Néron models in high dimension: Nodal curves, Jacobians and tame base change

Poiret, T.

Author: Poiret, T.
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Part II

Néron models of nodal curves and their Jacobians

5 Generalities about Néron models

5.1 Definitions

Definition 5.1. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a U-algebraic space. An *S-model* of Z (or just *model* if there is no ambiguity) is an S-algebraic space X together with an isomorphism $X_U = Z$. A *morphism of S-models* between two models X and Y of Z is an S-morphism $X \to Y$ that commutes over U with the given isomorphisms $X_U = Z$ and $Y_U = Z$.

Definition 5.2. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a smooth separated U-scheme. An *ns-S-Néron model* of Z (or just *ns-Néron model* if there is no ambiguity) is a smooth S-model N satisfying the following universal property, called the *Néron mapping property*:
For each smooth S-algebraic space Y, the restriction map

$$\mathrm{Hom}_S(Y, N) \to \mathrm{Hom}_U(Y_U, Z)$$

is bijective.

If N is separated, we call it a *S-Néron model*, or just *Néron model*, of X_U.

Remark 5.2.1. In the litterature, Néron models are often required to be of finite type over the base, and what we called Néron model here is referred to as a Néron-lft model, where "lft" stands for "locally of finite type". The use of this terminology is not systematical anymore, so we prefer the more flexible definition above. To the author's knowledge, however, the separatedness hypothesis is usually never omitted, so we use the prefix "ns" (not necessarily separated) to avoid generating unnecessary confusion.

Remark 5.2.2. As an immediate consequence of the universal property, a ns-Néron model, when it exists, is unique up to a unique isomorphism. A fortiori, the same holds for Néron models.

Remark 5.2.3. Let S, U, Z be as above, and N be a smooth, separated S-model of Z. Consider a smooth S-algebraic space Y/S and two morphisms $f_1, f_2\colon Y \to N$ that coincide over U. The separatedness of N/S implies that the equalizer of f_1 and f_2 is a closed subspace of Y containing Y_U, and flatness of Y/S implies that the open subscheme Y_U of Y is scheme-theoretically dense (see [13], théorème 11.10.5). Thus, we automatically have *uniqueness in the Néron mapping property*, i.e. injectivity of the restriction map

$$\mathrm{Hom}_S(Y, N) \to \mathrm{Hom}_U(Y_U, Z).$$

Therefore, we can try to construct Néron models as separated S-spaces satisfying existence in the Néron mapping property (i.e. surjectivity of the restriction map).

5.2 Base change and descent properties

Proposition 5.3. *The formation of ns-Néron models (resp. Néron models) is compatible with smooth base change, i.e. given a smooth morphism $S' \to S$, a scheme-theoretically dense open $U \subset S$ and an S-algebraic space X which is a ns-Néron model (resp. Néron model) of X_U, the base change $X_{S'}$ is a ns-Néron model (resp. Néron model) of $X_{U'}$.*

Proof. First, note that $X_{S'}/S'$ is smooth since X/S is, separated if X/S is, and that U' is scheme-theoretically dense in S' by [13], théorème 11.10.5. Thus, we only need to check that X'/S' has the Néron mapping property.

Let Y' be a smooth S'-scheme and $u' : Y'_{U'} \to X_{U'}$ a U'-morphism. Composing with the projection: $X_{U'} \to X_U$, we get a U-morphism $Y'_{U'} \to X_U$, which extends to a unique S-morphism $Y' \to X$ by the Néron mapping property since Y'/S is smooth. Then the induced morphism $Y' \to X'$ extends u', and this extension is unique since a morphism $Y' \to X'$ is uniquely determined by the two composites $Y' \to X$ and $Y' \to S'$. □

Corollary 5.4. *If S'/S is a cofiltered limit of smooth S-schemes (indexed by a cofiltered partially ordered set, e.g. a localization, a henselization when S is local...), and X is the (ns-)S-Néron model of X_U, then $X_{S'}$ is the (ns-)S'-Néron model of $X_{U'}$.*

Lemma 5.5 (Néron models are compatible with disjoint unions on the base)**.** *Let I be a set, $(S_i)_{i\in I}$ a family of schemes, and $(N_i \to S_i)_{i\in I}$ a family of* *morphisms of algebraic spaces. Write* $S = \coprod_{i \in I} S_i$ *and* $N = \coprod_{i \in I} N_i$*. Let* U *be a scheme-theoretically dense open of* S*, and write* $U_i = U \times_S S_i$ *for every* i*. Then* N *is the* S*-ns-Néron model of* N_U *(resp. the* S*-Néron model of* N_U*) if and only if for all* i *in* I*,* N_i *is the* S_i*-ns-Néron model of* $N_i \times_{S_i} U_i$ *(resp. the* S_i*-Néron model of* $N_i \times_{S_i} U_i$*).*

Proof. Suppose N is the ns-Néron model of N_U. Then, by Proposition 5.3, for all i in I, N_i is the ns-Néron model of $N_i \times_{S_i} U_i$. Conversely, suppose that for every i in I, N_i/S_i is the ns-Néron model of its restriction to U, and consider a smooth S-algebraic space Y with a morphism $f_u \colon Y_U \to N_U$. For each i, we write $Y_i = Y \times_S S_i$. We have

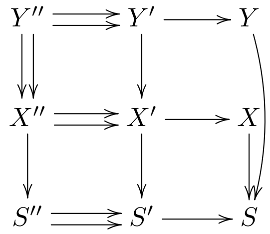
$$\begin{aligned}\mathrm{Hom}_S(Y, N) &= \prod_{i \in I} \mathrm{Hom}_{S_i}(Y_i, N_i) \\ &= \prod_{i \in I} \mathrm{Hom}_{U_i}(Y_i \times_{S_i} U_i, N_i \times_{S_i} U_i) \\ &= \mathrm{Hom}_U(Y_U, N_U),\end{aligned}$$

where the first and third equalities hold since Y is the disjoint union of the Y_i, and the second one because each Y_i/S_i is smooth. Since N/S is smooth (resp. smooth and separated) if and only if all N_i/S_i are smooth (resp. smooth and separated), we are done. □

Proposition 5.6 (Néron models descend along smooth covers)**.** *Let* S *be a scheme and* U *a scheme-theoretically dense open of* S*. Let* $S' \to S$ *be a smooth surjective morphism and* $U' = U \times_S S'$*. Let* X_U *be a smooth* U*-algebraic space, and suppose* $X_{U'}$ *has a (ns-)*S'*-Néron model* X'*. Then* X_U *has a (ns-)*S*-Néron model* X *satisfying* $X' = X \times_S S'$*.*

Proof. We first show X' comes via base change from an S-algebraic space X. Call p_1, p_2 the two projections $S'' := S' \times_S S' \to S'$. They are smooth morphisms, so by Proposition 5.3 and uniqueness of the Néron model, we know $p_1^* X' = p_2^* X'$ is the S''-Néron model of $X_{U''}$ with $U'' = U \times_S S''$. It follows from effectiveness of fppf descent for algebraic spaces ([30, Tag 0ADV]) that X' comes from an S-algebraic space X/S.

The morphism $X \to S$ is smooth since X'/S' is, and separated if X'/S' is (both properties are even fpqc local on the base, see [30, Tag 02KU] and [30, Tag 02VL]). Therefore, we only need to show X/S has the Néron mapping property. Take Y a smooth S-algebraic space with a generic morphism $f_U \colon Y_U \to X_U$, and write Y' (resp. f'_U) for the pullbacks of Y (resp. f_U) under $S' \to S$. Then Y'/S' is smooth so f'_U extends to a unique $f' \colon Y' \to X'$. We have a cartesian diagram



where $S'' := S' \times_S S'$ and the arrows $S'' \to S'$ are the two projections p_1, p_2, so that all horizontal rows are equalizers. We only need to show that $p_1^* f' = p_2^* f'$, which follows from uniqueness in the Néron mapping property of X''/S'' since they coincide over U''. □

Proposition 5.7. *Let S be a scheme, U a scheme-theoretically dense open subscheme of S, X_U/U a smooth U-scheme and N/S a model of X_U of finite type. Then N is the (ns-)Néron model of X_U if and only if for all $s \in S$, $N \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ is a (ns-)$\operatorname{Spec} \mathcal{O}_{S,s}^{et}$-Néron model of its restriction to U.*

Proof. If all the $N \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{et} / \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ are separated, then N/S is also separated, and the "only if" part is a special case of Corollary 5.4. All that remains to do is prove N/S is the ns-Néron model of N_U/U, assuming that for all $s \in S$, N is the ns-Néron model over $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ of its restriction to U. Let Y/S be a smooth S-algebraic space and $f_u \colon Y_U \to X_U$ a U-morphism. Since Y/S is locally of finite presentation, by [13], théorème 8.8.2, every point $s \in S$ has an étale neighbourhood $V_s \to S$ such that f_u extends uniquely to a morphism $Y \times_S V_s \to X \times_S V_s$. By [13], théorème 11.10.5, U remains scheme-theoretically dense in every V_s, so these maps glue as in the proof of Proposition 5.6 and f_U extends to a morphism $Y \to X$. □

5.3 Schemes vs algebraic spaces

Here we introduce the (perhaps more standard) definition of a Néron model as a scheme and not an algebraic space, and go for a little sanity check by showing both notions coincide under conditions of existence.

Definition 5.8. Let S be a scheme and U a dense open subscheme of S. Let Z/U be a smooth separated U-scheme. A *ns-S-Sch-Néron model* of Z is a smooth S-scheme N, with an identification $N_U = Z$, satisfying the *Sch-Néron mapping property*:

For each scheme Y with a smooth morphism $Y \to S$, the restriction map

$$\operatorname{Hom}_S(Y, N) \to \operatorname{Hom}_U(Y_U, Z)$$

is bijective.

Remark 5.8.1. • The ns-Sch-Néron model, if it exists, is unique up to a unique isomorphism.

- Again, when U is scheme-theoretically dense, given a smooth separated S-scheme N with $N_U = Z$, it is the ns-Sch-Néron model if and only if it satisfies existence in the mapping property.
- When a ns-Néron model is a scheme, it is automatically the ns-Sch-Néron model since it satisfies the Sch-Néron mapping property.

Proposition 5.9. *Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a smooth U-scheme. Suppose Z admits a ns-Sch-Néron model N. Then N is also a ns-Néron model of Z.*

Proof. We show that N has the Néron mapping property. Let Y be a smooth S-algebraic space, together with a morphism $Y_U \to Z$ of algebraic spaces. We can choose a presentation of Y as a quotient of a scheme by an étale equivalence relation ([30, Tag 0262]), i.e. S-schemes R and V with an étale covering map $V \to Y$ and an equivalence relation $R \to V \times_S V$ such that the two induced maps $R \to V$ are étale, and such that the diagram

$$R \rightrightarrows V \to Y$$

is a coequalizer of sheaves of sets on $(Sch/S)_{fppf}$. This presentation is compatible with the base change $U \to S$ ([30, Tag 03I4]), so we get a coequalizer

$$R_U \rightrightarrows V_U \to Y_U$$

in the category of sheaves of sets on $(Sch/U)_{fppf}$. Thus $Y_U \to Z$ can be seen as a map $V_U \to Z$ such that both composites $R_U \to Z$ coincide. Then, since V and R are smooth over S by composition, applying the Sch-Néron mapping property, we can extend uniquely $V_U \to Z$ to an S-map $V \to N$. The two composites $R \to N$ both extend the same $R_U \to Z$, so they are equal by uniqueness in the Sch-Néron mapping property. So we have a unique morphism $Y \to N$ of algebraic spaces extending $Y_U \to Z$, as required. $\square$

Corollary 5.10. *If Z admits a ns-Néron model N and a ns-Sch-Néron model N', then $N = N'$ is a scheme.*

6 Néron models of Jacobians

6.1 Alignment and its relation to the Picard space

This subsection summarizes the main results of [21] and introduces a few definitions to adapt them to our context. From now on, given a local ring R, we will note R^{sh} for a strict henselization of R.

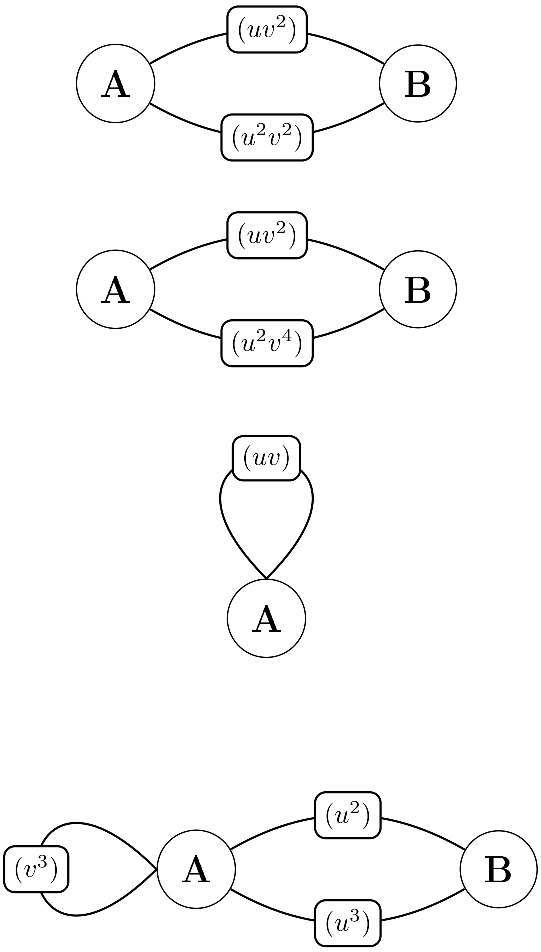
6.1.1 Definition and examples

Definition 6.1. Suppose S is a regular scheme. Let s be a geometric point of S and $R = \mathcal{O}^{et}_{S,s}$. Following [21], Definition 2.11, we say that a labelled graph Γ

is *aligned* when for every cycle Γ^0 in Γ, all the labels figuring in Γ^0 are positive powers of the same principal ideal; and that a nodal curve X/S is *aligned at s* when its dual graph Γ_s at s is aligned. We say X/S is *aligned* if it is aligned at every geometric point of S.

