f} & G \times Y \\ \sigma_X \downarrow & & \downarrow \sigma_Y \\ X & \xrightarrow{f} & Y \end{array}$$

For a fixed G -space X , the category $G\text{-Top}/X$ of G -spaces over X has as its objects continuous G -equivariant maps $p: E \rightarrow X$ and as morphisms continuous G -equivariant maps over X

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \swarrow p' \\ & X & \end{array}$$

For a G -space over X given by $p: E \rightarrow X$, the corresponding sheaf of sections \mathcal{F} carries the following extra datum. For each open subset $U \subset X$ and each $g \in G$ there is a bijection of sets

$$\begin{aligned} \alpha_{g,U}: \mathcal{F}(U) &\xrightarrow{\cong} \mathcal{F}(g \cdot U), \\ (s: U \rightarrow E) &\mapsto \left(\begin{array}{l} g \cdot U \rightarrow E, \\ g \cdot u \mapsto g \cdot s(u) \end{array} \right), \\ \left(\begin{array}{l} U \rightarrow E, \\ u \mapsto g^{-1} \cdot s(g \cdot u) \end{array} \right) &\leftarrow (s: g \cdot U \rightarrow E). \end{aligned}$$

Using the fact that p is G -equivariant, one checks that $\alpha_{g,U}$ indeed sends sections over U to sections over $g \cdot U$. We also see that the bijections $\alpha_{g,U}$ satisfy the following properties:

- 1) compatibility with restrictions: for open subsets $V \subset U$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow[\cong]{\alpha_{g,U}} & \mathcal{F}(g \cdot U) \\ \text{res}_{V,U} \downarrow & & \downarrow \text{res}_{g \cdot V, g \cdot U} \\ \mathcal{F}(V) & \xrightarrow[\cong]{\alpha_{g,V}} & \mathcal{F}(g \cdot V) \end{array}$$

commutes;

- 2) for the identity element $1 \in G$ and each open subset $U \subset X$ we have

$$\alpha_{1,U} = \text{id}: \mathcal{F}(U) \rightarrow \mathcal{F}(U);$$

- 3) the cocycle condition: for each open subset $U \subset X$ and $g, h \in G$ the diagram

$$\begin{array}{ccc} & \mathcal{F}(h \cdot U) & \\ \alpha_{h,U} \nearrow & & \searrow \alpha_{g,h \cdot U} \\ \mathcal{F}(U) & \xrightarrow[\alpha_{gh,U}]{} & \mathcal{F}(gh \cdot U) \end{array}$$

commutes.

For a morphism of G -spaces over X

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \swarrow p' \\ & X & \end{array}$$

the corresponding morphism of sheaves of sections $\phi: \mathcal{F} \rightarrow \mathcal{F}'$ is easily seen to be compatible with the maps $\alpha_{g,U}$ and $\alpha'_{g,U}$: for each $U \subset X$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{F}'(U) \\ \alpha_{g,U} \downarrow & & \downarrow \alpha'_{g,U} \\ \mathcal{F}(g \cdot U) & \xrightarrow{\phi_{g \cdot U}} & \mathcal{F}'(g \cdot U) \end{array}$$

commutes.

Now hopefully, the last example makes the following definition look natural.

0.6.3. Definition. Let G be a discrete group and let X be a G -space. Then a **G -equivariant presheaf** (of sets) on X is a presheaf \mathcal{F} equipped with bijections of sets

$$\alpha_{g,U}: \mathcal{F}(U) \xrightarrow{\cong} \mathcal{F}(g \cdot U)$$

for each $g \in G$ and open subset $U \subset X$ that satisfy the following axioms:

- 1) these bijections are compatible with restrictions:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow[\cong]{\alpha_{g,U}} & \mathcal{F}(g \cdot U) \\ \text{res}_{V,U} \downarrow & & \downarrow \text{res}_{g \cdot V, g \cdot U} \\ \mathcal{F}(V) & \xrightarrow[\cong]{\alpha_{g,V}} & \mathcal{F}(g \cdot V) \end{array}$$

- 2) $\alpha_{1,U} = \text{id}: \mathcal{F}(U) \rightarrow \mathcal{F}(U)$;
- 3) for $g, h \in G$ the cocycle condition holds:

$$\begin{array}{ccc} & \mathcal{F}(h \cdot U) & \\ \alpha_{h,U} \nearrow & & \searrow \alpha_{g,h \cdot U} \\ \mathcal{F}(U) & \xrightarrow[\cong]{\alpha_{gh,U}} & \mathcal{F}(gh \cdot U) \end{array}$$

A **G -equivariant sheaf** is a G -equivariant presheaf satisfying the usual sheaf axiom: for each open covering $U = \bigcup_i U_i$ we have an equalizer

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

A **morphism of G -equivariant (pre)sheaves** $\mathcal{F} \rightarrow \mathcal{F}'$ is a morphism of (pre)sheaves which is compatible with the maps $\alpha_{g,U}$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{F}'(U) \\ \alpha_{g,U} \downarrow & & \downarrow \alpha'_{g,U} \\ \mathcal{F}(g \cdot U) & \xrightarrow{\phi_{g \cdot U}} & \mathcal{F}'(g \cdot U) \end{array}$$

We denote the category of G -equivariant presheaves (resp. sheaves) on X by $\mathbf{PSh}(G, X)$ (resp. $\mathbf{Sh}(G, X)$).

We may summarize 0.6.2 by saying that taking the sheaf of sections is a functor

$$\Gamma: G\text{-Top}_{/X} \rightarrow \mathbf{Sh}(G, X).$$

It commutes with the forgetful functors:

$$\begin{array}{ccc} G\text{-Top}_{/X} & \xrightarrow{\Gamma} & \mathbf{Sh}(G, X) \\ \downarrow & & \downarrow \\ \mathbf{Top}_{/X} & \xrightarrow{\Gamma} & \mathbf{Sh}(X) \end{array}$$

0.6.4. Remark. Despite the extra datum coming from the action of G , the category $\mathbf{Sh}(G, X)$ is still a Grothendieck topos. This can be deduced from Giraud's characterization of Grothendieck toposes [SGA 4, Exposé IV, 1.2] (see e.g. [MLM1994, Appendix] for details). However, the underlying Grothendieck site is not obvious.

0.6.5. Observation. The global sections $\mathcal{F}(X)$ of a G -equivariant (pre)sheaf is a G -set with the action of G given by

$$\alpha_{g,X}: \mathcal{F}(X) \xrightarrow{\cong} \mathcal{F}(g \cdot X) = \mathcal{F}(X).$$

Taking the global sections is a functor

$$\mathbf{PSh}(G, X) \rightarrow G\text{-Set}.$$

Proof. The axioms $\alpha_{1,X} = \text{id}$ and $\alpha_{gh,X} = \alpha_{g,h \cdot X} \circ \alpha_{h,X}$ correspond to the axioms of a group action. ■

0.6.6. Example. Let \mathcal{F} be the sheaf of sections of a G -space over X given by $p: E \rightarrow X$. Then the action of $g \in G$ on $\mathcal{F}(X)$ sends a global section $s: X \rightarrow E$ to the global section

$$\begin{aligned} X &\rightarrow E, \\ x &\mapsto g \cdot s(g^{-1} \cdot x). \end{aligned}$$

(see the formula for $\alpha_{g,U}$ in 0.6.2). ▲

0.6.7. Definition. Let S be a G -set. For a G -space X , consider the presheaf S_X defined by $S_X(U) = S$ for each open subset $U \subset X$ with the identity restriction maps. The morphisms

$$\begin{aligned} \alpha_{g,U} = \sigma_g: S_X(U) &\rightarrow S_X(g \cdot U), \\ x &\mapsto g \cdot x. \end{aligned}$$

define a structure of a G -equivariant presheaf on S_X , called the **constant G -equivariant presheaf** associated to S .

0.6.8. Observation. Formation of the constant G -equivariant presheaf is a functor

$$G\text{-Set} \rightarrow \mathbf{PSh}(G, X),$$

which is left adjoint to the global section functor:

$$\text{Hom}_{\mathbf{PSh}(G, X)}(S_X, \mathcal{P}) \cong \text{Hom}_{G\text{-Set}}(S, \mathcal{P}(X)).$$

Proof. A morphism of G -equivariant presheaves $S_X \rightarrow \mathcal{P}$ is given by a collection of maps $\phi_U: S \rightarrow \mathcal{P}(U)$ that are compatible with the restriction maps and the G -equivariant structure morphisms:

$$\begin{array}{ccc} S & \xrightarrow{\phi_X} & \mathcal{P}(X) \\ & \searrow \phi_U & \downarrow \text{res}_{UX} \\ & & \mathcal{P}(U) \\ & \searrow \phi_V & \downarrow \text{res}_{VU} \\ & & \mathcal{P}(V) \end{array} \quad \begin{array}{ccc} S & \xrightarrow{\phi_U} & \mathcal{P}(U) \\ \sigma_g \downarrow & & \downarrow \alpha_{g,U} \\ S & \xrightarrow{\phi_{g \cdot U}} & \mathcal{P}(g \cdot U) \end{array}$$

res_{VX}

From the first diagram we see that $\phi_U = \text{res}_{UX} \circ \phi_X$, so that the map $\phi_X: S \rightarrow \mathcal{P}(X)$ defines the rest, and from the second diagram we see that it

is G -equivariant. This shows that the bijection in question is given by

$$\begin{aligned} \{\phi_U\} &\mapsto \phi_X, \\ \{\phi_U := \text{res}_{UX} \circ \phi\} &\leftarrow \phi. \end{aligned}$$

■

Alternative definition via G -equivariant espaces étalés

One says that a continuous map $p: E \rightarrow X$ is **étale**^{*} if it is a local on the source homeomorphism (for each $e \in E$ there exists an open neighborhood $V \ni p$ such that $p(V)$ is open in X and $p|_V: V \rightarrow p(V)$ is a homeomorphism). We have a full subcategory

$$G\text{-}\acute{\text{E}}t/X \subset G\text{-}\text{Top}/X$$

formed by G -spaces that are étale over X . We note that if p and p' are étale and we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \swarrow p' \\ & X & \end{array}$$

then f is étale as well, so that the morphisms in $G\text{-}\acute{\text{E}}t/X$ are automatically étale. The importance of étale spaces over X is explained by the following well-known result, which we state G -equivariantly.

