Néron models in high dimension: Nodal curves, Jacobians and tame base change

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Part III

Base change of Néron models along finite tamely ramified maps

8 Motivation

Given an open immersion $U \subset S$ and a smooth algebraic space over U, we can sometimes get informations about the Néron model of this space (existence, nonexistence, explicit construction if it exists) only after base change to some finite, locally free extension S'/S. Examples include smooth curves acquiring nodal reduction over S', Jacobians of smooth curves acquiring nodal reduction, and to some extent abelian varieties acquiring semi-abelian reduction (see Theorem 6.20, Theorem 7.40 and Theorem 7.48, as well as [28]). Therefore, it is interesting to have tools to turn this into information on the Néron model over S. In [4], one studies the base change behaviour of Néron models of abelian varieties over discrete valuation rings along finite tamely ramified extensions. We are interested in what happens when the base is higher-dimensional. The first complication appearing in this setting is that the existence of Néron models is no longer guaranteed. We address this problem by proving

Theorem 8.1 (Theorem 10.5)**.** *Let $S' \to S$ be a finite, locally free, tamely ramified map between regular schemes and U the complement of a strict normal crossings divisor $D \to S$. Suppose $U' := U \times_S S'$ is étale over U.*

Let X_U be a smooth U-algebraic space, such that $X_U \times_U U'$ has a Néron model X' over S'. Then the scheme-theoretical closure of X_U in $\prod_{S'/S} X'$ is the Néron model of X_U over S (where X_U maps to $\prod_{S'/S} X'$ as in Example 9.3).

Moreover, if $S' \to S$ is a quotient for the right-action of a finite group G, then G acts on X' via its action on S': it follows that G acts on $\prod_{S'/S} X'$ as in Remark 9.4.1. In that case, the Néron model of X_U is $(\prod_{S'/S} X')^G$.

Then, when the Néron models N and N' exist, we introduce filtrations of certain strata of N (quite similar to the filtration of the closed fiber described in [4]), and describe explicitly the successive quotients of these filtrations in terms of N'. Namely, after reducing to its hypotheses by working étale-locally on the base (see Lemma 10.12), we prove the following result:

Theorem 8.2 (Theorem 10.13)**.** *Let $S = \operatorname{Spec} R$ be a regular affine scheme,*

$f_1, ..., f_r$ regular parameters of R, $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where the n_j are invertible on S, and let $S' = \operatorname{Spec} R'$. Suppose R contains the group μ_{n_j} of n_j-th roots of unity for all j. Let U be the locus in S where all f_j are invertible, and Z the locus where all f_j vanish. Let X_U be a proper smooth U-group algebraic space with a Néron model N' over S'. Then X_U has a Néron model N over S, and we have sub-Z-group spaces $(F^d N_Z)_{d\in\mathbb{N}}$ of N_Z (see Definition 10.7 and Definition 10.10) such that:

- *For all $d \in \mathbb{N}$, $F^{d+1}N_Z \subset F^d N_Z$.*
- *$F^0 N_Z = N_Z$.*
- *If $d > \prod_{j=1}^{r} (n_j - 1)$ then $F^d N_Z = 0$.*
- *$F^0 N_Z / F^1 N_Z$ is the subspace of N'_Z invariant under the action of $G = \prod_{j=1}^{r} \mu_{n_j}$, where $(\xi_j)_{1\le j\le r}$ acts by multiplying T_j by ξ_j.*
- *If $d > 0$, $F^d N_Z / F^{d+1} N_Z$ is isomorphic to the fiber product over Z of the $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ where $\mathbf{k}$ ranges through all r-uples of integers $(k_1, ..., k_r)$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j; $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ is the subspace of $\mathrm{Lie}_{N'_Z/Z}$ where all $(\xi_j)_{1\le j\le r}$ in G act by multiplication by $\prod_{j=1}^{r} \xi_j^{k_j}$; and the map $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}] \to Z$ is given by identifying $\mathrm{Lie}_{N'_Z/Z}$ with $\mathrm{Lie}_{N'_Z/Z}(P.R)$, where $P = \prod_{i=1}^{r} T_i^{k_i}$.*

9 Prerequisites

9.1 Weil restrictions

As in [4], we introduce the Weil restriction functor and give some well-known representability properties.

Definition 9.1. Let S'/S be a morphism of schemes, and X' a contravariant functor from (Sch/S') to (Set). We call *Weil restriction* of X'/S' to S, and we note $\prod_{S'/S} X'$, the contravariant functor from (Sch/S) to (Set) sending $T \to S$ to $X'(T \times_S S')$. If $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ are affine, we will sometimes write $\prod_{R'/R} X'$.

