Néron models in high dimension: Nodal curves, Jacobians and tame base change

Poiret, T.

Author: Poiret, T.
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Part I

Nodal curves, dual graphs and resolutions

1 Local structure of nodal curves and their dual graphs

1.1 First definitions

The results of this subsection are mostly either well-known facts about nodal curves, or come from [21]. When the proofs are short enough, we reproduce them for convenience.

Definition 1.1. A *graph* G is a pair of finite sets (V, E), together with a map $f : E \to (V \times V)/\mathcal{S}_2$. We call V the set of vertices of G and E its set of edges. We think of f as the map sending an edge to its endpoints. We call *loop* any edge in the preimage of the diagonal of $(V \times V)/\mathcal{S}_2$. We will often omit f in the notations and write $G = (V, E)$.

Let v, v' be two vertices of G. A *path between v and v' in G* is a finite sequence $(e_1, ..., e_n)$ of edges, such that there are vertices $v_0 = v, v_1, ..., v_n = v'$ satisfying $f(e_i) = (v_{i-1}, v_i)$ for all $1 \leq i \leq n$. We call n the *length* of the path. A *chain* is a path as above, with positive n, where the only repetition allowed in the vertices $(v_i)_{0 \leq i \leq n}$ is $v_0 = v_n$. A *cycle* is a chain from a vertex to itself. The cycles of length 1 of G are its loops.

Let M be a semigroup. A *labelled graph over M* (or *labelled graph* if there is no ambiguity) is the data of a graph $G = (V, E)$ and a map $l \colon E \to M$, called *edge-labelling*. The image of an edge by this map is called the *label* of that edge.

Definition 1.2. Let X be an algebraic space. We call *geometric point* of X a morphism $\operatorname{Spec} \bar{k} \to X$ where the image of $\operatorname{Spec} \bar{k}$ is a point with residue field k, and $\bar{k}$ is a separable closure of k (notice the "separable" instead of "algebraic"). Given a geometric point x' over a point x of X, we will call *étale local ring of X at x'*, and note $\mathcal{O}_{X,x'}$, the strict henselization of $\mathcal{O}_{X,x}$ determined by the residue extension $k(x')/k(x)$. Given two geometric points s, t of X, we say that t is an *étale generization* of s (or that s is an *étale specialization* of t) when the morphism $t \to X$ factors through $\operatorname{Spec} \mathcal{O}_{X,s}$. We will often omit the word "étale" and just call them specializations and generizations.

Definition 1.3. A *curve* over a separably closed field k is a proper morphism $X \to \operatorname{Spec} k$ with X of pure dimension 1. It is called *nodal* if it is connected, and for every point x of X, either X/k is smooth at x, or x is an ordinary double point (i.e. the completed local ring of X at x is isomorphic to $k[[u, v]]/(uv)$).

A *curve* (resp. a *nodal curve*) over a scheme S is a proper, flat, finitely presented morphism $X \to S$ such that all its geometric fibers are curves (resp. nodal curves).

Remark 1.3.1. By [22], Proposition 10.3.7, our definition of nodal curves is unchanged if one defines geometric points with the standard algebraic closures instead of separable closures.

Definition 1.4. Let S be a scheme, s a point of S, and $\bar{s}$ a geometric point mapping to s. We will call *étale neighbourhood of $\bar{s}$ in S* the data of an étale morphism of schemes $V \to S$, a point v of V, and a factorization $\bar{s} \to v \to s$ of $\bar{s} \to s$. Etale neighbourhoods naturally form a codirected system, and we call *étale stalk of S at s* the limit of this system. The étale stalk of S at s is an affine scheme, and we call *étale local ring at s*, and note $\mathcal{O}_{S,s}^{et}$, its ring of global sections. We will sometimes keep the choice of geometric point $\bar{s}$ implicit and abusively call (V, v), or even V, an étale neighbourhood of s in S.

Remark 1.4.1. The étale local ring of S at s is a strict henselization of the Zariski local ring $\mathcal{O}_{S,s}$. The étale local ring of S at $\bar{s}$ is the strict henselization determined by the separable closure $k(s) \to k(\bar{s})$.

1.2 The local structure

Proposition 1.5. *Let S be a locally noetherian scheme and X/S be a nodal curve. Let s be a geometric point of S and x be a non-smooth point of X_s. There exists a unique principal ideal (Δ) of the étale local ring $\mathcal{O}_{S,s}^{et}$, called the* singular ideal of x*, such that*

$$\widehat{\mathcal{O}_{X,x}^{et}} \simeq \widehat{\mathcal{O}_{S,s}^{et}}[[u,v]]/(uv - \Delta)$$

Proof. This is [21], Proposition 2.5. □

Remark 1.5.1. The singular ideal of x is generated by a nonzerodivisor if and only if X/S is generically smooth in a neighbourhood of x.

1.3 The dual graph at a geometric point

Definition 1.6. Let X, S be as above and s be a geometric point of S. We define the *dual graph* of X at s to be the graph whose vertices are the irreducible components of X_s, and whose edges are the singular points of X_s: the two vertices an edge connects are the two (not necessarily distinct) irreducible components the singular point belongs to. We also make it a labelled graph over the commutative semigroup of nontrivial principal ideals of $\mathcal{O}_{S,s}^{et}$: the label of an edge is the singular ideal of the corresponding singular point.

When S is strictly local, we will sometimes refer to the dual graph of X at the closed point as simply "the dual graph of X".

Definition 1.7. A nodal curve X over a field k is said to be *split* if its singular points are rational, and all its irreducible components are geometrically irreducible and smooth. A nodal curve is *split* when all its fibers are split. This implies that there is no geometric point of the base over which the dual graph of the curve has loops.

Proposition 1.8. *Let $S' \to S$ be a morphism of locally noetherian schemes, X/S a nodal curve, s a geometric point of S, and s' a geometric point of S' such that $s' \to S$ is a generization of $s \to S$. Let X' be the base change of X to S', Γ and Γ' be the dual graphs, respectively of X at s and of X' at s'.*

Let $R := \mathcal{O}^{et}_{S,s}$; $R' := \mathcal{O}^{et}_{S',s'}$, and ϕ be the natural map $\operatorname{Spec} R' \to \operatorname{Spec} R$. Then Γ' is obtained from Γ by contracting all edges whose label becomes invertible in R', and pulling back the labels of the other edges by ϕ.

In particular, if s' has image s, Γ and Γ' are isomorphic as non-labelled graphs, and the labels of Γ' are obtained by pulling back those of Γ.

Proof. This is [21], Remark 2.12. We reprove it here.

We can reduce to $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ affine and strictly local (i.e. isomorphic to spectra of strictly henselian local rings), of respective closed points s and s'.

Let x be a singular point of X of image s, and Δ be a generator of its (principal) singular ideal. Then we can choose an isomorphism $\widehat{\mathcal{O}_{X,x}} = \widehat{R}[[u,v]]/(uv - \Delta)$.

This yields $\widehat{\mathcal{O}_{X,x}} \otimes_R R' = \widehat{R} \otimes_R R'[[u,v]]/(uv - \Delta)$. The ring $\widehat{R} \otimes_R R'$ is local, with completion $\widehat{R'}$ with respect to the maximal ideal: as desired, if Δ is invertible in R', then X' is smooth above a neighbourhood of x, and otherwise, X' has exactly one singular closed point of image x, with singular ideal $\Delta R'$. □

Example 1.9. With notations as above, in the case $S = S'$, we have defined the *specialization maps* of dual graphs: take s, ξ geometric points of S with s specializing ξ, we have a canonical map from the dual graph at s to the dual graph at ξ, contracting an edge if and only if its label becomes the trivial ideal in $\mathcal{O}^{et}_{S,\xi}$.

