CHAPTER 1

General Introduction

1.1 Introduction

Algebra is difficult for many secondary school students, and even beyond secondary school (Kieran, 2006). In literature again and again (e.g. Arcavi, Drijvers, & Stacey, 2017; Chazan & Yerushalmy, 2003; Drijvers, Goddijn, & Kindt, 2011; Kieran, 2006; Arcavi, 1994; Ayalon, Watson, & Lerman, 2015; Hoch & Dreyfus, 2005, 2010; Oehrtman, Carlson, & Thompson, 2008) it is found that:

- for many secondary school students' algebraic formulas with their symbols are very abstract
- students have difficulties to give meaning to algebraic formulas
- students have serious cognitive and affective difficulties with algebra, resulting in a lack of confidence to engage in algebra
- algebra is often taught through series of similar exercises, with a focus on basic skills with algebraic calculations.

On all educational levels, students have problems reading through algebraic formulas, that is, to see a formula as a whole rather than a concatenation of letters, and to recognize its global characteristics: they lack symbol sense (Arcavi, 1994). Arcavi introduced the concept of symbol sense as "an intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools". Drijvers et al. (2011) see symbol sense as complementary to basic skills. Basic skills in algebra are about procedural work, with a local focus and an emphasis on algebraic calculations. Symbol sense is about taking a global view, adopting a strategic approach and algebraic reasoning, and forms a compass for basic skills. For example, expanding brackets is a basic skill, but whether it is efficient to expand brackets in a problem situation is a matter of symbol sense. Symbol sense is especially important if a task is not recognized as a standard algebraic task and basic skills cannot be used immediately. In such situations, symbol sense is needed to know how basic skills can be used and which. Therefore, symbol sense is indispensable in solving non-routine algebraic tasks, and vice versa, students' performances in non-routine tasks is a measure of their symbol sense.

Symbol sense is very broad and is involved in three phases of the problem-solving cycle. Pierce and Stacey (2004) used the concept "algebraic insight" to capture the symbol sense involved in the solving phase when using computer algebra (CAS), that is, proceeding from the mathematical problem to the mathematical solution (see Figure 1.1).

Real world problem	**Formulate** →	**Mathematical problem**
Check ↑	⇧ **Symbol sense** ⇨ ⇩	**Solve** ↓
Real world solution	← **Interpret**	**Mathematical solution**

Figure 1.1 Symbol sense in problem-solving cycle, based on Pierce & Stacey (2004)

Algebraic insight has to do with the ability to plan, monitor, estimate and interpret algebraic calculations, and has two aspects: algebraic expectation and ability to link representations. Algebraic expectation is about skills to scan expressions for clues that allow one to see and predict patterns and make sense of symbolic operations. A more detailed description is shown in Table 1.1.

Table 1.1 Algebraic insight framework (Pierce and Stacey, 2004)

1	Algebraic expectation
1.1	Recognition of conventions and basic properties
1.1.1.	know meaning of symbols
1.1.2.	know order of operations
1.1.3.	know properties of operations
1.2.	Identification of structure
1.2.1.	identify objects
1.2.2.	identify strategic groups of components
1.2.3.	recognize simple factors
1.3.	Identification of key features
1.3.1.	identify form
1.3.2.	identify dominant term
1.3.3.	link form with solution type
2	Ability to link representations
2.1	Linking of symbolic and graphic representations
2.1.1	link form with shape
2.1.2	link key features with likely positions
2.1.3	link key features with intercepts and asymptotes

In our research we aimed at the development of students' abilities to read though algebraic formulas and to make sense of them. In their algebraic insight, Pierce and Stacey (2004) focused on interpreting and making sense of algebraic calculations that are performed via

CAS, and they included manipulations of formulas, for instance, to determine equivalence of formulas. As our research was not on manipulating algebraic formulas, but exclusively on reading through algebraic formulas and making sense of them, we focused on a subset of algebraic insight. Therefore, we use the term insight into algebraic formulas: to identify the structure of a formula and its components, and to reason with and about formulas. Identifying structure in algebra includes abilities such as seeing an algebraic expression as an entity, recognizing the expression as a previously met structure, dividing the entity into sub-structures, and recognizing the connection between structures (Hoch & Dreyfus, 2010). Teaching symbol sense is not straightforward (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In this thesis we investigated how to promote this insight into algebraic formulas for students in grades 11 and 12.