Remark 9.1.1. The Weil restriction of a presheaf $X' : (\mathrm{Sch}/S')^{op} \to \mathrm{Set}$ to S is its pushforward to $\mathrm{Hom}((\mathrm{Sch}/S)^{op}, \mathrm{Set})$.

Proposition 9.2. *If $S' \to S$ is a flat and proper morphism between locally noetherian schemes, and X' is a quasi-projective S-algebraic space with X'/S*

factoring through S', then $\prod_{S'/S} X'$ is (representable by) an algebraic space. If in addition X' is a scheme, then $\prod_{S'/S} X'$ is a scheme.

Proof. The general statement follows from the case where X' is a scheme, which is treated in [9], 4.c. □

Example 9.3. Let $S' \to S$ be a morphism of schemes and Y/S be an S-algebraic space. Let Y' be the S'-space $Y \times_S S'$. Then for all T/S we have $\mathrm{Hom}_S(T, \prod_{S'/S} Y') = \mathrm{Hom}_{S'}(T \times_S S', Y') = \mathrm{Hom}_S(T \times_S S', Y)$. In particular, there is a natural map $Y \to \prod_{S'/S} Y'$.

Proposition 9.4. *If $S' \to S$ is a flat and finite morphism of schemes, and X' is a smooth quasi-projective S'-algebraic space, then $\prod_{S'/S} X'$ is smooth over S.*

Proof. The formation of $\prod_{S'/S} X'$ is étale-local on both X' and S, so we can assume $X' = \operatorname{Spec} A'$ and $S = \operatorname{Spec} R$ are affine schemes. Then S' is a disjoint union of affine schemes of the form $\operatorname{Spec} R'$, with R'/R finite and flat, and $\prod_{S'/S} X'$ is a scheme by Proposition 9.2. By hypothesis, each $R' \to A'$ is formally smooth and locally of finite presentation. But then $\prod_{S'/S} X'/S$ is also formally smooth, and it follows from [13], Proposition 8.14.2, that it is locally of finite presentation as well. □

Remark 9.4.1. Suppose given equivariant right-actions of a group G on a morphism of schemes $S' \to S$ and on a morphism from an algebraic space X' to S'. Suppose moreover that G acts trivially on S. Then $\prod_{S'/S} X' \to S$ carries a natural G-action, defined as follows: for any S-scheme T, define $T' := T \times_S S'$, every $g \in G$ induces an automorphism $\rho_{X'}(g)$ of X' and an automorphism $\rho_{T'}(g)$ of T' (obtained by extending the automorphism on S' by the identity on T). The action takes $f \in \mathrm{Hom}(T, \prod_{S'/S} X') = \mathrm{Hom}(T', X')$ to

$$f.g = \rho_{X'}(g) \circ f \circ \rho_T(g)^{-1}.$$

When $\prod_{S'/S} X'$ is representable, this action is equivariant.

9.2 Fixed points

We will show later that under certain hypotheses, we can construct a Néron model by considering the Weil restriction of a Néron model over a bigger base, and looking at its subspace of fixed points under a Galois action: here we define the functor of fixed points and talk about its representability and possible smoothness. This is all contained in [4], to which we refer for the proofs unless they are short enough.

Definition 9.5. Let $\pi\colon S' \to S$ be a morphism of schemes and G a finite group, acting on the right on S'. We say that π is a *quotient* for this action if it is affine, and for every affine open subscheme $\operatorname{Spec} A \subset S$ of pullback $\operatorname{Spec} A' \subset S'$ by π, A is the subring A'^G of G-invariants of A'.

Definition 9.6. Let S be a scheme and X an algebraic space. Suppose a group G acts equivariantly on $X \to S$ with the trivial action on S. We define the *subfunctor of fixed points* $X^G\colon (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ by $X^G(T) = (X(T))^G$.