It can be somewhat inconvenient to always have to look at geometric points. We can often avoid it as in [18], by reducing to a case in which the dual graphs already make sense without working étale-locally on the base.

1.4 Quasisplitness, dual graphs at non-geometric points

Definition 1.10 (see [18], Definition 4.1)**.** We say a nodal curve $X \to S$ is *quasisplit* if the two following conditions are met:

1. for any point $s \in S$ and any irreducible component E of X_s, there is a smooth section $S \to (X/S)^{sm}$ intersecting E;
2. the singular locus $\operatorname{Sing}(X/S) \to S$ is of the form

$$\coprod_{i \in I} F_i \to S,$$

where the $F_i \to S$ are closed immersions.

Example 1.11. Consider the real conic

$$X = \mathrm{Proj}(\mathbb{R}[x, y, z]/(x^2 + y^2)).$$

It is an irreducible nodal curve over $\mathrm{Spec}\,\mathbb{R}$, but the base change $X_{\mathbb{C}}$ has two irreducible components: X is not quasisplit over $\mathrm{Spec}\,\mathbb{R}$.

On the other hand, consider the real projective curve

$$Y = \mathrm{Proj}(\mathbb{R}[x, y, z]/(x^3 + xy^2 + xz^2)).$$

It has two irreducible components (respectively cut out by x and by $x^2+y^2+z^2$), both geometrically irreducible. The singular locus of $Y/\mathbb{R}$ consists of two $\mathbb{C}$-rational points, with projective coordinates $(0 : i : 1)$ and $(0 : -i : 1)$, at which $Y_{\mathbb{C}}$ is nodal. Since $\mathrm{Sing}(Y/\mathbb{R})$ is not a disjoint union of $\mathbb{R}$-rational points, Y is not quasisplit over $\mathrm{Spec}\,\mathbb{R}$. However, both X and Y become quasisplit after base change to $\mathrm{Spec}\,\mathbb{C}$.

Remark 1.11.1. Our definition of quasisplitness is similar to that of [18], but more restrictive.

Remark 1.11.2. For a quasisplit nodal curve X/S and a point $s \in S$, for any geometric point s' of S of image s, the irreducible components of $X_{s'}$ are in canonical bijection with those of X_s by the first condition defining quasisplitness, and the singular ideals of X at s' come from principal ideals of the Zariski local ring $\mathcal{O}_{S,s}$ by the second condition. Thus we can define without ambiguity the dual graph of X at s: its vertices are the irreducible components of X_s, and it has an edge for every non-smooth point $x \in X_s$, with endpoints the two components x meets, labelled by the principal ideal of $\mathcal{O}_{S,s}$ that gives rise to the singular ideal when we base change to a strict henselization.

From now on, we will call the latter the singular ideal of X at x, and talk freely about the dual graphs of quasisplit curves at (not necessarily geometric) field-valued points of S. This can clash with Definition 1.6 when x is a singular point of a geometric fiber of X/S. Unless specified otherwise, when there is an ambiguity, we always privilege Definition 1.6.

Lemma 1.12. *Quasisplit curves are stable under arbitrary base change.*

Proof. Both conditions forming the definition of quasisplitness are stable under base change. □

Lemma 1.13. *Let S be a strictly local scheme and X/S a nodal curve. Then X/S is quasisplit.*

Proof. There is a section through every closed point in the smooth locus of X/S, so in particular there is a smooth section through every irreducible component of every fiber. Proposition 1.5 implies the map $\mathrm{Sing}(X/S) \to S$ is a disjoint union of closed immersions. □

Lemma 1.14. *Let S be a locally noetherian scheme and X/S a nodal curve. Then every point $s \in S$ has an étale neighbourhood (V, v) such that X_V/V is quasisplit.*

Proof. By Lemma 1.13, and using the fact X/S is of finite presentation, there are étale neighbourhoods V_1 and V_2 of s such that the singular locus of X_{V_1}/V_1 is a disjoint union of closed immersions and there are smooth sections through all irreducible components of all fibers of X_{V_2}/V_2. Both conditions are stable under further étale localization, so the base change of X/S to $V_1 \times_S V_2$ is quasisplit. $\square$

Corollary 1.15. *Let S be a locally noetherian scheme and X/S a nodal curve. There is an étale covering morphism $V \to S$ such that X_V/V is quasisplit.*

Proof. By the preceding lemma, every point $s \in S$ admits an étale neighbourhood V_s such that X_{V_s}/V_s is quasisplit, and we can pick $V = \coprod_{s \in X} V_s$. $\square$

Lemma 1.16. *Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Quasisplitness of X/S gives a factorization*

$$x \to F \to \mathrm{Sing}(X/S) \to X \to S,$$

where $F \to S$ is a closed immersion and $F \to \mathrm{Sing}(X/S)$ is the connected component containing x. Then, there is an étale neighbourhood (V, y) of x in X; two effective Cartier divisors C, D on V; and an isomorphism $V \times_X F = C \times_V D$ such that $V \times_S F$ is the union of C and D.

Proof. Let $\bar{s}$ be a geometric point of S mapping to s, and $\bar{x} = x \times_s \bar{s}$. By Proposition 1.5, we have an isomorphism $\widehat{\mathcal{O}^{et}_{X,\bar{x}}} = \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta)$, where Δ is a generator of the singular ideal of x in $\mathcal{O}^{et}_{S,\bar{s}}$. The base change of F/S to $\operatorname{Spec} \mathcal{O}^{et}_{S,\bar{s}}$ is cut out by Δ, and the zero loci C_u of u and C_v of v are effective Cartier divisors on $\widehat{\mathcal{O}^{et}_{X,\bar{x}}}$, intersecting in $\widehat{\mathcal{O}^{et}_{X,\bar{x}}}/(u,v) = F \times_X \operatorname{Spec} \widehat{\mathcal{O}^{et}_{X,\bar{x}}}$. The union of C_u and C_v is $\widehat{\mathcal{O}^{et}_{S,\bar{s}}}[[u,v]]/(\Delta, uv)$, so the proposition follows by a limit argument. $\square$

2 Primality and base change

In this section, we will discuss questions of permanence of primality (of an element of an integral regular ring) under étale maps, smooth maps and completions. One of the reasons these considerations are important to talk about Néron models of nodal curves is that a nodal curve over an excellent and regular base admits local sections through a singular point if and only if the label of this singular point is reducible (Lemma 4.3).

An element Δ of a regular local ring R is prime in R^{sh} when the quotient $R/(\Delta)$ is *unibranch* (i.e. has integral strict henselization), so we are interested in questions of permanence of unibranch rings under tensor product. For a more detailed discussion on unibranch rings or counting geometric branches in general, see [29], chapitre IX. We also refer to [12], 23.2.

In [31], Sweedler gives a necessary and sufficient condition for the tensor product of two local algebras over a field to be local. Since we are interested in how étale stalks behave under base-change, what we would like is a suitable sufficient condition for algebras over a strictly local ring. Sweedler's proof adapts well to this situation: this is the object of the following lemma.