1.2 Using graphs to learn about formulas

Many studies have suggested how students might learn about linear formulas and make sense of them by linking them to realistic contexts. In lower secondary school it is easy to link linear and exponential formulas to realistic contexts, but for more complex formulas in upper secondary school, like logarithmic, root, rational functions, and compositions of functions, the link to realistic contexts is in general difficult. There is less research how the students in upper secondary school might learn to develop insight into these more complex formulas. Besides linking formulas to realistic contexts, Kieran (2006) and Radford (2004) have suggested using multiple representations to make sense of formulas. A formula is one of the representations of a mathematical function, besides others like a table, a graph, a verbal description. A mathematical object like a function can only be studied through its representations. Different representations give different information about a function (Arzarello, Bazzini, & Chiappini, 2001). Formulas stress the input-output dependency, whereas graphs give a Gestalt-view of the function, visualizing the “story” a function tells in a single picture. Interpreting graphs seems easier than interpreting algebraic formulas for students, as graphs seem to be more concrete for them.

In mathematics education, to use graphing tools such as graphic calculators for learning about functions and their multiple representations is recommended (Hennessy, Fung, & Scanlon, 2001; Kieran & Drijvers, 2006; Heid, Thomas, & Zbiek, 2013; Philipp, Martin, & Richgels, 1993; Yerushalmy & Gafni, 1992). However, Goldenberg (1988) found that students established the connection between formula and graph more effectively when they did graph by hand than if they only performed computer graphing. Others have confirmed the need for pen and paper activities when learning about formulas (Kieran & Drijvers, 2006). Therefore, we focused on graphing formulas by hand, without technology. In this thesis we refer to this by *graphing formulas*. In the past, graphing a formula was a time-consuming goal in itself. Via a fixed step-by-step plan, a function was investigated by calculating its domain, zeroes, extreme values (via the derivative), and asymptotes. This approach caused students to focus on the many calculations, and not to reason about the functions. As our aim is to promote students' insight into algebraic formulas, our approach of graphing formulas does not focus on calculations and on detailed graphing but on reading through formulas, reasoning and rough sketches of a graph.

1.3 Students' difficulties learning about formulas

Using graphing formulas to promote insight into formulas might address in a natural and integrated way several aspects that seem problematic in learning about functions.

First, mathematical objects like functions are not directly accessible as physical objects. Only through representations and combining the information obtained through different representations can one understand a function and the rich concept image of the function (Thomas, Wilson, Corballis, Lim, & Yoon, 2010; Tall & Vinner, 1981). Thus, the translation from one representation of a concept to another is, like in graphing formulas, at the core of doing and understanding mathematics (Duval, 2006; Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009; Lesh, 1999; Thomas & Hong, 2001).

Second, students are often found to have difficulties with the so-called process—object character of a function, that is, seeing a function both as an input-output machine and as an object, which can be used e.g. to reason about and to categorize (Ayalon et al., 2015; Breidenbach, Dubinsky, Hawks, & Nichols; 1992; Gray & Tall, 1994; Moschkovich, Schoenfeld, & Arcavi, 1993; Oehrtman et al., 2008; Sfard, 1991). Formulas stress the function's process character, but graphs appeal to a Gestalt-view, and stress the object character (Kieran, 2006; Moschkovich et al., 1993; Schwartz & Yerushalmy, 1992).

Third, graphing formulas is related to covariational reasoning. Students have difficulties with this kind of reasoning. Covariational reasoning is the ability to coordinate an image of two varying quantities and to note how they change in relation to each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) and is found to be essential to understand major concepts of calculus: functions, limits, derivatives, rates of change, concavity, inflection points, and their real world interpretations (Carlson et al., 2002; Oehrtman et al., 2008). Covariational reasoning is often used in realistic contexts, but it is also used with algebraic functions when, “imagining running through all input-output pairs simultaneously and so reason about how a function is acting on an entire interval of input values” (Carlson et al., 2002). Carlson, Madison, and West (2015) found that in an exam only 37% of the university students were able to select the correct graph (out of five alternatives) of $f(x) = 1/(x-2)^2$, indicating, according to the authors, that many students were not able to reason “as the value of x gets larger the value of y decreases, and as the value of x approaches 2, the value of y increases.”

Fourth, students have difficulties to recognize the structure of formulas. Ernest (1990) suggested to construct a syntactical tree: via an iterative procedure, algebraic expressions are decomposed into meaningful parts (building blocks) by identifying the main operator of the expression.

In sum, through graphing formulas attention is paid to switching between representations of functions, to the process—object character of functions, to covariational reasoning, and to the structure of formulas and their components.

1.4 Research questions

To engage in algebra, one needs a combination of basic skills and symbol sense. In regular education there is an overemphasis on basic skills. It has been acknowledged that it is hard to teach symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In this research we focused on one aspect of symbol sense, that is, insight into formulas. Insight has been defined as the ability to recognize the structure of a formula and its key features and to reason with and about a formula. To teach this aspect of symbol sense, we chose graphing formulas by hand. Graphing formulas by hand requires reading through formulas and includes many other aspects that students find difficult when learning about functions. This led to the overall research question of this thesis:

How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?

