CHAPTER 6

General Conclusions

6.1 Introduction

Students, even beyond secondary school, have cognitive and affective difficulties with algebra and its abstract symbols (Arcavi, 1994; Arcavi et al., 2017; Ayalon et al., 2015; Chazan & Yerushalmy, 2003; Drijvers et al., 2011; Kieran, 2006; Hoch & Dreyfus, 2005, 2010; Oehrtman et al., 2008). In regular algebra education, the focus is often on manipulations, starting with all kinds of basic skills, like expanding brackets, factorizing, calculating zeroes, extreme values, etc. However, many students do not know how to use these basic skills in solving algebraic problems, and find it hard to look through algebraic formulas and make sense of them: they lack symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005; 2010; Oehrtman et al., 2008; Thompson, 2013). Symbol sense concerns a very general notion of "when and how" to use symbols (Arcavi, 1994), and involves strategic work, taking a global view, and algebraic reasoning, whereas basic skills involves a local view, procedural working, and algebraic calculations. In this way, symbol sense functions as a compass when using basic skills (Drijvers et al., 2011). When students lack symbol sense, they have problems with giving meaning to and reading through formulas, resulting in a lack of confidence and in reluctance to engage in algebraic reasoning; so, students will focus on just learned methods in algebra lessons, in particular on basic skills, and on the symbolic representations (Arcavi et al., 2017; Kieran, 2006; Knuth, 2000; Eisenberg & Dreyfus, 1994; Pierce & Stacey, 2007). It is not clear how symbol sense can be taught effectively and efficiently in a systematical way (Arcavi, 2005; Hoch & Dreyfus, 2005).

In this research we investigated how to learn aspects of symbol sense, in particular reading through algebraic formulas and making sense of them, that is, to recognize structure and key features, and to reason with and about formulas. We called these aspects of symbol sense insight into algebraic formulas. Although identifying equivalent formulas is also an aspect of symbol sense, this aspect was not our first concern. Our research focused on grade 11 and 12 students, so students who in regular education already have learned about functions.

We chose to use graphing formulas with one variable by hand, so without technology, as a context to teach insight into formulas. In graphing formulas, all kinds of formulas can be involved and linking formulas to graphs can give students the opportunity to make sense of these formulas (Kieran, 2006; Radford, 2004). We chose to use graphing formulas *by hand* because the connection between formula and graph is more effectively established via graphing by hand than via computer graphing (Goldenberg, 1988). As our aim was students learning to read through formulas and to make sense of them, graphing formulas by hand does not here focus on a detailed graph in itself, but rather on making rough sketches of graphs.

To explore what knowledge and skills are needed to perform a complex skill like graphing formulas (complex because of the large variety of different formulas), it is recommended to study expert behavior (Kirschner & Van Merriënboer, 2008; Schoenfeld, 1978), as experts are supposed to use symbol sense when graphing formulas by hand. An analysis of expert behavior is crucial because guidelines for both what and how to teach on graphing formulas by hand can be partly derived from such an analysis. The overall research question in this thesis was: *How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?*

6.2 Results of partial studies

First, we present the main findings of the four separate studies of this thesis, followed by a discussion with implications, limitations and directions for future research.

6.2.1 Findings from study 1 (chapter 2)

In this study, we investigated experts' strategies in graphing formulas. Expertise literature indicates that problem solving could be described in terms of recognition and heuristic search. A two-dimensional framework with the dimensions recognition and heuristics was developed. The research questions addressed in this study were: Does the framework describe strategies in graphing formulas appropriately and discriminatively? Which strategies do experts use in formula-graphing tasks? In a case study, five experts and three teachers had to graph a more complex function ($y = 2x\sqrt{8-x} - 2x$) and had to find a formula that would fit a given graph, while thinking aloud. The protocols were transcribed and were cut into fragments which contained crucial steps of explanations.