0.6.9. Proposition. *Let \mathcal{F} be a G -equivariant presheaf on X . Consider the disjoint union of stalks*

$$\coprod_{x \in X} \mathcal{F}_x, \quad \mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U).$$

It carries a natural action of G . For each section $s \in \mathcal{F}(U)$ such that $U \ni x$, denote by $s_x \in \mathcal{F}_x$ the corresponding germ at x . This defines a map between sets (which we again denote by s)

$$\begin{aligned} s: U &\rightarrow \coprod_{x \in X} \mathcal{F}_x, \\ x &\mapsto s_x. \end{aligned}$$

Consider now the topology on $\coprod_{x \in X} \mathcal{F}_x$ generated by $s(U)$ for all open subsets $U \subset X$ and all $s \in \mathcal{F}(U)$. Then the action of G is continuous with respect to this

^{*}This is in fact the topological counterpart of étale morphisms of schemes.

topology, and the natural projection

$$p: \coprod_{x \in X} \mathcal{F}_x \rightarrow X,$$

$$\mathcal{F}_x \ni s_x \mapsto x.$$

is an étale G -equivariant map.

Proof. This is a well-known, basic result (see e.g. [MLM1994, Chapter II]); one just has to check the G -equivariance. ■

This leads to an equivalent definition of G -equivariant sheaves.

0.6.10. Alternative definition. Let G be a group and X be a G -space. Then a **G -equivariant sheaf** on X is an étale G -space over X

$$p: E \rightarrow X,$$

and a morphism of G -equivariant sheaves is a morphism over X

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p \searrow & & \swarrow p' \\ & X & \end{array}$$

0.6.11. Remark. Note that the above definition looks more natural than 0.6.3. It also generalizes to the case a topological group G acting on E and X continuously. This is not possible in 0.6.3, because there we consider only how each separate element $g \in G$ acts on X .

0.6.12. Example. In these terms, it is easier to describe equivariant sheafification and what a constant sheaf is. If S is a G -set and X is a G -space, we may endow S with the discrete topology and consider the G -space $S \times X$ with the component-wise action of G (which is the product in the category of G -spaces). Then the projection $S \times X \rightarrow X$ is an étale G -equivariant map, so it corresponds to some G -equivariant sheaf. We call it the **constant G -equivariant sheaf** associated to S . This construction is obviously functorial: a G -equivariant map $S \rightarrow S'$ induces a morphism in $G\text{-}\mathring{\text{Et}}/X$

$$\begin{array}{ccc} S \times X & \longrightarrow & S' \times X \\ & \searrow & \swarrow \\ & X & \end{array}$$

▲

Abelian G -equivariant sheaves and their cohomology

0.6.13. Proposition. *Let X be a G -space. Consider the category $\mathbf{Sh}(G, X)^{\mathbf{Ab}}$ of G -equivariant sheaves of abelian groups on X (defined, for instance, as abelian group objects in the category of G -equivariant sheaves of sets). It is an abelian category with enough injectives.*

Proof. The usual argument of Grothendieck works: any abelian category which satisfies the axiom AB5) and has generators has enough injectives [Tôhoku, Ch. I, 1.10]. This is the case for $\mathbf{Sh}(G, X)^{\mathbf{Ab}}$ (as for the generators, see [MLM1994, Appendix]). ■

0.6.14. Example. Let A be a G -set (resp. G -module). Then the associated constant sheaf \underline{A} has a canonical G -equivariant abelian sheaf structure. ▲

0.6.15. Example. Consider some topological space with an action of the Galois group $G_{\mathbb{R}} := \text{Gal}(\mathbb{C}/\mathbb{R})$; for instance, the set of complex points of a scheme $X(\mathbb{C})$ equipped with the analytic topology. Then the complex m -th roots of unity $\mu_m(\mathbb{C})$ (reviewed above in §0.5) give us a constant $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$. This is the only example we will be interested in. ▲

0.6.16. Definition. The equivariant global section functor

$$\begin{aligned} \Gamma(G, X, -) : \mathbf{Sh}(G, X)^{\mathbf{Ab}} &\rightarrow \mathbf{Ab}, \\ \mathcal{F} &\rightsquigarrow \mathcal{F}(X)^G \end{aligned}$$

is left exact. Here the global sections

$$\mathcal{F}(X) := \{s : X \rightarrow \text{Ét}(\mathcal{F}) \mid \pi \circ s = \text{id}_X\}$$

come with an action of G by

$$(g \cdot s)(x) := g \cdot s(g^{-1} \cdot x).$$

(Note that in general, $\mathcal{F}(U)$ carries such an action of G , whenever $U \subset X$ is closed under the action of G .) The fixed points of this action are precisely the G -equivariant sections, i.e. sections that satisfy $s(g \cdot x) = g \cdot (s(x))$. The right derived functors of $\Gamma(G, X, -)$ are by definition $R\Gamma(G, X, \mathcal{F})$.

This is related to the usual sheaf cohomology by

$$(0.6.2) \quad R\Gamma(G, X, \mathcal{F}) \cong R\Gamma(G, R\Gamma(X, \mathcal{F})),$$

where the right hand side is the group cohomology. Indeed, $\Gamma(G, X, -)$ is a composition of two left exact functors: the usual global section functor and the fixed points functor

$$\mathbf{Sh}(G, X)^{\mathbf{Ab}} \xrightarrow{\mathcal{F} \rightsquigarrow \mathcal{F}(X)} \mathbf{G}\text{-Mod} \xrightarrow{A \rightsquigarrow A^G} \mathbf{Ab}$$

and (0.6.2) are the derived functors of a composition of functors (this is known as the **Grothendieck spectral sequence**; see e.g. [Wei1994, §10.8]). On the level of cohomology, we have a spectral sequence

$$E_2^{pq} = H^p(G, H^q(X, \mathcal{F})) \implies H^{p+q}(G, X, \mathcal{F}).$$

0.7 From étale to analytic sheaves (the morphism α^*)

The canonical reference for comparison between étale and singular cohomology is [SGA 4, Exposé XI, §4], so let us to borrow some definitions and notation from there. Let X be an arithmetic scheme (separated, of finite type over $\text{Spec } \mathbb{Z}$).

1. The base change from $\text{Spec } \mathbb{Z}$ to $\text{Spec } \mathbb{C}$

$$\begin{array}{ccc} X_{\mathbb{C}} & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ \text{Spec } \mathbb{C} & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

gives us a morphism of sites

$$\gamma: X_{\mathbb{C}, \text{ét}} \rightarrow X_{\text{ét}}.$$

2. We denote by $X(\mathbb{C})$ the set of complex points of X equipped with the usual analytic topology.

Let X_{cl} be the site of étale maps $f: U \rightarrow X(\mathbb{C})$. A covering family in X_{cl} is a family of maps $\{U_i \rightarrow U\}$ such that U is the union of images of U_i . The notation “cl” comes from SGA 4 and stays for “classique”.

As the inclusion of an open subset $U \subset X(\mathbb{C})$ is trivially an étale map, we have a fully faithful functor $X(\mathbb{C}) \subset X_{cl}$, and the topology on $X(\mathbb{C})$ is induced by the topology on X_{cl} . This gives us a morphism of sites

$$\delta: X_{cl} \rightarrow X(\mathbb{C}),$$

which by the well-known “comparison lemma” [SGA 4, Exposé III, Théorème 4.1] induces an equivalence of the corresponding categories of sheaves

$$\delta_*: \mathbf{Sh}(X_{cl}) \rightarrow \mathbf{Sh}(X(\mathbb{C})).$$

3. A morphism of schemes $f: X'_\mathbb{C} \rightarrow X_\mathbb{C}$ over $\text{Spec } \mathbb{C}$ is étale if and only if $f(\mathbb{C}): X'(\mathbb{C}) \rightarrow X(\mathbb{C})$ is étale in the topological sense [SGA 1, Exposé XII, Proposition 3.1], and therefore the functor $X'_\mathbb{C} \rightsquigarrow X'(\mathbb{C})$ gives us a morphism of sites

$$\epsilon: X_{cl} \rightarrow X_{\mathbb{C}, \text{ét}}.$$

We may now consider the composite functor

$$\mathbf{Sh}(X_{\text{ét}}) \xrightarrow{\gamma^*} \mathbf{Sh}(X_{\mathbb{C}, \text{ét}}) \xrightarrow{\epsilon^*} \mathbf{Sh}(X_{cl}) \xrightarrow[\simeq]{\delta_*} \mathbf{Sh}(X(\mathbb{C}))$$

where γ^* is given by the base change from $\text{Spec } \mathbb{Z}$ to $\text{Spec } \mathbb{C}$, the functor ϵ^* is the comparison, and δ_* is an equivalence of categories. As we start from a scheme over $\text{Spec } \mathbb{Z}$ and base change to $\text{Spec } \mathbb{C}$, the resulting sheaf on $X(\mathbb{C})$ is in fact equivariant with respect to the complex conjugation, and the above composition gives us an “inverse image” functor

$$\alpha^*: \mathbf{Sh}(X_{\text{ét}}) \rightarrow \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C})).$$

0.8 Cohomology with compact support on $X_{\text{ét}}$ and $X(\mathbb{C})$

For any arithmetic scheme $f: X \rightarrow \text{Spec } \mathbb{Z}$ (separated, of finite type) there exists a **Nagata compactification** $f = g \circ j$ where j is an open immersion and g is a proper morphism:

$$\begin{array}{ccc} X & \xrightarrow{j} & \mathfrak{X} \\ & \searrow f & \swarrow g \\ & \text{Spec } \mathbb{Z} & \end{array}$$

This is a result of Nagata, and a modern exposition (following Deligne) may be found in [Con2007, Con2009]. See also [SGA 4, Exposé XVII].

0.8.1. Definition. Let X be an arithmetic scheme and let \mathcal{F}^\bullet be a complex of abelian torsion sheaves on $X_{\text{ét}}$. Then we define the **cohomology of \mathcal{F}^\bullet with compact support** via the complex

$$(0.8.1) \quad R\Gamma_c(X_{\text{ét}}, \mathcal{F}^\bullet) := R\Gamma(\mathfrak{X}_{\text{ét}}, j_! \mathcal{F}^\bullet).$$

For torsion sheaves, this does not depend on the choice of $j: X \hookrightarrow \mathfrak{X}$, but here we would like to fix this choice to be able to compare j with the corresponding morphism $j(\mathbb{C}): X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. Note that thanks to the **Leray spectral sequence** $R\Gamma(\mathfrak{X}_{\text{ét}}, -) \cong R\Gamma(\text{Spec } \mathbb{Z}_{\text{ét}}, -) \circ Rg_*$ (that is, the

Grothendieck spectral sequence coming from $\Gamma(\mathfrak{X}_{\acute{e}t}, -) = \Gamma(\text{Spec } \mathbb{Z}_{\acute{e}t}, -) \circ g_*$, we have

$$(0.8.2) \quad R\Gamma_c(X_{\acute{e}t}, \mathcal{F}^\bullet) \cong R\Gamma(\text{Spec } \mathbb{Z}_{\acute{e}t}, Rf_! \mathcal{F}^\bullet),$$

where by definition

$$Rf_! \mathcal{F}^\bullet := Rg_* j_! \mathcal{F}^\bullet$$

(this is just a piece of notation, standard and quite unfortunate; “ $Rf_!$ ” does not mean that we are deriving $f_!$).