Proposition 9.7. *With notations as above, X^G is an algebraic space, and its formation commutes with base change. If $X \to S$ is locally separated (resp. separated), then X^G is a subspace (resp. a closed subspace) of X.*

Proof. Compatibility with base change is immediate. Each $g \in G$ gives an automorphism $\rho_X(g)$ of X, thus a graph $\Gamma_g : X \to X \times_S X$. We can write Γ_g as the composition

$$X \xrightarrow{\Delta} X \times_S X \xrightarrow{(p_1, \rho_X(g) \circ p_2)} X \times_S X$$

where Δ is the diagonal map, and p_1, p_2 are the two projections from $X \times_S X$ onto X. Since $\rho_X(g)$ is an automorphism of X, $(p_1, \rho_X(g) \circ p_2)$ is an automorphism of $X \times_S X$. Write $Z \to X \times_S X$ for the fiber product of all Γ_g. Then Z is an algebraic space, which represents X^G. Suppose X is locally separated (resp. separated) over S. Then, $\Delta = \Gamma_0$ is an immersion (resp. a closed immersion), so Z is a subspace (resp. closed subspace) of X. $\square$

Proposition 9.8. *With the same hypotheses and notations as in Definition 9.6, if $f\colon X \to S$ is smooth and $n := \#G$ is invertible on X, then $X^G \to S$ is smooth.*

Proof. This follows from [4], Proposition 3.4 and Proposition 3.5. $\square$

Corollary 9.9. *Let G be a finite group acting equivariantly on a smooth morphism of algebraic spaces $X \to S$. If $\#G$ is invertible on X, then $X^G \to S^G$ is smooth.*

9.3 Twisted Lie algebras

We will make use of a slightly broader than usual notion of tangent space and Lie algebra of a group algebraic space over a base scheme, so we present the definition and a few properties here. These objects are studied in much more detail in [3].

Definition 9.10. Let S be a scheme and X an S-group algebraic space. Let $\mathcal{M}$ be a free $\mathcal{O}_S$-module of finite type. We write $\mathrm{T}_{X/S}(\mathcal{M})$ for the functor $(\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ taking T/S to $\operatorname{Hom}_S(\operatorname{Spec}(\mathcal{O}_T \oplus \mathcal{M}_T), X)$, where the $\mathcal{O}_T$-module $\mathcal{O}_T \oplus \mathcal{M}_T$ is endowed with the $\mathcal{O}_T$-algebra structure making $\mathcal{M}_T$ a square-zero ideal. The morphism of $\mathcal{O}_S$-modules $\mathcal{M} \to 0$ induces a morphism

$$\mathrm{T}_{X/S}(\mathcal{M}) \to X = \mathrm{T}_{X/S}(0),$$

and we write $\mathrm{Lie}_{X/S}(\mathcal{M})$ for the pullback of $\mathrm{T}_{X/S}(\mathcal{M})$ by the unit section $S \to X$. In particular, when $\mathcal{M} = \mathcal{O}_S$, they are the usual tangent bundle and Lie algebra of X over S, that we write $\mathrm{T}_{X/S}$ and $\mathrm{Lie}_{X/S}$.

Proposition 9.11. *With hypotheses and notations as above,* $\mathrm{T}_{X/S}(\mathcal{M})$ *and* $\mathrm{Lie}_{X/S}(\mathcal{M})$ *are representable by group S-algebraic spaces, and the canonical morphisms*

$$\mathrm{Lie}_{X/S}(\mathcal{M}) \to \mathrm{T}_{X/S}(\mathcal{M}) \to X$$

are morphisms of S-groups.

Proof. Representability when X is a scheme is [3], exposé 2, Proposition 3.3. The case of algebraic spaces is similar. Existence of the group structure, and the fact the canonical maps respect them, is [3], exposé 2, Corollaire 3.8.1. □

Proposition 9.12. *Let $(e_1, ..., e_n)$ be a basis for the free $\mathcal{O}_S$-module $\mathcal{M}$. Then, we have a natural isomorphism between $T_{X/S}(\mathcal{M})$ and the fiber product over X of the $T_{X/S}(\mathcal{O}_S.e_i)$, which we write $\prod_{1\leq i\leq n, X} T_{X/S}(\mathcal{O}_S.e_i)$; as well as between* $\mathrm{Lie}_{X/S}(\mathcal{M})$ *and the fiber product* $\prod_{1\leq i\leq n, S} \mathrm{Lie}_{X/S}(\mathcal{O}_S.e_i)$.

Proof. When X is a scheme, this is [3], exposé 2, Proposition 2.2. The case of algebraic spaces is similar. □

10 The morphism of base change for tame extensions

10.1 Compatibility with Weil restrictions

In this section, $S' \to S$ will be a finite locally free morphism between regular schemes, X'/S' an algebraic space, U a scheme-theoretically dense open subscheme of S, and $U' = U \times_S S'$. Note that U' is scheme-theoretically dense in S' by [13], théorème 11.10.5.