We define Γ_s to be *strictly aligned*, or X to be *strictly aligned at s*, when it satisfies the following condition: for any cycle $\Gamma^0 \subset \Gamma_s$, there exists a *prime* element $\Delta \in R$ such that all the labels of Γ^0 are powers of the principal ideal (Δ) of R. We say that X is *strictly aligned* if it is strictly aligned at every geometric point of S.

Example 6.2. Over $S = \operatorname{Spec} \mathbb{C}[[u, v]]$, at the closed point, among the 4 following dual graphs, the first is non-aligned; the second and the third are aligned but not strictly, and the last one is strictly aligned.



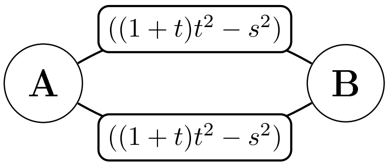
Remark 6.2.1. Strict alignment implies alignment, and is equivalent to strict alignment in the sense of [21], Definition 3.4. In particular, when S is regular, excellent and separated, and X is split and smooth over the complement of a strict normal crossings divisor (i.e. the singular ideals are generated by products of the elements of some regular system of parameters), using [21], Proposition 3.6, we see strict alignment is equivalent to the existence of a Néron model for the Jacobian. We want to investigate what happens when X is a generically smooth nodal curve, but not necessarily smooth over the complement of a normal crossings divisor.

We have to be a little careful about the fact that alignment and strict alignment both deal with étale neighbourhoods. Let us consider two examples.

Example 6.3. The curve over $R = \mathbb{C}[s,t]_{(s,t)}$, given in the weighted projective space $\mathbb{P}_S(1,2,1)$ (in affine coordinates (x,y)) by

$$y^2 = \left((x-1)^2 - (1+t)t^2 + s^2\right)\left((x+1)^2 + (1+t)t^2 - s^2\right)$$

is quasisplit, and its dual graph at the closed point is the following 2-gon:

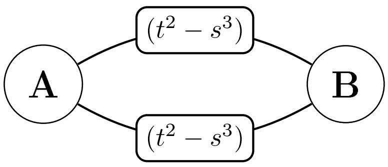


but it is not strictly aligned, even though $(1+t)t^2 - s^2$ is a prime element of R. Indeed, $(1+t)t^2 - s^2$ is prime in R but has two distinct prime factors in a strict henselization, since $(1+t)$ becomes a square, and it is the prime factor decomposition in the étale local rings that counts in the definition of strict alignment.

Example 6.4. On the other hand, the equation

$$y^2 = \left((x-1)^2 - t^2 + s^3\right)\left((x+1)^2 + t^2 - s^3\right)$$

defines a nodal curve over $\operatorname{Spec} R$ with dual graph



which is strictly aligned at the closed point, because $t^2 - s^3$ remains prime in R^{sh}.

6.1.2 Alignment and Néron models

A classical way of obtaining a Néron model for the Jacobian of a proper smooth curve X_U/U with a nodal model X/S, when X is "nice enough", is to consider the biggest separated quotient of the subspace $\mathrm{Pic}^{[0]}_{X/S}$ of $\mathrm{Pic}_{X/S}$ consisting of line bundles of total degree 0 (see for example [1]). In other words, a good "candidate Néron model" is the quotient of $\mathrm{Pic}^{[0]}_{X/S}$ by the closure of its unit section. This works well when three conditions are met: $\mathrm{Pic}^{[0]}_{X/S}$ is representable by an S-algebraic space; the closure of its unit section is flat over S (so that the quotient is also representable); and $\mathrm{Pic}^{[0]}_{X/S}$ satisfies existence in the Néron mapping property (i.e. X is semifactorial after every smooth base change). These are the ideas behind the main result of [21], that we will recall here, and behind the notion of alignment.

Proposition 6.5. *Let S be a regular scheme, $U \subset S$ a dense open, and X/S a nodal curve, smooth over U. Let $P = \mathrm{Pic}^{[0]}_{X/S}$ be the subsheaf of $\mathrm{Pic}_{X/S}$ consisting of line bundles of total degree 0. It is representable by a smooth quasi-separated algebraic space, that we call P again ([1], 8.3.1 and 9.4.1). Let E be the scheme-theoretical closure in P of its unit section. Then the following conditions are equivalent:*

1. *E/S is flat.*
2. *E/S is étale.*
3. *X/S is aligned.*

Proof. This is [21], Theorem 5.17. □

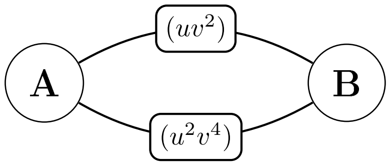
Proposition 6.6. *With the same hypotheses and notations as in Proposition 6.5 above, let J be the Jacobian of X_U. If a Néron model N for J exists, then E/S is flat. Conversely, if $X \times_S S'$ is locally factorial for every smooth base change $S' \to S$ and E/S is flat, then P/E is an S-Néron model for J.*

Proof. This is [21], Theorem 6.2 and Remark 6.3. The idea is that, when the regularity condition we give on X is satisfied, we can use the correspondence between Weil divisors and Cartier divisors to show that line bundles over U extend to the whole base, so P satisfies existence in the Néron mapping property. It follows that its biggest separated quotient P/E (which exists as an algebraic space if and only if E/S is flat) also does. □

We want to investigate the in-between zone, i.e. what happens if we are given an aligned nodal curve that is not locally factorial.

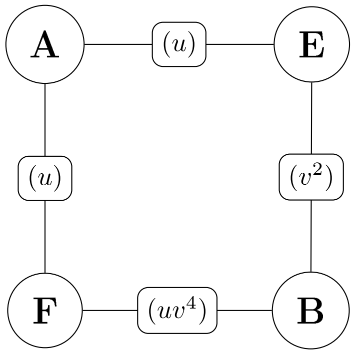
A consequence of Proposition 6.6 is that if J has a Néron model, then *every* nodal model of X_U must be aligned. This is stronger than just alignment of X since alignment is not stable under modifications of nodal curves (see the example below).

Example 6.7. Consider a nodal curve X over $S = \mathrm{Spec}\,\mathbb{C}[[u, v]]$ having the following dual graph at the closed point:

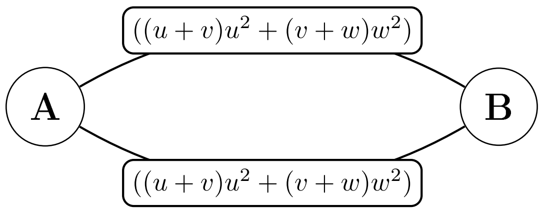


This graph is aligned, and X is smooth over the complement U in S of $\mathrm{Div}(uv)$, but X is not locally factorial (see 4.2), so Proposition 6.5 and Proposition 6.6 do not allow us to immediately conclude to either existence or nonexistence of a Néron model.

The zero locus of u in X has two irreducible components, one containing A and one containing B, and explicit computation shows that if we blow up X in the one containing A, the result is still a nodal curve, with dual graph



This new curve coincides with X over U, and it is not aligned: the Jacobian of X_U cannot have a Néron model. Similarly, if $S = \mathrm{Spec}\, R$ with $R = \mathbb{C}[[u, v, w]]$ and X has dual graph



then, again, we cannot immediately apply 6.5 and 6.6: the curve X is aligned, but its (smooth) base change to $R' = \mathbb{C}[[u, v, w]][v^{-1}, \sqrt{u+v}, \sqrt{v+w}]$ fails to be locally factorial, since in R' the labels become sums of two squares and factor into a product of two primes.

However, we can observe that if the Jacobian of X_U/U had a Néron model, it would still be a Néron model over R', and after that base change, we are now in a case similar to that of the previous example! The curve $X \times_R R'$ can be blown up into a non-aligned nodal curve over R', which means its Jacobian does not have a Néron model.

In conclusion, asking for X to remain aligned after smooth base changes and birational morphisms of nodal curves is a strictly stronger condition than just asking for X to be aligned, yet it remains necessary for a Néron model of the Jacobian to exist. We will see that strict alignment is precisely the closure of alignment under those operations, and that it is the right notion to talk about Néron models of Jacobians in terms of dual graphs.

6.2 Étale-universally prime elements

Here, we will elaborate on a phenomenon illustrated in Example 6.7, namely the fact that some prime elements of a regular, strictly henselian local ring can have several prime factors in a further étale localization. If such an element labels a cycle of an aligned nodal curve, then there is an étale base change after which this curve has a non-aligned refinement (which forbids the existence of a Néron model for the generic Jacobian). We will also show this is actually the *only* possible reason for the smooth base change of an aligned curve to have non-aligned refinements.

Definition 6.8. Let R be a regular local ring and Δ be a non-invertible element of R. We say Δ is *étale-universally prime* when, for all prime ideals $\mathfrak{p} \subset R$ containing Δ, the image of Δ in a strict henselization of $R_{\mathfrak{p}}$ is prime.

Example 6.9. Take $R = \mathbb{C}[s,t]_{(s,t)}$ and $\Delta = (t^2 - s^3)$. Then Δ is étale-universally prime, since Δ remains prime in both R^{sh} and $(R_{(\Delta)})^{sh}$, and the maximal ideal and (Δ) are the only primes of R containing Δ.

Example 6.10. On the other hand, some prime elements are not étale-universally prime even if R is strictly henselian: let $R = (\mathbb{C}[u,v,w]_{(u,v,w)})^{sh}$ and $\Delta = u^2(v+w) - v^2(v-w)$. Then Δ is prime in R (it is even prime in $\mathbb{C}[[u,v,w]]$), but if we consider the prime ideal $\mathfrak{p} = (u,v)$ of R, which contains Δ, we see that Δ has a nontrivial factorization in $R_{\mathfrak{p}}^{sh}$ since the units $v+w$ and $v-w$ of $R_{\mathfrak{p}}$ become squares in $R_{\mathfrak{p}}^{sh}$.

Remark 6.10.1. An element $\Delta \in R$ is étale-universally prime if and only if all localizations of $R/(\Delta)$ are geometrically unibranch in the sense of [12], 23.2.1 or [29], IX, Définition 2.

We are interested in those étale-universal primes to study Néron models because they behave well with respect to the smooth topology. Their key property is Lemma 6.11.

Lemma 6.11. *Let $S = \operatorname{Spec} R$ be an affine regular scheme and Δ be an element of R. Then Δ is étale-universally prime in R if and only if for every smooth morphism $Y \to \operatorname{Spec} R$ and every geometric point $y \in Y$, the image of Δ in $\mathcal{O}_{Y,y}^{et}$ is either invertible or prime.*

Proof. The "if" sense is immediate since the identity $\operatorname{Spec} R \to \operatorname{Spec} R$ is smooth. For the converse, suppose Δ is étale-universally prime. Since smoothness is a Zariski-local property, it is enough to prove that for each smooth map of affines $\operatorname{Spec} A \to \operatorname{Spec} R$, Δ is étale-universally prime in A. Let $\mathfrak{p} \subset A$ be a prime ideal containing Δ and $\mathfrak{m}$ the preimage of $\mathfrak{p}$ in R, the map $R \to (A_{\mathfrak{p}})^{sh}$

factors as $R \to (R_{\mathfrak{m}})^{sh} \to A \otimes_R (R_{\mathfrak{m}})^{sh} \to (A_{\mathfrak{p}})^{sh}$. Since the middle arrow $(R_{\mathfrak{m}})^{sh} \to A \otimes_R (R_{\mathfrak{m}})^{sh}$ is smooth, and since $A \otimes_R (R_{\mathfrak{m}})^{sh} \to (A_{\mathfrak{p}})^{sh}$ is a strict localization at a prime containing the image of $\mathfrak{m}$, we can conclude by quotienting by (Δ) and applying Lemma 2.2. □

Proposition 6.12. *Take X/S a generically smooth nodal curve with S regular. Take s a geometric point of S such that X is strictly aligned at s. Then X is strictly aligned at every étale generization t of s if and only if for every cycle Γ_0 of Γ_s, the only prime of $\mathcal{O}_{S,s}^{et}$ appearing as a factor of the labels of Γ_0 is étale-universally prime.*

Proof. It follows from Proposition 1.8 applied to the specialization morphism: $\operatorname{Spec} \mathcal{O}_{S,t}^{et} \to \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ and the definitions. □

This allows us to detect strict alignment, only looking at the dual graph of the closed fiber:

Definition 6.13. Let X/S be a generically smooth nodal curve with S regular.

We say that Γ_s is *étale-strictly aligned*, or that X is *étale-strictly aligned at s*, when it satisfies the following condition: for any cycle $\Gamma^0 \subset \Gamma_s$, including loops, there exists an étale-universally prime element $\Delta \in R$ such that all the labels of Γ^0 are powers of the principal ideal (Δ) of R. We say that X is *étale-strictly aligned* if it is étale-strictly aligned at every geometric point of S.

Proposition 6.14. *If X/S is a nodal curve with S regular, the following conditions are equivalent:*

1. *X is strictly aligned.*
2. *X is étale-strictly aligned at the closed geometric points of S.*
3. *X is étale-strictly aligned.*

Proof. (3) $\implies$ (2) and (2) $\implies$ (1) are clear. (2) $\implies$ (3) follows from observing that in a locally noetherian scheme, every point specializes to a closed point, and if R is a local ring, Δ an étale-universally prime element of R, and $\mathfrak{p}$ a prime ideal of R containing Δ, then Δ is also étale-universally prime in $R_{\mathfrak{p}}$. We will show (1) $\implies$ (2).

Take X/S a strictly aligned generically smooth nodal curve with S regular. We will show it is étale-strictly aligned at the closed geometric points of S. We can assume $S = \operatorname{Spec} R$ is local and strictly henselian, with closed point s. Let Γ be the dual graph of X at s, and Γ^0 be a cycle of Γ. There is a prime $\Delta \in R$ such that all labels of Γ_0 are powers of Δ, and we have to show Δ is étale-universally prime in R.

