Néron models in high dimension: Nodal curves, Jacobians and tame base change

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Abstract

In several areas of mathematics, one encounters families of objects (groups, varieties, schemes, graphs...) parametrized by one or several unknowns, that are well-behaved and easy to define except for a few specific values of these unknowns. Think, for example, of an elliptic curve over the field of rational numbers: starting with an equation with rational coefficients, one can clear denominators and get an equation with integer coefficients, and this equation reduces to the equation of an elliptic curve modulo p for all but a finite number of primes p. Even then, it is often convenient to be able to extend our family into a global, compact one, or at least satisfying some good continuity properties with respect to the parameters. A *model* of a family of objects defined for all but certain values of the parameters is a way of extending it to all possible values. Incomplete smooth families of schemes (or, more generally, of stacks) sometimes admit a "best possible smooth model", the *Néron model*. This thesis deals with questions of existence and explicit construction of Néron models. It is divided in three parts.

In the first part, we study generically smooth families of nodal curves (i.e. curves with at worst ordinary double points) over a regular base scheme. We define certain birational modifications of such nodal (relative) curves, which we call *refinements*. We prove that refinements always exist étale-locally on the base. We define invariants measuring the complexity of the singularities of a nodal curve, and explain how refinements can be used to find the nodal models of a generic smooth curve with the simplest singularities.

In the second part, we are interested in the existence of Néron models for (families of) Jacobians and curves, over a regular base with no restriction of dimension. First, we introduce a condition on nodal curves called *strict alignment*. Strict alignment can be read on the *dual graph*, a simple combinatorial invariant summarizing information about the global structure of the curve and how its singularities vary in families. We show that the generic Jacobian of a generically smooth nodal curve has a Néron model if and only if the curve is strictly aligned. Then, we prove that for a smooth curve to have a Néron model, it is necessary that the singular locus of any nodal model be locally irreducible for the étale topology. Using the contraction morphisms of $\mathcal{M}_{g,n}$ stacks, we deduce an even stronger necessary condition in terms of dual graphs (equivalent to the closure of this local irreducibility of the singular locus under étale base change and contraction), and we show that this new necessary condition is also sufficient under some technical hypotheses.

In the third part, we study the base change behavior of Néron models under finite, tamely ramified morphisms $S' \to S$ between regular schemes. We show that if an abelian variety defined generically over S has a Néron model N'/S' after such a base change, then it admits a Néron model N/S, and we make explicit the successive quotients of a certain filtration of N in terms of N'.

Keywords: Nodal curves, Néron models, Jacobians, high dimension.