Definition 10.1. Let t be a point of $S\backslash U$ of codimension 1 in S, so that $\mathcal{O}_{S,t}$ is a discrete valuation ring. We call *ramification index of S'/S at t* the lcm of ramification indexes of all valuation ring maps $\mathcal{O}_{S,t} \to \mathcal{O}_{S',t'}$ with $t' \in S'$ of image t. Let $\mathbf{t} = (t_1, ..., t_r)$ be a r-uple of generic points of $S\backslash U$, we call *ramification index of S'/S at* $\mathbf{t}$ the r-uple $(n_1, ..., n_r)$, where n_i is the ramification index of S'/S at t_i for all i.

Proposition 10.2 (see [4], Proposition 4.1). *Let X' be a smooth quasi-projective S'-algebraic space. Suppose X' is the S'-Néron model of $X'_{U'}$. Then $\prod_{S'/S}(X')$ is the S-Néron model of $\prod_{U'/U}(X'_{U'})$.*

Proof. The restriction $\prod_{S'/S}(X')$ is smooth and separated over S by Proposition 9.4 and, if Y is a smooth S-algebraic space, then

$$\begin{aligned}\mathrm{Hom}\left(Y, \prod_{S'/S}(X')\right) &= \mathrm{Hom}(Y', X') \\ &= \mathrm{Hom}(Y'_{U'}, X'_{U'}) \\ &= \mathrm{Hom}\left(Y_U, \prod_{U'/U}(X'_{U'})\right)\end{aligned}$$

where Y' denotes the base change $Y \times_S S'$. □

Proposition 10.3 (Abhyankar's lemma)**.** *Let $S = \mathrm{Spec}\, R$ be a regular local scheme and $S' \to S$ a finite locally free tamely ramified morphism of schemes, étale over the complement U of a strict normal crossings divisor D of S. Let $(f_1, ..., f_r)$ be part of a regular system of parameters of R such that $D = \mathrm{Div}(f_1...f_r)$. Then there are integers $n_1, ..., n_r$, prime to the residue characteristic p of R, such that if $\tilde{S} = \mathrm{Spec}\, R[T_1, ..., T_r]/(T_i^{n_i} - f_i)_{1\leq i\leq r}$, the normalization of $\tilde{S}$ in the total ring of fractions of $S' \times_S \tilde{S}$ is étale over $\tilde{S}$.*

Proof. This is [15], exposé XIII, Proposition 5.2. □

Proposition 10.4. *Under the hypotheses of Proposition 10.3, suppose that S is strictly local, and that S' is connected and regular. Then there are integers $n_1, ..., n_r$, prime to p, such that $S' = \mathrm{Spec}\, R[T_1..., T_r]/(T_i^{n_i} - f_i)$.*

Proof. By Proposition 10.3, there are integers $m_1, ..., m_r$ prime to p such that if we call $\tilde{S}$ the spectrum of

$$\tilde{R} = R[T_1, ..., T_r]/(T_i^{m_i} - f_i)_{1\leq i\leq r},$$

then the normalization Y of $\tilde{S}$ in the fraction field of $S' \times_S \tilde{S}$ is finite étale over $\tilde{S}$. Since $\tilde{S}$ is strictly local and S' is connected, $Y \to \tilde{S}$ must be an isomorphism. Since $S' \to S$ is integral, we get a factorization $\tilde{S} \to S' \to S$. Let G be the group of S'-automorphisms of $\tilde{S}$. As both S' and $\tilde{S}$ are spectra of free R-algebras, the map $\tilde{S} \to S'$ is a quotient for the G-action. We can see G as a subgoup of the group of S-automorphisms of $\tilde{S}$, which is the product $\prod_{i=1}^{r} \mu_{m_i}$ of the groups μ_{m_i} of m_i-th roots of unity of R (where $\xi \in \mu_{m_i}$ acts by sending T_i to ξT_i). We know that $R' = \tilde{R}^G$ is generated as a R-module by all of the G-invariant monomials in $(T_1, ..., T_r)$. Let n_i be the minimal integer such that T_i is G-invariant. The $T_i^{n_i}$ are irreducible - hence prime - elements of the regular local ring $R' = \tilde{R}^G$, and we have $n_i | m_i$ for each i. We claim that G is of the form $< \xi_1, \xi_2, ..., \xi_r >$, where $\xi_i \in \mu_{m_i}$ is a primitive $d_i := \frac{m_i}{n_i}$-th root of unity. We will now prove the claim. Let $M = \prod_{i=1}^{r} T_i^{k_i}$ be a G-invariant unitary monomial. It suffices to show that for all i, we have $n_i | k_i$. But M divides some positive power of $\prod_{i=1}^{r} f_i = \prod_{i=1}^{r} T_i^{m_i}$ in

$R' = \tilde{R}^G$, and by uniqueness of the prime factor decomposition in R' it follows that $n_i | k_i$ for all i. This concludes the proof of the claim.