The formulas (0.8.1) and (0.8.2) give two equivalent definitions. We are going to use (0.8.2) in the next section to introduce a slightly different version of cohomology with compact support, denoted by $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^\bullet)$, which is needed for arithmetic duality theorems. In this section, we need to use (0.8.1) to define cohomology with compact support on $X(\mathbb{C})$, in a way that allows us to compare it with cohomology with compact support on $X_{\acute{e}t}$.

0.8.2. Definition. If $j: X \hookrightarrow \mathfrak{X}$ is a Nagata compactification, then we have the corresponding open immersion

$$j(\mathbb{C}): X(\mathbb{C}) \rightarrow \mathfrak{X}(\mathbb{C}),$$

and for a sheaf \mathcal{F} on $X(\mathbb{C})$ we define

$$\Gamma_c(X(\mathbb{C}), \mathcal{F}) := \Gamma(\mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \mathcal{F}).$$

Similarly, for a $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$ we define

$$\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathcal{F}) := \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \mathcal{F}).$$

0.8.3. Proposition. *Let \mathcal{F} be a sheaf on $X_{\acute{e}t}$.*

1) *There exists a morphism*

$$\Gamma(X_{\acute{e}t}, \mathcal{F}) \rightarrow \Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}),$$

which is natural in the sense that every morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ gives a commutative diagram

$$\begin{array}{ccc} \Gamma(X_{\acute{e}t}, \mathcal{F}) & \longrightarrow & \Gamma(X_{\acute{e}t}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}) & \longrightarrow & \Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{G}) \end{array}$$

2) *Similarly for cohomology with compact support, there is a natural morphism*

$$\Gamma_c(X_{\acute{e}t}, \mathcal{F}) \rightarrow \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}).$$

The same holds for abelian sheaves on $X_{\acute{e}t}$.

Proof. This is standard and follows from the functoriality of α^* , but it is easier to recall the construction than to find the relevant point in SGA 4. The morphism in 1) is given by

$$\begin{aligned} \Gamma(X_{\acute{e}t}, \mathcal{F}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Sh}(X_{\acute{e}t})}(\underline{\{\ast\}}, \mathcal{F}) \rightarrow \mathrm{Hom}_{\mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))}(\alpha^* \underline{\{\ast\}}, \alpha^* \mathcal{F}) \\ &\xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))}(\underline{\{\ast\}}, \alpha^* \mathcal{F}) \xrightarrow{\cong} \Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}). \end{aligned}$$

For abelian sheaves, in the above formula one has to replace the constant sheaf $\underline{\{\ast\}}$ with $\underline{\mathbb{Z}}$. The naturality is easily seen from the above definition.

In 2), if $j: X \hookrightarrow \mathfrak{X}$ is Nagata compactification, then we have the corresponding compactification $j(\mathbb{C}): X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. The extension by zero morphism $j(\mathbb{C})_!: \mathbf{Sh}(X(\mathbb{C})) \rightarrow \mathbf{Sh}(\mathfrak{X}(\mathbb{C}))$ restricts to the subcategory of $G_{\mathbb{R}}$ -equivariant sheaves: if \mathcal{F} is a $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$, then $j(\mathbb{C})_! \mathcal{F}$ is a $G_{\mathbb{R}}$ -equivariant sheaf on $\mathfrak{X}(\mathbb{C})$ (this is evident from the definition of equivariant sheaves as equivariant spaces étalés). It should be clear from the definition of α^* that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{Sh}(X_{\acute{e}t}) & \xrightarrow{\alpha^*} & \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C})) \\ j_! \downarrow & & \downarrow j(\mathbb{C})_! \\ \mathbf{Sh}(\mathfrak{X}_{\acute{e}t}) & \xrightarrow{\alpha_{\mathfrak{X}}^*} & \mathbf{Sh}(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C})) \end{array}$$

(For instance, note that this diagram commutes for representable étale sheaves, and then every étale sheaf is a colimit of representable sheaves, and α^* , $j_!$, $\alpha_{\mathfrak{X}}^*$, $j(\mathbb{C})_!$ preserve colimits, as left adjoints.)

Now the morphism in question is now given by

$$\begin{aligned} \Gamma_c(X_{\acute{e}t}, \mathcal{F}) &:= \Gamma(\mathfrak{X}_{\acute{e}t}, j_! \mathcal{F}) \rightarrow \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! \mathcal{F}) \\ &= \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_! \alpha^* \mathcal{F}) =: \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}). \end{aligned}$$

■

Finally, we will need the fact that the morphisms

$$\Gamma_c(X_{\acute{e}t}, \mathcal{F}) \rightarrow \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F})$$

are compatible with the distinguished triangles associated to open-closed decompositions. To check this compatibility, let us recall how such triangles arise. If we have an open subscheme $U \subset X$ and its closed complement $Z := X \setminus U$:

$$Z \xleftarrow{i_Z} X \xleftarrow{j_U} U$$

then there are the following six functors between the corresponding categories of abelian sheaves:

$$\mathbf{Sh}(Z_{\acute{e}t})^{\mathbf{Ab}} \begin{array}{c} \xleftarrow{i_Z^*} \\ \xrightarrow{-i_{Z^*}} \\ \xleftarrow{i_Z^*} \end{array} \mathbf{Sh}(X_{\acute{e}t})^{\mathbf{Ab}} \begin{array}{c} \xleftarrow{j_U!} \\ \xrightarrow{-j_U^*} \\ \xleftarrow{j_{U^*}} \end{array} \mathbf{Sh}(U_{\acute{e}t})^{\mathbf{Ab}}$$

(see e.g. [SGA 4, Exposé 4, §14]). Here each arrow is left adjoint to the arrow depicted below it. For an abelian sheaf \mathcal{F} on $X_{\acute{e}t}$, there is a natural short exact sequence

$$0 \rightarrow j_{U!}j_U^*\mathcal{F} \rightarrow \mathcal{F} \rightarrow i_{Z^*}i_Z^*\mathcal{F} \rightarrow 0$$

(naturality means that the two arrows are counit and unit of the corresponding adjunctions). Now if $j: X \rightarrow \mathfrak{X}$ is a Nagata compactification, then the above short exact sequence gives us a short exact sequence of abelian sheaves on $\mathfrak{X}_{\acute{e}t}$ (the functor $j_!$ is exact):

$$0 \rightarrow j_!j_{U!}j_U^*\mathcal{F} \rightarrow j_!\mathcal{F} \rightarrow j_!i_{Z^*}i_Z^*\mathcal{F} \rightarrow 0$$

and finally, this gives the distinguished triangle

$$R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!j_{U!}j_U^*\mathcal{F}) \rightarrow R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!\mathcal{F}) \rightarrow R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!i_{Z^*}i_Z^*\mathcal{F}) \rightarrow R\Gamma(\mathfrak{X}_{\acute{e}t}, j_!j_{U!}j_U^*\mathcal{F})[1]$$

which we may write as

$$R\Gamma_c(U_{\acute{e}t}, \mathcal{F}|_U) \rightarrow R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) \rightarrow R\Gamma_c(Z_{\acute{e}t}, \mathcal{F}|_Z) \rightarrow R\Gamma_c(U_{\acute{e}t}, \mathcal{F}|_U)[1]$$

For ($G_{\mathbb{R}}$ -equivariant) sheaves on $X(\mathbb{C})$, such triangles arise in the same manner.

0.8.4. Proposition. *For an open-closed decomposition*

$$Z \xleftarrow{i_Z} X \xleftarrow{j_U} U$$

the morphism α^* gives a morphism of distinguished triangles

$$(0.8.3) \quad \begin{array}{ccc} R\Gamma_c(U_{\acute{e}t}, \mathcal{F}|_U) & \longrightarrow & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \alpha^*\mathcal{F}|_{U(\mathbb{C})}) \\ \downarrow & & \downarrow \\ R\Gamma_c(X_{\acute{e}t}, \mathcal{F}) & \longrightarrow & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^*\mathcal{F}) \\ \downarrow & & \downarrow \\ R\Gamma_c(Z_{\acute{e}t}, \mathcal{F}|_Z) & \longrightarrow & R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \alpha^*\mathcal{F}|_{Z(\mathbb{C})}) \\ \downarrow & & \downarrow \\ R\Gamma_c(U_{\acute{e}t}, \mathcal{F}|_U)[1] & \longrightarrow & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \alpha^*\mathcal{F}|_{U(\mathbb{C})})[1] \end{array}$$

Proof. Since α^* is essentially the inverse image functor associated to a continuous morphism of sites, it is exact, and therefore the short exact sequence on $\mathfrak{X}_{\text{ét}}$

$$0 \rightarrow j_! j_U! j_U^* \mathcal{F} \rightarrow j_! \mathcal{F} \rightarrow j_! i_{Z*} i_Z^* \mathcal{F} \rightarrow 0$$

gives a short exact sequence of equivariant sheaves on $\mathfrak{X}(\mathbb{C})$

$$0 \rightarrow \alpha_{\mathfrak{X}}^* j_! j_U! j_U^* \mathcal{F} \rightarrow \alpha_{\mathfrak{X}}^* j_! \mathcal{F} \rightarrow \alpha_{\mathfrak{X}}^* j_! i_{Z*} i_Z^* \mathcal{F} \rightarrow 0$$

This gives us the corresponding morphism of triangles

$$\begin{array}{ccc} R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}|_U) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! j_U! j_U^* \mathcal{F}) \\ \downarrow & & \downarrow \\ R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! \mathcal{F}) \\ \downarrow & & \downarrow \\ R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}|_Z) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! i_{Z*} i_Z^* \mathcal{F}) \\ \downarrow & & \downarrow \\ R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}|_U)[1] & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! j_U! j_U^* \mathcal{F})[1] \end{array}$$

