SUMMARY

Students in, and even beyond, secondary school continue to have serious problems with algebra, in particular in giving meaning to algebraic formulas which are very abstract for them (e.g. Kieran, 2006). Many students lack symbol sense, that is, they have trouble with reading through formulas, recognizing the structure of formulas, and making sense of formulas. In many curricula, the importance of symbol sense is acknowledged (e.g., NCTM, 2000). The main aim of the present research was to promote aspects of students' symbol sense that enable students in grade 11 and 12 to read through formulas and to make sense of these formulas, and to deal with non-routine algebraic problems.

In chapter 1, we elaborate on the concept symbol sense, describe our strategy to teach this symbol sense to students in upper secondary school and give an outline of the different studies. Symbol sense is a very broad concept, which was described by Arcavi as "an intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools" (Arcavi, 1994, p. 25). Drijvers et al. (2011) see symbol sense as complementary to basic skills, like procedural work, with a local focus and algebraic calculations. Symbol sense forms a compass for basic skills and is about taking a global view, algebraic reasoning, and adopting a strategic approach. Pierce and Stacey (2004) use the concept 'algebraic insight' for interpreting and making sense of algebraic calculations that are performed via computer algebra systems and therefore include manipulations of formulas to determine equivalence of formulas. As our focus was exclusively on reading through and making sense of algebraic formulas and not on manipulating them, we use the term insight into algebraic formulas, defined as the ability to recognize the structure of a formula and its components, and to reason with and about formulas.

To give meaning to algebraic formulas, Kieran (2006) and Radford (2004) have suggested to use linking multiple representations, like table, graph, formula, and realistic context. However, except for linear and exponential formulas, linking formulas to realistic contexts is in general difficult. For our research, we chose to link formulas to graphs. Although it is recommended to use graphing tools such as graphic calculators for learning about functions and their representations (Hennessy et al., 2001; Heid et al., 2013), Goldenberg (1988) suggested to use graphing by hand to establish a better connection between formulas and graphs. The need for pen-and-paper activities was later found by others (Kieran & Drijvers, 2006; Arcavi et al., 2017). As students in upper secondary school have experience with graphing tools, we followed Goldenberg's suggestion and focused on graphing formulas by hand, without technology (graphing formulas). When graphing formulas, the formulas are linked to its graphs. Graphs give a Gestalt-view of a function, visualizing the "story" a function tells in a single picture, and so emphasize the function object character and show how the dependent and independent variables covary in relation to each other. In this way, several aspects that seem problematic in learning about functions and formulas are addressed: mathematical objects like functions are not directly accessible as physical objects, switching between the process—object character of a function (seeing a function both as an input-output machine and as an object (Moschkovich et al., 1993)), and covariational reasoning (coordinating how two varying quantities change in relation to each other (Carlson et al., 2002)).

To engage in algebra, a combination of basic skills and symbol sense is needed. However, it is hard to teach symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In this dissertation, we tried to promote students' insight into formulas and chose to teach graphing formulas for this purpose. The overall research question of this thesis is:

How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?

We conducted four studies to investigate this overall research question. Because it was not clear what knowledge and skills are needed to graph formulas effectively and efficiently, in studies 1 and 2 (chapter 2 and 3) we first investigated expert behavior and thinking in graphing formulas. The findings resulted in a framework that guided the intervention in study 3. In study 3 (chapter 4), we designed an intervention to teach grade 11 students' expertise in graphing formulas, that is, graphing through a combination of recognition and qualitative reasoning and investigated whether students' insight into algebraic formulas was promoted. In study 4 (chapter 5), we focused on the relation between students' symbol sense involved in graphing formulas and in solving algebraic problems.

In chapter 2, we investigate experts' strategies in graphing formulas. Expertise literature indicates that problem solving can be described in terms of recognition and heuristic search (Chi, 2011; Gobet, 1998; Gobet & Simon, 1996). To describe experts' strategies in graphing formulas, a two-dimensional framework was proposed, using levels of recognition and heuristics. The levels of recognition reflect the levels of awareness formulated by Mason (2003): from complete recognition and instantly knowing the graph, to decomposing the formula into manageable sub-formulas, to perceiving graph properties, to no recognition at all and only calculating some points. On each level of recognition, domain-specific heuristics were described and ordered from strong to weak. Strong heuristics give information about large parts of a graph, like using qualitative reasoning about the function's infinity behavior or when adding or multiplying two sub-graphs. Weak heuristics only give local information about the graph, like calculating a point of a graph. Two research questions guided this study: Does the framework describe strategies in graphing formulas appropriately and discriminatively? Which strategies do experts use in tasks graphing formulas?