The results show that all these steps from the protocols of all eight participants could be encoded within the two-dimensional framework. The solution process generated a path in the framework. Different strategies by the participants gave different paths in the framework. Therefore, we concluded that the framework was also discriminative. The experts used a range of strategies in graphing formulas. The main strategies seemed to be: recognizing function families and using their prototypical graphs, recognizing key graph features, using qualitative reasoning when composing two sub-graphs after decomposing a formula into two sub-formulas, and when exploring parts of the graph, e.g., infinity behavior. For recognition, a repertoire of basic functions (Eisenberg and Dreyfus, 1994) which can be instantly visualized by a graph is important. Expertise in graphing formulas does not involve calculations of derivatives, as all our experts seemed to hesitate to start such calculations and made mistakes when they did.

6.2.2 Findings from study 2 (chapter 3)

In the second study, we investigated experts' recognition in graphing formulas and addressed the research questions: Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families? What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning? IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. These building blocks can be combined (addition, multiplication, chaining, etc.) into new and more complex building blocks (e.g. IGFs $y = -x^4$ and $y = 6x^2$ combining to polynomial function $y = -x^4 + 6x^2$). Experts are expected to have more, and more complex, IGFs than novices, which generally enables them to graph formulas with fewer demands on working memory (Sweller, 1994).

The same five experts as in the first study worked on a card-sorting task to investigate what function families experts use, a matching task to investigate experts' recognition, and a thinking aloud multiple-choice task to portray experts' recognition processes. The experts' results in the card-sorting showed that the categories they constructed, and the category descriptions, were very similar, although some experts made more sub-categories (e.g., differences between parabolas with a max versus with a min). These descriptions were closely related to the basic function families that are taught in secondary school: linear functions, polynomial functions, exponential and logarithmic functions, broken functions, and power functions.

To analyze the thinking aloud protocols of the multiple-choice tasks, Barsalou's model of organized hierarchical knowledge with categories was used (Barselou, 1992). To portray students' concept image of functions, Schwarz and Hershkowitz (1999) used prototypicality (the use of prototypical members of a category or function family), attribute understanding (the ability to recognize attributes of a function across representations), and part-whole reasoning (the ability to recognize that different formulas or different graphs relate to the same entity). We combined both to formulate a Barsalou model for recognizing IGFs with function families, attributes and values, and graphs, to analyse how experts solved the multiple-choice task.

We found that experts' recognition of IGFs could be described with the Barselou model, in which function families, prototypes, a set of attributes and values of the attributes, and graphs are linked. For instance, given a logarithmic formula such as $y = \log_3(2x + 4)$, a prototype $y = \log_3(x)$ or $y = \log(x)$ was instantly identified and attribute reasoning (translation, domain $x > -2$, and/or vertical asymptote at $x = -2$) resulted in a graph. We also found that experts could easily work from a graph to a formula. For instance, a graph with attributes like domain $x > a$, a vertical asymptote at $x = a$ and concave down was instantly identified as a logarithmic function.

The findings show what knowledge experts used in recognizing IGFs: they used the basic functions to organize the function families, they used prototypes to handle other exemplars of function families, and also used prototypes and attributes to link graphs and formulas of function families. Our study suggests that only learning and practicing basic functions is not enough to become proficient in linking the formulas and graphs of functions. Students need to learn how to handle parameters in formulas and they need opportunities to integrate their knowledge of prototypes and attributes of function families into well-connected hierarchical mental networks. Through this study, we were able to adjust our Barsalou model based on Schwarz and Hershkowitz (1999) with linkages between attribute and value sets, prototypes and function families and with linkages from graph to attributes, prototypes, and function families. Our study gives an impression of an expert "library" of properties that may be helpful to further describe the recognition and identification of objects, forms, key features, and dominant terms used in Pierce and Stacey's algebraic insight (Pierce and Stacey, 2004; Kenney, 2008).