Therefore, $\prod_{i=1}^{r} \mu_{m_i}/G$ is a product of quotients of the μ_{m_i}, i.e. there are integers $n_1, ..., n_r$ such that n_i divides m_i for all i and S' itself is of the form $\operatorname{Spec} R[T_1, ..., T_r]/(T_i^{n_i} - f_i)_{1 \leq i \leq r}$. □

Remark 10.4.1. The integer n_i is the ramification index of S'/S at the generic point of $\{f_i = 0\}$ in S.

Remark 10.4.2. This proof means that, étale-locally on any regular base, a finite tamely ramified morphism either does nothing more than adding roots of regular parameters, or must have a scheme that is not locally factorial as a source. Since in many practical situations, the behaviour of Néron models is only well-known over (at least) locally factorial bases, we will only be considering the "adding roots" case. For the same reason, we always take D to be strict: indeed, suppose D is a (non-strict) normal crossings divisor, and suppose there is an étale morphism $\tilde{S} \to S$ and an irreducible component D_0 of D which breaks into multiple irreducible components in $\tilde{S}$, then no extension S'/S with ramification index > 1 over the generic point of D_0 can have factorial étale local rings.

Theorem 10.5 (see [4], Theorem 4.2. for the case $\dim S = 1$)**.** *Let $S' \to S$ be a finite, locally free, tamely ramified map between regular schemes and U the complement of a strict normal crossings divisor $D \to S$. Suppose $U' := U \times_S S'$ is étale over U.*

Let X_U be a smooth U-algebraic space, such that $X_U \times_U U'$ has a S'-Néron model X'. Then the scheme-theoretical closure of X_U in $\prod_{S'/S} X'$ is the S-Néron model of X_U (where X_U maps to $\prod_{S'/S} X'$ as in Example 9.3).

Moreover, if $S' \to S$ is a quotient for the right-action of a finite group G, then G acts on X' via its action on S': it follows that G acts on $\prod_{S'/S} X'$ as in Remark 9.4.1. In that case, the Néron model of X_U is $(\prod_{S'/S} X')^G$.

Proof. By Corollary 5.4 and Proposition 5.7, we can assume $S = \operatorname{Spec} R$ is strictly local. We can also assume S' is connected, in which case Proposition 10.4 shows that $S' \to S$ is a quotient for the action of a finite group G. Call Z the restriction $\prod_{S'/S} (X')$ and N the scheme-theoretical closure of X_U in Z. The theorem now reduces to the floowing claims: Z^G is a smooth S-model of X_U; there is a canonical isomorphism $N = Z^G$; and N is a separated S-space satisfying existence in the Néron mapping property. We will now prove these claims in order.

The group G acts on $Z = \prod_{S'/S} X_{S'}$ via its right-action on S', and $S' \to S$ is a quotient for the latter. As seen in Proposition 10.4, $\#G$ is prime to the residue characteristic of R, so by Proposition 9.8, Z^G is S-smooth. We have a pullback diagram of algebraic spaces

$$\begin{tikzcd} X'_{U'} \arrow[r] \arrow[d] & X_U \arrow[d] \\ U' \arrow[r] & U \end{tikzcd}$$

where both horizontal arrows are quotients for the action of G. We will show $X_U = Z_U^G$, which can be checked Zariski-locally: let V be an affine open subscheme of U, $V' = \operatorname{Spec} A$ its pullback to U', X_0 an affine open of X_U with image contained in V and $X'_0 = \operatorname{Spec} B$ its pullback to $X'_{U'}$. We have $V = \operatorname{Spec}(A^G)$ and $X_0 = \operatorname{Spec}(B^G)$, and there is a pushout diagram of rings

$$\begin{tikzcd} B & B^G \arrow[l] \\ A \arrow[u] & A^G \arrow[l] \arrow[u] \end{tikzcd}$$

Let W be the Weil restriction of X'_0 to V. Then we see that $W = \operatorname{Spec} R$ is affine, and for any A^G-algebra C, we have $\operatorname{Hom}_{A^G}(R, C) = \operatorname{Hom}_A(B, C \otimes_{A^G} A)$. But the G-invariant A-maps from B to $C \otimes_{A^G} A$ are precisely those lying in the image of $\operatorname{Hom}_{A^G}(B^G, C)$. Therefore $W^G = \operatorname{Spec}(B^G)$, and it follows from the sheaf property that $(Z_U)^G = X_U$: Z^G is a smooth S-model of X_U as claimed.