1.5 Outline of the thesis

To investigate this main question, we designed an intervention in which a group of 21 students were taught how to graph formulas. However, it was not clear what knowledge and which skills are needed to graph formulas. In studies 1 and 2, we studied expert behavior in graphing formulas and their recognition processes. In study 3, the intervention, based on expert strategies in graphing formulas, was designed and tested. In study 4 we investigated whether there is a positive relation between students' abilities to graph formulas and their abilities to solve non-routine algebraic problems with symbol sense. In chapter 6, we summarize the findings of the studies presented and the practical and scientific implications.

1.6 Characterization of studies 1–4

Study 1, described in chapter 2, was about identifying a framework for graphing formulas from expert strategies. Although graphing formulas is a well-described task, it can be complex because of the large variety of functions that may be involved. To investigate what is needed for such a complex task, to examine expert behavior has been recommended (Schoenfeld, 1978; Kirschner & Van Merriënboer, 2008). In expertise research it has been found that experts, compared to novices, have more structured knowledge, which enable them to make a more sophisticated problem representation and reach higher levels of recognition, and to search more efficiently in a problem space (Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Chi, 2011; De Groot, 1965; De Groot, Gobet, & Jongman, 1996; Gobet, 1998). Berliner and Ebeling (1989) formulated a model in which performance is a function of two variables: recognition and heuristic search. In this model, the degree of recognition determines the problem space and, as a consequence, the heuristic search. Thus, although expertise is described in terms of recognition and heuristic search in the literature (Chi, 2011; Gobet, 1998; Gobet & Simon, 1996), it appears this interplay has never been used in designing concrete teaching. We formulated a two-dimensional framework to describe strategies in graphing formulas, using levels of recognition and at each level of recognition, heuristics, that is, reasoning with and about formulas, to graph formulas. The levels of recognition in this framework reflect the different levels of awareness that have been formulated by Mason (2003): from complete recognition and instantly knowing the graph, to decomposing the formula into manageable sub-formulas, to perceiving graph properties, to no recognition at all and only calculating some points. We had two research questions: Does the framework describe strategies in graphing formulas appropriately and discriminatively? Which strategies do experts use in tasks graphing formulas? Five experts and three secondary-school math teachers were asked to solve two complex graphing tasks while thinking aloud.

The second study, in chapter 3, was about unraveling experts' recognition processes. Our research questions were: Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families? What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning?

Three different tasks were developed to elicit the experts' repertoires of IGFs and to explore the experts' recognition processes: a card-sorting task, a matching task, and a thinking-aloud multiple-choice task. The tasks were administered to the same five experts from study 1. The participants' categorizations of the card-sorting were compared to an expert categorization. The data analysis of the multiple-choice task was based on Barsalou (1992) and Schwarz and Hershkowitz (1999), using prototype, attribute, and part-whole reasoning.

The third study, in chapter 4, we investigated how to teach grade 11 students expertise in graphing formulas, that is, using a combination of recognition and qualitative reasoning to graph formulas. The research question addressed was: How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas? An intervention consisting of a series of five 90 minutes lessons was designed, using principles of teaching complex skills and the meta-heuristic "questioning the formula". The teaching focused on a repertoire of basic function families with their characteristics and on qualitative reasoning using prototypes, graph features, and exploration of parts of a graph, like infinity behavior. A group of 21 grade 11 students were involved in the intervention and made a pre, post, and retention tests, and filled in as well a post-intervention questionnaire. During the pre and post tests, six students were asked to think aloud.

The fourth study, in chapter 5, explored the relation between students' graphing abilities and their symbol sense abilities to solve non-routine algebraic tasks. We investigated whether students might be able to use insight learned in the domain of graphing formulas, in broader domains of algebra, with problems like: How many solutions does this equation have? What y-values can this formula have?

We limited the algebraic problems to those that can be solved with graphs and reasoning. Besides the symbol sense involved in graphing formulas, the students needed also another aspect of symbol sense, namely, to abandon the symbolic representation, and to use graphs and/or reasoning, instead of starting calculations. This led to the fourth research question: How do grade

12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks? We formulated two sub-questions: To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebraic tasks with symbol sense? Is students' use of symbol sense in graphing formulas similar or different from their use of symbol sense in solving non-routine algebraic tasks? A symbol sense test was administered to a group of 114 grade 12 students, including 21 students who had participated in a previous intervention described in study 3. The test consisted of 8 graphing tasks and 12 non-routine algebraic tasks, which could be solved by graphing and reasoning. The results of the written test were graded, and the symbol sense use was analyzed using four categories: blank, calculations, making a graph, recognition and reasoning. To get a more detailed picture of students' symbol sense, six students, all involved in the intervention of the third study, were asked to think aloud during the symbol sense test.