6.2.3 Findings from study 3 (chapter 4)

In the third study, we investigated how graphing formulas based on recognition and reasoning could be taught to grade 11 students and whether this graphing could improve students' insight into algebraic formulas. The research question addressed was: How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas? The two-dimensional framework was the base for the design of an intervention consisting of a series of five lessons of 90 minutes. As graphing formulas can be considered as a complex task, a whole task approach, with support and reflection tasks, is recommended (Collins, 2006; Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002). The importance of the meta-heuristic “questioning the formula” is stressed by Landa (1983), Arievitch and Haenen (2005), and Pierce and Stacey (2007). The five whole tasks reflected the levels of recognition in the two-dimensional framework. First, attention was paid to a repertoire of basic function families with their characteristics. Then to translations of the prototypes of the function families. In the third whole task, students practiced decomposing a formula into two instantly graphable sub-formulas, graphing the sub-formulas, and composing the sub-graphs. In a subsequent whole task, the focus was on the recognition of graph features from a formula, e.g., the zeroes and extreme values. In the last whole task, students explicitly practiced to reason qualitatively about infinity behavior, weaker and stronger components of a formula, in- and decreasing of functions, etc. This way of reasoning is often used by experts and is characterized by its focus on the global shape of the graph and global descriptions and ignoring what is not relevant.

The 21 grade 11 students from the first author’s school who participated in the intervention made a written pre-, post-, and retention test after four months, which contained a graphing task and a matching task that was similar to the one used in study 2. Six students were asked to think aloud during the graphing tasks in the pre- and post-test. We found that in the pre-test the students lacked insight into formulas, and the thinking-aloud protocols suggested a lack of recognition and reasoning skills. The post-test results showed that students had improved their recognition of function families and graph features and their qualitative reasoning abilities. The students themselves indicated that their recognition and performances in graphing formulas had improved and that they understood formulas better. We interpreted that as the students having improved their insight into formulas. In the retention test, the scores on the graphing task and multiple-choice task were, as expected, lower than in the post-test, but higher than in the pre-test. This suggests a long-lasting effect of the intervention.

The findings of this study suggested that, although many students still had problems with more complex formulas, teaching graphing formulas to grade 11 students, based on recognition and qualitative reasoning, might be a means to promote student insight into algebraic formulas in a systematical way.

6.2.4 Findings from study 4 (chapter 5)

In study 4, the research question was: How do grade 12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks? We used the sub-questions: (1) To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebra tasks with symbol sense? And (2) How is students' symbol sense use in graphing formulas similar or different from their symbol sense use in solving non-routine algebra tasks?

The 21 students who were involved in the intervention in the third study made a written symbol sense test, together with 91 grade 12 students from five different schools throughout the Netherlands. The test consisted of eight graphing tasks and twelve non-routine algebra tasks, which could be solved by graphing and reasoning. We determined from the written test whether students could solve the tasks and what strategies they used (using symbol sense strategies like graphing or reasoning, or non-symbol sense strategy like calculations). Six students who participated in the intervention were asked to think aloud during the test.

With respect to the first sub-question, we found a positive correlation between students' graphing abilities and their abilities to solve algebra tasks and their symbol use when solving these tasks, also when corrected for students' general math abilities. High scoring students more often used strategies like making a graph and reasoning, and less often started calculations than students who were less successful.

With respect to the second sub-question, we found that 16 of the 21 students who were involved in the teaching of graphing formulas by hand in the intervention of study 3, belonged to the 25% highest scoring students on the graphing tasks. These students used more symbol sense when solving non-routine algebra tasks than the other students. Among these 16 students were the six thinking-aloud students, who showed that they used similar aspects of symbol sense in both the graphing tasks and the algebra tasks, including combinations of recognition function families and key graph features and qualitative reasoning. As symbol sense involved in graphing formulas is a subset of symbol sense involved in solving non-routine algebra tasks, these findings seem to confirm our expectations that students who are able to graph formulas by hand can use these abilities in a broader domain of non-routine algebra tasks. This suggests that teaching to graph formulas by hand might be an approach to promote students' symbol sense to solve non-routine algebraic problems.