Lemma 2.1. *Let R be a strictly henselian local ring, $R \to A$ an integral morphism of local rings with purely inseparable residue extension, and $R \to B$ any morphism of local rings. Then $A \otimes_R B$ is local, and its residue field is purely inseparable over that of B.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $A \otimes_R B$. The map $B \to A \otimes_R B$ is integral so it has the going-up property ([30, Tag 00GU]), therefore the inverse image of $\mathfrak{m}$ in B is a maximal ideal: it must be $\mathfrak{m}_B$. Thus $\mathfrak{m}$ contains $A \otimes_R \mathfrak{m}_B$.

In particular, $\mathfrak{m}$ also contains the image of $\mathfrak{m}_R$ in $A \otimes_R B$: it corresponds to a maximal ideal of $A \otimes_R B/(\mathfrak{m}_R A \otimes_R B)$, that we will still call $\mathfrak{m}$. We have a commutative diagram

$$\begin{array}{ccc} k_R & \longrightarrow & B/\mathfrak{m}_R B \\ \downarrow & & \downarrow \\ A/\mathfrak{m}_R A & \longrightarrow & A \otimes_R B/(\mathfrak{m}_R A \otimes_R B). \end{array}$$

Since $A/\mathfrak{m}_R A$ is local and integral over the field k_R, its maximal ideal m_A is nilpotent and is its only prime. The inverse image of $\mathfrak{m}$ in $A/\mathfrak{m}_R A$ is a prime ideal, so it can only be $\mathfrak{m}_A$. This shows that, as an ideal of $A \otimes_R B$, $\mathfrak{m}$ also contains $\mathfrak{m}_A \otimes_R B$.

Every maximal ideal of $A \otimes_R B$ contains both $\mathfrak{m}_A \otimes_R B$ and $A \otimes_R \mathfrak{m}_B$, so the maximal ideals of $A \otimes_R B$ are in bijective correspondence with those of $k_A \otimes_{k_R} k_B = A \otimes_R B/(\mathfrak{m}_A \otimes_R B + A \otimes_R \mathfrak{m}_B)$. We will now show the latter is local, with purely inseparable residue extension over k_B.

By hypothesis, the extension k_A/k_R is purely inseparable. If k_R has characteristic 0, then $k_A = k_R$ and we are done. Suppose k_R has characteristic $p > 0$. For any $x \in k_A \otimes_{k_R} k_B$, we can write x as a finite sum $\sum_{i \in I} \lambda_i \otimes \mu_i$ with the λ_i, μ_i in k_A, k_B respectively. There is an integer $N > 0$ such that for all i, $\lambda_i^{p^N}$ is in k_R. Therefore $x^{p^N} = \sum_{i \in I} \lambda_i^{p^N} \mu_i^{p^N}$ is in k_B, and x is either nilpotent or invertible. It follows that $k_A \otimes_{k_R} k_B$ is local, with maximal ideal its nilradical, and that its residue field is purely inseparable over k_B as claimed. □

Lemma 2.2. *Let $(R, \mathfrak{m})$ be an integral and strictly local noetherian ring, and $R \to R'$ a smooth ring map. Let $\mathfrak{p}$ be a prime ideal of R' containing $\mathfrak{m}R'$. Call $\tilde{R}'$ a strict henselization of $R'_\mathfrak{p}$. Then $R'_\mathfrak{p}$ is geometrically unibranch, i.e. $\tilde{R}'$ is an integral domain.*

Proof. We know $\tilde{R}'$ is reduced since it is a filtered colimit of smooth R-algebras. Let B, B' be the integral closures of $R, R'_\mathfrak{p}$ in their respective fraction fields. The ring $R'_\mathfrak{p}$ is integral so by [29], chapitre IX, corollaire 1, $\tilde{R}'$ is an integral domain if and only if B' is local and the extension of residue fields of $R'_\mathfrak{p} \to B'$

is purely inseparable. But any smooth base change of B/R remains normal (see [22], Corollary 8.2.25), so $B \otimes_R R'_{\mathfrak{p}}$ is normal as a filtered colimit of normal B-algebras. Moreover, any normal algebra over $R'_{\mathfrak{p}}$ must factor through $B \otimes_R R'_{\mathfrak{p}}$, so we have $B' = B \otimes_R R'_{\mathfrak{p}}$. Applying Lemma 2.1, we find that B' is local and the extension of residue fields of $R'_{\mathfrak{p}} \to B'$ is purely inseparable, which concludes the proof. □

Corollary 2.3. *Let S be a regular scheme, $Y \to S$ a smooth morphism and $y \to Y$ a geometric point. Then for any prime element Δ of $\mathcal{O}^{et}_{S,y}$, the image of Δ in $\mathcal{O}^{et}_{Y,y}$ is prime.*

Proof. Base change to $\operatorname{Spec} \mathcal{O}^{et}_{S,y}/(\Delta)$, replace Y by an affine neighbourhood of y in Y, and apply Lemma 2.2. □

Lemma 2.4. *Let R be a strictly henselian excellent local ring and $\widehat{R}$ its completion with respect to the maximal ideal. Then an element Δ of R is prime in R if and only if it is in $\widehat{R}$.*

Proof. The nontrivial implication is the direct sense. Suppose Δ is prime in R. Since R is excellent, the morphism $R \to \widehat{R}$ is regular. Then by Popescu's thorem ([30, Tag 07GB]), $\widehat{R}$ is a directed colimit of smooth R-algebras. We conclude by writing $R/(\Delta) \to \widehat{R}/(\Delta)$ as a colimit of smooth $R/(\Delta)$-algebras and applying Lemma 2.2. □

Lemma 2.5. *Let S be an excellent and regular scheme, X/S an S-scheme of finite presentation, $\bar{s}$ a geometric point of S, and x a closed point of $X_{\bar{s}}$ with an isomorphism $\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{\mathcal{O}^{et}_{S,\bar{s}}}[[u,v]]/(uv - \Delta)$ for some $\Delta \in \mathfrak{m}_{\bar{s}} \subset \mathcal{O}^{et}_{S,\bar{s}}$. For every $t_1, t_2 \in \mathfrak{m}_{\bar{s}}$ such that $t_1 t_2 = \Delta$, there exists an étale neighbourhood $S' \to S$ of $\bar{s}$ and a section $S' \to X$ through x such that the induced map $\widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{\mathcal{O}^{et}_{S',\bar{s}}}$ sends u, v respectively to a generator of (t_1) and a generator of (t_2).*

Proof. Put $R = \mathcal{O}^{et}_{S,\bar{s}}$ and consider the map $\widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{R}$ that sends u, v to t_1, t_2 respectively. Compose it with $\mathcal{O}^{et}_{X,x} \to \widehat{\mathcal{O}^{et}_{X,x}}$ to get a map $f_0 : \mathcal{O}^{et}_{X,x} \to \widehat{R}$.

