Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology
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Introduction

Let X be an **arithmetic scheme**, i.e. separated and of finite type over $\operatorname{Spec}\mathbb{Z}$. The corresponding **zeta function** is defined by the infinite product

$$\zeta(X,s) := \prod_{x\in X_0} \frac{1}{1-N(x)^{-s}},$$

where X_0 denotes the set of closed points of X, and $N(x)$ denotes the cardinality of the residue field at $x \in X_0$. This infinite product converges for $\operatorname{Re} s > \dim X$, and conjecturally, it has a meromorphic continuation to the whole complex plane. I refer to [Ser1965] for the basic results and conjectures.

This thesis is concerned with studying the special values of $\zeta(X,s)$: the goal is to interpret in cohomological terms the vanishing orders and leading Taylor coefficients at $s = n \in \mathbb{Z}$. This is a part of the program that was envisioned by Stephen Lichtenbaum and initiated in [Lic2005, Lic2009a, Lic2009b], and the conjectural underlying cohomology theory is known as Weil-étale cohomology. Later on Matthias Flach and Baptiste Morin gave a construction of Weil-étale cohomology using Bloch cycle complexes $\mathbb{Z}(n)$ to study $\zeta(X,s)$ at $s = n \in \mathbb{Z}$, see [Mor2014] and [FM2016]. Their work concerns proper regular arithmetic schemes, and the goal of this thesis is to relax these restrictions while studying the case $n < 0$.

From now on n denotes a strictly negative integer.

In chapter 0 I collect various definitions and results that are used in the constructions. Most of this material is quite standard. This chapter is lengthy, but it is needed to set up the stage.

Chapter 1 is dedicated to a construction of Weil-étale complexes

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)).$$

This will be done in two steps: first I construct complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$, which by definition give a cone of certain morphism

$$\alpha_{X,n}\colon R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \to R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n))$$

in the derived category of complexes of abelian groups:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \\ \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \end{aligned}$$

Then I construct yet another morphism

$$i_\infty^*\colon R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

in the derived category and declare its mapping fiber to be $R\Gamma_{W,c}(X,\mathbb{Z}(n))$:

$$\begin{aligned} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \\ \to R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \end{aligned}$$

Finally, in chapter 2 I formulate the main conjecture. I use the regulator construction from [KLMS2006]. After reviewing the necessary preliminaries about Deligne cohomology and homology in §2.1, I define in §2.2 a morphism

$$Reg^\vee\colon R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}),$$

under the assumption that $X_\mathbb{C}$ is smooth and quasi-projective. Then $Reg^\vee$ is conjectured to be a quasi-isomorphism. This allows us to construct an ad hoc "cup product"

$$\smile\theta\colon R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \to R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]$$

that gives a long exact sequence of Weil-étale cohomology groups with real coefficients

$$\begin{aligned} \cdots \to H^i_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \\ \xrightarrow{\smile\theta} H^{i+2}_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R} \to \cdots \end{aligned}$$

Then the general theory of determinants of complexes of Knudsen and Mumford implies the existence of a canonical trivialization morphism

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} (\det\nolimits_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n)))\otimes_\mathbb{Z}\mathbb{R}.$$

Our main conjecture $\mathbf{C}(X,n)$, formulated in §2.3, says that the leading Taylor coefficient of $\zeta(X,s)$ at $s=n$ is given by

$$\lambda(\zeta^*(X,n)^{-1})\cdot\mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)),$$

while the corresponding vanishing order is

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

If X is proper and regular, then this is equivalent to Conjecture 5.12 and Conjecture 5.13 from [FM2016]. In particular, it is showed in [FM2016, §5.6] that if X is projective and smooth over a number ring, then the special value conjecture is equivalent to the Tamagawa number conjecture.

Finally, I verify in §2.4 that the conjecture is compatible with the operations of taking disjoint unions of schemes, gluing schemes from an open and closed part, and passing from X to the affine space $\mathbb{A}^r_X$. This means that taking as an input the schemes for which the conjecture $\mathbf{C}(X,n)$ is known, it is possible to construct new schemes, possibly singular, for which the conjecture $\mathbf{C}(X,n)$ holds as well. This is the main unconditional outcome of the machinery developed in this thesis.

Contents