Let $\mathfrak{p}$ be a prime ideal of R containing Δ, and choose a strict henselization $R_{\mathfrak{p}}^{sh}$ of R_p. It gives an étale generization t of s, at which X is strictly aligned, so the cycle pulled back from Γ^0 in the dual graph of X at t has all its labels generated by powers of some prime of $R_{\mathfrak{p}}^{sh}$. Thus, the image of Δ in $R_{\mathfrak{p}}^{sh}$ is a power of a prime. Therefore it is enough to show $R_{\mathfrak{p}}^{sh}/(\Delta)$ is reduced.

But $R_{\mathfrak{p}}^{sh}/(\Delta)$ is a strict henselization of $R_{\mathfrak{p}}/(\Delta)$ since the quotient of a henselian ring is henselian, so it is reduced as a directed colimit of reduced $R_{\mathfrak{p}}/(\Delta)$-algebras. □

6.3 Strict alignment is necessary and sufficient for Néron models to exist

The goal of this subsection is to prove Theorem 6.20. It is to be noted that a variant of the theorem probably holds under a weaker assumption than regularity of S: what we care about is extending generic line bundles on X after étale base change, so having S parafactorial along the complement of the discriminant locus after every smooth base change, plus some other minor assumptions, should suffice. Of course, strict alignment would then have to be defined in that context, since the étale local rings of S would not be unique factorization domains anymore. In addition, if one were in need of such generality, they would need to verify that the material we use (e.g. in [21]) also works after weakening the hypotheses.

6.3.1 The necessity of strict alignment

We start with the easy implication: we will show that if a nodal curve is not strictly aligned, then over some étale local ring of the base, we can find a non-aligned refinement of it (which means there can be no Néron model for the generic Jacobian).

Proposition 6.15. *Let S be a regular scheme and X/S a nodal curve, smooth over a dense open $U \subset S$. If the Jacobian of X_U/U has a Néron model over S, then X/S is strictly aligned.*

Proof. We will work by contradiction, assuming there is a geometric point $s \in S$ at which X is not strictly aligned. Using Corollary 5.4, we can assume S is a strictly local scheme $\operatorname{Spec} R$, with closed point s. Remember that R is regular, thus a unique factorization domain. Note Γ the dual graph of X at s and l the edge-labelling of Γ. By assumption, there is a cycle Γ^0 in Γ and two (not necessarily distinct) edges e and e' of Γ^0 such that $l(e)l(e')$ has at least two distinct prime factors.

We know X is aligned by Proposition 6.5 and Proposition 6.6: there is an element $\Delta \in R$ such that all edges of Γ^0 are labelled by positive powers of ΔR. This applies in particular to e and e', so we can write Δ as a product $(\Delta_1\Delta_2)$, where Δ_1, Δ_2 are non-invertible elements of R with no common factor.

Let x be the singular point of X corresponding to e. Since S is strictly local with closed point s, we can pick an orientation (C, D) of X/S at x. Call $X' \to X$ the (Δ_1)-refinement of X at x relatively to (C, D). By Lemma 4.6, the dual graph of X' at s contains a cycle, refining Γ^0, such that the edge corresponding to x has been replaced by a chain of two edges, one of label (Δ_1) and one of label (Δ_2). In particular, X' is not aligned at s. However, $X'_U = X_U$, so the jacobian of X'_U has a Néron model: we get a contradiction by virtue of Proposition 6.5 and Proposition 6.6. □

6.3.2 Fiberwise-disconnecting locus of nodal curves and closure of the unit section of the Picard scheme

As strict alignment is only a condition on the cycles of the dual graphs, we have to show that "labels of disconnecting points do not matter", in a sense that will made precise by Proposition 6.19. The idea is that when one blows up a nodal curve over a strictly local base in a section through a disconnecting singular point, all "new" line bundles are killed by the growth of the closure of the unit section, and the quotient P/E does not change. We start with a few technical lemmas.

Lemma 6.16. *Let $S = \operatorname{Spec} R$ be a trait (i.e. the spectrum of a discrete valuation ring) and $f : X \to S$ a generically smooth quasisplit nodal curve. Let $\pi : X' \to X$ be the blowing-up in a closed non-smooth point x of X/S. Let $\mathfrak{L}$ be a line bundle on X', trivial over the exceptional fiber of π. Then $\pi_*\mathfrak{L}$ is a line bundle on X.*

Proof. This is [26], Proposition 4.2. □

Lemma 6.17. *Let $S = \operatorname{Spec} R$ be a regular and strictly local scheme. Let $f\colon X \to S$ be a quasisplit nodal curve, smooth over some dense open $U \subset S$. Let $\pi\colon X' \to X$ be a refinement and $\mathfrak{L}$ be a line bundle on X'. Let $Y \subset X'$ be the exceptional locus of π and suppose $\mathfrak{L}|_Y \simeq \mathcal{O}_Y$. Then $\pi_*\mathfrak{L}$ is a line bundle on X.*

Proof. $\pi_*\mathfrak{L}$ is a coherent $\mathcal{O}_X$-module and X is reduced, so it is enough to check that, for all $y \in X$, we have $\dim_{k(y)} \pi_*\mathfrak{L} \otimes_{\mathcal{O}_X} k(y) = 1$. It is obvious for all y such that π is a local isomorphism at y, so we only need to check it when y is in the image of the exceptional locus of π.

Take a section $\sigma\colon S \to X$ such that π is the blowing-up in the sheaf of ideals of σ. Let x be in the image of the exceptional locus of π and s its image in S: we have $x = \sigma(s)$. The base change of π to $\operatorname{Spec} \mathcal{O}^{sh}_{S,s}$ is still the blowing-up in the sheaf of ideals of σ, and the condition $\dim_{k(x)} \pi_*\mathfrak{L} \otimes_{\mathcal{O}_X} k(x) = 1$ can also be checked after base change to $\operatorname{Spec} \mathcal{O}^{sh}_{S,s}$, so we can assume s is the closed point of S. Iterating the prime avoidance lemma, we see that $\mathcal{O}_S(S)$ admits a quotient D, that is a discrete valuation ring, such that the generic point of $T = \operatorname{Spec} D$ lands in U. Pick a uniformizer t of D. We have $\Delta_x D = t^n D$ for some $n \geq 1$, where Δ_x is a generator of the singular ideal of x.

The base change X_T/T is a nodal curve. The point corresponding to x, that we still call x, has singular ideal $t^n D$. The sheaf of ideals $\mathcal{I}$ of σ in X_T is trivial away from x, and given at the completed étale local ring $\widehat{\mathcal{O}^{et}_{X_T,x}} = \widehat{D^{sh}}[[u,v]]/(uv - t^n)$

by the ideal $(u - t^k, v - t^l)$ with $k + l = n$ for a good choice of isomorphism. If $n = 1$, then $\mathcal{I}$ is trivial (and a fortiori Cartier) so $X'_T = X_T$.

Suppose $n \geq 2$ and pick $d = \lfloor \frac{n}{2} \rfloor$. There is a sequence

$$X'' = X_d \to X_{d-1} \to \ldots \to X_0 = X_T$$

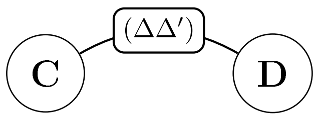
where each $X_{i+1} \to X_i$ is a blowing-up in a closed point of image x, namely the only closed non-T-smooth point of X_i of image x and of singular ideal $\neq tD$, and the preimage of x in X_d is a chain of $n-1$ copies of $\mathbb{P}^1_{k(x)}$, intersecting in $n-2$ non-smooth points of ideal tD. The sheaf of ideals $\mathcal{I}$ on X_T is Cartier in X'', which, by the universal property of blowing-ups, implies that $X'' \to X_T$ factors through $X'_T \to X_T$.

Now, the restriction $\mathfrak{L}|_{X''}$ is a line bundle on X'', trivial on the exceptional locus of $X'' \to X_T$. Thus, using the preceding lemma, we see inductively that its pushforward to every X_i, and in particular to $X_0 = X_T$, is a line bundle. This, in turn, gives us $\dim_{k(x)} \pi_* \mathfrak{L} \otimes_{\mathcal{O}_X} k(x) = 1$: $\pi_*\mathfrak{L}$ is a line bundle on X. □

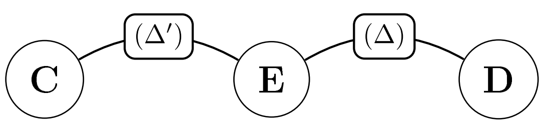
Lemma 6.18. *Let $S = \operatorname{Spec} R$ be a regular and strictly local scheme with closed point s. Let $f\colon X \to S$ be a nodal curve, smooth over some dense open $U \subset S$. Let $\pi\colon X' \to X$ be a refinement such that the exceptional locus of π is disconnecting in the closed fiber. Let $\mathfrak{L}$ be a line bundle of total degree 0 on X'. There exists a line bundle $\mathfrak{L}'$ on X', trivial over U, such that $(\mathfrak{L} \otimes \mathfrak{L}')|_Z \simeq \mathcal{O}_Z$, where Z is the exceptional locus of $X' \to X$.*

Proof. The morphism $X' \to X$ is the blowup in a section $\sigma\colon S \to X$. Set $x = \sigma(s)$, and call Γ, Γ' the respective dual graphs of X and X' at s. By hypothesis, x is a singular point of X, disconnecting in its fiber.

We only look at the local picture at x, since π is an isomorphism away from x. Pick an isomorphism $\widehat{\mathcal{O}_{X,x}} \simeq \widehat{R}[[u,v]]/uv - \Delta_x$ where Δ_x is a generator of the singular ideal of x. The map $\widehat{\mathcal{O}_{X,x}} \to \widehat{R}$ given by σ sends u, v to elements Δ, Δ' of R with $\Delta\Delta' = \Delta_x$. Lemma 4.6 shows that the edge



corresponding to x in Γ (where C, D are necessarily distinct since x is disconnecting) is replaced in Γ' by a chain



Where we still write C, D for the respective strict transforms of C and D in X'.

The new nodal curve $f'\colon X' \to S$ is quasisplit since S is strictly local. Call z the singular point $E \cap D$ of X' (of ideal (Δ)) and Y the connected component of the non-smooth locus of X'/S containing z. The map $Y \to S$ is a closed immersion, cut out by Δ.

Now, since $Y \times_S Y$ is disconnecting in $X \times_S Y$, we know $(X \backslash Y) \times_S Y$ has two distinct connected components Y_1^0 and Y_2^0, respectively containing the images of C and D in $(X' \backslash Y) \times_S Y$. Call Y_2 the scheme-theoretical closure of Y_2^0 in X'. We will show it is a Cartier divisor on X'.

The sheaf of ideals $\mathcal{J}$ defining Y_2 in X' is locally principal away from Y, cut out by Δ in Y_2^0 and by 1 in $X \backslash (Y \cup Y_2^0)$: we only need to check it is invertible on $\mathcal{O}_{X',z}$, which is a consequence of Lemma 1.16 (or can be seen explicitly in $\operatorname{Spec} \widehat{\mathcal{O}^{et}_{X',z}}$). Thus $\mathcal{J}$ is Cartier, and $\mathfrak{L}'' := \mathcal{O}(Y_2)$ is a line bundle on X', trivial over U.

Let V be the closed subscheme of S cut out by the ideal $(\Delta, \Delta') \subset R$. The exceptional locus Z of π is a $\mathbb{P}^1$-bundle on V. But Z and Y_2 intersect transversally at one double point in each fiber over V, so $\deg \mathfrak{L}''|_Z = 1$. Let d be the degree of $\mathfrak{L}$ on Z and $\mathfrak{L}' = \mathfrak{L}''^{\otimes -d}$, then $\mathfrak{L} \otimes_{\mathcal{O}_{X'}} \mathfrak{L}'$ has degree zero on Z, hence is trivial on Z since $Z \simeq \mathbb{P}^1_V$. □

Proposition 6.19. *Let $f : X \to S$ be a nodal curve with $S = \operatorname{Spec} R$ regular and strictly local. Let $\pi : X' \to X$ be a refinement, such that its exceptional locus is disconnecting in the closed fiber. Set $P = \operatorname{Pic}^{[0]}_{X/S}$ and $P' = \operatorname{Pic}^{[0]}_{X'/S}$ and call E and E' the scheme-theoretical closures of the unit sections of P and P' respectively. Then the canonical morphism of algebraic spaces $P \to P'$ induces an isomorphism $P/E \to P'/E'$.*

Proof. First, we show $P \to P'$ is an open immersion.

The fiberwise-connected component of unity $\operatorname{Pic}^0_{X/S}$ is an open neighbourhood of the unit section in P, and same goes for $\operatorname{Pic}^0_{X'/S} \to P'$. We have a commutative diagram of S-spaces

$$\begin{array}{ccccc} \operatorname{Pic}^0_{X/S} & \hookrightarrow & P & \twoheadrightarrow & \Phi_{X/S} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Pic}^0_{X'/S} & \hookrightarrow & P' & \twoheadrightarrow & \Phi_{X'/S} \end{array}$$

where both horizontal rows are exact, and $\operatorname{Pic}^0_{X/S} \to \operatorname{Pic}^0_{X'/S}$ is an isomorphism by Lemma 6.17, so $P \to P'$ is locally on the source an open immersion: to deduce it is an actual open immersion, we only need to show it is set-theoretically injective, which can be checked on its fibers over S. Let s be a point of S and k its residue field. The smooth locus of X_s^{sm} has a k-rational point in every irreducible component by quasisplitness of X (which follows from the fact S is strictly local), and π is an isomorphism above X_s^{sm}, so $\Phi_s \to \Phi'_s$ is set-theoretically injective. It follows $P \to P'$ is set-theoretically injective, so it is an open immersion.