In this case study, five experts in mathematics and three secondary-school math teachers thought aloud while graphing a more complex function ($y = 2x\sqrt{8-x} - 2x$) and had to find a formula that would fit a given graph. The video recordings were transcribed, cut into fragments which contained crucial steps of explanations, and analyzed. The results showed that all these steps could be encoded within the two-dimensional framework, generating paths in the framework. We concluded that the framework was discriminative, because different strategies by the participants gave different paths in the framework. The experts used various strategies when graphing formulas: some focused on their repertoire of formulas they could instantly visualize by a graph; others relied on strong heuristics, such as qualitative reasoning. The experts' main strategies were: recognizing function families and using their prototypical graphs, recognizing key graph features, using qualitative reasoning when exploring parts of the graph, e.g., infinity behavior or when composing two sub-graphs after decomposing a formula into two sub-formulas. The teachers hardly used function families and more often used weaker heuristics. It was concluded that expertise in graphing formulas does not involve calculations of derivatives, as all our experts seemed to hesitate to start such calculations and made mistakes when they did.

In chapter 3, we report on the study in which we investigated experts' recognition processes in graphing formulas. We focused on instantly graphable formulas. An instantly graphable formula (IGF) is a formula that a person can instantly visualize by a graph. IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. These building blocks can be combined (addition, multiplication, chaining, etc.) into new and more complex building blocks (e.g., IGFs $y = -x^4$ and $y = 6x^2$ can be combined into a 4-degree polynomial function $y = -x^4 + 6x^2$). The research questions in this study were: Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families? What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning? The five experts of study 1 worked on a card-sorting task to investigate what function families experts use, a matching task to investigate experts' recognition, and a thinking aloud multiple-choice task to portray experts' recognition processes. The experts' results in the card-sorting task showed that the categories they constructed, and the category descriptions, were very similar, although some experts made more sub-categories (e.g., differences between parabolas with a max versus with a min). These descriptions were closely related to the basic function families that are taught in secondary school: linear functions, polynomial functions, exponential and logarithmic functions, broken functions, and power functions. The experts had no problems with the matching task, in which they had to match formulas with one of the 21 alternative graphs.

The analyses of the thinking aloud protocols of the multiple-choice tasks were based on both Barsalou's model of organized hierarchical knowledge (Barsalou, 1992) and on Schwarz and Hershkowitz's (1999) descriptions of concept images, using prototypicality (the use of prototypical members of a category or function family), attribute understanding (the ability to recognize attributes of a function across representations), and part–whole reasoning (the ability to recognize that different formulas and/or different graphs relate to the same entity). The findings of these analyses suggested that experts' recognition of IGFs can be described with the Barselou model in which formulas, function families, prototypes, a set of attributes and values, and graphs are linked in well-connected hierarchical mental networks.

In chapter 4, we investigate how graphing formulas based on recognition and reasoning could be taught to grade 11 students with the aim to promote students' insight into algebraic formulas. The research question addressed was: How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas? In an intervention of five 90-minute lessons, 21 grade 11 students were taught to graph formulas by hand. The intervention's design was based on experts' strategies in graphing formulas, that is, using a combination of recognition and qualitative reasoning. We used the principles of teaching complex skills, that is, using a whole task approach, with support and reflection tasks (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002), and also included the meta-heuristic "questioning the formula" (Landa, 1983; Pierce & Stacey, 2007). The five whole tasks reflected the levels of recognition of the two-dimensional framework. First, attention was paid to a repertoire of basic function families with their characteristics. Then, single transformations of prototypes of the function families were addressed. In the third whole task, students practiced decomposing a formula into two sub-formulas and composing the sub-graphs. In the fourth whole task, the focus was on the recognition of graph features from a formula, e.g., the zeroes and extreme values. In the last whole task, students explicitly practiced qualitative reasoning about infinity behavior, weaker and stronger components of a formula, in- and decreasing of functions, etc. Qualitative reasoning is often used by experts and is characterized by its focus on the global shape of the graph, with global descriptions and ignoring what is not relevant.