Now, observe that $Z^G \to Z$ is a closed immersion through which X_U factors, so $N \to Z$ factors through a closed immersion $N \to Z^G$. But Z^G is S-smooth, hence S-flat, so $Z_U^G = X_U$ is scheme-theoretically dense in Z^G by [13], théorème 11.10.5, which means N is scheme-theoretically dense in Z^G: the closed immersion $N \to Z^G$ is an isomorphism.

The map $N \to S$ is separated since $Z \to S$ is. Let Y be a smooth S-algebraic space with a map $f_U \colon Y_U \to X_U$. Call Y' the base change $Y \times_S S'$. The map $Y'_{U'} \to X'_{U'}$ obtained by base change extends uniquely to a map $Y' \to X'$, which induces a map $Y \to Z$ extending f_U. By definition of the scheme-theoretical closure, $Y \to Z$ factors through a map $f \colon Y \to N$ extending f_U. We have shown $\operatorname{Hom}_S(Y, N) \to \operatorname{Hom}_U(Y_U, X_U)$ is surjective, so $N/S = Z^G/S$ is the Néron model of X_U. $\square$

10.2 A filtration of the Néron model over the canonical stratification

By Proposition 5.3 and Proposition 5.6, Néron models can always be described over an étale covering of the base. Therefore, in this section, unless mentioned otherwise, we will work assuming that $S = \operatorname{Spec} R$ is an affine regular connected scheme, that R contains all roots of unity of order invertible on S, that D is a strict normal crossings divisor on S cut out by regular parameters $f_1, ..., f_r$ of R, and that $S' = \operatorname{Spec} R'$ with $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)_{1 \leq j \leq r}$. Note that in our previous setting (where $S' \to S$ was a finite, locally free and tamely ramified morphism between regular schemes, étale over the complement of a strict normal crossings divisor), all these assumptions hold in an étale neighbourhood of any given point of S.

We put $A = R/(f_j)_{1\le j\le r}$ and $A' = A \otimes_R R' = A[T_1, ..., T_r]/(T_j^{n_j})_{1\le j\le r}$. The closed subscheme $Z = \operatorname{Spec} A$ of S is the closed stratum of D. We let X_U be a proper and smooth U-group space with a Néron model N' over S'. It follows from Theorem 10.5 that X_U has a Néron model N/S, and that N is the subspace of G-invariants of the Weil restriction of N' to S, where the action of $G = \prod_{j=1}^{r} \mu_{n_j}$ on S' is given by multiplying T_j by the j-th coordinate of an element of G.

In [4], section 5, when S is a discrete valuation ring, one computes the successive quotients of a filtration of the closed fiber of N. We adapt this construction to our context to get a filtration of N_Z and express its successive quotients in terms of N'.

For all $d \in \mathbb{N}^*$, we write Λ_d the set of monomials of the form $\prod_{j=1}^{r} T_j^{k_j}$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j. The set $A'_d \subset A'$ of homogenous polynomials of degree d in the T_j is a finite free A-module with basis Λ_d.

Definition 10.6. For $d \in \mathbb{N}^*$, we define a sheaf

$$\operatorname{Res}^d N'_Z \colon (\mathrm{Sch}/Z)^{op} \to \mathrm{Set}$$

as follows: for any A-algebra C, we put $\operatorname{Res}^d N'_Z(C) = N'(C \otimes_A A'/(\Lambda_d))$.

Remark 10.6.1. The functor $\operatorname{Res}^d N'_Z$ is (representable by) the Z-algebraic space $\prod_{(A'/(\Lambda_d))/A} N'_{A'/(\Lambda_d)}$. We have $\operatorname{Res}^1 N'_Z = N'_Z$, and for any $d > \prod_{j=1}^{r} (n_j - 1)$, we have $\operatorname{Res}^d N'_Z = \left(\prod_{S'/S} N'\right) \times_S Z$ since Λ_d is empty. There are natural maps $\operatorname{Res}^{d+1} N'_Z \to \operatorname{Res}^d N'_Z$.

Definition 10.7. For $d \in \mathbb{N}^*$, we define $F^d N'_Z$ as the kernel of the canonical morphism $\left(\prod_{S'/S} N'\right) \times_S Z \to \operatorname{Res}^d N'_Z$ of Z-group spaces. We also put $F^0 N'_Z = \left(\prod_{S'/S} N'\right) \times_S Z$. The $F^d N'_Z$ form a descending filtration of $\left(\prod_{S'/S} N'\right) \times_S Z$ by Z-subgroup spaces, stationary at 0 starting from $d = 1 + \prod_{j=1}^{r} (n_j - 1)$. We call $\operatorname{Gr}^d N'_Z$ the quotient $F^d N'_Z / F^{d+1} N'_Z$.