For noetherian local rings, quotients commute with completion with respect to the maximal ideal, so two distinct ideals are already distinct modulo some power of the maximal ideal. Let $\prod_{i=1}^{n} \Delta_i^{\nu_i}$ be the prime factor decomposition of Δ in R. Principal ideals of R of the form $(\Delta_i^{\mu_i})$ with $0 \leq \mu_i \leq \nu_i$ are pairwise distinct and in finite number, so there exists some $N \in \mathbb{N}$ such that their images in $R/\mathfrak{m}_R^N$ are pairwise distinct. Since R is henselian and excellent, it has the Artin approximation property ([30, Tag 07QY]), so there exists a map $f \colon \mathcal{O}^{et}_{X,x} \to R$ that coincides with f_0 modulo m_R^N. This f induces a map $\widehat{f} : \widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{R}$. Call a, b the respective images of u, v by $\widehat{f}$, we have $a = t_1$ and $b = t_2$ in $R/\mathfrak{m}_R^N$. But $ab = \Delta$ in $\widehat{R}$ and, by Lemma 2.4, Δ has the same prime factor decomposition in R and $\widehat{R}$, so the only principal ideals of $\widehat{R}$ containing Δ are of the form $(\Delta_i^{\mu_i})$ with $0 \leq \mu_i \leq \nu_i$. By definition of N we get $a\widehat{R} = t_1\widehat{R}$ and $b\widehat{R} = t_2\widehat{R}$. Since

X/S is of finite presentation, f comes from an S-morphism $S' \to X$ where S' is an étale neighbourhood of $\bar{s}$ in S. □

3 Sections of nodal curves

We present a few technicalities regarding sections of a quasisplit nodal curve X/S. There is a double interest in this. First, blow-ups of nodal curves in the ideal sheaves of sections will be an important tool to construct ns-Néron models of smooth curves with nodal reduction. Second, we can view the Néron mapping property as a condition of extension of rational points into sections after smooth base change. Thus, we will be interested in conditions under which a section of a nodal curve factors through the blowing-up in the ideal sheaf of another section.

The basis for this formalism was thought of together with Giulio Orecchia, and these notions should appear again in a future joint work describing the sheaf of regular models of a nodal curve, smooth over the complement of a normal crossings divisor.

3.1 Type of a section

We will define a combinatorial invariant, the *type* of a section, summarizing information about the behavior of said section around the singular locus of a nodal curve X/S. Later on, we will show that sections of all types exist étale-locally on the base (Proposition 3.9) and that the type of a section locally characterizes the blowing-up of X in the ideal sheaf of that section (Corollary 4.7).

Definition 3.1. Let S be a regular scheme and $s \to S$ a geometric point. Let X/S be a nodal curve, smooth over a dense open subscheme U of S. Let x be a singular point of X_s, we call *thickness* of x (in X/S) the image of the singular ideal of x in the monoid $\mathcal{O}^{et}_{S,s}/(\mathcal{O}^{et}_{S,s})^*$ of principal ideals of $\mathcal{O}^{et}_{S,s}$.

Definition 3.2. Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let F be the connected component of $\mathrm{Sing}(X/S)$ containing x. Then the set of connected components of $(X \backslash F) \times_X \operatorname{Spec} \mathcal{O}^{et}_{X,x} \times_S F$ is a pair $\{C, D\}$ (see Proposition 1.5 or Lemma 1.16), on which the Galois group $\mathrm{Aut}_{O_S}(\mathcal{O}^{et}_{S,s}) = \mathrm{Gal}(k(s)^{sep}/k(s))$ acts. If this action is trivial, we say X/S is *orientable at* x, and we call *orientations of* X/S *at* x the ordered pairs (C, D) and (D, C). The scheme-theoretical closures of C and D in $\operatorname{Spec} \mathcal{O}^{et}_{X,x}$ are effective Cartier divisors, and we will often also call them C and D.

Remark 3.2.1. The curve X/S is orientable at x if and only if the preimage of x in the normalization of X_s consists of two $k(s)$-rational points, in which case an orientation is the choice of one of these points. Roughly speaking, this also corresponds to picking an orientation of the edge corresponding to x in the dual graph of X at s. The "roughly speaking" is due to the case of loops: there is an ambiguity on how to orient them. We could get rid of this ambiguity by using a heavier notion of dual graphs (such as the tropical curves often used in log geometry), but this work does not require it.

Remark 3.2.2. Since the base change of X/S to $\operatorname{Spec}\mathcal{O}^{et}_{S,s}$ is orientable at x, there is an étale neighbourhood (V,v) of s in S such that X_V/V is orientable at $x \times_s v$.

Lemma 3.3. *Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let x' be a singular point of X/S specializing to x and s' the image of x' in S. Suppose X/S is orientable at x, then it is orientable at x'. Moreover, in that case, there is a canonical bijection between orientations at x and orientations at x'.*

Proof. Suppose X/S is orientable at x and let (C_1, C_2) be an orientation at x. Let F be the connected component of $\operatorname{Sing}(X/S)$ containing x and x'. Pick a (non-canonical) isomorphism

$$\mathcal{O}^{et}_{X,x} = \mathcal{O}^{et}_{S,s}[[u,v]]/(uv - \Delta). \tag{1}$$

Permuting u and v if necessary, we can assume C_1 and C_2 come from the zero loci of u and v respectively. Pick a (canonical up to the residue extension $k(s)^{sep} \to k(s')^{sep}$) factorization

$$\operatorname{Spec}\mathcal{O}^{et}_{S,s'} \to \operatorname{Spec}\mathcal{O}^{et}_{S,s} \to S.$$

Then, tensoring by $\mathcal{O}^{et}_{S,s'}$ in equation (1), we get an isomorphism

$$\mathcal{O}^{et}_{X,x'} = \mathcal{O}^{et}_{S,s'}[[u,v]]/(uv - \Delta),$$

and the two connected components C'_1 and C'_2 of $(X \backslash F) \times_X \operatorname{Spec}\mathcal{O}^{et}_{X,x'} \times_S F$ come from the zero loci of u and v. Therefore, in order to prove that X/S is orientable at x' and that the order (C'_1, C'_2) is canonically induced by the order (C_1, C_2), we only need to know that the action of $G = \operatorname{Aut}_{\mathcal{O}^{et}_{S,s}}(\mathcal{O}^{et}_{S,s'})$ on $\{C'_1, C'_2\}$ is trivial, which is true since the action of G on $\mathcal{O}^{et}_{S,s'}[[u,v]]/(uv - \Delta)$ preserves the zero loci of u and v. $\square$

From now on, given an orientation (C_1, C_2) at a singular point x, we will also write (C_1, C_2) for the induced orientation at a singular generization of x.

Definition 3.4 (type of a section)**.** Let X/S be a quasisplit nodal curve with S regular. Suppose X is smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s at which X/S is orientable. Pick an orientation (C_1, C_2) at x and an isomorphism

$$\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta_x),$$

where C_1 corresponds to $u = 0$ and Δ_x is a generator of the singular ideal of x in $\mathcal{O}^{et}_{S,s}$. We call *type at x* any element of the monoid $\mathcal{O}^{et}_{S,s}/(\mathcal{O}^{et}_{S,s})^*$ strictly comprised between 1 and the thickness of x (for the order induced by divisibility). There are only finitely many types at x, given by the association classes of the strict factors of Δ_x in $\mathcal{O}^{et}_{S,s}$ (which is a unique factorization domain).

Let σ be a section of X/S through x. It induces a morphism

$$\widehat{\sigma\#} : \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta_x) \to \widehat{\mathcal{O}^{et}_{S,s}}$$

By Lemma 2.4, Δ_x has the same prime factor decomposition in $\mathcal{O}^{et}_{S,s}$ and in $\widehat{\mathcal{O}^{et}_{S,s}}$, so there is a canonical embedding of the submonoid of $\widehat{\mathcal{O}^{et}_{S,s}}/(\widehat{\mathcal{O}^{et}_{S,s}})^*$ generated by the factors of Δ_x into $\mathcal{O}^{et}_{S,s}/(\mathcal{O}^{et}_{S,s})^*$. We call *type of σ at x relatively to (C_1, C_2)* the image of u in in $\mathcal{O}^{et}_{S,s}/(\mathcal{O}^{et}_{S,s})^*$. It is a type at x, and since X/S is orientable at x, it does not depend on our choice of isomorphism $\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv-\Delta_x)$ as long as C_1 is given by $u = 0$. When they are clear from context, we will omit x and (C_1, C_2) from the notation and just call it the *type of σ*. In general, given a type T at x, there need not exist a section of type T.