The students did a written pre- and post-test, followed by a retention test after four months, which contained a graphing task and a matching task that was similar to the one used in chapter 3. Six students were asked to think aloud during the graphing tasks in the pre- and post-test. The pre-test results showed that the students lacked insight into formulas, and the thinking-aloud protocols suggested a lack of recognition and reasoning skills. The post-test results showed that students had improved their recognition of function families and graph features as well as their qualitative reasoning abilities. In a post-intervention questionnaire, the students themselves indicated that they understood formulas better. In the retention test, the scores on the graphing task and multiple-choice task were, as expected, lower than in the post-test, but significantly higher than in the pre-test. This suggested a long-lasting effect of the intervention. The findings of this study suggested that, although many students still had problems with more complex formulas, teaching graphing formulas to grade 11 students, based on recognition and qualitative reasoning, might be a means to promote student insight into algebraic formulas in a systematical way.

In chapter 5, we explore the relation between students' graphing abilities and their symbol sense abilities to solve non-routine algebraic tasks, like: How many solutions does this equation have? What y-values can this formula have? To solve these kinds of problems, students could use their graphing abilities, but also other aspects of symbol sense, like abandoning the symbolic representation, and using graphs and/or reasoning, instead of starting calculations. So, the symbol sense involved in graphing formulas is a subset of the symbol sense involved in solving these algebra tasks. We investigated whether students might be able to use the symbol sense involved in graphing formulas in other non-routine algebraic problems that could be solved with graphs and reasoning. The main research question of this study was: How do grade 12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks? Two sub-questions were formulated: To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebraic tasks with symbol sense? Is students' use of symbol sense in graphing formulas similar or different from their use of symbol sense in solving non-routine algebraic tasks? A written symbol sense test was administered to a group of 114 grade 12 students, including 21 students who had participated in the intervention described in chapter 4, and 93 students from five other schools across the Netherlands. Six students who were involved in the intervention were asked to think aloud during the symbol sense test, which consisted of 8 graphing tasks and 12 non-routine algebraic tasks. The results of the written test were graded, and the symbol sense use was analyzed and graded using four categories: blank, calculations, making a graph, recognition, and reasoning.

A positive correlation was found between students' graphing abilities and their abilities to solve algebra tasks and their symbol use when solving these tasks, also when corrected for students' general math abilities. Students who scored high on the graphing tasks did more often use the strategy "making a graph" when working on the algebra tasks. With respect to the second sub-question, we found that 16 of the 21 students involved in the teaching of graphing formulas by hand in the intervention of study 3 belonged to the 25% highest scoring students on the graphing tasks. These high scoring students used more symbol sense when solving non-routine algebra tasks than the other students. The six thinking-aloud students who were among these 16 students showed that they used similar aspects of symbol sense in both the graphing tasks and the algebra tasks, including combinations of recognition of function families and key graph features and qualitative reasoning. These findings seemed to confirm our expectations that students who are able to graph formulas by hand can use these abilities in a broader domain of non-routine algebra tasks.