6.3 Discussion and Conclusion

The aim of this research was to promote aspects of students' symbol sense, that is, students' abilities to read through formulas, to make sense of formulas, and to use this symbol sense when solving algebraic problems. The overall research question that led the research was: *How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?*

To answer this question, we assumed that it is extremely important that students can make sense of their algebra activities, and that students need some flexibility in their algebraic reasoning. Algebraic problems are not always represented in such way that students can instantly use their basic skills, and even if this is the case, they have to be able to recognize and select correct basic skill(s). If students cannot make sense of their algebra activities, they will not develop confidence in their algebraic reasoning, which result in a reluctancy to engage in algebraic reasoning, leading to inflexibility. Therefore, we chose a small but rich domain in algebra, namely graphing formulas. In graphing formulas, many different kinds of algebraic formulas are involved, it requires students to read through formulas, and it allows students to make sense of formulas. As our aim was to foster insight into formulas, we restricted the tasks to interpreting formulas and ignored algebraic manipulations, which are often at the core of regular algebra education. These restrictions would allow students to learn expertise in such a small domain and to make sense of algebraic formulas. We chose to graph formulas *by hand* because connections between formula and graph established via by hand activities seem to be more effective than via computer graphing. As experts are supposed to use insight into formulas, we investigated expert behavior in graphing formulas by hand and detected essential thinking processes. We described expert thinking in terms of recognition and reasoning in a two-dimensional framework. This gave us a clue about what to teach: a repertoire of function families with their characteristics and prototypical graphs, recognizing key graph features, and qualitative reasoning. We designed an intervention of five lessons of 90 minutes based on the two-dimensional framework: the GQR-design (Graphing based on Qualitative reasoning and Recognition). Through whole tasks, with help and reflection questions, and using "questioning the formula" as a leading meta-heuristic, graphing formulas was taught step-by-step and in a systematical way. In this GQR-design, explicit attention is paid to the interplay between recognition and reasoning by using combinations of function families with their prototypical graphs as building blocks, key graph features, and qualitative reasoning. The whole tasks approach forces students to take a global view for recognition, to reason and argue, and to consider their strategies, which are essential aspects of symbol sense (Drijvers et al., 2011). We expected that students could use these aspects of symbol sense learned through graphing formulas while solving non-routine algebra tasks. We designed a symbol sense test with non-routine algebra tasks that could be solved via recognition, reasoning, and making a graph. Results in a symbol sense test suggest that the students involved in the intervention were able to use their symbol sense in graphing formulas and were able to use graphs as visualizations while solving the non-routine algebra tasks. We conclude that teaching graphing formulas by hand with our GQR-design could be an effective means to teach students in the higher grades of secondary school aspects of symbol sense, like insight into algebraic formulas, that can be used to solve non-routine algebra tasks.

6.3.1 Contributions to Theory and Practice

The contributions of this research to the knowledge about symbol sense and teaching symbol sense are (1) that it describes the nature of expertise in terms of recognition, reasoning and its interplay, and shows how this can be elaborated for the domain of graphing formulas, and (2) that it shows how grade 11 students can acquire insight into algebraic formulas through an innovative intervention about graphing formulas, and (3) that it explores how symbol sense might be taught to students. We elaborate these three aspects.

As a first contribution of this research, we described the nature of expertise in terms of recognition and heuristics. From expertise research, it is known that experts have more structured knowledge compared to novices. This enables them to recognize more and make more sophisticated problem representations, which allow for more efficient searching in a problem space (Chi et al., 1981; Chi et al., 1982; Chi, 2011; De Groot, 1965; De Groot et al., 1996; Gobet, 1998). The level of recognition determines the problem space and, as a consequence, the heuristic search: recognition guides heuristic search. Based on this, we identified a two-dimensional framework to describe strategies in graphing formulas with different levels of recognition and heuristics, like qualitative reasoning about, e.g., infinity behavior, weaker/stronger components of a formula, etc. This two-dimensional framework stresses the interplay between recognition and reasoning. This approach differs from, for instance, descriptions of knowledge bases, in which mathematical competences are described in terms of conceptual and procedural knowledge, together with strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, & Findell, 2001). Conceptual knowledge refers to knowledge of concepts including principles and definitions which are connected in a network, and procedural knowledge refers to knowledge of procedures, including action sequences and algorithms used in problem solving (Star & Stylianides, 2013). The integration of the five different strands of mathematical competence has been stressed (Kilpatrick, Swafford, & Findell, 2001), but to our knowledge, has not led to models in which these components are actually integrated. The contribution of this research is that it shows how expertise in graphing formulas could be described through an interplay between recognition and domain-specific heuristic search. Recognition can be related to conceptual knowledge about function families with their characteristics and graph features. Domain-specific heuristics in graphing formulas like qualitative reasoning when composing two sub-graphs (after decomposing a formula into sub-formulas) and when exploring infinity behaviour, weaker/stronger components of a formula can be related to procedural knowledge. Also, strategic competence and adaptive reasoning can be related to the two-dimensional framework. The strategic component can be related to different routes in the framework that might lead to the graph of a formula, so to different strategies. Adaptive reasoning is included in the framework because, on each level of recognition, the framework gives suggestions to make progress in the graphing.