Then it is possible to verify that the right triangle coincides with the one obtained from the short exact sequence of $G_{\mathbb{R}}$ -equivariant sheaves on $X(\mathbb{C})$

$$0 \rightarrow j_U(\mathbb{C})! j_U(\mathbb{C})^* \alpha^* \mathcal{F} \rightarrow \alpha^* \mathcal{F} \rightarrow i_Z(\mathbb{C})_* i_Z(\mathbb{C})^* \alpha^* \mathcal{F} \rightarrow 0$$

by applying $j(\mathbb{C})_! : X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$ and $R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), -)$, i.e. the right column in (0.8.3). \blacksquare

0.9 Étale cohomology with compact support à la Milne

Let us first recall the definition of Tate cohomology (see e.g. [Bro1994, Chapter VI]). Let G be a finite group. Then the trivial $\mathbb{Z}G$ -module \mathbb{Z} admits a resolution by *finitely generated* free $\mathbb{Z}G$ -modules

$$(0.9.1) \quad (P_{\bullet} \twoheadrightarrow \mathbb{Z}): \quad \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

(for instance, the bar-resolution will do). The group cohomology of G with coefficients in a G -module A is the cohomology of the complex of abelian groups

$$R\Gamma(G, A) := \text{Hom}_{\mathbb{Z}G}(P_{\bullet}, A),$$

i.e.,

$$H^i(G, A) = H^i(R\Gamma(G, A)).$$

If we dualize (0.9.1) by applying the functor $(-)^{\vee} := \underline{\mathrm{Hom}}(-, \mathbb{Z}G)$, then P_i^{\vee} are also finitely generated free $\mathbb{Z}G$ -modules, and we obtain a “backwards resolution”, which is an acyclic complex

$$(0.9.2) \quad (\mathbb{Z} \twoheadrightarrow P_{\bullet}^{\vee}): \quad 0 \rightarrow \mathbb{Z} \rightarrow P_0^{\vee} \rightarrow P_1^{\vee} \rightarrow P_2^{\vee} \rightarrow \cdots$$

We may splice together (0.9.1) and (0.9.2) to obtain a so-called **complete resolution** (with homological numbering)

$$\widehat{P}_{\bullet}: \quad \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$$

where $P_i := P_{-i-1}^{\vee}$ for $i < 0$, and the morphism $P_0 \rightarrow P_{-1}$ is given by the composition of $P_0 \twoheadrightarrow \mathbb{Z}$ and $\mathbb{Z} \twoheadrightarrow P_0^{\vee}$. Then the **Tate cohomology** of G with coefficients in a G -module A is given by the cohomology of the complex

$$R\widehat{\Gamma}(G, A) := \mathrm{Hom}_{\mathbb{Z}G}(\widehat{P}_{\bullet}, A);$$

that is,

$$\widehat{H}^i(G, A) := H^i(R\widehat{\Gamma}(G, A)).$$

This corresponds to the usual cohomology in positive degrees $i > 0$ and homology in degrees $i < -1$:

$$\widehat{H}^i(G, A) = \begin{cases} H^i(G, A), & i > 0, \\ H_{-i-1}(G, A), & i < -1, \end{cases}$$

while the groups $\widehat{H}^{-1}(G, A)$ and $\widehat{H}^0(G, A)$ are given by the exact sequence

$$0 \rightarrow \widehat{H}^{-1}(G, A) \rightarrow H_0(G, A) \xrightarrow{\overline{N}} H^0(G, A) \rightarrow \widehat{H}^0(G, A) \rightarrow 0$$

where $\overline{N}: H_0(G, A) \rightarrow H^0(G, A)$ is the norm map induced by $N := \sum_{g \in G} g$.

Slightly more generally, if A^{\bullet} is a bounded below (cohomological) complex of G -modules, we obtain a *double complex* of abelian groups $\mathrm{Hom}^{\bullet\bullet}(P_{\bullet}, A^{\bullet})$ (resp. $\mathrm{Hom}^{\bullet\bullet}(\widehat{P}_{\bullet}, A^{\bullet})$), and it makes sense to define the corresponding **group hypercohomology** (resp. **Tate hypercohomology**) by the complex

$$\begin{aligned} R\Gamma(G, A^{\bullet}) &:= \mathrm{Tot}^{\oplus}(\mathrm{Hom}^{\bullet\bullet}(P_{\bullet}, A^{\bullet})), \\ R\widehat{\Gamma}(G, A^{\bullet}) &:= \mathrm{Tot}^{\oplus}(\mathrm{Hom}^{\bullet\bullet}(\widehat{P}_{\bullet}, A^{\bullet})). \end{aligned}$$

Note that there is an obvious morphism of complexes $\widehat{P}_{\bullet} \rightarrow P_{\bullet}$.

$$\begin{array}{cccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & P_{-1} & \longrightarrow & P_{-2} & \longrightarrow & \cdots \\
& & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

which after applying the contravariant functor $\text{Tot}^{\oplus} \text{Hom}^{\bullet\bullet}(-, A^{\bullet})$ gives a morphism from the usual cohomology to Tate cohomology:

$$(0.9.3) \quad R\Gamma(G, A^{\bullet}) \rightarrow \widehat{R}\Gamma(G, A^{\bullet}).$$

0.9.1. Example. If G is a finite cyclic group of order m generated by an element t , then it admits a periodic free resolution

$$\cdots \rightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

where

$$N := \sum_{g \in G} g = 1 + t + t^2 + \cdots + t^{m-1}$$

is the norm element, and

$$\epsilon: \sum_{g \in G} n_g g \mapsto \sum_{g \in G} n_g$$

is the augmentation morphism. If we dualize the above resolution, we get the acyclic complex

$$0 \rightarrow \mathbb{Z} \xrightarrow{\epsilon^{\vee}} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \rightarrow \cdots$$

It is easily seen that the morphism ϵ^{\vee} is given by $1 \mapsto N$, and the composition $\epsilon^{\vee} \circ \epsilon$ is the action by N on $\mathbb{Z}G$. The corresponding complete resolution is

$$(0.9.4) \quad \widehat{P}_{\bullet}: \cdots \rightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \rightarrow \cdots$$

After applying $\underline{\text{Hom}}_{\mathbb{Z}G}(-, A)$, we obtain a periodic cohomological complex

$$\cdots \rightarrow A \xrightarrow{N} A \xrightarrow{t-1} A \xrightarrow{N} A \xrightarrow{t-1} A \xrightarrow{N} A \xrightarrow{t-1} A \rightarrow \cdots$$

So that

$$\widehat{H}^i(G, A) \cong \begin{cases} A^G / NA, & i \text{ even,} \\ \{a \in A \mid N \cdot a = 0\} / (t-1)A, & i \text{ odd.} \end{cases}$$

Recall that if G is any finite group, then its homology $H_i(G, A)$ and cohomology $H^i(G, A)$ groups are annihilated by multiplication by $\#G$ for $i > 0$. In fact, this follows from a stronger result: if $P_\bullet \rightarrow \mathbb{Z}$ is the bar resolution, then the morphism

$$\begin{aligned} \text{"}\#G\text{"}: P_\bullet &\rightarrow P_\bullet, \\ (\#G - N): P_0 &\rightarrow P_0, \\ \#G: P_i &\rightarrow P_i \quad \text{for } i > 1, \end{aligned}$$

which induces multiplication by $\#G$ on $H_i(G, A)$ and $H^i(G, A)$ for $i > 0$, is null-homotopic—see e.g. [Wei1994, Theorem 6.5.8]. In our case, when G is cyclic of order m , for the 2-periodic complete resolution (0.9.4), it is easy to see that the multiplication by m on \widehat{P}_\bullet is null-homotopic. Indeed, such a null homotopy is also 2-periodic, and should be given by a family of morphisms

$$h^0: \mathbb{Z}G \rightarrow \mathbb{Z}G, \quad h^1: \mathbb{Z}G \rightarrow \mathbb{Z}G$$

Satisfying

$$(0.9.5) \quad h^0 \circ (t-1) + N \circ h^1 = m, \quad h^1 \circ N + (t-1) \circ h^0 = m.$$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z}G & \xrightarrow{t-1} & \mathbb{Z}G & \xrightarrow{N} & \mathbb{Z}G & \longrightarrow & \cdots \\ & & & & \swarrow h^1 & \#G \downarrow & \swarrow h^0 & \#G \downarrow & \swarrow h^1 & & \\ \cdots & \longrightarrow & \mathbb{Z}G & \xrightarrow{N} & \mathbb{Z}G & \xrightarrow{t-1} & \mathbb{Z}G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \cdots \end{array}$$

Let h^0 be the multiplication by $-x \in \mathbb{Z}G$, where

$$x := (m-1) + (m-2)t + (m-3)t^2 + \cdots + t^{m-2},$$

and let h^1 be the identity map. Then

$$\begin{aligned} x \cdot (t-1) &= (m-1)t + (m-2)t^2 + (m-3)t^3 + \cdots + t^{m-1} \\ &\quad - (m-1) - (m-2)t - (m-3)t^2 - \cdots - t^{m-1} \\ &= -m + 1 + t + t^2 + \cdots + t^{m-1} = -m + N, \end{aligned}$$

so that

$$(-x) \cdot (t-1) + N = m,$$

which means that (0.9.5) is satisfied. This implies that the groups $\widehat{H}^i(G, A)$ are annihilated by m for all i , and in general, for any bounded below complex of G -modules A^\bullet , the groups $\widehat{H}^i(G, A^\bullet)$ are annihilated by m . The latter is evident from our argument and not so evident from the spectral sequence

$$E_2^{pq} = \widehat{H}^q(G, H^p(A^\bullet)) \implies \widehat{H}^{p+q}(G, A^\bullet).$$

▲

We use Tate cohomology to define étale cohomology with compact support à la Milne [Mil2006, §II.2]. If \mathcal{F}^\bullet is a bounded below complex of abelian sheaves on $\mathrm{Spec} \mathbb{Z}_{\acute{e}t}$, then by definition, $R\widehat{\Gamma}_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet)$ is the complex sitting in the distinguished triangle

$$R\widehat{\Gamma}_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\Gamma(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\widehat{\Gamma}(G_{\mathbb{R}}, \mathcal{F}_{\mathbb{C}}^\bullet) \rightarrow R\widehat{\Gamma}_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet)[1]$$

where $R\widehat{\Gamma}(G_{\mathbb{R}}, \mathcal{F}_{\mathbb{C}}^\bullet)$ is the Tate cohomology defined above, and $\mathcal{F}_{\mathbb{C}}^\bullet$ is the complex of $G_{\mathbb{R}}$ -modules obtained by taking the stalks at $\mathrm{Spec} \mathbb{C} \rightarrow \mathrm{Spec} \mathbb{R}$. The morphism $R\Gamma(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\widehat{\Gamma}(G_{\mathbb{R}}, \mathcal{F}_{\mathbb{C}}^\bullet)$ arises as follows.