Proposition 10.8. *We have* $\operatorname{Gr}^0 N'_Z = N'_Z$, *and for any* $d \ge 1$, $\operatorname{Gr}^d N'_Z$ *is canonically isomorphic to* $\operatorname{Lie}_{N'_Z/Z}(A'_d) = \prod_{P \in \Lambda_d, Z} \operatorname{Lie}_{N'_Z/Z}(PA)$.

Proof. The proof of [4], 5.1. carries over without much change: let $d \ge 1$, and let C be an A-algebra. Let $\lambda_1, ..., \lambda_k$ be parameters for the formal group of N'

over R'. An element $a \in F^d N'_Z(C)$ corresponds to a ring map

$$\phi\colon R'[[\lambda_1, ..., \lambda_k]] \to C[T_1, ..., T_r]/(T_j^{n_j})_{1 \leq j \leq n}$$

such that for all $1 \leq i \leq k$, $\phi(\lambda_i)$ is in the ideal generated by Λ_d, i.e. is of the form $\sum_P a_{i,P} P$ where the $a_{i,P}$ are in C and P runs over all nonzero monomials $\prod_{j=1}^r T^{k_j}$ with $\sum_j k_j \geq d$. Thus, we can associate to a an element of $\mathrm{Lie}_{N'_Z/Z}(A_d)(C)$ by truncature, sending λ_i to $\sum_{P \in \Lambda_d} a_{i,P} P$. This gives a surjective morphism of Z-groups $F^d N'_Z \to \mathrm{Lie}_{N'_Z/Z}(A_d)$, with kernel $F^{d+1} N'_Z$. The identification $\mathrm{Lie}_{N'_Z/Z}(A'_d) = \prod_{P \in \Lambda_d, Z} \mathrm{Lie}_{N'_Z/Z}(PA)$ is Proposition 9.12. $\square$

Proposition 10.9. *With the same hypotheses and notations as in Proposition 10.8, the action of G on $\mathrm{Lie}_{N'_Z/Z}(A_d)$ is obtained from equivariant actions of G on each factor $\mathrm{Lie}_{N'_Z/Z}(PA) \to Z$, where P ranges through Λ_d. Moreover, for any $P = \prod_{j=1}^r T_j^{k_j}$ in Λ_d, the bijection $\mathrm{Lie}_{N'_Z/Z}(PA) = \mathrm{Lie}_{N'_Z/Z}$ induced by $P \mapsto 1$ identifies the subspace $\mathrm{Lie}_{N'_Z/Z}(PA)^G$ with the subspace of $\mathrm{Lie}_{N'_Z/Z}$ where all $\xi = (\xi_j)_{1 \leq j \leq r}$ in G act by multiplication by $\prod_{j=1}^r \xi_j^{k_j}$. We will write this subspace $\mathrm{Lie}_{N'_Z/Z}[k_1, ..., k_r]$ or $\mathrm{Lie}_{N'_Z/Z}[P]$.*

Proof. For any A-algebra C, the action of ξ on $\mathrm{Hom}(R'[[\lambda_1, ..., \lambda_k]], C \otimes_A A')$ makes the following diagram commute:

$$\begin{tikzcd} R'[[\lambda_1, ..., \lambda_k]] \arrow[r, "\xi.\psi"] \arrow[d, "\xi"] & C \otimes_A A' \\ R'[[\lambda_1, ..., \lambda_k]] \arrow[r] & C \otimes_A A' \arrow[u, "T_j \mapsto \xi_j^{-1} T_j"'] \end{tikzcd}$$

where the map $R'[[\lambda_1, ..., \lambda_k]] \xrightarrow{\xi} R'[[\lambda_1, ..., \lambda_k]]$ is given by the G-action on N'. Therefore, the G-action on $\mathrm{Lie}_{N'_Z/Z}(A_d)$ comes from G-actions on the factors $\mathrm{Lie}_{N'_Z/Z}(PA)$, given for $P = \prod_{j=1}^r T_j^{k_j}$ by

$$\begin{tikzcd} R'[[t_1, ..., t_d]] \arrow[r, "\xi.\psi"] \arrow[d, "\xi"] & C \oplus PC \\ R'[[t_1, ..., t_d]] \arrow[r] & C \oplus PC \arrow[u, "P \mapsto \prod_j \xi_j^{-k_j} P"'] \end{tikzcd}$$

from which the proposition follows. $\square$

Definition 10.10. For any integer $d \in \mathbb{N}$, we define

$$F^d N_Z := (F^d N'_Z)^G$$

and

$$G^d N_Z := F^d N_Z / F^{d+1} N_Z.$$

Remark 10.10.1. The F^dN_Z form a descending filtration of sub-Z-group spaces of N_Z, with $F^0N_Z = N_Z$ and $F^dN_Z = 0$ when $d > \prod_{j=1}^{r}(n_j - 1)$.