Describing expertise in terms of recognition and heuristics in a two-dimensional framework, also seems possible in other domains of algebra. Pouwelse, Janssen and Kop (submitted) proposed a framework with recognition and heuristics for finding indefinite integrals in calculus. The framework could be used as an instrument for designing teaching material but also as an instrument in teacher professional development, as it might allow teachers to reflect on their current teaching and inspire them to adjust it. Further research is needed to explore how the interplay between recognition and heuristic search in other domains could be described and used in designing teaching and/or in teacher professional development.

As a second contribution of the research, we showed how grade 11 students can acquire insight into algebraic formulas through graphing formulas. We used the two-dimensional framework as a base for our GQR-design (Graphing based on Qualitative reasoning and Recognition), an innovative series of lessons on graphing formulas. The levels of recognition in the framework were used as the meta-heuristic “questioning the formula”, stimulating students to take a global view before starting their graphing work. These levels of recognition were also used to formulate five whole tasks, with help and reflection questions. Our GQR-design is an innovative approach to teach about functions in a systematical and structural way in grade 11, but also in grade 12. Our research focused on students in the higher grades of secondary school, who learned in grade 8 and 9 about basic functions, like linear, quadratic, exponential functions, and in grade 10, with using graphic calculators, about power, rational, logarithmic functions. Much research is known about learning linear and exponential functions in lower secondary school, and how students might make sense of these functions and acquire insight into the formulas by linking them to realistic contexts. However, in higher secondary school, students must deal with many more different functions, which cannot easily be linked to realistic contexts.

The GQR-design differs from both regular and other innovative approaches to learning about functions, in particular regarding the link between formulas and graphs. Our approach differs from regular education about functions that often focuses on the manipulation of algebraic expressions (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992) and on using graphing tools to explore function families and to work on calculus problems. In comparison to regular approaches, in the GQR-design, explicit attention is paid to recognition of function families and key graph features and to reasoning with and about functions. The first two whole tasks focus on a repertoire of function families with their characteristics, which are used as building blocks of formulas in the other whole tasks. In the fourth whole task, students learn to read key features from the formulas, e.g., reading the zeroes or extreme values from a formula, and in each whole task, attention is paid to reasoning, for instance about parameters of function families (in the second whole task), when composing two sub-graphs (in the third whole task), when exploring parts of the graph (e.g., about infinity behavior of the function in the fifth whole task).

Other innovative approaches often focus on reasoning about functions, using graphing tools; for instance, about the composition and translation of graphs (Schwartz & Yerushalmy, 1992; Yerushalmy & Gafni, 1992; Yerushalmy, 1997), about the role of parameters (Drijvers, 2003; Heid et al., 2013), and about special function families (Heid et al., 2013). In comparison to our approach, these approaches do not explicitly pay attention to qualitative reasoning and to the recognition of function families. Our GQR-design focuses on qualitative reasoning, with its focus on the global shape and on global descriptions, and with ignoring what is not relevant in the problem situation. The importance of qualitative reasoning and its omission in a mathematics curriculum was already stressed by Leinhardt et al. (1990), Goldenberg et al. (1992), Yerushalmy (1997), and Duval (2006), but to our knowledge, this idea has never been implemented in concrete and systematic teaching approaches. In our GQR-design, we use basic function families as building blocks for formulas, which, according to Davis (1983) could help students to recognize the structure of a formula. Davis (1983) has suggested that students learn to use larger thinking units, because they often work on an atomic level, that is, the role of each number and variable is analyzed, which makes it difficult to recognize any structure. (Davis, 1983). The larger thinking units and the use of qualitative reasoning might relieve working memory (Sweller et al., 2019), and might account for the results on the pre-, post-, and retention tests in study 3.