The canonical morphism $v: \mathrm{Spec} \mathbb{R} \rightarrow \mathrm{Spec} \mathbb{Z}$ induces a morphism

$$(0.9.6) \quad R\Gamma(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\Gamma(\mathrm{Spec} \mathbb{R}_{\acute{e}t}, v^* \mathcal{F}^\bullet),$$

and the cohomology on $\mathrm{Spec} \mathbb{R}_{\acute{e}t}$ corresponds to the cohomology of the Galois group $G_{\mathbb{R}}$: specifically, we have an equivalence of categories

$$\begin{aligned} \mathbf{Sh}(\mathrm{Spec} \mathbb{R}_{\acute{e}t})^{\mathbf{Ab}} &\xrightarrow{\cong} G_{\mathbb{R}}\text{-}\mathbf{Mod}, \\ \mathcal{F} &\rightsquigarrow \mathcal{F}_{\mathbb{C}} \end{aligned}$$

—see [SGA 4, Exposé VII, 2.3]. We may thus see (0.9.6) as a morphism^{*}

$$R\Gamma(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\Gamma(G_{\mathbb{R}}, \mathcal{F}_{\mathbb{C}}^\bullet),$$

which we may compose with the morphism (0.9.3) to the Tate cohomology $R\widehat{\Gamma}(G_{\mathbb{R}}, \mathcal{F}_{\mathbb{C}}^\bullet)$.

The notation “ $R\widehat{\Gamma}_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, -)$ ” is not standard; for instance, Geisser in [Gei2010] writes “ $R\Gamma_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, -)$ ” for the same thing. We will use the notation “ $R\widehat{\Gamma}_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, -)$ ” to avoid any confusion with the usual étale cohomology with compact support, as defined in §0.8.

Note that by definition, we have a morphism of complexes

$$(0.9.7) \quad R\widehat{\Gamma}_c(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\Gamma(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathcal{F}^\bullet).$$

^{*}Indeed, let $v^* \mathcal{F}^\bullet \xrightarrow{\cong} I^\bullet$ be a resolution of $v^* \mathcal{F}^\bullet$ by injective sheaves on $\mathrm{Spec} \mathbb{R}_{\acute{e}t}$, and let $P_\bullet \rightarrow \mathbb{Z}$ be a resolution of \mathbb{Z} by finitely generated free $\mathbb{Z}G$ -modules. Then, thanks to the equivalence of categories $\mathbf{Sh}(\mathrm{Spec} \mathbb{R}_{\acute{e}t})^{\mathbf{Ab}} \xrightarrow{\cong} G_{\mathbb{R}}\text{-}\mathbf{Mod}$, the complex of $G_{\mathbb{R}}$ -modules $I_{\mathbb{C}}^\bullet$ is an injective resolution of $(v^* \mathcal{F}^\bullet)_{\mathbb{C}} = \mathcal{F}_{\mathbb{C}}$. We have canonical quasi-isomorphisms of complexes

$$\mathrm{Hom}_{\mathbf{Sh}(\mathrm{Spec} \mathbb{R}_{\acute{e}t})}(\mathbb{Z}, I^\bullet) \rightarrow \mathrm{Tot}^{\oplus} \mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, I_{\mathbb{C}}^\bullet) \leftarrow \mathrm{Tot}^{\oplus} \mathrm{Hom}^{\bullet\bullet}(P_\bullet, \mathcal{F}_{\mathbb{C}}^\bullet),$$

so in the derived category (!), there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Sh}(\mathrm{Spec} \mathbb{R}_{\acute{e}t})}(\mathbb{Z}, I^\bullet) \xrightarrow{\cong} \mathrm{Tot}^{\oplus} \mathrm{Hom}^{\bullet\bullet}(P_\bullet, \mathcal{F}_{\mathbb{C}}^\bullet).$$

Now if \mathcal{F}^\bullet is a bounded below complex of abelian sheaves on $X_{\acute{e}t}$, then we pick a Nagata compactification of X

$$\begin{array}{ccc} X & \xrightarrow{j} & \mathfrak{X} \\ & \searrow f & \swarrow g \\ & & \text{Spec } \mathbb{Z} \end{array}$$

and set

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^\bullet) := R\widehat{\Gamma}_c(\text{Spec } \mathbb{Z}_{\acute{e}t}, Rf_! \mathcal{F}^\bullet),$$

where $Rf_! := Rg_* j_!$. In particular, the morphism (0.9.7) gives us for any bounded below complex of abelian sheaves \mathcal{F}^\bullet on $X_{\acute{e}t}$ a morphism

$$(0.9.8) \quad R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow R\Gamma_c(X_{\acute{e}t}, \mathcal{F}^\bullet),$$

where $R\Gamma_c(X_{\acute{e}t}, \mathcal{F}^\bullet) := R\Gamma(\text{Spec } \mathbb{Z}_{\acute{e}t}, Rf_! \mathcal{F}^\bullet)$. By definition of $R\widehat{\Gamma}_c(\text{Spec } \mathbb{Z}_{\acute{e}t}, -)$, we have a long exact sequence in cohomology

$$(0.9.9) \quad \cdots \rightarrow \widehat{H}^{i-1}(G_{\mathbb{R}}, (Rf_! \mathcal{F}^\bullet)_{\mathbb{C}}) \rightarrow \widehat{H}_c^i(X_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow H_c^i(X_{\acute{e}t}, \mathcal{F}^\bullet) \\ \rightarrow \widehat{H}^i(G_{\mathbb{R}}, (Rf_! \mathcal{F}^\bullet)_{\mathbb{C}}) \rightarrow \cdots$$

The groups $\widehat{H}^i(G_{\mathbb{R}}, (Rf_! \mathcal{F}^\bullet)_{\mathbb{C}})$ are annihilated by multiplication by $2 = \#G_{\mathbb{R}}$, which means that the morphism

$$\widehat{H}_c^i(X_{\acute{e}t}, \mathcal{F}^\bullet) \rightarrow H_c^i(X_{\acute{e}t}, \mathcal{F}^\bullet)$$

is identity, except for possible 2-torsion.

0.9.2. Remark. If $X(\mathbb{R}) = \emptyset$, then the canonical map

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathcal{F}^*) \rightarrow R\Gamma_c(X_{\acute{e}t}, \mathcal{F}^*)$$

is the identity.

0.10 Singular cohomology of complex varieties

We will need the following result.

0.10.1. Proposition. *Let X be an arithmetic scheme (separated, of finite type over $\text{Spec } \mathbb{Z}$). Consider the corresponding space of complex points $X(\mathbb{C})$ equipped with the analytic topology. Then*

- 1) *the singular cohomology groups with compact support $H_c^i(X(\mathbb{C}), \mathbb{Z})$ are finitely generated for all i ;*

2) the groups $H_c^i(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ are of cofinite type (\mathbb{Q}/\mathbb{Z} -dual of finitely generated groups).

The above groups vanish for $i \gg 0$.

The statement is very plausible, but I could not find a good reference, so I outline a proof.

Proof. Everything relies on the fact that $X(\mathbb{C})$ has homotopy type of a finite CW-complex. This is a well-known classical result, due to van der Waerden (see [vdW1930] and [LW1933]).

If $X(\mathbb{C})$ is smooth, then we may reduce the problem to the case of pure dimension d , and by Poincaré duality,

$$H_c^i(X(\mathbb{C}), \mathbb{Z}) \cong H_{2d-i}(X(\mathbb{C}), \mathbb{Z}),$$

where $H_{2d-i}(X(\mathbb{C}), \mathbb{Z})$ are finitely generated groups, trivial for all but finitely many i , as $X(\mathbb{C})$ is homotopy equivalent to a finite CW-complex, and the homology $H_\bullet(X(\mathbb{C}), \mathbb{Z})$ is homotopy invariant.

To deal with the general case, we use induction on the dimension. If the dimension is 0, then the statement is obvious. For induction step, we may consider the open-closed decomposition

$$U(\mathbb{C}) \hookrightarrow X(\mathbb{C}) \leftarrow Z(\mathbb{C})$$

where $Z(\mathbb{C})$ is the singular locus, having smaller dimension. This gives us a distinguished triangle

$$R\Gamma_c(U(\mathbb{C}), \mathbb{Z}) \rightarrow R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \rightarrow R\Gamma_c(Z(\mathbb{C}), \mathbb{Z}) \rightarrow R\Gamma_c(U(\mathbb{C}), \mathbb{Z})[1]$$

where $R\Gamma_c(U(\mathbb{C}), \mathbb{Z})$ is a perfect complex by the above argument, and the complex $R\Gamma_c(Z(\mathbb{C}), \mathbb{Z})$ is perfect by induction. This implies that $R\Gamma_c(X(\mathbb{C}), \mathbb{Z})$ is a perfect complex.

As for \mathbb{Q}/\mathbb{Z} -coefficients, the statement follows from the distinguished triangle (keep in mind that tensoring with \mathbb{Q} is exact)

$$R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \rightarrow R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow R\Gamma_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \rightarrow R\Gamma_c(X(\mathbb{C}), \mathbb{Z})[1]$$

Indeed, the associated long exact sequence in cohomology

$$\begin{aligned} \cdots \rightarrow H_c^i(X(\mathbb{C}), \mathbb{Z}) \rightarrow H_c^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_c^i(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \rightarrow \\ H_c^{i+1}(X(\mathbb{C}), \mathbb{Z}) \rightarrow H_c^{i+1}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \cdots \end{aligned}$$

shows that $H_c^i(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ is an extension of a finite group by a group of cofinite type, hence it is of cofinite type as well (see 0.1.3):

$$\begin{aligned} 0 \rightarrow \operatorname{coker}(H_c^i(X(\mathbb{C}), \mathbb{Z}) \rightarrow H_c^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow \\ H_c^i(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \rightarrow \\ \ker(H_c^{i+1}(X(\mathbb{C}), \mathbb{Z}) \rightarrow H_c^{i+1}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow 0 \end{aligned}$$

Finally, $H_c^i(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ vanishes for $i \gg 0$, because $H_c^i(X(\mathbb{C}), \mathbb{Z})$ does. ■

0.11 Cycle complexes and motivic cohomology

Bloch's cycle complexes were introduced in [Blo1986a] to define higher Chow groups (there was a gap in the proof of the "moving lemma" that was fixed later in [Blo1994]). A good modern survey of cycle complexes may be found in [Gei2005], and there is also a useful text [Blo2005] available from Bloch's home page.