In the concluding chapter 6, we first present the findings of the four separate studies, followed by discussion and conclusions with the main contributions and limitations of the studies. The main aim of this research was to promote aspects of students' symbol sense, that is, students' abilities to read through formulas, to make sense of formulas, and to use this symbol sense when solving algebraic problems. The premise in our study was that students have to make sense of algebraic formulas and, therefore, have to be able to read through them. If students cannot make sense of their algebra activities, they will not develop confidence in their algebraic work, which result in a reluctancy to engage in algebraic reasoning and thinking. To enable students to develop expertise and confidence in reading algebraic formulas, we selected a small but rich domain in algebra, namely graphing formulas. Graphing formulas requires students to read through many kinds of formulas, and it allows them to make sense of these formulas by linking formulas to their graphs. As our aim was to foster insight into formulas, we restricted the tasks to interpreting formulas and ignored algebraic manipulations, which are often at the core of regular algebra education and a source of problems for many students. We chose to graph formulas by hand because connections between formula and graph established via by hand activities are more effective than via computer graphing (Goldenberg, 1988). As experts are supposed to use insight into formulas, we investigated expert behavior in graphing formulas, identified essential thinking processes, and described these in terms of recognition and reasoning in a two-dimensional framework. This gave us a clue about what to teach. Based on the two-dimensional framework, we designed an intervention of five lessons of 90 minutes, the so-called GQR-design (Graphing based on Qualitative reasoning and Recognition). Through whole tasks, with help and reflection questions, and using "questioning the formula" as a leading meta-heuristic, graphing formulas was taught step-by-step and in a systematical way. In this GQR-design, explicit attention is paid to the interplay between recognition and reasoning by using combinations of function families with their prototypical graphs as building blocks, key graph features, and qualitative reasoning. The whole task approach forces students to take a global view for recognition, to reason and argue, and to consider their strategies, which are essential aspects of symbol sense (Drijvers et al., 2011). We expected that students could use these aspects of symbol sense once they had learned these through graphing formulas while solving non-routine algebra tasks. We designed a symbol sense test with non-routine algebra tasks that could be solved via recognition, reasoning, and making a graph. Results with this symbol sense test suggested that the students involved in the intervention were able to use their symbol sense in graphing formulas and were able to use graphs as visualizations while solving the non-routine algebra tasks. We concluded that teaching graphing formulas by hand with our GQR-design could be an effective means to teach students in the higher grades of secondary school aspects of symbol sense, like insight into algebraic formulas, that can be used to solve non-routine algebra tasks.

Inevitably, the studies reported have their limitations. In chapter 2 and 3, only five experts participated and worked on only two tasks, due to the labor-intensive method for strategy assessment. Although we expected that most common strategies were captured in the two-dimensional framework, testing a larger group of potential experts before describing expertise in graphing formulas might give another, more detailed picture of expertise. In the intervention in study 3, only one class of students from the Netherlands was involved, and no comparison group was included. However, one year and two years later, the same series of lessons from the intervention was used in two other groups in the same school, both of 23 students. Both groups made the same post-test that was used in our study 3, and the scores showed similar results. Although this might be a confirmation that students can develop insight into formulas via our GQR-design, we suggest future research including more students and teachers to further investigate whether and how students can improve their insight into formulas through GQR-design.

In the symbol sense test in study 4, we used a combination of graphing tasks and algebra tasks for research purposes. This might have suggested to use “making a graph” when working on the algebra tasks. In the test, we often used the variables x and y to make the test more recognizable for the students. In future studies, we suggest using other variables than x and y more often. The algebra tasks in the current test were limited to those that could be solved using graphing and reasoning. In future tests, we suggest broadening this scope. Algebraic problems often require a combination of reasoning, graphing and calculations, and a next step might be to also include tasks in which one has to consider whether calculations are required or not, like “how many solutions does the equation $3.6(1 - \mathrm{e}^{-2.5t}) = 10t$ have?” In the symbol sense test, we found that many students had problems with solving inequalities such as $x(x - 1) > 4x$. Instead of using their graphing abilities, half of the best graphing students started calculations and were often unsuccessful. These findings seemed to suggest that an inequality triggered previously learned associations, and that such associations might hinder later learned symbol sense. Further research is needed to investigate how just learned symbol sense can be incorporated in students’ strategies and habits to deal with algebraic problems. In study 4, we suggested that graphing formulas based on recognition and reasoning might be a means to teach symbol sense in upper secondary school that could be used by students to solve non-routine algebra tasks. More research is needed to clarify this suggestion. A next step might be to set up a quasi-experimental study, in which a group of students is taught to graph formulas like in the intervention, using a control group and a pre-test and post-test. As we expect that difficulties with insight into algebraic formulas and symbol sense are not exclusive to Dutch students, students and teachers from other countries should also be included in future studies.

This dissertation contributes to our knowledge about symbol sense and teaching symbol sense. Firstly, it describes the nature of expertise in terms of recognition, reasoning, and its interplay, and shows how this can be elaborated for the domain of graphing formulas. Secondly, it shows how grade 11 students can acquire insight into algebraic formulas through an innovative intervention about graphing formulas. And, thirdly, it explores how symbol sense might be taught to students.