The third contribution of this research is that it shows how symbol sense to solve non-routine algebra tasks might be taught to students. Symbol sense is difficult to teach (Arcavi et al., 2017; Hoch & Dreyfus, 2005), probably because symbol sense is a very broad concept, involved in many aspects of algebraic thinking and working. Therefore, it seems hard to teach symbol sense in a systematical way, and consequently, students have problems with symbol sense. Students, also in upper secondary school, seem to avoid engaging in algebraic thinking and reasoning, and to focus on just learned methods (Arcavi et al., 2017; Kieran, 2006; Knuth, 2000; Eisenberg & Dreyfus, 1994; Pierce & Stacey, 2007). In regular education, many teachers and students focus on basic skills and manipulating formulas and expressions (Arcavi et al., 2017), expecting that students will develop symbol sense through this kind of practice. Innovative approaches focus more on reasoning, and give suggestions how to teach this, for instance, through using productive practices, such as reverse thinking and constructing examples (Friedlander & Arcavi, 2012; Kindt, 2011), using rich, collaborative tasks (Swan, 2008), and snapshots for classroom discussions (Pierce & Stacey, 2007).

At the core of the current research is the idea of teaching symbol sense in a small domain of algebra, graphing formulas by hand, allowing students to develop expertise in this domain. Graphing formulas can be considered a small domain in algebra because the task is clear and easily recognizable (make a sketch of the graph), but it is also a complex task because of the many different (types of) formulas that can be involved. The teaching of graphing formulas should focus on essential aspects of symbol sense, among them taking a global view for recognition, (qualitative) reasoning, and strategic work, which would allow some transfer of these essential aspects of symbol sense to a broader domain of algebra. In the GQR-design, students can learn these essential aspects of symbol sense in a systematical way. They learn how to use recognition, reasoning, and the interplay between recognition and reasoning, with thinking tools like the meta-heuristic "questioning the formula", a repertoire of function families, and qualitative reasoning. The GQR-design differs from other approaches by explicitly teaching these thinking tools, which are often implicit in other approaches. Our research shows that students obtained insight into formulas, and learned essential aspects of symbol sense, which they could later use while solving non-routine algebraic problems in the symbol sense test. The students involved in the intervention indicated that they thought they understood functions better, could visualize formulas better, in particular basic functions, and indicated that qualitative reasoning was very new and motivating for them ("we now use global reasoning; it is fun, this kind of reasoning"). This suggests that the GQR-design is a motivating and systematical way to teach students aspects of symbol sense.

6.3.2 Limitations and suggestions for future research

In this section, we address the limitations of the different studies and suggest directions for further research. In study 1 and 2, only five experts participated: two mathematicians who had been teaching calculus and analysis to first-year students at university, an author of a mathematics textbooks, who was also a teacher in secondary school, a math teacher who was involved in the National Math Exams and was a secondary school teacher, and a math teacher educator in university. All had a master's degree in mathematics, and two had a PhD in mathematics and had been working as a teacher for more than 20 years. During their career, they had been graphing many formulas without technology. Therefore, we considered them experts in graphing formulas, since we did not know other criteria for expertise in this domain of graphing formulas. We realized that this criterion for expertise was a bit vague. Testing a larger group of potential experts before describing expertise in graphing formulas might give another, more detailed picture of expertise. The experts worked on only two tasks, due to the labor-intensive method for strategy assessment: graphing one complex formula ($y = 2x\sqrt{8-x} - 2x$) and finding a formula fitting a given graph. Although we expected that most common strategies were captured in the two-dimensional framework, future research, involving more and other functions could provide information on whether alternative strategies not mentioned in the framework are used regularly.