If the type of a section σ at x relatively to (C_1, C_2) is equal to the type of a section σ' at x relatively to (C_2, C_1), we say σ and σ' are of *opposite type* at x.

Lemma 3.5. *Let X/S, s, x, (C_1, C_2) and U be as in Definition 3.4, and σ be a section $S \to X$ of type T at x. Let s' be a generization of s. Then there is a singular point of $X_{s'}$ specializing to x if and only if the thickness of x does not map to 1 in $\mathcal{O}^{et}_{S,s'}/(\mathcal{O}^{et}_{S,s'})^*$. Suppose it is the case and write x' this singular point, then*

- *if the image of T in $\mathcal{O}^{et}_{S,s'}/(\mathcal{O}^{et}_{S,s'})^*$ is either 1 or the thickness of x', then $\sigma(s')$ is a smooth point of $X_{s'}$;*
- *otherwise, the image of T is a type at s', that we still write T, and σ is of type T at x' relatively to (C_1, C_2).*

Proof. By Proposition 1.5, if the thickness of x maps to 1 in $\mathcal{O}^{et}_{S,s'}/(\mathcal{O}^{et}_{S,s'})^*$, then all points of X mapping to s' and specializing to x are S-smooth, and otherwise there is a unique singular point x' of $X_{s'}$ specializing to s. Suppose the latter holds, then by Lemma 3.3, (C_1, C_2) induces an orientation of X/S at x', and the lemma follows from the definition of types. □

Remark 3.5.1. One can think of the thickness of x as the relative version of a length, and of the type of a section σ relatively to an orientation (C_1, C_2) as a measure of the intersection of σ with C_1, seen as an effective Cartier divisor locally around x as in Lemma 1.16. In other words, the type is a measure of "how close to C_1" the section is.

Proposition 3.6. *Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let σ and σ' be two S-sections of X. Then the union of $(X/S)^{sm}$ with the set of singular points x of X/S at which σ and σ' have the same type (resp. opposite types) is an open subscheme of X.*

Proof. Since the singular locus $\mathrm{Sing}(X/S)$ is finite over S, every singular point x of X/S has a Zariski-open neighbourhood $V \subset X$ containing only S-smooth points and singular generizations of x. Thus, since the smooth locus of X/S is open in X, the proposition reduces to the following claim: if σ and σ' have the same type (resp. opposite types) at a singular geometric point $x \to X$, then they have the same type (resp. opposite types) at every singular generization of x. This claim is true by Lemma 3.5. □

Definition 3.7. The open subschemes of X described in Proposition 3.6 above are called respectively the *same type locus* and the *opposite type locus* of σ and σ'.

3.2 Admissible neighbourhoods

Here we will show that, when one works étale-locally on the base (in a sense that we will make precise), one can always assume sections of all types exist.

Definition 3.8. Let S be a regular scheme and X/S a nodal curve, smooth over a dense open U of S. Let s be a point of S and (V, v) an étale neighbourhood of s in S. We say (V, v) is an *admissible neighbourhood of* s (relatively to X/S) when the following conditions are met:

1. X_V/V is quasisplit;
2. X_V/V is orientable at all singular points of X_v;
3. for any singular point x of X_v with singular ideal $(\Delta_x) \subset \mathcal{O}^{et}_{S,s}$, all prime factors of Δ_x in $\mathcal{O}^{et}_{S,s}$ come from global sections of $\mathcal{O}_V$;
4. for every singular point x of X_v (however oriented), there are sections $V \to X_V$ of all types at x.

When $\bar{s} \to S$ is a geometric point with image s and (V, v) an admissible neighbourhood of s with a factorization $\bar{s} \to v$, we will also sometimes call V an *admissible neighbourhood of* $\bar{s}$.

Remark 3.8.1. In the situation of Definition 3.8, if S is strictly local, then it is an admissible neighbourhood of its closed point.

Proposition 3.9. *Let X/S be a nodal curve, where S is a regular and excellent scheme. Then any point $s \in S$ has an admissible neighbourhood.*

Proof. Replacing S by an étale neighbourhood, we can assume X/S is quasisplit (for example using Lemma 1.13 and the fact X/S is of finite presentation). By Remark 3.2.2 and since X_s has finitely many singular points $x_1, ..., x_n$, we can assume X/S is orientable at all the x_i. Each x_i has only finitely many prime factors in its singular ideal in $\mathcal{O}^{et}_{S,s}$, so we can shrink S again into a neighbourhood satisfying condition 3. of the definition of admissibility. The fact this V can be shrinked again until it meets all four conditions follows from Lemma 2.5. □

Remark 3.9.1. If (V, v) is an admissible neighbourhood of a point s of S, then V need not be an admissible neighbourhood of all of its points (even if V is strictly local with closed point v, condition 3. of the definition may fail, see Example 6.10). Thus, it is not easy a priori to find a good global notion of admissible cover.

The next proposition states that admissible neighbourhoods are compatible with smooth (and not just étale) morphisms.

Proposition 3.10. *Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let $Y \to S$ be a smooth morphism and $y \to Y$ a geometric point. Let V be an admissible neighbourhood of y in S, then $V \times_S Y$ is an admissible neighbourhood of y in Y.*

Proof. This follows from Corollary 2.3 and the definition of admissible neighbourhoods. □

4 Refinements and resolutions

This section is dedicated to techniques aiming at constructing inductively nodal models of a smooth curve with prime singular ideals, starting from an arbitrary nodal model. When Néron models are concerned, the interest of nodal models with prime labels lies in two facts: they are locally factorial, which is a crucial hypothesis in the existence result of [21] for Néron models of Jacobians, and all their sections factor through their smooth locus, which will allow us to construct Néron models of curves as gluings of smooth loci of nodal models. We will now make these two statements precise and prove them, and the consequences will be developped in part II.

4.1 Arithmetic complexity and motivation for refinements

We start by defining what will be our recursion parameter, the *arithmetic complexity* of a nodal curve.

4.1.1 Arithmetic complexity

Definition 4.1. Let M be the free commutative semigroup over a set of generators G. We call *word length* of $m \in M$, and note $wl(m)$, the (unique) $n \in \mathbb{N}^*$ such that we can write $m = \prod_{i=1}^{n} g_i$ with all the g_i in G.

Given a labelled graph $\Gamma = (V, E, l)$ over M and an edge $e \in E$, we call *arithmetic complexity* of e and note n_e the natural integer $wl(l(e)) - 1$. We call *arithmetic complexity* of Γ and note n_Γ the sum of the arithmetic complexities of all its edges.

Given a nodal curve X/S where $S = \operatorname{Spec} R$ is a local unique factorization domain, the semigroup of nontrivial principal ideals of R is the free commutative semigroup over the set of primes of height 1. From now on, we will talk freely about arithmetic complexities of edges of dual graphs, always implicitly referring to this set of generators.

Thus, when X/S is quasisplit, we define the *arithmetic complexity* of a closed singular point x, noted n_x, as the arithmetic complexity of the corresponding edge of the dual graph: If S is local, we define the *arithmetic complexity* of X, noted n_X, to be that of its dual graph.

Note that X is of arithmetic complexity 0 if and only if every singular ideal is prime: it is an integer measuring "how far away from being prime" the singular ideals are. Arithmetic complexity is not stable even under étale base change.