In this section I will go through various definitions that will be used later on in the constructions. Let X be an arithmetic scheme (separated, of finite type over $\operatorname{Spec} \mathbb{Z}$) or a variety over a field k (a separated scheme of finite type over $\operatorname{Spec} k$). Let $n \in \mathbb{Z}$ be some fixed integer. Then to X we may associate the following objects:

- 1a) a homological complex of abelian groups $z_n(X, \bullet)$, defined in terms of cycles of dimension $n + i$ in $X \times \Delta^i$;
- 1b) the corresponding cohomological complex of étale and Zariski sheaves

$$\mathbb{Z}^c(n) := z_n(-, - \bullet - 2n);$$

- 2a) a homological complex of abelian groups $z^n(X, \bullet)$, defined in terms of cycles of codimension n in $X \times \Delta^i$, where Δ^i is the algebraic i -simplex;
- 2b) the corresponding cohomological complex of étale and Zariski sheaves

$$\mathbb{Z}(n) := z^n(-, 2n - \bullet);$$

- 2c) some variation of 2a): a homological complex of abelian groups $z_{\square}^n(X, \bullet)$, defined in terms of cycles of codimension n in $X \times \square^i$, where \square^i is the algebraic i -cube.

In fact, we will use only 1a) and 1b) in our constructions. I discuss 2a), 2b), 2c) simply because at some point (namely, in chapter 2) we will need to refer to the literature where 2a), 2b), 2c) are used instead of 1a) and 1b).

Simplicial and cubical complexes

Let us briefly recall some definitions regarding simplicial objects (see [May1992] and [GJ2009]) and cubical objects (see e.g. [Cis2006] and [BHS2011]). I do this mostly because of the cubical objects that seem to be less common.

0.11.1. Definition. The **simplicial category** Δ is the category where the objects are finite ordered sets

$$\mathbf{n} := \{0 < 1 < \cdots < n\}$$

for $n = 0, 1, 2, \dots$ and the morphisms are nondecreasing maps $\mathbf{n} \rightarrow \mathbf{m}$.

A **simplicial** (resp. **cosimplicial**) **object** in a category \mathcal{C} is a contravariant functor $X: \Delta^\circ \rightarrow \mathcal{C}$ (resp. covariant functor $X: \Delta^\circ \rightarrow \mathcal{C}$).

For $0 \leq i \leq n$, let us denote by

$$\partial_n^i: \mathbf{n} - \mathbf{1} \hookrightarrow \mathbf{n}$$

the increasing map that skips i :

$$\partial_n^i(j) := \begin{cases} j, & j < i, \\ j+1, & j \geq i; \end{cases}$$

and let us denote by

$$\sigma_n^i: \mathbf{n} + \mathbf{1} \rightarrow \mathbf{n}$$

the nondecreasing map that applies two elements to i :

$$\sigma_n^i(j) := \begin{cases} j, & j \leq i, \\ j-1, & j > i. \end{cases}$$

Sometimes ∂^i 's are called **coface morphisms** and σ^i 's are called **codegeneracy morphisms**. It is easy to see that every morphism in Δ may be written as a composition of such maps, and they satisfy the so-called **cosimplicial identities**:

$$(0.11.1) \quad \sigma_n^j \circ \partial_{n+1}^i = \begin{cases} \partial_n^i \circ \sigma_{n-1}^{j-1}, & \text{if } i < j, \\ \text{id}_{\mathbf{n}}, & \text{if } i = j \text{ or } i = j + 1, \\ \partial_n^{i-1} \circ \sigma_{n-1}^j, & \text{if } i > j + 1; \end{cases}$$

$$(0.11.2) \quad \sigma_n^j \circ \sigma_{n+1}^i = \sigma_n^i \circ \sigma_{n+1}^{j+1} \quad \text{if } i \leq j;$$

$$(0.11.3) \quad \partial_n^j \circ \partial_{n-1}^i = \partial_n^i \circ \partial_{n-1}^{j-1} \quad \text{if } i < j;$$

in fact, (0.11.1), (0.11.2), (0.11.3) give all possible relations between morphisms in Δ . This means that a simplicial object $X: \Delta^\circ \rightarrow \mathcal{C}$ is equivalent to a collection of objects

$$X_n := X(\mathbf{n}) \in \text{Ob}(\mathcal{C}) \quad (n = 0, 1, 2, \dots)$$

and a collection of morphisms

$$\partial_i^n: X_n \rightarrow X_{n-1}, \quad \sigma_i^n: X_n \rightarrow X_{n+1} \quad (0 \leq i \leq n),$$

called **face** and **degeneracy morphisms** that satisfy the **simplicial identities** (dual to the identities (0.11.1), (0.11.2), (0.11.3)):

$$(0.11.4) \quad \partial_i^{n+1} \circ \sigma_j^n = \begin{cases} \sigma_{j-1}^{n-1} \circ \partial_i^n, & \text{if } i < j, \\ \text{id}_{X_n}, & \text{if } i = j \text{ or } i = j + 1, \\ \sigma_j^{n-1} \circ \partial_{i-1}^n, & \text{if } i > j + 1, \end{cases}$$

$$(0.11.5) \quad \sigma_i^{n+1} \circ \sigma_j^n = \sigma_{j+1}^{n+1} \circ \sigma_i^n \quad \text{if } i \leq j.$$

$$(0.11.6) \quad \partial_i^{n-1} \circ \partial_j^n = \partial_{j-1}^{n-1} \circ \partial_i^n \quad \text{if } i < j.$$

A simplicial object may be visualized as a diagram

$$\begin{array}{ccccccc} & & & \longleftarrow \partial_0^2 & \longrightarrow & & \\ & & & \longleftarrow \partial_0^1 & \longrightarrow & \longleftarrow \partial_0^2 & \longrightarrow \\ X_0 & \xrightarrow{\sigma_0^1} & X_1 & \xleftarrow{\partial_1^2} & X_2 & \xrightarrow{\sigma_2^2} & \cdots \\ & & & \longleftarrow \partial_1^1 & \longrightarrow & \longleftarrow \partial_1^2 & \longrightarrow \\ & & & \longleftarrow \partial_2^2 & \longrightarrow & & \end{array}$$

0.11.2. Lemma (Complex of alternating face maps). *Let $A: \Delta^\circ \rightarrow \mathbf{Ab}$ be a simplicial abelian group. Then the morphisms of abelian groups*

$$d_n := \sum_{0 \leq i \leq n} (-1)^i \partial_i^n: A_n \rightarrow A_{n-1}$$

satisfy

$$d_{n-1} \circ d_n = 0,$$

i.e.

$$(A_\bullet, d_\bullet): \quad \cdots \rightarrow A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \rightarrow 0$$

is a chain complex.

Proof. Easily follows from the simplicial identity (0.11.6). ■

0.11.3. Definition ([Cis2006]). The **cubical category** \square is the category where the objects are finite sets

$$\square^n := \{0, 1\}^n = \{(x_1, \dots, x_n) \mid x_i \in \{0, 1\}\}$$

for $n = 0, 1, 2, \dots$ and the morphisms are compositions of the following two kinds of maps:

- 1) for $n \geq 1$ and $1 \leq i \leq n$ the inclusion

$$\partial_n^{i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$$

that inserts $\epsilon \in \{0, 1\}$ into the i -th position:

$$(0.11.7) \quad \partial_n^{i,\epsilon}(x_1, \dots, x_{n-1}) := (x_1, \dots, x_{i-1}, \epsilon, x_i, \dots, x_{n-1}).$$

- 2) for $n \geq 0$ and $1 \leq i \leq n + 1$ the projection

$$\sigma_n^i : \square^{n+1} \twoheadrightarrow \square^n$$

that forgets the i -th coordinate:

$$(0.11.8) \quad \sigma_n^i(x_1, \dots, x_{n+1}) := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}).$$

A **cubical** (resp. **cocubical**) **object** in a category \mathcal{C} is a contravariant functor $X : \square^\circ \rightarrow \mathcal{C}$ (resp. covariant functor $X : \square \rightarrow \mathcal{C}$).

All relations between the morphisms in \square follow from the so-called **cocubical identities**:

$$(0.11.9) \quad \sigma_n^j \circ \partial_{n+1}^{i,\epsilon} = \begin{cases} \partial_n^{i,\epsilon} \circ \sigma_{n-1}^{j-1}, & \text{if } i < j, \\ \text{id}_{\square^n}, & \text{if } i = j, \\ \partial_n^{i-1,\epsilon} \circ \sigma_{n-1}^j, & \text{if } i > j; \end{cases}$$

$$(0.11.10) \quad \sigma_n^j \circ \sigma_{n+1}^i = \sigma_n^i \circ \sigma_{n+1}^{j+1} \quad \text{if } i \leq j;$$

$$(0.11.11) \quad \partial_n^{j,\eta} \circ \partial_{n-1}^{i,\epsilon} = \partial_n^{i,\epsilon} \circ \partial_{n-1}^{j-1,\eta} \quad \text{if } i < j.$$

This means that a cubical object $X : \square^\circ \rightarrow \mathcal{C}$ is just a collection of objects

$$X_n := X(\square^n) \in \text{Ob}(\mathcal{C}), \quad n = 0, 1, 2, \dots$$

and morphisms

$$\partial_{i,\epsilon}^n: X_n \rightarrow X_{n-1}, \quad \sigma_i^n: X_n \rightarrow X_{n+1}$$

that satisfy the **cubical identities**, i.e. the ones dual to (0.11.9), (0.11.10), (0.11.11):

$$(0.11.12) \quad \partial_{i,\epsilon}^{n+1} \circ \sigma_j^n = \begin{cases} \sigma_{j-1}^{n-1} \circ \partial_{i,\epsilon'}^n & \text{if } i < j, \\ \text{id}_{X_n} & \text{if } i = j, \\ \sigma_j^{n-1} \circ \partial_{i-1,\epsilon'}^n & \text{if } i > j; \end{cases}$$

$$(0.11.13) \quad \sigma_i^{n+1} \circ \sigma_j^n = \sigma_{j+1}^{n+1} \circ \sigma_i^n \quad \text{if } i \leq j;$$

$$(0.11.14) \quad \partial_{i,\epsilon}^{n-1} \circ \partial_{j,\eta}^n = \partial_{j-1,\eta}^{n-1} \circ \partial_{i,\epsilon}^n \quad \text{if } i < j.$$

0.11.4. Lemma (Reduced cubical complex). *Let $A: \square^\circ \rightarrow \mathbf{Ab}$ be a cubical abelian group. Consider the morphisms*

$$(0.11.15) \quad d_n := \sum_{1 \leq i \leq n} (-1)^i (\partial_{i,1}^n - \partial_{i,0}^n): A_n \rightarrow A_{n-1}.$$

Then

1) $d_{n-1} \circ d_n = 0$, i.e. (A_\bullet, d_\bullet) is a chain complex.