In study 3, we investigated the GQR-design, a series of lessons on graphing formulas by hand, that is based on the two-dimensional framework and focuses on teaching expert strategies in graphing formulas, that is, a combination of recognition and qualitative reasoning. We used the theory of teaching complex skills to formulate three design principles: the use of whole tasks, to support students when working on these whole tasks, and the use of the meta-heuristic “questioning the formula”. The levels of recognition of the two-dimensional framework form the backbone of the series of lessons, as they reflect the five whole tasks and the meta-heuristic “questioning the formula”. The GQR-design is meant for higher grades in secondary school, when students already have learned about basic function families, about transformations, and graph features. Thus, this series has a formative character. Therefore, ideas of Swan and Burkhardt about formative assessment were used (Swan, 2005; Burkhardt & Swan, 2013), resulting in whole tasks about differences and similarities of two graphs or formulas (whole task 2) and about categorizing functions according to their infinity behavior (whole task 5). Through whole tasks, students are confronted from the start with the full complexity of graphing formulas, that is, the interplay between recognition and reasoning. Because of time constraints, on each level of recognition, only one whole task was used. Although the limited time demands of this series is a strong point, we would recommend to consider Kirschner and Van Merriënboer’s (2008) suggestion to use more variability in the whole tasks (so, more whole tasks on each level of recognition), with more practice of the integration and coordination of all sub-skills. A second design principle was to support students when working on the whole tasks. For each whole task, help is offered in the teaching material, as well as reflection questions in which own examples are demanded. Other aspects of support were students cooperating in pairs or groups of three, and the modeling of expert behavior in graphing formulas by the teacher. A suggestion might be to use video to show the modeling of expert thinking processes in graphing formulas. To improve students’ reflection, one might consider the implementation of cumulative reflection tasks, which require students to reflect not only on the just completed task but also on all previous tasks. The third design principle was using the meta-heuristic “questioning the formula”. The students improved their recognition, as was shown in the post-test and retention test, but the thinking aloud protocols did not show that students had started to consciously question the formula. This might mean that the students had already automatized the habit of questioning the formula, as was our purpose. However, because only some of the better performing students showed that they considered their strategies, we believe that more attention should be paid to the habit of consciously questioning the formula.

Several aspects in the series of lessons might be adjusted when the series of lessons is used a next time. Although we thought that transformations of basic functions should be familiar to the students, the whole task on transformations (whole task 2) took more time and was more difficult for them than we had expected. We suggest taking more time for this whole task. Explicit use of qualitative reasoning was new to the students, and this kind of reasoning was demonstrated several times by the teacher. The results of the pre- and post-test suggested that the students had started to use qualitative reasoning, but many of them still had problems with using qualitative reasoning to compose sub-graphs and to explore parts of a graph. We suggest paying more attention to this qualitative reasoning, in particular in whole task 3 and 5. Another point for consideration is to pay more attention to third- and fourth-degree polynomials as function families. Probably because polynomial functions were not explicitly considered as a function family in the teaching material and because of practicing to decompose a formula into sub-formulas, many students used decomposing $y = -x^4 + 2x^2$ into $y = -x^4$ and $y = 2x^2$, but then had problems with the composition of the two sub-graphs. Recognizing $y = -x^4 + 2x^2$ as a member of the fourth degree polynomial function family with its characteristics would be helpful. In such situation, one might consider factorizing the formula ($y = x^2(-x^2 + 2)$), which would enable one to easily find the zeroes of the function. These findings also suggest that small manipulations, like factorizing might be helpful and needed, and we suggest to include these in a next series of lessons.

In the intervention, only one class from the Netherlands was involved, and no comparison group was included. However, one year and two years later, the same series of lessons from the intervention was used in two other groups in the same school, both of 23 students. Both groups made the same post-test that was used in study 3. The scores of both groups showed similar results to those of our 21 students in the study 3. Although this might be an indication that students can develop insight into formulas via this series of lessons, we suggest future research including more students and teachers to further investigate whether and how students can improve their insight into formulas through GQR-design.