4.1.2 Factoriality of completed étale local rings

Lemma 4.2. *Let R be a regular complete local ring, and Δ be an element of $\mathfrak{m}_R$. Let $\widehat{A} = R[[u,v]]/(uv - \Delta)$, then $\widehat{A}$ is a unique factorization domain if and only if Δ is prime in R.*

Proof. Suppose that $\widehat{A}$ is a unique factorization domain, and let d be a prime factor of Δ in R. Call S the complement of the prime ideal (u,d) in $\widehat{A}$. Let $\mathfrak{p}$ be a nonzero prime ideal of $S^{-1}\widehat{A}$. Then, p contains a nonzero element $x = ux_u + x_v$, with x_u and x_v in $R[[u]]$ and $R[[v]]$ respectively. Since $\mathfrak{p} \neq S^{-1}\widehat{A}$, we have $d|x_v$. Call n and m respectively the maximal elements of $\mathbb{N}^* \cup \{+\infty\}$ such that $u^n|ux_u$ and $d^n|x_v$. Since x is nonzero, we know either n or m is finite. If $n \leq m$, then $v^n x = \Delta^n \frac{x_u}{u^{n-1}} + d^n \frac{v^n x_v}{d^n}$ is in $\mathfrak{p}$, and is associated to d^n in $S^{-1}\widehat{A}$, so we obtain $d \in \mathfrak{p}$, from which it follows that $\mathfrak{p} = (u,d)$. If $m < n$, a similar argument shows that $\mathfrak{p}$ contains u^m and thus equals (u,d). Therefore, $S^{-1}\widehat{A}$ has Krull dimension one, i.e. (u,d) has height 1 in $\widehat{A}$. Since $\widehat{A}$ is a unique factorization domain, it follows that (u,d) is principal in it, from which we deduce that Δ and d are associated in $\widehat{A}$. In particular, Δ is prime in R.

The interesting part is the converse: let us assume that Δ is prime in R. We want to show that $\widehat{A}$ is a unique factorization domain. We first prove that $A := R[u,v]/(uv - \Delta)$ is a unique factorization domain: let p be a prime ideal of A of height 1, we have to show p is principal in A. We observe that u is a prime element of A, since the quotient $A/(u) = R/(\Delta)[v]$ is an integral domain. Therefore, if p contains u, then $p = (u)$ is principal. Otherwise, p gives rise to a prime ideal of height 1 in $A_u := A[u^{-1}]$, which is principal since $A[u^{-1}] = R[u,u^{-1}]$ is a unique factorization domain. In that case, write $pA_u = fA_u$ for some $f \in A_u$. Multiplying by a power of the invertible u of A_u, we can choose the generator f to be in $A \backslash uA$. Since p is a prime ideal of A not containing u, we know p contains f and thus fA. We will now prove the reverse inclusion. Let x be an element of p. The localization $pA_u = fA_u$ contains x, so x satisfies a relation of the form $u^n x = fy$ for some $n \in \mathbb{N}$ and some $y \in A$. But since u is prime in A, we know u^n divides y and x is in fA.

Now, we will deduce the factoriality of $\widehat{A}$ from that of A. The author would like to thank Ofer Gabber for providing the following proof. Let q be a prime ideal of $\widehat{A}$ of height 1, we will show q is principal. We put $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, $\widehat{X} = \operatorname{Spec} \widehat{A}$, and $Z = \operatorname{Spec}(\widehat{A}/q)$, so that Z is a prime Weil divisor on $\widehat{X}$. Let η, η' be the generic points of the respective zero loci of u, v in the closed fiber $\operatorname{Spec} k_R[[u,v]]/(uv)$ of $\widehat{X} \to S$. Since u and v are prime elements of $\widehat{A}$, we can once again assume Z contains neither η nor η'. It follows that the closed fiber of $Z \to S$ is of dimension 0: the morphism $Z \to S$ is quasi-finite, hence finite by [10], chapter 0, 7.4. A fortiori, $\widehat{A}/q$ is finite over A, so by Nakayama's lemma, the morphism $A \to \widehat{A}/q$ is surjective. Call p its kernel. Then $A/p = \widehat{A}/q$ is $\mathfrak{m}_A$-adically complete and separated, so it maps isomorphically to its completion $\widehat{A}/p\widehat{A}$. The prime ideal p of A is of height 1 since $\widehat{X} \to X$ is a flat map of normal noetherian schemes. Therefore, p is principal in A, and $q = p\widehat{A}$ is principal in $\widehat{A}$. □

4.1.3 Factoring sections through the smooth locus

Lemma 4.3. *Let X/S be a quasisplit nodal curve, where $S = \operatorname{Spec} R$ is a regular, strictly local and excellent scheme. Let σ be a section of X/S. Let $s \in S$ be the closed point and $x = \sigma(s)$, then x is either smooth over S, or singular of arithmetic complexity ≥ 1.*

Proof. Suppose by contradiction that x is singular of arithmetic complexity 0. The section σ factors through $\operatorname{Spec} \mathcal{O}_{X,x}$, and gives rise to a $\operatorname{Spec} \widehat{R}$-section of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$. As $\widehat{\mathcal{O}_{X,x}}$ is of the form $\widehat{R}[[u,v]]/(uv-\Delta)$ for some prime $\Delta \in R$, this section is given by a morphism $\widehat{R}[[u,v]]/(uv-\Delta) \to \widehat{R}$, which is fully described by the images of u and v. But Δ is prime in $\widehat{R}$ by Lemma 2.4, so either u or v has invertible image in $\widehat{R}$. Thus the image of s factors through the complement of one of the two irreducible components of $\operatorname{Spec} \widehat{X}_{s,x}$, a contradiction. □

4.2 Refinements of graphs

Now, to reap the benefits of the properties of nodal curves with prime labels, all we need is an algorithm that takes a generically smooth nodal curve as an input, and returns a nodal curve, birational to the first, with strictly lower arithmetic complexity.

Definition 4.4. As in [21], Definition 3.2, for a graph $\Gamma = (V, E, l)$ with edges labelled by elements of a semigroup M, we call *refinement* of Γ the data of another labelled graph $\Gamma' = (V', E', l')$ labelled by M and two maps

$$E' \to E$$
$$V' \to E \coprod V$$

such that:

- every vertex v in V has a unique preimage v' in V';
- for every edge $e \in E$ with endpoints $v_1, v_2 \in V$, there is a chain $C(e)$ from v_1', v_2' in Γ' such that the preimage of $\{e\}$ in $V' \coprod E'$ consists of all edges and intermediate vertices of $C(e)$;
- for all $e \in E$, the length of e is the sum of the lengths of all edges of $C(e)$.

We will often keep the maps implicit in the notation, in which case we call Γ' a *refinement* of Γ and write $\Gamma' \preceq \Gamma$. We say Γ' is a *strict refinement* of Γ and note $\Gamma' \prec \Gamma$, if in addition the map $E' \to E$ is not bijective.

Remark 4.4.1. Informally, a refinement of a graph is obtained by "replacing every edge by a chain of edges of the same total length". Suppose $\Gamma' \preceq \Gamma$, then $\Gamma' \prec \Gamma$ if and only if at least one of the chains $C(e)$ is of length ≥ 2, i.e. if and only if Γ' has strictly more edges than Γ.

Now we want to blow up X in a way that does not affect X_U, but refines the dual graph. We will define *refinements* of curves (Definition 4.5). We can obtain any refinement of a dual graph of X by iterating these refinements of curves, but they exist only étale-locally on the base.