2) The **degenerate cubes** defined by

$$(A_n)_{\text{degn}} := \sum_{1 \leq i \leq n} \sigma_i^{n-1}(A_{n-1}) \subset A_n$$

form a subcomplex of (A_\bullet, d_\bullet) .

3) We also have the subcomplex of **reduced cubes** given by

$$(A_n)_0 := \bigcap_{1 \leq i \leq n} \ker \partial_{i,1}^n \subset A_n.$$

4) There is a canonical splitting

$$A_n = (A_n)_{\text{degn}} \oplus (A_n)_0.$$

Sketch of the proof. Writing out all the involved combinatorial identities might not be very illuminating, but the reader should note how everything resembles the simplicial setting. 1) is deduced from the cubical identity (0.11.14);

in 2), to show that $d_n((A_n)_{\text{degn}}) \subseteq (A_{n-1})_{\text{degn}}$, one should use the cubical identity (0.11.12); in 3), to show that $d_n((A_n)_0) \subseteq (A_{n-1})_0$, one should again use (0.11.14). Finally, to show 4), one may consider the endomorphism $\pi_n: A_n \rightarrow A_n$ defined by

$$\begin{aligned} \pi_n := & (\text{id} - \sigma_n^{n-1} \circ \partial_{n,1}^n) \circ (\text{id} - \sigma_{n-1}^{n-1} \circ \partial_{n-1,1}^n) \circ \cdots \\ & \circ (\text{id} - \sigma_2^{n-1} \circ \partial_{2,1}^n) \circ (\text{id} - \sigma_1^{n-1} \circ \partial_{1,1}^n). \end{aligned}$$

Then π_n defines the splitting

$$0 \rightarrow (A_n)_{\text{degn}} \rightarrow A_n \xrightarrow{\pi_n} (A_n)_0 \rightarrow 0$$

Namely, it is clear from the definition that $\pi_n|_{(A_n)_0} = \text{id}_{(A_n)_0}$, and one deduces from the cubical identities that $\ker \pi_n = (A_n)_{\text{degn}}$ and $\text{im } \pi_n = (A_n)_0$. ■

0.11.5. Definition. In the setting of 0.11.4, the **reduced cubical complex** associated to a cubical abelian group $A: \square^\circ \rightarrow \mathbf{Ab}$ is the quotient complex

$$(A_\bullet / (A_\bullet)_{\text{degn}}, d_\bullet) \cong ((A_\bullet)_0, d_\bullet).$$

0.11.6. Remark (Cubical singular complex in topology). It is worth noting why quotienting out the degenerate cubes is necessary. Everything is motivated by cubical (co)homology in algebraic topology (see e.g. [Mas1977] and [EM1953]). We consider the **geometric cubes** defined for each $n = 0, 1, 2, 3, \dots$ by

$$\square^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1\}.$$

We naturally have inclusions $\partial_n^{i,\varepsilon}: \square^{n-1} \hookrightarrow \square^n$ and projections $\sigma_n^i: \square^{n-1} \rightarrow \square^n$, defined by the same formulas (0.11.7) and (0.11.8). This gives us a cocubical topological space $\square^\bullet: \square \rightarrow \mathbf{Top}$. Now for a topological space X , the sets

$$\text{Sing}^\square(X)_n := \text{Hom}_{\mathbf{Top}}(\square^n, X)$$

form a cubical set $\text{Sing}^\square(X)_\bullet: \square^\circ \rightarrow \mathbf{Set}$, which is the composition of functors $\square^\bullet: \square \rightarrow \mathbf{Top}$ and $\text{Hom}_{\mathbf{Top}}: \mathbf{Top}^\circ \rightarrow \mathbf{Set}$. Namely, for a continuous map $\phi: \square^n \rightarrow X$, we may consider its restrictions to $\square^{n-1} \subset \square^n$ given by setting $x_i = 0$ or $x_i = 1$ for $i = 1, \dots, n$, and also extensions to $\square^{n+1} \supset \square^n$ given by putting 0 or 1 in i -th position. This gives us face and degeneracy maps

$$\begin{aligned} \partial_{i,\varepsilon}: \text{Sing}^\square(X)_n &\rightarrow \text{Sing}^\square(X)_{n-1}, \\ \sigma_i: \text{Sing}^\square(X)_n &\rightarrow \text{Sing}^\square(X)_{n+1} \end{aligned}$$

that satisfy the cubical identities. By composing our functor $\text{Sing}^\square(X)_\bullet: \square^\circ \rightarrow \mathbf{Set}$ with the free abelian group functor $\mathbf{Set} \rightarrow \mathbf{Ab}$, we obtain a cubical

abelian group $\mathbb{Z}\langle \text{Sing}^\square(X)_\bullet \rangle : \square^\circ \rightarrow \mathbf{Ab}$. As in 0.11.4, we may build from it a chain complex.

Now if $X = *$ is just a point, then

$$\text{Sing}^\square(*)_n = \text{Hom}_{\mathbf{Top}}(\square^n, *)$$

are one-element sets, so that the complex will look like

$$\cdots \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

However, note that in this case we have $\partial_{i,1}^n = \partial_{i,0}^n$ for all n and i , therefore the differentials (0.11.15) are all trivial, and the point has homology $\cong \mathbb{Z}$ in all degrees, which is not very desirable. However, the cubes of dimension $n > 0$ are all degenerate, so the corresponding *reduced* cubical complex looks like

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0$$

Note that for the usual singular complex defined using simplices instead of cubes (replace \square^n with Δ^n in all the above), the degenerate simplices also form a subcomplex, but it is easily seen from the simplicial identities that passing to the corresponding reduced complex does not affect the homology. E.g. the simplicial singular complex for a point will look like

$$\cdots \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

For comparison of the simplicial and cubical approach to defining singular (co)homology, see [EM1953].

Bloch's cycle complexes $z^n(X, \bullet)$ and $z_n(X, \bullet)$

Now we define several versions of Bloch's cycle complexes; we refer to [Gei2005] and [Blo2005] for details; our reference for intersection theory is [Ful1998].

Let X be a separated scheme of finite type over a base scheme S . For our particular purposes, we only consider the cases $S = \text{Spec } k$ for a field k or $S = \text{Spec } \mathbb{Z}$. For each $n = 0, 1, 2, \dots$ the **algebraic n -simplex** is given by

$$\Delta^n := \text{Spec } \mathbb{Z}[t_0, \dots, t_n] / (1 - \sum_i t_i), \quad \Delta_S^n := \Delta^n \times_{\text{Spec } \mathbb{Z}} S.$$

This is isomorphic to the affine space \mathbb{A}^n , but not canonically; instead, it comes with canonical "simplicial" coordinates. Each nondecreasing map

$\rho: \mathbf{n} \rightarrow \mathbf{m}$ induces functorially a morphism of schemes $\tilde{\rho}: \Delta_S^n \rightarrow \Delta_S^m$ given by

$$\tilde{\rho}(t_i) := \begin{cases} 0, & \text{if } \rho^{-1}(i) = \emptyset, \\ \sum_{\rho(j)=i} t_j, & \text{otherwise.} \end{cases}$$

Similarly, for an S -scheme X , a nondecreasing map $\rho: \mathbf{n} \rightarrow \mathbf{m}$ induces functorially a morphism of schemes $(\text{id} \times \tilde{\rho}): X \times_S \Delta_S^n \rightarrow X \times_S \Delta_S^m$. This defines a cosimplicial S -scheme

$$X \times_S \Delta_S^\bullet: \mathbf{\Delta} \rightarrow \mathbf{Sch}/_S.$$

Now for a fixed $n \in \mathbb{Z}$, one considers the following two series of free abelian groups indexed by $i \in \mathbb{Z}$:

- 1) we let $z_n(X, i)$ be the free abelian group generated by closed integral subschemes $Z \subset X \times_S \Delta_S^i$ of dimension $n + i$ that meet all faces of Δ_S^i properly.
- 2) if X is an equidimensional scheme, we let $z^n(X, i)$ be the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^i$ of codimension n that meet all faces of Δ_S^i properly.

We note that the first definition is in fact more natural in some sense: it does not require X to be equidimensional. If X is of pure dimension d , then we see that

$$(0.11.16) \quad z_n(X, i) = z^{d-n}(X, i).$$

Both $z_n(X, \bullet)$ and $z^n(X, \bullet)$ are simplicial abelian groups $\mathbf{\Delta}^\circ \rightarrow \mathbf{Ab}$. Namely, for a morphism $\rho: \mathbf{i} \rightarrow \mathbf{j}$ in $\mathbf{\Delta}$,

- 1) if ρ is injective, then $\text{id} \times \tilde{\rho}: X \times_S \Delta_S^i \rightarrow X \times_S \Delta_S^j$ is a closed immersion, and for a cycle $V \subset X \times_S \Delta_S^j$ we may consider the intersection

$$(\text{id} \times \tilde{\rho})(X \times_S \Delta_S^i) \cdot V;$$

- 2) if ρ is surjective, then $\text{id} \times \tilde{\rho}: X \times_S \Delta_S^i \rightarrow X \times_S \Delta_S^j$ is a flat morphism, for which we have the corresponding flat pullback of cycles;

in both cases, we obtain morphisms

$$\rho^*: z_n(X, j) \rightarrow z_n(X, i), \quad \rho^*: z^n(X, j) \rightarrow z^n(X, i).$$

In particular, as we noted in 0.11.2, $z_n(X, \bullet)$ and $z^n(X, \bullet)$ give us chain complexes

$$\cdots \rightarrow z_n(X, i) \xrightarrow{d_i} z_n(X, i-1) \xrightarrow{d_{i-1}} z_n(X, i-2) \rightarrow \cdots$$

and

$$\cdots \rightarrow z^n(X, i) \xrightarrow{d_i} z^n(X, i-1) \xrightarrow{d_{i-1}} z^n(X, i-2) \rightarrow \cdots$$

with the differentials

$$d_i := \sum_{0 \leq \ell \leq i} (-1)^\ell \partial_\ell.$$

Let us recall Bloch's definition of higher Chow groups, for which he introduced the complexes $z^n(X, \bullet)$.