In study 4, a symbol sense test was designed to investigate whether grade 12 students' abilities to graph formulas by hand were related to their abilities to solve non-routine algebra tasks and to their use of symbol sense. These algebra tasks were limited to those that could be solved using graphing and reasoning (e.g., discussing the number of solutions of a given equation or the *y*-values of a function), as our research focused on reading through formulas and making sense of them, and not on algebraic calculations. In the symbol sense test, we used a combination of graphing tasks and algebra tasks. Such a combination in a single test might suggest using graphs when working on the algebra tasks. As "making a graph" is considered a symbol sense strategy, we suggest being careful with explicit graphing tasks in future symbol sense tests. In the current test, the variables *x* and *y* were used predominantly to make the test more recognizable for the students. In future studies, we suggest using other variables than *x* and *y* more often, because working with such variables is also an aspect of symbol sense. Another issue is the selection of non-routine algebra tasks in the symbol sense test. In the current symbol sense test, we had several types of tasks: about the number of solutions of equations, about the *y*-values of functions, about inequalities, about approximations of functions when *x* is very large, about what information a formulas tell about a give situation, about the location of a maximum of functions. In future tests, we want to broaden the scope of these non-routine algebra tasks that can be solved with combinations of recognition and reasoning, so without algebraic calculations, like tasks about integrals and graph features, for example, "calculate $\int_{-4}^{4} x^3 e^{-x^2}\, dx$", and "how many zeroes, extreme values, and points of inflection has $y = (x-3)^2(x-5)^2$". As indicated above, the tasks in the current test could be solved with reasoning and graphing, and algebraic calculations were not needed. However, algebraic problems often require a combination of reasoning, graphing *and* calculations. Solving algebraic problems with symbol sense includes recognizing when reasoning is sufficient and when calculations are required to solve the problem. A next step is also to include tasks in which one has to consider whether calculations are required or not. Two examples to illustrate these kinds of tasks. First, "how many solutions has the equation $3.6(1 - \mathrm{e}^{-2.5t}) = 10t$?". Second, "consider the quadratic function $y = x^2 - 3$ and a family of linear functions $y = ax + 3$; for which value of a is the area bounded by parabola and by the line minimal?" (Stylianou & Silver, 2004).

In the symbol sense test, we found that students had problems with solving inequalities like $x(x-1) > 4x$. The 25% best graphing students were less successful on this task (score of .57) than the second 25% best graphing students (score of .70). Instead of using their graphing abilities, half of the best graphing students started calculations and were often unsuccessful. These findings seem to suggest that an inequality triggered previously learned associations, and that such associations might hinder later learned symbol sense. Further research is needed to investigate how just learned symbol sense can be incorporated in students' strategies and habits to deal with algebraic problems.

In study 4, we suggested that graphing formulas based on recognition and reasoning might be a means to teach symbol sense in grade 11 that could be used by students to solve non-routine algebra tasks. More research is needed to clarify this suggestion. A next step might be to set up a quasi-experimental study, in which a group of students is taught to graph formulas like in the intervention, using a control group and a pre-test and post-test. As we expect that difficulties with insight into algebraic formulas and symbol sense are not exclusive to Dutch students, students and teachers from other countries should also be included in future studies.

The ability to read through formulas and make sense of them is an important aspect of symbol sense and will remain important in the future. We expect that technology will take over many algebraic manipulations. However, to be able to use this technology, people have to interpret results, make global estimations about results, and understand what is going on. For this purpose, they will need some symbol sense, have to be able to question the problem and to read through formulas in models, to use visualizations in problem solving, and to have confidence in their own algebraic reasoning. Therefore, students in school need to develop some formula sense, that includes:

- making sense of a formula
- using function families as building blocks of formulas
- identifying and using the structure of a formula
- interpreting the role of parameters in a formula
- ignoring what is not relevant for a problem situation
- using a graph as a visualization of a function
- reasoning with and about formulas

These ideas might be interesting for developing curricula for secondary school. In many curricula the importance of symbol sense is acknowledged (e.g., NCTM, 2000). However, this is often in terms of understanding and not in concrete terms, such as in our list of formula sense above. In this research, we showed how reading through formulas and making sense of them can be taught to students via our GQR-design and that these aspects of symbol sense can be used by students when solving non-routine algebra tasks. We suggest that a similar approach might be successful in lower secondary school as well. Through such an approach, all students might be able to learn insight into formulas and develop some confidence in their own reasoning with and about algebraic formulas.