Now this implies the scheme-theoretical closure in P' of the unit section of P is E', so $E = E' \times_{P'} P$. Thus $P/E = P'/E' \times_{P'} P$, and $P/E \to P'/E'$ is an open immersion as a base change of the open immersion $P \hookrightarrow P'$. Moreover, the formation of P and P' commutes with base change, so $P/E \to P'/E'$ will be surjective (thus an isomorphism) if it is surjective on S-points: take a section $\sigma\colon S \to P'/E'$, Lemma 6.18 shows that σ can be represented by a line bundle $\mathfrak{L}$ on X', trivial over the exceptional locus of π. But then by Lemma 6.17, $\pi_*\mathfrak{L}$ is a line bundle on X, so it gives a section $S \to P/E$. Composing with $P/E \to P'/E'$, we obtain the S-point of P'/E' corresponding to the line bundle $\pi^*\pi_*\mathfrak{L}$, which is none other than σ since $\pi^*\pi_*\mathfrak{L} \otimes_{\mathcal{O}_{X'}} \mathfrak{L}^{\otimes -1}$ is trivial over U. Thus σ comes from an S-point of P/E and we are done. □

6.3.3 The main theorem

Theorem 6.20. *Let S be an excellent regular scheme, $U \subset S$ a dense open subscheme, and X a nodal S-curve, smooth over U. The following conditions are equivalent:*

(i) The Jacobian J of X_U admits a Néron model over S.

(ii) X is strictly aligned.

(iii) X is étale-strictly aligned.

(iv) X is étale-strictly aligned at all closed points of S.

If these conditions are met, the Néron model is of finite type. If in addition X/S has a partial resolution $X' \to X$, the Néron model of J is P/E, where $P = \mathrm{Pic}^{[0]}_{X'/S}$ and E is the scheme-theoretical closure of the unit section in P.

Proof. The Néron model is of finite type if it exists by [19], Theorem 2.1.

Conditions (ii), (iii) and (iv) are equivalent by Proposition 6.14, and $(i) \to (ii)$ is Proposition 6.15.

We will show $(iii) \to (i)$. As both (iii) and (i) can be checked after base change to an étale neighbourhood of an arbitrary point s of S, we can assume X/S is quasisplit (using Lemma 1.14). Base-changing to a further étale cover, we can assume X/S has a partial resolution (see Proposition 4.14). Replacing X by this partial resolution, we can assume X is square-free and étale-strictly aligned. Thus, every non-smooth point of X/S that is not disconnecting in its fiber has étale-universally prime label. Take $P = \mathrm{Pic}^{[0]}_{X/S}$, we will show P/E is a Néron model for J.

The base change of X to any étale local ring of S still has étale-universally prime labels in all cycles of all dual graphs, so our claim that P/E is a Néron model for J can be checked over the étale local rings of S using 5.7 and the fact that the formation of P and E commutes with base change to the étale local rings of S. Thus we also assume S is strictly local, with closed point s.

The quotient P/E is a smooth and separated model of J (since $J = \mathrm{Pic}^0_{X_U/U}$) so we only have to show it satisfies existence in the Néron mapping property. Let Y be a smooth S-algebraic space, together with a generic morphism $f_u : Y_U \to J$. We want to show f_u extends to a morphism $Y \to P/E$. Using the uniqueness in the Néron mapping property and the effectiveness of fppf descent for algebraic spaces ([30, Tag 0ADV]), we can work étale-locally on Y: it is enough to extend f_u to $\mathcal{O}_{Y,y}$ for every geometric point y of Y. Thus we can and will work assuming $Y = \mathrm{Spec}\, A$ is a strictly local scheme, and replacing the hypothesis that Y/S is smooth by the hypothesis that A is a filtered colimit of smooth R-algebras.

The map $Y_U \to J$ corresponds to a line bundle $\mathfrak{L}_U$ on $(X \times_S Y)_U$. We only have to show $\mathfrak{L}_U$ comes from a line bundle on $X \times_S Y$: indeed, such a line bundle would have total degree zero and give a morphism $Y \to P$. Composing with $P \to P/E$, we would get a map extending f_U as desired.

The base change $X \times_S Y$ is still a nodal curve whose non-disconnecting singular points have étale-universally prime label by Lemma 6.11. Take a Y-resolution $X_0 \to X \times_S Y$. There is a Cartier divisor D_U on $(X \times_S Y)_U = (X_0)_U$ such that $\mathfrak{L}_U = \mathcal{O}(D_U)$. Write it as a finite sum $D_U = \sum_{i=1}^{k} n_i D_i$ where the n_i are integers and the D_i are primitive Weil divisors on $(X_0)_U$, and take $D = \sum_{i=1}^{k} n_i \overline{D_i}$, where $\overline{D_i}$ is the scheme-theoretical closure of D_i in X_0. By definition D is only a Weil divisor on X_0, but, by Lemma 4.2, X_0 is locally factorial, so D is automatically Cartier and the line bundle $\mathfrak{L} = \mathcal{O}(D)$ on X_0 restricts to $\mathfrak{L}_U$. Moreover, $E \times_S Y$ is still the closure of the unit section in $P \times_S Y$, so the quotient of $\mathrm{Pic}^{[0]}_{X_0/Y}$ by the closure of its unit section is equal to $P/E \times_S Y$ by Proposition 6.19, and we get the desired line bundle on $X \times_S Y$ extending $\mathfrak{L}_U$. □

7 Néron models of curves with nodal models

Let S be a regular base scheme, U a dense open subscheme of S, and X/S a nodal relative curve, smooth over U. In what follows, we are interested in the existence of a Néron model over S for the curve X_U/U.

We will end up getting a very restrictive condition on the local structure of singularities for an actual Néron model to exist. When X/S is quasisplit, almost all connected components of its singular locus need to be irreducible. However, we will also see one can often exhibit a smooth (bot not necessarily separated) S-algebraic space with the Néron mapping property. Our condition of local irreducibility of the singular locus of X/S then becomes a condition for separability of this object, i.e. a condition for it to be a true Néron model. More precisely, the main results of this section are:

Theorem 7.1 (Theorem 7.40). *Let S be a regular excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Suppose X has no rational loops, and suppose no geometric fiber of X contains a rational component meeting the non-exceptional other irreducible components in three points or more. Then X_U/U has a ns-Néron model N/S. If in addition*

X/S is quasisplit, then N is the smooth aggregate (Construction 7.5) of the stable model X^{stable} of X_U (Definition 7.26).

Theorem 7.2 (Theorem 7.48)**.** *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. If X_U has a Néron model over S, then the two following conditions are met:*

- *The singular locus* $\mathrm{Sing}(X/S)$ *is irreducible around every non-exceptional singular geometric point of X/S.*
- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}^{et}_{S,s}$ have the same radical.*

Conversely, suppose these conditions are met. Suppose in addition that no geometric fiber of X/S contains either a rational cycle or a rational component meeting the non-exceptional other components in at least three points. Then X_U/U has a Néron model, i.e. the ns-Néron model of X_U/U exhibited in Theorem 7.40 is separated over S.

7.1 Factoring sections through refinements

A first question, easier to tackle than existence of a Néron model, is "given a U-point of X_U, can we extend it to a section of a smooth S-model of X_U".

We answer with a two-step strategy: first, when X has no rational component in its geometric fibers, all U-points of X extend to sections by the following result from [8]:

Proposition 7.3 ([8], Proposition 6.2)**.** *Let X/S be a proper morphism of schemes, where S is noetherian, regular and integral. Let K be the function field of S, and suppose that no geometric fiber of X/S contains a rational curve. Then every K-rational point of X_K extends to a section $S \to X$.*

The S-section of X we obtain might meet the singular locus. Our second step consists in finding a refinement of X such that the section comes (at least locally on S) from a smooth section of this refinement.

Lemma 7.4. *Let X be a quasisplit nodal curve over a regular scheme S. Suppose X is smooth over a scheme-theoretically dense open subscheme $U \subset S$. Let σ, τ be sections of X/S, and $\phi\colon X' \to X$ the blowing-up in the ideal sheaf of τ. Let s be a point of S and suppose $\tau(s)$ is a singular point x of X_s at which X/S is orientable. Then the three following conditions are equivalent:*

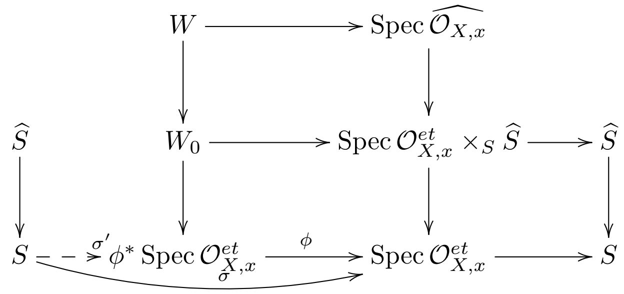
1. *The restriction of σ to the étale local ring* $\mathrm{Spec}\,\mathcal{O}^{et}_{S,s}$ *factors through a smooth section of* $X' \times_S \mathrm{Spec}\,\mathcal{O}^{et}_{S,s}/\mathrm{Spec}\,\mathcal{O}^{et}_{S,s}$.
2. *There exists an étale neighbourhood V of s such that the restriction of σ to V factors through a smooth section of $X' \times_S V/V$.*

3. *Either $\sigma(s)$ is a smooth point of X_s, or $\sigma(s) = x$ and σ and τ are of opposite types at x.*

Proof. Conditions (1) and (2) are equivalent since nodal curves are of finite presentation. We will now prove that (1) and (3) are equivalent. This can be done assuming $S = \operatorname{Spec} R$ is strictly local, with closed point s. We can also assume $\sigma(s) = x$ since otherwise, the equivalence of (1) and (3) follows from the fact ϕ is an isomorphism away from x.

Under these additional hypotheses, let us assume (3) and prove (1). We know σ factors uniquely through $\operatorname{Spec} \mathcal{O}^{et}_{X,x}$.

Let us note $\widehat{S} = \operatorname{Spec} \widehat{R}$. We have the following commutative diagram:



where $W_0 = \phi^* \operatorname{Spec} \mathcal{O}^{et}_{X,x} \times_S \widehat{S}$ and $W = W_0 \times_{X \times_S \widehat{S}} \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$, so that all squares are pullbacks, and σ' is the strict transform of σ in X'. Then σ' is a rational map (defined at least over U) and our goal is to prove that it is defined everywhere.

There are sections $\widehat{\sigma}$ and $\widehat{\tau}$ of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}} / \widehat{S}$ induced by σ and τ respectively. Pick an isomorphism

$$\operatorname{Spec} \widehat{\mathcal{O}_{X,x}} \simeq \widehat{R}[[u,v]]/(uv - \Delta\Delta'),$$

where the comorphism of $\widehat{\tau}$ sends u, v to Δ, Δ' respectively. The section $\widehat{\sigma}$ is fully described by the images t_1 of u and t_2 of v in $\widehat{R}$ by its comorphism. Since σ and τ have opposite types at x, there is a unit λ such that $t_1 = \lambda\Delta'$ and $t_2 = \lambda^{-1}\Delta$.

We claim that $\widehat{\sigma}$ factors through $W \to \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$. Since $W \to \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$ is the blow-up in the ideal $I_{\widehat{\tau}} = (u - \Delta, v - \Delta')$ defining $\widehat{\tau}$, by the universal property of blow-ups ([30, Tag 085U]), it suffices to show that the pull-back of $I_{\widehat{\tau}}$ to $\widehat{S}$ by $\widehat{\sigma}$ is Cartier. Blow-ups commute with completions, so our claim reduces to proving that the ideal $(u - \Delta, v - \Delta)$ of

$$A := \widehat{R}[u,v]/(uv - \Delta\Delta')$$

becomes invertible in $\widehat{R}$ when we map A to $\widehat{R}$ via

$$\begin{aligned} A &\to \widehat{R} \\ u &\mapsto \lambda\Delta' \\ v &\mapsto \lambda^{-1}\Delta. \end{aligned}$$

The image of $I_{\widehat{\tau}}$ under this map is the ideal $I = (\lambda\Delta' - \Delta)$. If $\lambda\Delta' \neq \Delta$, then I is invertible and the claim holds. Otherwise, we reduce to this case by observing that the blow-up of A in $(u-\Delta, v-\Delta')$ is canonically isomorphic to the blow-up of A in $(u - \mu\Delta, v - \mu^{-1}\Delta')$ for any unit μ of $\widehat{R}$, as can be seen in the proof of Corollary 4.7.

Now, let us check that σ factors through X' if and only if $\widehat{\sigma}$ factors through W. Looking at the diagram above, we see that a factorization of $\widehat{\sigma}$ through W yields a factorization of $\sigma \times_S \widehat{S}$ through W_0, which means the (faithfully flat) base change to $\widehat{S}$ of the rational map σ' is defined everywhere, so σ' itself is defined everywhere. Conversely, if σ' is an actual S-section, it yields a section from $\widehat{S}$ to a completed local ring of X', and all completed local rings of X' at points above x factor through W.

We have proven σ factors through a section $\sigma' \colon S \to X'$. We need to show this section is smooth. It suffices to show $\sigma'(s)$ is a smooth point of X'/S. Call E the preimage of x in X'_s. The point $\sigma'(s)$ must be in E since $\sigma(s) = x$. Looking at the local description of X' in the proof of Lemma 4.6, we see E contains exactly two non-smooth points y and y' of X_s, and there is an isomorphism $\widehat{\mathcal{O}_{X',y}} = \widehat{R}[[\beta, v]]/(\beta v + \Delta)$ such that the natural map $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{X',y}}$ sends u, v to $\beta(v - \Delta') + \Delta$ and v respectively. It follows that $\sigma'(s) = y$ if and only if t_2 strictly divides Δ, i.e. if and only if Δ' strictly divides t_1. Symmetrically, $\sigma'(s) = y'$ if and only if Δ strictly divides t_2. Thus, $\sigma'(x)$ is in the smooth locus of X'/S as claimed, and we have proven (3) implies (1).

For the converse, suppose σ comes from a section $\sigma' \colon (X'/S)^{sm}$. By our additional hypothesis that $\sigma(s) = x$, we know $\sigma'(s)$ is a point of E that is neither y nor y', and it follows from the discussion in the paragraph above that σ and τ are of opposite types at x. □

7.2 First construction of the ns-Néron model

In the previous subsection, we have seen how to factor (at least locally) one section of X to the smooth locus of some refinement of X. If we want to approach the Néron mapping property, we would rather have a smooth model of X_U, mapping to X, through which *all* sections will simultaneously factor. Intuitively speaking, we need this model to contain the smooth loci of all possible refinements of X, at all singular points and of all types, after any smooth base change. We will now present the formal construction.