0.11.7. Definition ([Blo1986a]). If X is an equidimensional scheme as above, its **higher Chow groups** are given by

$$CH^n(X, i) := H_i(z^n(X, \bullet)).$$

The usual Chow groups (algebraic cycles on X modulo rational equivalence) correspond to $i = 0$:

$$CH^n(X) = CH^n(X, 0).$$

Cubical cycle complexes $z_{\square}^n(X, \bullet)$

We briefly recall the cubical version of $z^n(X, \bullet)$, which is often used in the literature, e.g. in [Lev1994]. If k is a field, we consider the **algebraic cube**

$$\square_k^n := (\mathbb{P}_k^1 \setminus \{1\})^n$$

with coordinates (x_1, \dots, x_n) . Setting some x_i to 0 or ∞ gives us a **codimension 1 face of \square_k^n** . In general, setting x_{i_1}, \dots, x_{i_s} to 0 or ∞ gives a **codimension s face**. We have a cocubical variety \square_k^\bullet in the sense of 0.11.3. Namely,

- 1) for each $n \geq 1$ we have the inclusion maps

$$\begin{aligned} \partial_n^{i, \epsilon} : \square_k^{n-1} &\hookrightarrow \square_k^n, \quad (1 \leq i \leq n, \epsilon = 0, \infty) \\ (x_1, \dots, x_{n-1}) &\mapsto (x_1, \dots, x_{i-1}, \epsilon, x_i, \dots, x_{n-1}); \end{aligned}$$

- 2) for $n \geq 0$ we have the projection maps

$$\begin{aligned} \sigma_n^i : \square_k^{n+1} &\twoheadrightarrow \square_k^n \quad (1 \leq i \leq n+1), \\ (x_1, \dots, x_{n+1}) &\mapsto (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}); \end{aligned}$$

and these maps satisfy the cocubical identities.

Now if X is an equidimensional variety over k , we denote by $z_{\square}^n(X, i)$ the free abelian group generated by the irreducible subvarieties

$$V \subset X \times_k \square_k^i, \quad \text{codim}_k V = n,$$

meeting all faces properly. The maps $\text{id} \times \partial_i^{\ell, \varepsilon}$ and $\text{id} \times \sigma_i^\ell$ induce pullback morphisms

$$\begin{aligned} \partial_{\ell, \varepsilon}^i: z_{\square}^n(X, i) &\rightarrow z_{\square}^n(X, i-1), \\ \sigma_i^j: z_{\square}^n(X, i) &\rightarrow z_{\square}^n(X, i+1) \end{aligned}$$

which satisfy the cubical identities and thus turn $z_{\square}^n(X, \bullet)$ into a cubical abelian group. As in 0.11.5, we form from this a chain complex $z_{\square}^n(X, \bullet)$, where the differentials are given by

$$d_i := \sum_{1 \leq \ell \leq i} (-1)^\ell (\partial_{\ell, \infty}^i - \partial_{\ell, 0}^i): z_{\square}^n(X, i) \rightarrow z_{\square}^n(X, i-1),$$

and the degenerate cubes are quotiented out. The following is proved in [Lev1994].

0.11.8. Theorem. *There is an isomorphism in the derived category*

$$z_{\square}^r(X, \bullet) \cong z^r(X, \bullet).$$

Complexes of sheaves $\mathbb{Z}^c(n)$

The cycle complexes may be “sheafified” as follows. The presheaves

$$U \mapsto z_n(U, i), \quad U \mapsto z^n(U, i)$$

are in fact sheaves on $X_{\text{ét}}$ or X_{Zar} (this is verified e.g. in [Gei2004, Lemma 3.1]). We will use the opposite numbering and denote

$$Z_n^X := z_n(-, -\bullet), \quad Z_X^n := z^n(-, -\bullet).$$

These are cohomological complexes of abelian sheaves on $X_{\text{ét}}$ or X_{Zar} .

We will also need the following result, saying that the cohomology of the cycle complexes $z^n(X, -\bullet)$ coincides with the Zariski hypercohomology of Z_X^n .

0.11.9. Theorem. *If X is a scheme of finite type over a field, we have a quasi-isomorphism of complexes of abelian groups*

$$R\Gamma(X_{\text{Zar}}, Z_X^n) \simeq z^n(X, -\bullet).$$

Proof. See [Gei2005, §1.2.4.] for details. ■

Finally, in terms of Z_n^X , one defines complexes $\mathbb{Z}^c(n)$, which will be one of the most important objects in our constructions.

0.11.10. Definition ([Gei2010]). The **dualizing cycle complex** is given by

$$\mathbb{Z}^c(n) := Z_n^X[2n].$$

It is a cohomological complex of sheaves with $z_n(-, -i - 2n)$ sitting in i -th degree. In general, for any abelian group A , one defines

$$A^c(n) := \mathbb{Z}^c(n) \otimes_{\mathbb{Z}}^L A = \mathbb{Z}^c(n) \otimes_{\mathbb{Z}} A.$$

(As $\mathbb{Z}^c(n)$ is a complex of flat sheaves, the derived tensor product coincides with the usual tensor product.)

For the sake of completeness, I also recall the the related definition based on $z^n(-, \bullet)$:

$$\mathbb{Z}(n) := Z_X^n[-2n].$$

It is a cohomological complex of sheaves with $z^n(-, 2n - i)$ in degree i . If X is equidimensional of dimension d , then (0.11.16) gives us the corresponding relation for complexes of sheaves

$$Z_n^X = Z_X^{d-n},$$

which allows us to express $\mathbb{Z}^c(n)$ in terms of Z_X^n :

$$\mathbb{Z}^c(n) = Z_X^{d-n}[2n] = \mathbb{Z}(d - n)[2d].$$

Now the reader should actually forget about this $\mathbb{Z}(n)$, because later on “ $\mathbb{Z}(n)$ ” will denote a completely different complex of sheaves, to be defined in §1.2.

$\mathbb{Z}^c(n)$ as a dualizing complex

0.11.11. Topological digression. Let us recall that for a locally compact topological space X , one may define **Borel–Moore homology groups** $H_i^{BM}(X, \mathbb{Z})$ (see [Ive1986, Chapter IX]). These will make their appearance in §2.1, but now they will serve us as a motivating example of duality.

Local Verdier duality [Ive1986, §VII.5] tells that if $f: X \rightarrow Y$ is a continuous map between locally compact topological spaces of finite dimension, then there is a natural isomorphism in the derived category $\mathbf{D}^+(Y)$

$$R\mathrm{Hom}(Rf_! \mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong Rf_* R\mathrm{Hom}(\mathcal{F}^\bullet, f^! \mathcal{G}^\bullet)$$

where $\mathcal{F}^\bullet \in \mathbf{D}^+(X)$, $\mathcal{G}^\bullet \in \mathbf{D}^+(Y)$, and $f^!: \mathbf{D}^+(Y) \rightarrow \mathbf{D}^+(X)$ is the right adjoint functor to $Rf_!: \mathbf{D}^+(X) \rightarrow \mathbf{D}^+(Y)$. In particular, for the projection to the point $p: X \rightarrow *$ the above reads

$$R\mathrm{Hom}(R\Gamma_c(X, \mathcal{F}^\bullet), \mathcal{G}^\bullet) \cong R\Gamma(X, R\mathrm{Hom}(\mathcal{F}^\bullet, p^! \mathcal{G}^\bullet))$$

for $\mathcal{F} \in \mathbf{D}^+(X)$ and $\mathcal{G}^\bullet \in \mathbf{D}(\mathbf{Ab})$. If we take \mathcal{G}^\bullet to be the complex consisting of a single constant sheaf \mathbb{Z} , the object $p^!\mathbb{Z} \in \mathbf{D}^+(X)$ is called the **dualizing sheaf** on X , and **Borel–Moore homology** is defined by

$$H_i^{BM}(X, \mathbb{Z}) := H^{-i}(R\Gamma_{BM}(X, \mathbb{Z})),$$

$$R\Gamma_{BM}(X, \mathbb{Z}) := R\Gamma(X, p^!\mathbb{Z}) \cong R\mathrm{Hom}(R\Gamma_c(X, \mathbb{Z}), \mathbb{Z}).$$

This means that Borel–Moore homology is *covariantly* functorial for proper maps and *contravariantly* functorial for inclusions of open subsets $U \hookrightarrow X$:

- 1) a proper continuous map of locally compact topological spaces $f: X \rightarrow Y$ induces a morphism $R\Gamma_c(Y, \mathbb{Z}) \rightarrow R\Gamma_c(X, \mathbb{Z})$, and therefore on Borel–Moore homology we have the **proper pushforward** morphism

$$R\Gamma_{BM}(X, \mathbb{Z}) \rightarrow R\Gamma_{BM}(Y, \mathbb{Z}).$$

- 2) an inclusion of an open subset $U \hookrightarrow X$ induces a morphism $R\Gamma_c(U, \mathbb{Z}) \rightarrow R\Gamma_c(X, \mathbb{Z})$, and therefore the corresponding **pullback** on Borel–Moore homology

$$R\Gamma_{BM}(X, \mathbb{Z}) \rightarrow R\Gamma_{BM}(U, \mathbb{Z}).$$

Moreover, if $U \subset X$ is an open subset and $Z := X \setminus U$ is its closed complement, then the corresponding pushforwards and pullbacks fit into a distinguished triangle

$$R\Gamma_{BM}(Z, \mathbb{Z}) \rightarrow R\Gamma_{BM}(X, \mathbb{Z}) \rightarrow R\Gamma_{BM}(U, \mathbb{Z}) \rightarrow R\Gamma_{BM}(X, \mathbb{Z})[1]$$

This is dual to the triangle

$$R\Gamma_c(U, \mathbb{Z}) \rightarrow R\Gamma_c(X, \mathbb{Z}) \rightarrow R\Gamma_c(Z, \mathbb{Z}) \rightarrow R\Gamma_c(U, \mathbb{Z})[1]$$

The cycle complex $\mathbb{Z}^c(n)$ behaves similarly to Borel–Moore homology.

0.11.1. Fact ([Gei2010, Corollary 7.2]).

- 1) a proper morphism of schemes $f: X \rightarrow Y$ induces a pushforward morphism

$$Rf_*\mathbb{Z}_X^c(n) \rightarrow \mathbb{Z}_Y^c(n);$$

- 2) an open immersion of schemes $f: U \hookrightarrow X$ induces a flat pullback morphism

$$f^*\mathbb{Z}_X^c(n) \rightarrow \mathbb{Z}_U^c(n).$$

If $U \subset X$ is an open subscheme and $Z := X \setminus U$ is its closed complement, then the proper pushforward associated to $Z \hookrightarrow X$ and the flat pullback associated to $U \hookrightarrow X$ give a distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1]$$