Proposition 10.11. *Keeping the notations of Proposition 10.9, for all $d \in \mathbb{N}$, we have $G^dN_Z = (G^dN'_Z)^G$. In particular, $G^0N_Z = (N'_Z)^G$, and for all $d \geq 1$, $G^dN_Z = \prod_{P \in \Lambda_{d,Z}} \mathrm{Lie}_{N'_Z/Z}[P]$.*

Proof. (see [4], 5.2.) The Z-group spaces $F^dN'_Z$ are unipotent for $d \geq 1$ and the order of G is invertible on Z, so the exact sequence

$$0 \to F^dN'_Z \to F^{d+1}N'_Z \to G^dN'_Z \to 0$$

remains exact after taking the G-invariants. □

We summarize all this into Theorem 10.13 below, and justify its hypotheses by Lemma 10.12 and Proposition 5.6.

Lemma 10.12. *Let $S' \to S$ be a finite, locally free, morphism between regular connected schemes. Let D be a strict normal crossings divisor of S and put $U = S \backslash D$. Suppose $S' \to S$ is étale over U. Let s be a point of S, and $D_1, ..., D_r$ the irreducible components of D containing s. Then there is an affine étale neighbourhood $V = \operatorname{Spec} R \to S$ of s in S such that:*

- *For all $1 \leq j \leq r$, $D_j|_V$ is cut out by a regular parameter f_j of R.*
- *There is an isomorphism $V \times_S S' = \operatorname{Spec} R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where n_j is the ramification index of $S' \to S$ at the generic point of D_j (in particular, if $S' \to S$ is tamely ramified, then n_j is invertible on R).*
- *R contains all n_j-th roots of unity for all j.*

Proof. Immediate from Proposition 10.4. □

Theorem 10.13. *Let $S = \operatorname{Spec} R$ be a regular affine scheme, $f_1, ..., f_r$ regular parameters of R, $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where the n_j are invertible on S, and let $S' = \operatorname{Spec} R'$. Suppose R contains the group μ_{n_j} of n_j-th roots of unity for all j. Let U be the locus in S where all f_j are invertible, and Z the locus where all f_j vanish. Let X_U be a proper smooth U-group algebraic space with a Néron model N' over S'. Then X_U has a Néron model N over S, and we have sub-Z-group spaces $(F^dN_Z)_{d \in \mathbb{N}}$ of N_Z (see Definition 10.7 and Definition 10.10) such that:*

- *For all $d \in \mathbb{N}$, $F^{d+1}N_Z \subset F^dN_Z$.*
- *$F^0N_Z = N_Z$.*
- *If $d > \prod_{j=1}^{r}(n_j - 1)$ then $F^dN_Z = 0$.*

- *$F^0 N_Z / F^1 N_Z$ is the subspace of N'_Z invariant under the action of $G = \prod_{j=1}^{r} \mu_{n_j}$, where $(\xi_j)_{1 \leq j \leq r}$ acts by multiplying T_j by ξ_j.*
- *If $d > 0$, $F^d N_Z / F^{d+1} N_Z$ is isomorphic to the fiber product over Z of the $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ where $\mathbf{k}$ ranges through all r-uples of integers $(k_1, ..., k_r)$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j; $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ is the subspace of $\mathrm{Lie}_{N'_Z/Z}$ where all $(\xi_j)_{1 \leq j \leq r}$ in G act by multiplication by $\prod_{j=1}^{r} \xi_j^{k_j}$; and the map $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}] \to Z$ is given by identifying $\mathrm{Lie}_{N'_Z/Z}$ with $\mathrm{Lie}_{N'_Z/Z}(P.R)$, where $P = \prod_{i=1}^{r} T_i^{k_i}$.*

Remark 10.13.1. Our choice of quotienting by all monomials of the same degree in Definition 10.6 is somewhat arbitrary, other choices could perhaps lead to interesting things as well.