Construction 7.5. Let S be a regular and excellent scheme and X/S a quasisplit nodal curve, smooth over a dense open U of S. For each point s of S, pick an admissible neighbourhood $V^{(s)}$ of s in S as in Definition 3.8. We will write $V^{(s,s')}$ the fiber product $V^{(s)} \times_S V^{(s')}$. For each s and each singular point x of

X_s, pick an orientation of $X_{V^{(s)}}$ at x. For each type T at x, pick a $V^{(s)}$-section $\tau^{(x,T)}$ of $X_{V^{(s)}}$ of type T at x, and write $X^{(x,T)} \to X_{V^{(s)}}$ the blowing-up in that section. Write

$$X^{tot} = \coprod_{(s,x,T)} (X^{(x,T)}/V^{(s)})^{sm}.$$

Then X^{tot} is a X-scheme, smooth over S. Consider two index triples (s,x,T) and (s',x',T') and call R' the same type locus of $\tau^{(x,T)}|_{V^{(s,s')}}$ and $\tau^{(x',T')}|_{V^{(s,s')}}$. Then R' is an open subscheme of $X_{V^{(s,s')}}$ by Proposition 3.6, and the pull-back $R^{(x,T,x',T')}$ of R' to $(X^{(x,T)}_{V^{(s,s')}}/V^{(s,s')})^{sm}$ is canonically isomorphic to the pull-back of R' to $(X^{(x',T')}_{V^{(s,s')}}/V^{(s,s')})^{sm}$ by Corollary 4.7. Therefore, we have étale maps

$$R^{(x,T,x',T')} \to (X^{(x,T)}/V^{(s)})^{sm}$$
$$R^{(x,T,x',T')} \to (X^{(x',T')}/V^{(s')})^{sm}.$$

These maps define an étale equivalence relation on X^{tot}. We write N the quotient algebraic space (see [30, Tag 02WW]), and call it the *smooth aggregate* of X.

Proposition 7.6. *With the same hypotheses and notations as in Construction 7.5, N is well-defined, smooth over S, and depends only on X (i.e. if one makes different choices of admissible neighbourhoods $V^{(s)}$ and of sections $\tau^{(x,T)}$, the resulting smooth aggregate N' is canonically isomorphic to N). The map $N \to X$ is an isomorphism above the smooth locus of X/S.*

Proof. First, let us prove that N is well-defined, i.e. that we have indeed given an étale equivalence relation on X^{tot}. For any pair of index triples (s,x,T) and (s',x',T'), the maps

$$R^{(x,T,x',T')} \to (X^{(x,T)}/V^{(s)})^{sm},$$
$$R^{(x,T,x',T')} \to (X^{(x',T')}/V^{(s')})^{sm}$$

are étale since $V^{(s)} \to S$ and $V^{(s')} \to S$ are. These maps jointly form an étale equivalence relation since for any quasisplit nodal curve Y/R with R regular, and any singular point y at which Y/R is orientable, "having the same type at y" is an equivalence relation on the set of sections $R \to Y$. Since X^{tot} is S-smooth, N is also S-smooth. The fact that $N \to X$ is an isomorphism above the smooth locus of X/S follows from observing that all $X^{(x,T)} \to X_{V^{(s)}}$ are isomorphisms above said smooth locus, and that the $V^{(s)}$ form an étale cover of S.

Now, we have to show N only depends on X. For every (s,x,T), consider another admissible neighbourhood $W^{(s)}$ of s and a section $\sigma^{(x,T)}$ of $X_{W^{(s)}}$ of type T at x. This gives rise to another smooth aggregate N', and we will prove N and N' are canonically isomorphic. We can assume $V^{(s)}$ and $W^{(s)}$ are admissible neighbourhoods of the same geometric point $\bar{s} \to S$ mapping to s.

First, we will do so assuming that the $W^{(s)}$ are smaller than the $V^{(s)}$ and that the $\sigma^{(x,T)}$ are obtained from the $\tau^{(x,T)}$ via pullback. In that case, there is a canonical map from $X'^{tot} := \coprod_{(s,x,T)} (X^{(x,T)}/V^{(s)})^{sm}$ to X^{tot}, compatible with the étale equivalence relations defining N and N', so we get a canonical map $N' \to N$ of S-algebraic spaces. This map restricts to an isomorphism over the étale stalks of all geometric points $\bar{s} \to S$, so it is an isomorphism.

Now, let us drop the assumption that the $\sigma^{(x,T)}$ are obtained from the $\tau^{(x,T)}$ via pullback. For all (s, x, T), by the special case proven above, we can assume $V^{(s)} = W^{(s)}$. Using Proposition 3.6 and the special case proven above, we can assume (shrinking $V^{(s)}$ if necessary) that $\sigma^{(x,T)}$ and $\tau^{(x,T)}$ have the same type everywhere. It follows from Corollary 4.7 that the blowing-ups in the sheaves of ideals of $\sigma^{(x,T)}$ and $\tau^{(x,T)}$ are canonically isomorphic, and this holds for all (s, x, T), so $N = N'$ by construction.

Finally, we also drop the assumption that there exist maps of étale neighbourhoods $W^{(s)} \to V^{(s)}$. Then, N and N' are still canonically isomorphic by the special cases above since $W^{(s)} \times_S V^{(s)}$ is an admissible neighbourhood of s that factors through both $V^{(s)}$ and $W^{(s)}$. □

Proposition 7.7. *The formation of smooth aggregates commutes with smooth base change, i.e. if S is a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, N the smooth aggregate of X/S, and Y/S a smooth morphism of schemes, then $N \times_S Y$ is the smooth aggregate of X_Y/Y.*

Proof. Immediate from Proposition 3.10 and Proposition 7.6. □

Corollary 7.8. *If S is a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, N the smooth aggregate of X/S, and Y/S a cofiltered limit of smooth morphisms, then $N \times_S Y$ is the smooth aggregate of X_Y/Y.*

Proposition 7.9. *Let S be a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, and N the smooth aggregate of X/S. Then every S-section of X/S factors uniquely through N.*

Proof. First, we prove uniqueness: suppose σ comes from two sections σ_0, σ_1 of N/S an let us show $\sigma_0 = \sigma_1$. Since σ_0 and σ_1 coincide on $N_U = X_U$, it is enough to show that for any $t \in S$ we have $\sigma_0(t) = \sigma_1(t)$. This can be done assuming S is strictly local with closed point t. Describe N as in Construction 7.5 using admissible neighbourhoods $V^{(s)}$ of every point s of S and sections $\tau^{(x,T)}$ of $X_{V^{(s)}}$ of type T at x for every singular point x of X_s and every type T at x. By Proposition 7.6, we can assume none of the $V^{(s)}$ contains t except $V^{(t)}$ and $V^{(t)} = S$. Put $y = \sigma(t)$. If y is a smooth point of X/S, then σ factors through the smooth locus of X/S, above which $N \to X$ is an isomorphism, so we are done. Otherwise, by Lemma 7.4, we see that σ_0 and σ_1 must both factor through the Zariski-open subscheme $X^{(t,y,T)}$ of N, where T is the type at y opposite to that of σ. Since $X^{(t,y,T)}$ is a nodal curve over S, it is S-separated, and we conclude using the fact σ_0 and σ_1 coincide over U.

Next, we have to show existence. We recycle the notations of Construction 7.5. By descent, using the uniqueness part we have already proven, it is enough to show that for all $s \in S$, the section $\sigma^{(s)} := (\sigma, \mathrm{Id})$ of $X_{V^{(s)}}/V^{(s)}$ comes from a map $V^{(s)} \to N$. Put $y = \sigma(s)$. By Lemma 7.4 and Proposition 7.6, we can assume (shrinking $V^{(s)}$ if necessary) that $\sigma^{(s)}$ factors through $X^{(y,T)}$, where T is the type at y opposite to that of σ, so we are done. □

The properties of smooth aggregates proven above allow us to see them as the solutions of a universal problem:

Proposition 7.10. *Let S be a regular, excellent scheme and X/S a quasisplit nodal curve. Then for any smooth S-algebraic space Y together with a morphism $f\colon Y \to X$, f factors uniquely through the canonical map $N \to X$.*

Proof. The section (f, Id) of X_Y/Y factors uniquely through N_Y by Proposition 7.9 since the latter is the smooth aggregate of X_Y/Y by Proposition 7.7. Projecting onto N, we get the unique map $Y \to N$ through which f factors. □

Corollary 7.11. *Let S be a regular and excellent scheme and $X' \to X$ a morphism between two quasisplit nodal curves over S. Let N be the smooth aggregate of X, then $N \times_X X'$ is the smooth aggregate of X'.*

Now, we are equipped to prove the following result, which is a weak version of our main theorem of existence for ns-Néron models of nodal curves:

Proposition 7.12. *Let S be a regular and excellent scheme and X/S a nodal curve, smooth over a dense open subscheme U of S, with no rational component in any geometric fiber. Then X_U has a ns-Néron model N/S, and there is a canonical morphism $N \to X$ of models of X_U. When X/S is quasisplit, N is the smooth aggregate of X.*

Proof. By Lemma 1.14, Proposition 5.6 and Lemma 5.5, we can assume X/S is quasisplit and S is integral. Let N be the smooth aggregate of X/S. Then N is a smooth S-model of X_U with a canonical S-map $N \to X$. Consider a smooth S-scheme Y, then we have

$$\begin{aligned} \mathrm{Hom}_U(Y_U, X_U) &= \mathrm{Hom}_S(Y, X) \\ &= \mathrm{Hom}_S(Y, N), \end{aligned}$$

where the first equality holds by Proposition 7.3 applied to the connected components of X_Y/Y, and the second by Proposition 7.7. Thus, N/S has the Néron mapping property. □

The remainder of this section will be dedicated to improving Proposition 7.12 by weakening the hypothesis that the geometric fibers of X/S have no rational components, and determining conditions under which N/S is separated, i.e. a Néron(-lft) model in the classical sense.

7.3 Exceptional components and minimal proper regular models

It is known that an elliptic curve over the fraction field of discrete valuation ring has a Néron model, given by the smooth locus of its minimal proper regular model. It is proven in [23] that the same holds for any smooth curve of positive genus. In particular, rational components of the special fiber that can be contracted to smooth points have a "special status": they must map to a mere point of the Néron model. Therefore, if one wants to weaken the hypotheses of Proposition 7.12 to allow for rational components, one must take this phenomenon into account. Over a discrete valuation ring, these components of the special fiber that can be contracted to smooth points, the so-called *exceptional components*, are characterized by Castelnuovo's criterion ([22], Theorem 9.3.8). This criterion uses intersection theory on fibered surfaces, so is not easy to generalize to higher-dimensional bases for arbitrary relative curves, but the nodal case is much simpler. We will discuss the analogue of the notion of exceptional components for nodal curves over arbitrary regular base schemes.

7.3.1 Definition

Definition 7.13. Let k be a separably closed field and X_k/k a nodal curve. Define a sequence of subsets of the (finite) set I of irreducible components of X_k by $J_0 = \emptyset$, and for all $n \in \mathbb{N}$, J_{n+1} is the subset of I consisting of components C meeting one of the following conditions:

- C is in J_n;
- C is rational and k-smooth, and intersects $\left(\bigcup_{D \in I - J_n - \{C\}} D \right)$ in exactly one point.

The sequence $(J_n)_{n \in N}$ is increasing, so it is stationary at some subset J of I, which we call the set of *exceptional components* of X.

We call *exceptional trees* the connected components of $\bigcup_{C \in J} C$.

A non-smooth point of X/k is called *exceptional* if it belongs to at least one exceptional component.

When X/S is a nodal relative curve, smooth over a schematic dense open $U \subset S$, we call *exceptional point* of X a singular point, exceptional in a fiber of X over a separably closed field-valued point of S.

If X/S is quasisplit, for any $s \in S$, we define the exceptional components (resp. exceptional points, resp. exceptional trees) of X_s as those giving rise to the exceptional components (resp. points, resp. trees) of $X_{\bar{s}}$ for some geometric point $\bar{s} \to s$.

Remark 7.13.1. Neither the components of X lying in a cycle of the dual graph, nor its components of genus ≥ 1 are exceptional. In particular, the exceptional trees correspond to actual trees of the dual graph, and they are not covering as soon as X is of genus ≥ 1.

7.3.2 The minimal proper regular model

Here, we discuss briefly the case of one-dimensional bases, where there is a canonical *minimal proper regular model* of which the Néron model is the smooth locus.

Proposition 7.14. *Let R be a discrete valuation ring, with field of fractions K and residue field k, and X_K a smooth K-curve of genus ≥ 1. Then X_K admits a unique minimal proper regular model X_{min} over S (i.e. X_{min} is a terminal object in the category of proper regular S-models of X_K).*

Moreover, if X_K has a regular nodal model X, then X_{min} is nodal and the map $X \to X_{min}$ is just a contraction of every exceptional tree of the special fiber of X into a smooth point (i.e. the image of an exceptional tree of the special fiber of X is a smooth point of X_{min}/S, and $X \to X_{min}$ restricts to an isomorphism over the rest of X_{min}).

Proof. The existence of the minimal proper regular model is [22], Theorem 9.3.21.

For the second part of the proposition, suppose X_K has a nodal regular model X/S. It follows from [22], Definition 3.1 and Theorem 3.8, that there exists a regular proper model X'/S of X_K and a map $X \to X'$ that is just a contraction of every exceptional tree into a smooth point. In particular, X'/S is nodal. But then X' is relatively minimal in the sense of [22], Definition 3.12, so it is X_{min} and we are done. □

Theorem 7.15 ([23], Theorem 4.1.)**.** *Let S be a connected Dedekind scheme (i.e. a regular scheme of dimension 1) with field of functions K. Let X_K/K be a proper regular connected curve of genus ≥ 1. Suppose either S is excellent, or X_K/K is smooth, and let X_{min} be the minimal proper regular S-model of X_K. Then $(X_{min}/S)^{sm}$ is the Néron model of $(X_K/K)^{sm}$.*

7.3.3 Van der Waerden's purity theorem

We will define the exceptional locus of a birational morphism, and cite a result of purity of this exceptional locus when the target is factorial. This will allow us to describe explicitly some open subsets of the ns-Néron model (when it exists) of a curve with a nodal model.