4.3 Refinements of curves

Definition 4.5. Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s at which X/S is orientable, and (C, D) an orientation of X/S at x. Let T be a type at x. We will call *T-refinement* of X (at x, relatively to (C, D)) the blowing-up of X in the sheaf of ideals of a section $S \to X$ through x, of type T. We will often omit x and (C, D) in the notation and just call these T-refinements of X.

A map $X' \to X$ is called a *refinement* if it is a T-refinement for some such $X, x, (C, D), T$.

Remark 4.5.1.

- If S is excellent, then any geometric point $s \in S$ has an admissible neighbourhood V by Proposition 3.9, so X_V/V has a T-refinement for any type T at any singular point of X_s.
- Consider any morphism $S' \to S$ where S' is still regular (e.g. any smooth map $S' \to S$). Let x be a singular point of X and x' a singular point of X' of image x. Then any orientation of X/S at x pulls back to an orientation of $X_{S'}/S'$ at x'; any type T at x pulls back to a type T' at x'; and any T-refinement pulls back to a T'-refinement.
- Let $x \in X$ be a singular point at which X/S is orientable, and y a generization of x. By Lemma 3.5, for any type T at x and any T-refinement $X' \to X$, either T corresponds to a type (still noted T) at y, in which case $X' \to X$ is a T-refinement at y, or T becomes trivial at y, in which case $X' \to X$ restricts to an isomorphism above a Zariski neighbourhood of y.

Lemma 4.6. *Let $f : X \to S$ be a quasisplit nodal curve with S regular and excellent. Suppose X is smooth over some dense open $U \subset S$. Let $\sigma \colon S \to X$ be a section and $\phi \colon X' \to X$ the blow-up in the ideal sheaf of σ.*

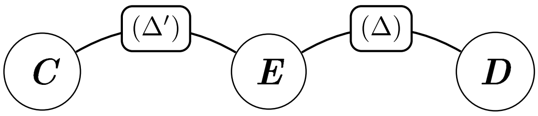
Then ϕ is an isomorphism above the complement in X of the intersection of $\mathrm{Sing}(X/S)$ *with the image of σ. In particular, it is an isomorphism above the smooth locus of X/S, which contains X_U, so X' is a model of X_U.*

Moreover, X' is a nodal curve, and its dual graphs are refinements of those of X. More precisely, let s be a point of S and suppose $\sigma(s)$ is a singular point x of X_s. Choose an orientation (C, D) of $X_{\mathcal{O}^{et}_{S,s}}$ at x, and call T the type of σ at x relatively to (C, D). Then, the singular ideal of x in $\mathcal{O}^{et}_{S,s}$ is generated by

$$\Delta_x = \Delta\Delta',$$

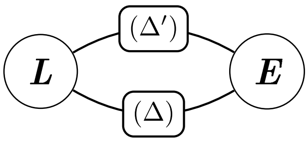
where Δ and Δ' are lifts to $\mathcal{O}^{et}_{S,s}$ of T and of the opposite type of T respectively. Let Γ, Γ' be the respective dual graphs of X and X' at s, and let e be the edge of Γ corresponding to x. Then e has label generated by $\Delta_x = \Delta\Delta'$, and one obtains Γ' from Γ as follows:

- *if e is not a loop, then C and D come from two distinct irreducible components of X_s (that we still call C and D). In that case, Γ' is obtained from Γ by replacing e by a chain*



  where the strict transforms of C and D are still called C and D, and E is the inverse image of x.

- *if e is a loop, i.e. x belongs to only one component L of X_s, then Γ' is obtained from Γ by replacing e by a cycle*



  where the strict transform of L is still called L and E is the inverse image of x.

Proof. The ideal sheaf of σ is already Cartier above the smooth locus of X/S and outside the image of σ, so by the universal property of blow-ups ([30, Tag 0806]), we only need to describe ϕ above the étale localizations $\operatorname{Spec} \mathcal{O}^{et}_{X,x}$, where $x, s, (C, D)$ are as in the statement of the lemma. We can assume $S = \operatorname{Spec} R$ is strictly local, with closed point s. Pick an isomorphism

$$\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{R}[[u, v]]/(uv - \Delta\Delta')$$

such that C is locally given by $u = 0$. The map

$$\widehat{\sigma} \colon \widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{R}$$

yielded by σ sends u to a generator of $\Delta\widehat{R}$ and v to a generator of $\Delta'\widehat{R}$. Scaling u and v by a unit of $\widehat{R}$ if necessary, we can assume $\widehat{\sigma}(u) = \Delta$ and $\widehat{\sigma}(v) = \Delta'$.

The completed local rings of $\operatorname{Spec} \mathcal{O}^{et}_{X,x} \times_X X'$ can be computed using the blowing-up of the algebra $B := R[u,v]/(uv - \Delta\Delta')$ in the ideal $(u - \Delta, v - \Delta')$ (since the completion of B at $(u, v, \mathfrak{m}_R)$ is $\widehat{\mathcal{O}_{X,x}}$).

The latter is covered by two affine patches:

- the patch where $u - \Delta$ is a generator, given by the spectrum of

$$R[u,v,\alpha]/((v - \Delta') - \alpha(u - \Delta), u\alpha + \Delta') \simeq R[u,\alpha]/(u\alpha + \Delta')$$

  since, in the ring $R[u,v,\alpha]/((v - \Delta') - \alpha(u - \Delta))$, the element $uv - \Delta\Delta'$ is equal to $(u - \Delta)(u\alpha + \Delta')$

- and the patch where $v - \Delta'$ is a generator, where we obtain symmetrically the spectrum of

$$R[v,\beta]/(v\beta + \Delta)$$

with the obvious gluing maps. Thus we see that X' remains nodal, and that the edge e of Γ (of label $(\Delta\Delta')$) is replaced in Γ' by a chain of two edges, one labelled (Δ) and one labelled (Δ'). It also follows from this description that the strict transform of C (resp. D) in $X' \times_X \operatorname{Spec} \widehat{\mathcal{O}^{et}_{X,x}}$ contains the singularity of label (Δ') (resp. (Δ)). □

Corollary 4.7. *With the same hypotheses and notations as in Lemma 4.6, for any two sections σ, σ' of X/S, the blow-ups $Y \to X$ and $Y' \to X$ in the respective ideal sheaves of σ and σ' are canonically isomorphic above the same type locus of σ and σ' in X.*