As a first contribution of this research, we described the nature of expertise in graphing formulas in terms of recognition and heuristics in a two-dimensional framework. We identified several levels of recognition which determines the problem space and, therefore, the heuristic search: recognition guides heuristic search. On each level of recognition, we formulated heuristics in the two-dimensional framework stressing the interplay between recognition and domain-specific heuristic search. This approach differs from, for instance, descriptions of knowledge bases, in which mathematical competences are described in lists of several components, like conceptual knowledge, procedural knowledge, and strategic competence. Although the need for integration of different components has been stressed (Kilpatrick, Swafford, & Findell, 2001), this has, to our knowledge, not led to models in which these components are actually integrated. Describing expertise in terms of recognition and heuristics in a two-dimensional framework also seems possible in other domains of algebra. Pouwelse, Janssen and Kop (submitted) proposed a framework with recognition and heuristics for finding indefinite integrals in calculus. The framework could be used as an instrument for designing teaching material but also as an instrument in teacher professional development. Further research is suggested to explore how the interplay between recognition and heuristic search in other domains like solving equations could be described and used in designing teaching and/or in teacher professional development.

As a second contribution of the research, we showed how grade 11 students can acquire insight into algebraic formulas through graphing formulas via our GQR-design. The GQR-design differs from both regular and other innovative approaches to learning about functions, in particular regarding the link between formulas and graphs. Our approach differs from regular education about functions, which often focuses on the manipulation of algebraic expressions (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992) and on using graphing tools to explore function families and to work on calculus problems. In comparison to regular approaches, in the GQR-design, explicit attention is paid to recognition and to reasoning with and about functions. In our design, we use basic function families as building blocks for formulas, following Davis' suggestion to use larger thinking units to allow for better recognition of the formula's structure (Davis, 1983), and we pay attention to read key features from the formulas. Explicit attention is paid to reasoning, e.g., about parameters of function families, about infinity behavior, and when composing two sub-graphs. Other innovative approaches often focus on reasoning about functions, using graphing tools. In comparison to our approach, these approaches do not explicitly pay attention to qualitative reasoning and to the recognition and use of function families. The importance of qualitative reasoning and its omission in a mathematics curriculum was already stressed by Leinhardt et al. (1990), Goldenberg et al. (1992), Yerushalmy (1997), and Duval (2006), but to our knowledge, this qualitative reasoning has never been implemented in concrete and systematic teaching approaches.

The third contribution of this research is that it shows how symbol sense to solve non-routine algebra tasks might be taught to students. Symbol sense seems difficult to teach (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In regular education, many teachers and students focus on basic skills and manipulating formulas and expressions (Arcavi et al., 2017), expecting that students will develop symbol sense through these kinds of practices. Innovative approaches focus more on reasoning, and give suggestions how to teach this, for instance through using productive practices, such as reverse thinking and constructing examples (Friedlander & Arcavi, 2012; Kindt, 2011), using rich, collaborative tasks (Swan, 2008), and snapshots for classroom discussions (Pierce & Stacey, 2007). At the core of our strategy is the idea of teaching symbol sense in a small domain of algebra, graphing formulas by hand, allowing students to develop expertise in this domain. If the teaching of graphing formulas focuses on essential aspects of symbol sense, like taking a global view for recognition, (qualitative) reasoning, and strategic work, then these essential aspects of symbol sense might be transferred to a broader domain of algebra. In our GQR-design, these essential aspects of symbol sense were explicitly and systematically taught as thinking tools, whereas in other approaches these thinking tools are often implicit. Our research shows that students obtained insight into formulas, and learned essential aspects of symbol sense, which they could later use while solving non-routine algebraic problems in the symbol sense test. The students involved in the intervention indicated that they thought they understood functions better, could visualize formulas better, in particular basic functions, and indicated that qualitative reasoning was very new and motivating for them ("we now use global reasoning; it is fun, this kind of reasoning"). This suggests that the GQR-design is a motivating and systematical way to teach students aspects of symbol sense.

The ability to read through formulas and make sense of them is an important aspect of symbol sense and will also remain important in the future when technology will take over manipulation of algebraic formulas even further. People will have to interpret results, make global estimations about results, and understand what is going on. For this purpose, they will need to develop some, what we might call, formula sense, that includes: making sense of a formula, using function families as building blocks of formulas, identifying and using the structure of a formula, and using qualitative reasoning. These ideas might be relevant for new curricula for secondary school. In this research, we showed how this symbol sense can be taught to students via our GQR-design.