Definition 7.16. Let $f : X \to Y$ be a morphism locally of finite type between two locally noetherian algebraic spaces. We say f is a *local isomorphism* at some $x \in X$ when f induces an isomorphism $\mathcal{O}_{Y,f(x)} = \mathcal{O}_{X,x}$ (or, equivalently, if x has a Zariski open neighbourhood $V \subset X$ such that f induces an isomorphism from V onto its image in Y). The set of all points at which f is a local isomorphism is an open subscheme W of X, and we call its complement the *exceptional locus* of f. If $W = X$, we say f is a *local isomorphism*.

Example 7.17. Let Y be a noetherian integral scheme and $X \to Y$ be the blowing-up along a closed subscheme $Z \to Y$ of codimension ≥ 1. Then the exceptional locus of $X \to Y$ is the preimage of the set of all $z \in Z$ around which Z is not a Cartier divisor.

Example 7.18. Let k be a field, and glue two copies of the identity $\mathbb{A}^1_k \to \mathbb{A}^1_k$ along the complement of the origin. The resulting map $A \to \mathbb{A}^1_k$, where A is the affine line with double origin, is a local isomorphism.

In Example 7.18, the birational map $f\colon A \to \mathbb{A}^1_k$ has empty exceptional locus, but it is not separated, so in particular not an open immersion. In the following lemma, we will see that non-separatedness is essentially the only possible obstruction preventing such maps from being open immersions.

Lemma 7.19. *Let $f\colon X \to Y$ be a separated local isomorphism between two locally noetherian integral algebraic spaces. Then f is an open immersion.*

Proof. We need to show f is injective. Call η the generic point of X. Since f is a local isomorphism at η, we know $f(\eta)$ is the generic point of Y. Consider two points x, x' of X with the same image y in Y. There are Zariski-open neighbourhoods U, U' of x and x' respectively, such that $U \to Y$ and $U' \to Y$ are open immersions. By separatedness of f, the canonical map $U \times_X U' \to U \times_Y U'$ is a closed immersion. But it follows from the fact f is a local isomorphism that $U \times_Y U'$ is integral, with generic point (η, η). Since this point is in the image of $U \times_X U'$, the map $U \times_X U' \to U \times_Y U'$ is an isomorphism, so the point $(x, x') \to X \times_Y X'$ factors through $U \times_X U'$, i.e. $x = x'$. □

Theorem 7.20 (Van Der Waerden)**.** *Let X, Y be locally noetherian integral schemes with Y locally factorial and $f : X \to Y$ a birational morphism of finite type. Then the exceptional locus of f is of pure codimension one in X.*

Proof. This is [14], Theorem 21.12.12. □

Lemma 7.21. *Let S be a regular scheme, U a dense open subscheme of S, and X/S a quasisplit nodal curve, smooth over U. Let E be the union in X of the exceptional components of all fibers X_s (which are well-defined by quasisplitness). Suppose that X_U admits a ns-Néron model N/S, then the map $(X \backslash E)^{sm} \to N$ extending the identity over U is an open immersion.*

Proof. The scheme $(X \backslash E)^{sm}$ is separated over S, hence separated over N. Therefore, using Lemma 7.19, we only need to prove the exceptional locus of $(X \backslash E)^{sm} \to N$ is empty. The subset E is Zariski-closed in X by 1.8. The unique morphism of algebraic spaces $g : X^{sm} \to N$ extending the identity over U is birational and of finite type, and the domain and codomain are S-smooth, hence regular. We only have to show that its exceptional locus E_0 is contained in E. Take an étale cover $V_0 \to N$ where V_0 is a scheme, it is enough to show $E_0 \times_N V_0 \subset E \times_N V_0$. Therefore, it is enough to prove that for any integral scheme V and any étale map $V \to N$, we have $E_0 \times_N V \subset E \times_N V$. The scheme V is smooth over S so it is regular, and $g_V : X^{sm} \times_N V \to V$ is birational and of finite type since g is. Furthermore, since the property "being an isomorphism" is local on the target for the fpqc topology, the exceptional locus of g_V

is precisely $E_0 \times_N V$. Thus $E_0 \times_N V$ is either empty or pure of codimension one in $X^{sm} \times_N V$ by Theorem 7.20. If it is empty, we are done. Otherwise, since $E \times_X X^{sm} \times_N V$ is closed in $X^{sm} \times_N V$, it is enough to prove that every point of $E_0 \times_N V$ of codimension 1 in $X^{sm} \times_N V$ is contained in $E \times_N V$. Since $V \to N$ is an étale cover, this is true if and only if every point of E_0 of codimension 1 in X^{sm} is contained in E.

Let x be a point of E_0 of codimension 1. Let ξ be the image of x in S, we have $\operatorname{codim}(x, X) = \operatorname{codim}(x, X_\xi) + \operatorname{codim}(\xi, S)$, so ξ has codimension ≤ 1 in S. Since X/S is smooth over U, ξ cannot be of codimension 0 in S, so it must be of codimension 1: $\mathcal{O}_S, \xi$ is a discrete valuation ring.

But then $N \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ is the $\operatorname{Spec} \mathcal{O}_{S,\xi}$-Néron model of its generic fiber by Proposition 5.4, so it is the smooth locus of the minimal proper regular model of $X \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ by Theorem 7.15. Now, by Proposition 7.14, the minimal proper regular model of $X \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ is the contraction of the exceptional trees of its special fiber into smooth points, and in particular contains $(X \backslash E) \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ as an open subscheme. This implies g is an isomorphism at every point of $(X \backslash E)^{sm} \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$, so x must be in E. □

Remark 7.21.1. With hypotheses and notations as in Lemma 7.21, if E is empty, it follows that the canonical morphism $N_0 \to N$, where N_0 is the smooth aggregate of X, is an open immersion. We will see in the next subsection that one can always reduce to this situation: if E is not empty, one can always contract X into a new nodal model of X_U with no exceptional components. However, we will also see that nodal models with no rational components at all do not always exist, so ns-Néron models cannot always be easily described in terms of smooth aggregates.

7.4 Contractions and stable models

So far, we have met two features of a nodal curve X/S, smooth over a dense open $U \subset S$, that can cause problems for us: one is the complexity of its singularities (for example because there can be sections through a singular point of positive arithmetic complexity, meaning we lose relevant information if we take the smooth locus and forget this point), and the other one is the presence of rational components in its geometric fibers (in the absence of such components, we can construct explicitly a ns-Néron model, see 7.2). Refinements allow us to "turn the first problem into the second": we get a new model of X_U with less complex singularities, but more rational components. Looking at Proposition 7.12, it is clear that we also have an interest in the inverse problem: if X/S has rational components, is it possible to blow them down and obtain a new nodal model with more complex singularities, but less rational components?

This question finds its answer in [7], in which the author introduces and studies contraction morphisms for the moduli stacks of n-pointed stable curves. In this subsection, we will see how this translates into the algorithm we need.

7.4.1 The stack of n-pointed stable curves and the contraction morphism

Definition 7.22 ([7], Definition 1.1.)**.** Let n, g be natural integers such that $2g-2+n > 0$. A *n-pointed stable curve of genus g over S* is a nodal relative curve X/S of genus g, together with n pairwise disjoint sections $\sigma_1, ..., \sigma_n \colon S \to X^{sm}$, such that for every geometric fiber X_s and every nonsingular rational component C of X_s, the sum of the number of intersection points between C and the union of all other irreducible components of X_s, and of the number of σ_i passing through C, is at least 3. When the sections are clear from context, we will sometimes omit them in the notation.

We define a *morphism* between two n-pointed stable curves $(X'/S', \sigma'_1, ..., \sigma'_n)$ and $(X/S, \sigma_1, ..., \sigma_n)$ as a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{g} & S \end{array}$$

such that $f\sigma'_i = \sigma_i g$ for all i.

Remark 7.22.1. The condition on the number of special points appearing on a rational component aims to guarantee that the S-automorphism group of X is finite.

Theorem 7.23. *Call $\mathcal{M}_{g,n}$ the category of stable n-pointed curves of genus g. As a category fibered in groupoids over schemes, it is a separated Deligne-Mumford stack, smooth and proper over* $\operatorname{Spec}\mathbb{Z}$.

Proof. This is [7], Theorem 2.7. □

Definition 7.24 ([7], Definition 1.3.)**.** Let S be a scheme, g a natural integer, and $f : X \to X'$ a morphism of S-schemes between stable pointed S-curves of genus g. It is called a *contraction*, or *contraction of X*, if:

- X is $n+1$-pointed and X' is n-pointed, with $2g-2+n > 0$, and their respective sections $(\sigma_i)_{1\leq i\leq n+1}, (\sigma'_i)_{1\leq i\leq n}$ satisfy $f \circ \sigma(i) = \sigma'(i)$ for all $0 \leq i \leq n$.
- For any geometric point $s \in S$, either $X_s \to X'_s$ is an isomorphism, or $\sigma_{n+1}(s)$ is in a rational component C of X_s such that $f(C)$ is a point $x \in X'_s$, and $X_s \backslash C \to X'_s \backslash \{x\}$ is an isomorphism.

Remark 7.24.1. We do not use the same notion of geometric point as [7], but the two subsequent definitions of contractions are equivalent by [22], Proposition 10.3.7.

Theorem 7.25. *Let S be a scheme and X/S a $n+1$-pointed stable curve of genus g with $2g-2+n > 0$. Then X admits a contraction, unique up to a canonical isomorphism.*

Proof. This is [7], Proposition 2.1. □

7.4.2 The stable model

Definition 7.26. Let S be a scheme and $U \subset S$ a scheme-theoretically dense open. Let X_U/U be a smooth curve of genus $g \geq 2$. We call *stable S-model* of X_U a 0-pointed stable curve X of genus g with an isomorphism $X \times_S U = X_U$.

Lemma 7.27. *Let S be a normal, noetherian and strictly local scheme, $U \subset S$ a scheme-theoretically dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Then X_U has a unique stable model X^{stable}, and there is a unique morphism of models $X \to X^{stable}$.*

Proof. Let $s \in S$ be the closed point. The fiber X_s has finitely many rational components, and there are infinitely many disjoint smooth sections of X/S through each of those. If a rational component E of X_s contains only one singular point, consider two disjoint smooth sections through E, and if it contains two singular points, consider one smooth section through E. This gives a finite number $\sigma_1, ..., \sigma_n$ of sections through X^{sm}. Applying Proposition 1.8, and using the fact that the irreducible components of X are geometrically irreducible by quasisplitness, we see that for any geometric fiber X_t of X/S, any rational component of X_t intersecting the other components in two points contains $\sigma_i(t)$ for some i, and any rational component of X_t intersecting the other components in one point contains $\sigma_i(t)$ and $\sigma_j(t)$ for some $i \neq j$.

Thus X/S endowed with the $(\sigma_i)_{0 \leq i \leq n}$ becomes a n-pointed stable curve of genus g, restricting over U to the data of X_U and the n U-sections $\sigma_i|_U : U \to X_U$, and we can apply repeatedly Theorem 7.25 to get a stable 0-pointed curve $X' \to S$ with a map $X \to X'$. Since X_U/U is smooth, the restriction of $X \to X'$ to U just forgets the sections (and induces an isomorphism on the curves), so $X^{stable} := X'$ is a stable model of X_U.

Consider two stable models of X_U, corresponding to two maps $a, b \colon S \rightrightarrows \mathcal{M}_{g,0}$ extending $U \to \mathcal{M}_{g,0}$. Call Z the equalizer of f and g, we have a cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & S \\ \downarrow & & \downarrow {\scriptstyle (a,b)} \\ \mathcal{M}_{g,0} & \longrightarrow & \mathcal{M}_{g,0} \times \mathcal{M}_{g,0} \end{array}$$

where the bottom arrow is the diagonal. Since $\mathcal{M}_{g,0}$ is Deligne-Mumford and separated, its diagonal is finite. Thus, $Z \to S$ is a finite and birational morphism of algebraic spaces, hence an isomorphism by Zariski's main theorem: the stable model X^{stable} is unique up to a unique isomorphism. As for uniqueness of the morphism $X \to X^{stable}$ of models of X_U, let f, g be two such morphisms, then their equalizer is a closed subscheme of X (by separatedness of X^{stable}/S), which contains X_U. But X_U is scheme-theoretically dense in X since U is scheme-theoretically dense in S and X/S is flat, so f and g must be equal. □

Proposition 7.28. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$. Then*

1. *X_U has a stable S-model X^{stable}, unique up to a unique isomorphism, and there is a canonical map $X \to X^{stable}$.*

2. *The formation of X^{stable} commutes with any base change $S' \to S$ such that S' is normal and locally noetherian and $U \times_S S'$ is scheme-theoretically dense in S'.*

Proof. (2) is a consequence of Proposition 1.8. In (1), uniqueness of X^{stable} and of the map $X \to X^{stable}$ holds by the same argument as in the proof of Lemma 7.27 above. We will now prove their existence. Let s be a point of S. By Lemma 7.27, $X_U \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ admits a stable model $X^{0,s}$ over $\operatorname{Spec} \mathcal{O}^{et}_{S,s}$. But then $X^{0,s}/\operatorname{Spec} \mathcal{O}^{et}_{S,s}$ is of finite presentation, thus comes via base change from a morphism $X^s \to V^s$, where V^s is an étale neighbourhood of s in S. Moreover, by [13], Proposition 8.14.2, restricting V^s if necessary, the map $X \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s} \to X^{0,s}$ extends to a V^s-map $X \times_S V^s \to X^s$. Restricting V^s once again if necessary, we take this map to be an isomorphism over U. Now, the locus on X^s where X^s/V^s is at-worst nodal is open in X^s and contains X_s so, restricting V_s again if necessary, we can assume X^s/V^s is a nodal curve. Finally, the union of all nonsingular rational components of fibers of X^s/V^s meeting the other components of their fiber in at most two points is closed in X^s, and does not meet X_s, so, restricting V_s one last time, we can assume X^s/V^s is stable. In particular, it corresponds to a morphism $V^s \to \mathcal{M}_{g,0}$.