Proof. It suffices to exhibit, for any point $s \to S$ and any singular point x of X_s such that $\sigma(s) = \sigma'(s) = x$ and σ, σ' have the same type T at x, a Zariski neighbourhood V of x in X and an isomorphism $Y \times_X V \to Y' \times_X V$ compatible with the canonical identifications $Y \times_X X^{sm} = X^{sm} = Y' \times_X X^{sm}$. Since X, Y, Y' are of finite presentation over S, this can be done assuming $S = \operatorname{Spec} R$ is strictly local, with closed point s. Using the universal property of blow-ups ([30, Tag 0806]), we reduce to proving that the pull-back of the ideal sheaf of σ' (resp. σ) to Y (resp. Y') is Cartier. The proofs are symmetric, so we will only show that the pull-back to Y of the ideal sheaf of σ' is Cartier. This, in turn, reduces to proving that the ideal sheaf of σ' in $\operatorname{Spec} \widehat{\mathcal{O}^{et}_{X,x}}$ becomes Cartier in $Y \times_X \operatorname{Spec} \widehat{\mathcal{O}^{et}_{X,x}}$. Pick an isomorphism

$$\widehat{A} := \widehat{R}[[u,v]]/(uv - \Delta_x) = \widehat{\mathcal{O}^{et}_{X,x}},$$

where $\Delta_x \in R$ is a generator of the singular ideal of x. The map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ sends u, v to elements Δ, Δ' of $\widehat{R}$ with $\Delta\Delta' = \Delta_x$. Since σ and σ' have the same type at x, there is a unit $\lambda \in \widehat{R}^\times$ such that the map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ' sends u and v to $\lambda\Delta$ and $\lambda^{-1}\Delta'$ respectively. We have reduced to proving that the sheaf given by the ideal $(u - \lambda\Delta, v - \lambda^{-1}\Delta')$ of $\widehat{A}$ becomes Cartier in the blow-up of $\widehat{A}$ in $(u - \Delta, v - \Delta')$. Put

$$A = \widehat{R}[u, v]/(uv - \Delta\Delta'),$$

then it is enough to prove that the ideal $I = (u - \lambda\Delta, v - \lambda^{-1}\Delta')$ of A becomes invertible in the two affine patches (as described in the proof of Lemma 4.6) forming the blowing-up of A in $(u - \Delta, v - \Delta')$. By symmetry, we only check it in the patch generated by $u - \Delta$, which is the spectrum of

$$A_1 = \widehat{R}[u, \alpha]/(u\alpha + \Delta'),$$

where v maps to $\Delta' + \alpha(u - \Delta)$. We have $I = (u - \lambda\Delta, \lambda v - \Delta')$, and in A_1 we can write

$$\begin{aligned} \lambda v - \Delta' &= \lambda(\Delta' + \alpha(u - \Delta)) + u\alpha \\ &= -\lambda\alpha\Delta + u\alpha \\ &= \alpha(u - \lambda\Delta). \end{aligned}$$

Thus, the preimage of I in A_1 is the invertible ideal $(u - \lambda\Delta)$, and we are done. □

4.4 Resolutions of nodal curves

Lemma 4.8. *Let Γ, Γ' be two labelled graphs over a free commutative semigroup. If $\Gamma' \preceq \Gamma$ (Definition 4.4), then $n_{\Gamma'} \leq n_\Gamma$. If $\Gamma' \prec \Gamma$, then $n_{\Gamma'} < n_\Gamma$.*

Proof. Suppose $\Gamma' \preceq \Gamma$. Then, by definition, the sum of lengths of edges of Γ is equal to the sum of lengths of edges of Γ' and Γ' has at least as many edges as Γ, so $n_{\Gamma'} \leq n_\Gamma$.

If equality holds in the latter, then Γ' and Γ have the same number of edges, which, combined with the fact $\Gamma' \preceq \Gamma$, implies they are isomorphic. □

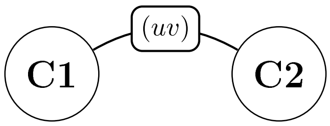
Now we want, starting from X, to find a model of X_U "refining Γ_s as much as possible", in the sense that it will be of arithmetic complexity 0 and its dual graph will be a refinement of Γ_s. This model's alignment will then determine the existence of a Néron model for the Jacobian of X_U. We do it following the ideas of [2], proposition 3.6, as follows:

Definition 4.9. Let X be a nodal curve over a regular and strictly local scheme S with closed point s, smooth over a dense open subscheme $U \subset S$. We call *resolution of* X any nodal S-model X' of X_U, obtained by a finite sequence of refinements, and of arithmetic complexity 0.

Proposition 4.10. *With notations and hypotheses as above, X admits a resolution.*

Proof. The base S is strictly local so it is an admissible neighbourhood of s. Take $X' \to X$ a finite sequence of refinements minimizing $n_{X'}$, then $X' \to X$ is a resolution. Indeed, suppose it was not, then there would be a closed singular point x of X' of arithmetic complexity ≥ 1. There would exist a type T at x, and a T-refinement $X'' \to X$. We would have $n_{X''} < n_{X'}$, a contradiction. □

Remark 4.10.1. Resolutions are not unique in general. For example, consider $S = \operatorname{Spec} \mathbb{C}[[u, v]]$, and suppose X is a nodal curve over S with dual graph



There are two types at the closed singular point x of X/S with respect to (C_1, C_2), namely the class T of u and the class T' of v. The T-refinement and the T'-refinement of X are both resolutions, but they are not isomorphic as models of X_U (they are not even isomorphic as schemes as soon as $C1, C2$ are not isomorphic, e.g. of distinct genera).

Definition 4.11. Let X be a quasisplit nodal curve over a regular scheme S, smooth over a dense open subscheme $U \subset S$. Let s be a point of S. We say X/S is *square-free at* s when all labels of edges of the dual graph of X' at s are square-free. We say X is *square-free* if it is square-free at every point of S.

Remark 4.11.1. In the definition above, we consider the dual graphs at points of S labelled by ideals of the Zariski local rings, see Remark 1.11.2. However, if $R \to R^{sh}$ is a strict henselization of a regular local ring, any square-free element $\Delta \in R$ has square-free image in R^{sh}, so the definition would be unchanged if we asked for the labels of the dual graphs at geometric points (which are ideals of the étale local rings of S) to be square-free.

Definition 4.12. Let X be a quasisplit nodal curve over a regular scheme S, smooth over a dense open subscheme $U \subset S$. Let s be a point of S. We call *partial resolution of* X *at* s any map $X' \to X$, composition of a finite number of refinements, such that X' is square-free at s. We will call *partial resolution of* X *over* S, or just *partial resolution of* X if there is no ambiguity, a map $X' \to X$ that is a partial resolution at every point of S.

Remark 4.12.1. The property "being a square-free principal ideal of the regular local ring R" is preserved by tensor product with R' for any morphism of local rings $R \to R'$ that is a directed colimit of étale morphisms. Therefore, a square-free quasisplit nodal curve remains square-free after base change to any codirected limit of étale maps, e.g. a localization or a strict localization. In particular, for any point $s \in S$ and any geometric point $\bar{s}$ above s, $X' \to X$ is a partial resolution at s if and only if $X' \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{s}}^{et} \to X \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{s}}^{et}$ is a partial resolution at the closed point.

Lemma 4.13. *Let X/S be a quasisplit nodal curve, smooth over a dense open $U \subset S$, with S regular. Then, the set of points $s \in S$ at which X/S is square-free is open in S.*

Proof. By quasisplitness, the singular locus of X/S is a disjoint union of closed immersions cut out by locally principal ideals $I_1, ..., I_n$ of $\mathcal{O}_S$. Thus, X is square-free at a point t if and only if the quotient $\mathcal{O}_S/I_i$ is reduced at t for every i, which is an open condition on S. □

Proposition 4.14. *Let X/S be a nodal curve, smooth over a dense open $U \subset S$, with S excellent and regular. Then, every point $s \in S$ admits an étale neighbourhood $V \subset S$ such that X_V/V has a partial resolution.*

Proof. Let s be a point of S, we will show it has such a neighbourhood. Shrinking S if necessary, we assume S is an admissible neighbourhood of s. If the arithmetic complexity of X at s is not 0, then there is always a refinement $X' \to X$ such that the arithmetic complexity of X' at s is strictly lower than that of X, so by induction we may assume all labels appearing in the dual graph of X at s are prime. In particular, X is square-free at s. The proposition now follows immediatelyfrom Lemma 4.13. □