For any $s, s' \in S$, the diagram of stacks

$$\begin{array}{ccc} V^s \times_S V^{s'} & \longrightarrow & V^s \\ \downarrow & & \downarrow \\ V^{s'} & \longrightarrow & \mathcal{M}_{g,0} \end{array}$$

commutes by uniqueness of the stable model of $X_{U \times_S V^s \times_S V^{s'}}$.

Applying a similar argument to the triple fibered products, we see the maps $V^s \to \mathcal{M}_{g,0}$ and their gluing isomorphisms satisfy the cocycle condition with respect to the (étale) covering of S by the V^s. Therefore, they come via base change from a map $S \to \mathcal{M}_{g,0}$ i.e. the X^s are obtained via base change from a stable curve X^{stable}/S (which is a model of X_U as desired). Likewise, the local maps $X \times_S V^s \to X^s$ glue to a morphism $X \to X^{stable}$. □

7.4.3 Rational components of the stable model

We will now determine conditions on X guaranteeing that X^{stable} has no rational components in any geometric fiber. When said conditions are met, this allows us to use Proposition 7.12 to describe explicitly the ns-Néron model of X_U.

Definition 7.29. Let k be a separably closed field and X/k a nodal curve. We say X *has rational cycles* if there is a union of rational components of X that is 2-connected, and *no rational cycles* otherwise. If S is a scheme and X/S a nodal curve, we say X/S *has rational cycles* if a fiber over some geometric point of S does.

Remark 7.29.1. The curve $X/\operatorname{Spec} k$ has rational cycles if and only if there is a cycle of its dual graph in which each vertex corresponds to a rational component. We call such cycles the *rational cycles* of the dual graph.

Remark 7.29.2. If $X/\operatorname{Spec} k$ is of genus $g \geq 2$ and has rational cycles, then every rational cycle of the dual graph either is a loop, or contains a rational component meeting the non-exceptional other components in at least three points.

Definition 7.30. Let k be a separably closed field and X/k a nodal curve. We call *rational loop* of X any singular rational irreducible component of X. If S is a scheme and X/S a nodal curve, we say *X/S has rational loops* if a fiber over some geometric point of S does.

Lemma 7.31. *Let k be a separably closed field, $(Y/k, y_1, ..., y_{n+1})$ a stable $n+1$-pointed curve of genus g over k with $2g-2+n > 0$, and $Y \to Z$ the contraction. Then Z has rational cycles if and only if Y does.*

Proof. If $Y \to Z$ is an isomorphism of schemes, it is obvious. Otherwise, there is a rational component C of Y whose image is a point $z \in Z$, and $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism. If C does not belong to a 2-connected union of rational components of Y, we are done. Otherwise, let Γ be said union, the image of Γ in Z is still 2-connected and still contains only rational components. □

Corollary 7.32. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$. Then X^{stable} has rational cycles if and only if X does.*

Rational cycles are an example of rational components we cannot get rid of by contracting. There is another family of such "problematic components": suppose for example that a rational component intersects three other non-rational components, then no contraction will get rid of it. The ones we *can* get rid of are described in the two following lemmas:

Lemma 7.33. *Let k be a separably closed field and $(Y/k, y_1, ..., y_{n+1})$ a stable $(n+1)$-pointed curve of genus g with $2g-2+n > 0$. Suppose Y/k has no rational cycles, and $Y \to Z$ is the contraction. Let F_Y, F_Z denote the union of all rational components meeting the non-exceptional other components in at most two points, in Y and Z respectively. Then $Y \backslash F_Y$ is isomorphic to its image in Z, and the image of F_Y in Z is either F_Z or the union of F_Z and a point.*

Proof. If $Y \to Z$ is an isomorphism of schemes, this is obvious. Otherwise, there is a rational component C of Y whose image is a point $z \in Z$, and $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism. We will prove that the image of F_Y in Z is $F_Z \cup \{x\}$.

The contracted component C meets the other irreducible components of Y in at most two points so C is in F_Y. There is a canonical bijection between irreducible components of Y that are not C and irreducible components of Z (namely, it sends a component of Y to its image in Z), so we just need to prove that a component $D \neq C$ of Y is in F_Y if and only if it is sent to a component of F_Z.

If D is not rational or does not meet C, this is true. Suppose D is rational and meets C. Then $D \cap C$ is exactly one point c_1 (otherwise $D \cup C$ would be 2-connected and Y would have rational cycles).

It is enough to show that D has as many intersection points with non-exceptional other irreducible components of Y than its image D' has in Z. Since the map $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism, it comes down to saying that if C is not exceptional, then the other irreducible component of Y that C meets is not exceptional either, which follows from the definition of exceptional components. □

Lemma 7.34. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$, with no rational cycles. Let s be a field-valued point of S, F the union of all rational components of X_s intersecting the non-exceptional other components in at most two points, and F' its image in X^{stable}. Then*

1. *$X \to X^{stable}$ induces an isomorphism between $X_s \backslash F$ and $X_s^{stable} \backslash F'$.*
2. *F' is a disjoint union of points, one for each connected component of F.*

Proof. By quasisplitness we can assume $k(s)$ is separably closed, and base-changing to $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ (which preserves X^{stable} by Proposition 7.28), we can assume S is strictly local, with closed point s. Then there are S-sections $\sigma_1, ..., \sigma_n$ of X making it a n-pointed stable curve. Consider the sequence $X = X_n \to X_{n-1} \to ... \to X_0 = X^{stable}$, where $X_i \to X_{i-1}$ is the contraction of σ_i. Call F_i the image of F in $(X_i)_s$ for all i, and call G_i the union of all rational components of $(X_i)_s$ meeting the non-exceptional other ones in at most two points. It follows from the preceding lemma that F_i is the union of G_i and a finite number of points. In particular, since $(X_i)_s \to (X_{i-1})_s$ induces an isomorphism from $(X_i)_s \backslash G_i$ to its image, we have $(X_i)_s \backslash F_i = (X_{i-1})_s \backslash F_{i-1}$, and (1) follows inductively. Then, (2) follows from observing that G_0 is empty: indeed, X_s^{stable} is a stable 0-pointed curve over $k(s)$, so it has no exceptional components, thus a component in G_0 would be rational, unmarked, and meet the other irreducible components in at most two points, which is forbidden by the definition of stable curves. □

Proposition 7.35. *Let S be a normal and locally noetherian scheme and X/S a quasisplit nodal curve of genus $g \geq 2$, smooth over a scheme-theoretically dense open $U \subset S$. Let X^{stable} be the stable model of X_U. Then the two following conditions are equivalent:*

1. *There is a geometric point $s \to S$ such that X_s^{stable} has a rational component.*
2. *One of the fibers of X over S contains either a rational loop, or a rational component meeting the non-exceptional other irreducible components in at least three points.*

Proof. If X has rational loops, then X^{stable} has a geometric fiber with a rational component by Corollary 7.32. If a fiber X_s has a rational component E meeting the other non-exceptional other components in at least three points, then any contraction of X_s is an isomorphism over the open subscheme $E \cap (X/S)^{sm}$ of

X_s, so X_s^{stable} has a rational component (thus some geometric fiber of X^{stable} also does).

Conversely, suppose X has no rational loops, and every rational component of a fiber X_s meets the other non-exceptional irreducible components of X_s in at most two points. By Remark 7.29.2, X has no rational cycles, so we conclude with Lemma 7.34. □

7.4.4 Singular ideals of the stable model

Now we want to understand precisely what the stable model looks like as a nodal curve, i.e. we want to compute its singular ideals. This will be made clear by Lemma 7.39. We build up to it with a few technicalities.

Lemma 7.36. *Let $S = \operatorname{Spec} R$ be a trait with generic point η. Let X be a nodal S-model of a smooth η-curve X_η. Then the sum of thicknesses of non-exceptional singular points of X is the number of singular points of the special fiber of X_{min}, where X_{min} is the minimal proper regular model of X_η.*

Proof. The sum of thicknesses of non-exceptional singular points does not change when one blows up in a singular point of the closed fiber, and after a finite sequence of such blow-ups, we obtain a regular model $X_{reg} \to X$ of X_η. Therefore, we can assume X is regular. But then, Proposition 7.14 allows us to conclude. □

Corollary 7.37. *The sum of thicknesses of non-exceptional singular points is a birational invariant for generically smooth nodal curves over a discrete valuation ring.*

Proposition 7.38. *Let $S = \operatorname{Spec} R$ be a regular local scheme with closed point s, U a dense open subscheme of S and X/S, Y/S two quasisplit nodal curves, with $X_U = Y_U$ smooth over U. Then the product of singular ideals of non-exceptional points of X_s is equal to that of Y_s.*

Remark 7.38.1. If we call *thickness* of a singular point its singular ideal, and note additively the monoid of principal prime ideals of R, we can rephrase this "the sum of thicknesses of non-exceptional points in the closed fiber is a birational invariant for generically smooth quasisplit nodal curves over a regular local ring".

Proof. Since R is regular, it is a unique factorization domain. Let $\Delta_1, ..., \Delta_k$ be the prime elements of R such that the generic point of $\{\Delta_i = 0\}$ is not in U. Every singular point of X_s has singular ideal of the form $\left(\prod_{i=1}^{k} \Delta_i^{\nu_i}\right)$ for some integers ν_i, not all zero, and the same goes for Y. Therefore, if we call λ, μ the products of all singular ideals of non-exceptional points of X_s and Y_s respectively, we have integers $n_1, .., n_k, m_1, ..., m_k$ with $\lambda = \left(\prod_{i=1}^{k} \Delta_i^{n_i}\right)$ and $\mu = \left(\prod_{i=1}^{k} \Delta_i^{m_i}\right)$, and we only need to show $n_i = m_i$ for all i.

Pick some $1 \leq i \leq k$, let t be the generic point of the zero locus of Δ_i in S and set $T = \operatorname{Spec} \mathcal{O}_{S,t}$. Base-changing to T, we get nodal curves X_T, Y_T, with the same smooth generic fiber. Proposition 1.8 implies that the sum of thicknesses of their non-exceptional singular points are respectively n_i and m_i, but they must be equal by Corollary 7.37, so $n_i = m_i$ for all i and we are done. □

The next lemma describes how to compute the singular ideals of X^{stable} from the singular ideals of X.

Lemma 7.39. *Let X/S be a quasisplit nodal curve with no rational cycles, of genus ≥ 2, with S regular. Suppose X is smooth over a dense open $U \subset S$. Let s be a point of S and $F \subset X_s$ be the union of all rational components of X_s intersecting the non-exceptional other irreducible components in at most two points. Call Z the set of non-exceptional singular points of X_s. The image in X^{stable} of a connected component G of F is a smooth point if all singular points in G are exceptional, and a singular point of label $\prod_{y \in Z \cap G} l(y)$ otherwise, where we note $l(y)$ the label of y.*

Remark 7.39.1. To put this in simpler words, in the formalism of Remark 7.38.1, with the additional convention that the points of thickness 0 are the S-smooth points, Lemma 7.39 says that the thickness of a point of X^{stable} is the sum of thicknesses of all non-exceptional singular points of X above it.

Proof. We can assume S is local, with closed point s. Let $\sigma_1, ..., \sigma_n$ be such that X/S endowed with the σ_i is in $\mathcal{M}_{g,n}$ (they exist by quasisplitness). Permuting the σ_i if necessary, we assume there is an index $0 \leq m \leq n$ such that for all $1 \leq i \leq m$, $\sigma_i(s)$ is in G, and for all $m < i \leq n$, $\sigma_i(s)$ is not in G. Consider the sequence $X_n \to ... \to X_0$, where X_n is X endowed with the σ_i, and each $X_{i+1} \to X_i$ is the contraction of σ_{i+1}: we have $X_0 = X^{stable}$. Call F_i, Z_i the images of F, Z in $(X_i)_s$. For all $y \in Z_i$, we call $l_i(y)$ the singular ideal of y in X_i.

Call G_i the image of G in X_i for every i. Since none of the $\sigma_i(s)$ with $1 \leq i \leq m$ are in G_i, we know that $G_m \to G_0$ is an isomorphism. But G_0 is a point, since it is connected, X_0 is stable, and all components of G are rational and meet the others in at most two points. Thus, G_m is a $k(s)$-point x of X_m.

Now, observe that for all $m < i \leq n$, $\sigma_i(s)$ is in G_i. Therefore, $X \to X_m$ induces an isomorphism $X \backslash G \to X_m \backslash G_m$. In particular, the product of all singular ideals of non-exceptional points of X outside of G is equal to the product of all singular ideals of non-exceptional points of X_m distinct from x. But we also know by Proposition 7.38 that the product of singular ideals of non-exceptional singular points of X and X_m are the same: it follows that if G consists only of exceptional components, then x is smooth over S, and otherwise, x is singular of label $\prod_{y \in Z \cap G} l(y)$. □

7.4.5 The main theorem

We can now generalize Proposition 7.12 by applying it to the stable model: since the latter is less likely to have rational components than the original nodal model, it is more likely to fall under the hypotheses of the proposition.

Theorem 7.40. *Let S be a regular excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Suppose X has no rational loops, and suppose no geometric fiber of X contains a rational component meeting the non-exceptional other irreducible components in three points or more. Then X_U/U has a ns-Néron model N/S. If in addition X/S is quasisplit, then N is the smooth aggregate of the stable model X^{stable} of X_U.*

Proof. We can assume X/S is quasisplit by Lemma 1.14, Proposition 5.6 and Lemma 5.5. Then X/S has a stable model X^{stable}, which has no rational components in any geometric fiber by Proposition 7.35. We conclude by applying Proposition 7.12 to X^{stable}/S. □

Remark 7.40.1. Our hypotheses on the rational components of the geometric fibers of X/S are quite unnatural, and merely come from the fact the rational components we allow are the only ones that can be contracted *while staying in the realm of nodal curves.* One could maybe get rid of these hypotheses in the following way:

1. Locally on the base if necessary, obtain a (not necessarily nodal) model of X_U with no rational components in any geometric fiber.
2. Try to see if the models obtained this way always fit into a category in which we can solve the universal problem 7.10.

Over one-dimensional bases, the standard way to contract a set E of rational components is to consider the projectivisation of the symmetric graded algebra of a very ample divisor that does not meet E. In higher dimension, however, it is not obvious that the resulting scheme would even remain flat over the base. In the case of nodal curves, this was proven for us in [7], so these subtleties are hidden behind Theorem 7.25, but in order to go beyond nodal curves, one would have to be careful about such matters.

7.5 Separatedness of the ns-Néron model

Ns-Néron models of nodal curves are often non-separated, which can make them a little difficult to work with. Here, we discuss (quite restrictive) criteria under which they *are* separated, i.e. under which a Néron model exists. Roughly speaking, the defect of separatedness of the ns-Néron model comes from the existence of non-isomorphic locally factorial models. In fact, we will show that a Néron model can only exist when there is a canonical "minimal étale-locally factorial model": this is quite similar to the case of one-dimensional bases studied in [23].

Definition 7.41. Let S be a regular scheme and $X \to S$ a nodal curve, smooth over a dense open $U \subset S$. Let s be a geometric point of S and x a singular point of X_s. We say $\mathrm{Sing}(X/S)$ is *irreducible around* x when the connected component of the singular locus of $X \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s} / \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ containing x is irreducible. We say $\mathrm{Sing}(X/S)$ is *étale-locally irreducible* if it is irreducible around every singular geometric point of X. We will sometimes omit the "étale" and just say $\mathrm{Sing}(X/S)$ is *locally irreducible.*

Lemma 7.42. *With hypotheses and notations as in Definition 7.41,* $\mathrm{Sing}(X/S)$ *is irreducible around* x *if and only if the singular ideal of* x *is of the form* $(\Delta)^n$*, where* n *is a positive integer and* Δ *a prime element of* $\operatorname{Spec} \mathcal{O}^{et}_{S,s}$.

Proof. The base change of X/S to $\operatorname{Spec} \mathcal{O}^{et}_{S,s}$ is quasisplit, so if we write Y the connected component of the singular locus of $X \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s} / \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ containing x, then the structural morphism $Y \to \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ is a closed immersion. Therefore, the irreducible components of Y are in bijection with the distinct irreducible factors of the singular ideal of x. ☐

Corollary 7.43. *Keeping the same hypotheses and notations,* $\mathrm{Sing}(X/S)$ *is irreducible around every étale generization of* x *if and only if the singular ideal of* x *is generated by an étale-universally prime element of* $\operatorname{Spec} \mathcal{O}_{S,s}$*. In particular,* X/S *has étale-locally irreducible singular locus if and only if all of its singular geometric points have a power of an étale-universally prime element as a label.*

Remark 7.43.1. With notations as above, $\mathrm{Sing}(X/S)$ is irreducible around x if and only if the radical of the singular ideal of x is generated by a prime element of $\mathcal{O}^{et}_{S,s}$: if the singular ideal of x is of the form (Δ^n) with Δ prime in $\mathcal{O}^{et}_{S,s}$ and $n > 0$, its radical is precisely (Δ).

Proposition 7.44. *Let* $S = \operatorname{Spec} R$ *be a strictly local unique factorization domain,* $U \subset S$ *a dense open subscheme, and take two non-units* Δ_1, Δ_2 *of* R *with no common prime factor. There exists a trait* $T = \operatorname{Spec} A \to S$ *such that*

- *The generic point of* T *is sent to a point of* U*.*
- *The special point of* T *is sent to the closed point of* S
- *The images in* A *of* Δ_1 *and* Δ_2 *are equal, nonzero, and not units.*

Proof. Take $\pi : S' \to S$ the blowing-up in the (non-invertible) ideal (Δ_1, Δ_2) of R. Call D_1 and D_2 the strict transforms of the divisors cut out in S by Δ_1 and Δ_2 respectively, and E the exceptional divisor. Let s be a closed point of S' in the zero locus of $\frac{\Delta_1}{\Delta_2} - 1$ (which is contained in $E \backslash (D_1 \cup D_2)$. By [11], Proposition 7.1.9, there exists a trait $T \to S'$ such that the closed point is mapped to s and the generic point to a point of U. The map $T \to S'$ factors through $\mathcal{O}_{S',s}$ and $\Delta_1 = \Delta_2$ in $\mathcal{O}_{S',s}$, so $T \to S$ satisfies all the desired properties. ☐

Remark 7.44.1. Though over one-dimensional bases, uniqueness in the Néron mapping property is already a weaker condition than separatedness, Proposition 7.44 illustrates the fact that the gap between these two conditions becomes much greater in higher base dimension. Indeed, as the base gets bigger, smooth morphisms of base change remain pretty rare and tame, while the quantity (and array of potentially wild behavior) of traits on the base gets much bigger. This is why non-separated ns-Néron models are so prevalent in higher dimension, even though, to the author's knowledge, no examples are known over a Dedekind scheme.

Lemma 7.45. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. Suppose X_U has a Néron model over S. Then* $\mathrm{Sing}(X/S)$ *is irreducible around every non-exceptional (Definition 7.13) singular geometric point x of X/S.*

Proof. We will work by contradiction: suppose there is some geometric point $s \in S$ and a singular point $x \in X_s$ around which $\mathrm{Sing}(X/S)$ is not irreducible. Since $X_U \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ has a Néron model by 5.4, we can assume x is a closed point of X and $S = \operatorname{Spec} R$ is strictly local with closed point s. In particular, S is an admissible neighbourhood of s relatively to X/S. Let $\prod_{i=1}^{r} \Delta_i^{\nu_i}$ be the decomposition in prime factors of a generator Δ_x of the singular ideal of x in R. By hypothesis, we have $r \geq 2$.

Let (C_1, C_2) be an orientation of X/S at x. Define T_1 and T_2 to be the images in the (multiplicative) monoid $R/R^\times$ of $\prod_{i=1}^{r-1} \Delta_i^{\nu_i}$ and $\Delta_r^{\nu_r}$ respectively. They are opposite types at x. For $j \in \{1,2\}$, consider a section $\sigma_j \colon S \to X$ of type T_j at x relatively to (C_1, C_2). We define X_j as the blowing-up of X in the ideal sheaf of σ_{1-j} (note the index). The X_j are refinements of X, in particular models of X_U, so by hypothesis there is a Néron model N for $(X_1)_U = (X_2)_U = X_U$. Call F the union of all exceptional components of X, and F_1, F_2 the preimages of F in X_1 and X_2 respectively. Since S is strictly local, X/S is quasisplit so F, F_1, F_2 are well-defined closed subsets of X, X_1, X_2 respectively. Using Lemma 4.6, we know F_i is the union of all exceptional components of X_i for $i = 1, 2$. By Lemma 7.21, the canonical maps

$$(X_i \backslash F_i)^{sm} \to N$$

are open immersions. Thus, they induce isomorphisms on open subspaces V_1, V_2 of N. Call V the open subspace $V_1 \cup V_2$ of N. Then V is isomorphic to the gluing of $(X_1 \backslash F_1)^{sm}$ and $(X_2 \backslash F_2)^{sm}$ along the preimages of $V_1 \cap V_2$ in each of them. We will conclude by proving V is not separated, which is absurd since it is an open subspace of N.

Using Lemma 7.4, we see that σ_1 factors through X_1^{sm}. Since x is not in F, σ_1 even factors through a section $\sigma_1' \colon S \to (X_1 \backslash F_1)^{sm}$. However, $X_2 \to X$ is precisely the blowing-up in the ideal sheaf of σ_1, so σ_1 does not factor through X_2. Symmetrically, σ_2 factors through a section $\sigma_2' \colon X_2^{sm} \backslash F_2$, but not through X_1. Let x_j be the image of $\sigma_j'(s)$ in N, the fact σ_j does not factor through X_{1-j} implies x_j is not in $V_1 \cap V_2$, so $x_1 \neq x_2$.

Proposition 7.44 gives a trait $T = \operatorname{Spec} A \to S$, with generic point η and closed point t, such that η is sent to a point of U and t to s, and such that the images in A of $\left(\prod_{i=1}^{r-1} \Delta_i^{\nu_i}\right)$ and $\Delta_r^{\nu_r}$ are equal, nonzero, and not units.

In particular, the T-sections $(\sigma_1)_T$ and $(\sigma_2)_T$ of X_T induced by σ_1 and σ_2 are equal (because $\left(\prod_{i=1}^{r-1} \Delta_i^{\nu_i}\right)$ and $\Delta_r^{\nu_r}$ are equal in A). Since η is sent to a point of U, it follows that $(\sigma_1')_T(\eta) = (\sigma_2')_T(\eta)$ in V_T. However, $(\sigma_1')_T(t) = \sigma_1'(s) = x_1 \neq x_2 = \sigma_2'(s) = (\sigma_2')_T(t)$ since x_1 is in $V_1 \backslash V_2$ and x_2 in $V_2 \backslash V_1$: V does not satisfy the valuative criterion for separatedness, a contradiction. □

Example 7.46.

- Local irreducibility of the singular locus is only necessary around non-exceptional points: for example, consider $S = \operatorname{Spec} \mathbb{C}[[a, b]]$ and $U = D(ab)$, and take the elliptic (so a fortiori nodal) curve X/S cut out in the weighted projective space $\mathbb{P}^2(2,3,2)$ by $y^2 = x(x^2 - z^2)$, so that X is the S-Néron model of X_U. Consider the blowing-up $X' \to X$ of X in the sheaf of ideals $\mathcal{I}$ given by (y, abz). Since $\mathcal{I}$ is Cartier outside of the zero locus of ab, we have $X'_U = X_U$: in particular, X'_U has a Néron model over S (namely X). However, computing the blowup explicitly, we find that X' is nodal and that its closed fiber consists of two irreducible components, intersecting in a point p of label (ab): the singular locus is not irreducible around p, but p is exceptional.

- Local irreducibility of the singular locus around non-exceptional points is not sufficient either: take X to be a nodal curve over $S = \operatorname{Spec} \mathbb{C}[[a, b]]$, whose closed fiber has two irreducible components C_1 and C_2 of genus 1, intersecting in a singular point of label (ab). In this case, X is smooth over the dense open $U = D(ab)$ of S and the singular point is not exceptional, so by (2), X_U has no Néron model over S. Take $X' \to X$ to be the (C_1, a)-refinement, this means X'_U has no Néron model over S, even though X' has étale-locally irreducible singular locus. However, we will see in Theorem 7.48 that if X has no rational components in any geometric fiber, the condition (which then just becomes "X has étale-locally irreducible singular locus") is necessary and sufficient.

As seen in Example 7.46, there are situations in which we cannot conclude to nonexistence of a Néron model by applying directly Lemma 7.45, but we can if we apply it to a different nodal model. This argument can be made systematic, and gives the following (more restrictive) necessary condition:

Lemma 7.47. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. Suppose X_U has a Néron model over S. Then the following two conditions are met:*

- *The singular locus* $\operatorname{Sing}(X/S)$ *is irreducible around every non-exceptional singular geometric point of X/S.*

- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}_{S,s}^{et}$ have the same radical.*

Proof. By Corollary 5.4, we can assume S is strictly local. In particular, X/S is quasisplit. If X/S is of genus 1, this is a special case of Proposition 6.15. Otherwise, we can apply Lemma 7.45 to the stable model X^{stable} and conclude using Lemma 7.39. □

Remark 7.47.1. In the genus 1 case, there is no 0-pointed stable model, but we could use a 1-pointed stable model instead of referring to our work on Jacobians.

One could wonder if the necessary conditions of Lemma 7.47 are still too weak. We will now show that, when we know the ns-Néron model exists, its separatedness (and therefore the existence of a Néron model) is equivalent to these conditions.

Theorem 7.48. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. If X_U has a Néron model over S, then the two following conditions are met:*

- *The singular locus* $\mathrm{Sing}(X/S)$ *is irreducible around every non-exceptional singular geometric point of* X/S.
- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}^{et}_{S,s}$ have the same radical.*

Conversely, suppose these conditions are met. Suppose in addition that no geometric fiber of X/S contains either a rational cycle or a rational component meeting the non-exceptional other components in at least three points. Then X_U/U has a Néron model, i.e. the ns-Néron model of X_U/U exhibited in Theorem 7.40 is separated over S.

Proof. The first part of the theorem is Lemma 7.47. We will now prove the "conversely" part. Let N/S be the ns-Néron model of X_U exhibited in Theorem 7.40. Separatedness of N/S can be checked over the étale stalks of S: we can and will assume $S = \mathrm{Spec}\, R$ is strictly local, and we call s its closed point. In particular, X/S is quasisplit. By Proposition 7.35, we know X^{stable} has no rational components in any geometric fiber, and by Lemma 7.39, we can assume $X = X^{stable}$ while preserving all hypotheses made on X. Then, N is the smooth aggregate of X/S. By Corollary 7.43, all prime factors of all singular ideals of X/S are étale-universally prime, so S is an admissible neighbourhood of *all* its geometric points (and not just of s). Therefore, if for every pair (x, T) where x is a singular point of X_s and T a type at x, we write $X^{(x,T)}$ the blowing-up of X in a section of type T at x, N is the gluing of the $(X^{(x,T)}/S)^{sm}$ along the strict transforms of $(X/S)^{sm}$ in each of them.

Now, for any singular point y of X_s, the singular ideal of y in R is of the form $(\Delta_y^{\nu_y})$, where Δ_y is an étale-universally prime element of R and ν_y a positive integer. Consider the morphism $X' \to X$ obtained as a composition of $\nu_y - 1$ blowing-ups in sections through points of positive arithmetic complexity above y, it follows that all the $X^{y,T}$ factor uniquely through X'. Repeating the process for every x, we find a nodal model X_{min} of X_U of arithmetic complexity 0, with a map $X_{min} \to X$, such that N is the smooth locus of X_{min}. In particular, N is separated. □

Remark 7.48.1. Here, we only constructed X^{min} locally, but when X/S is quasisplit and the hypotheses of the "conversely" part of Theorem 7.48 are met, these local models always flue into a canonical "minimal étale-locally factorial model" of X_U, of which the Néron model is the smooth locus.