Néron models in high dimension: Nodal curves, Jacobians and tame base change

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General introduction

Néron models

Given an integral scheme S with generic point η, a lot of proper and smooth schemes over $k(\eta)$ have no proper and smooth model over S. However, they can sometimes still have a canonical smooth S-model, the Néron model. Néron models were first introduced in 1964 by André Néron in his article [25] for abelian varieties over a Dedekind base scheme. The Néron model of X_η/η is defined as a smooth, separated scheme N/S restricting to X_η over η, satisfying the following universal property, called the *Néron mapping property*: for every smooth scheme $T \to S$ and morphism $\phi_K \colon T_\eta \to X_\eta$, there exists a unique morphism $\phi \colon T \to N$ extending ϕ_K.

The mapping property has a lot of nice consequences, making the Néron model the "best possible smooth model": among other things, it ensures that Néron models are unique up to a unique isomorphism, and inherit a group structure from X_η when the latter has one.

Néron models are always unique, but their existence is not trivially guaranteed. Néron proved in the original article [25] that abelian varieties over the fraction field of an integral Dedekind scheme always have Néron models. Recently, people have taken interest in constructing Néron models in different settings. For example, it was proved in 2013 by Qing Liu and Jilong Tong in [23] that smooth and proper curves of positive genus over a Dedekind scheme always have Néron models. This does not apply to genus 0: Proposition 4.12 of [23] shows that if S is the spectrum of a discrete valuation ring with field of fractions K, then $\mathbb{P}^1_K$ does not have a Néron model over S.

In the cases mentioned above, the Néron model is of finite type. This condition is even often included in the definition of Néron models, in which case authors refer to smooth, separated schemes satisfying the mapping property as *Néron-lft models* (where "*lft*" stands for "locally of finite type"), to emphasize the absence of quasi-compactness hypothesis. The use of this terminology is not systematic anymore in the litterature, so we will use a more flexible definition, essentially equivalent to the one given in the first paragraph. If R is an excellent and strictly henselian discrete valuation ring with field of fractions K, the simplest example of a K-group scheme with a R-Néron(-lft) model that is not of finite type is the multiplicative group $\mathbb{G}_m$ over K. In fact, it is shown in [1] (Theorem 10.2.1 and Theorem 10.2.2) that a smooth commutative K-group G_K has a Néron model if and only if it has no subgroup isomorphic to the additive group $\mathbb{G}_a$, in which case the Néron model is of finite type if and only if G_K has no subgroup isomorphic to the multiplicative group $\mathbb{G}_m$.

Among the concrete applications of the theory of Néron models, we can cite the semi-stable reduction theorem (an abelian variety over the fraction field of a discrete valuation ring acquires semi-abelian reduction after a finite extension of the base field); the Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties; the computation of canonical heights on Jacobians; as well as the linear and quadratic Chabauty methods to determine whether or not a list of rational points on a curve is exhaustive. For a geometric description of the quadratic Chabauty method, see [5]. Parallels can also be drawn to some problems in which Néron models do not explicitly intervene, such as extending the double ramification cycle on the moduli stack of smooth curves to the whole moduli stack of stable curves as in [20]. Here, one is interested in models in which one given section extends, instead of all sections simultaneously, but the two problems are closely related.

Existence in higher dimension

As we can see from the synthesis above, we understand reasonably well when Néron models exist over regular one-dimensional bases, especially for group schemes. Finding criteria for existence of Néron models over higher-dimensional bases and constructing them explicitly when they exist is a more recent and open area of research. In [21], David Holmes exhibits a necessary condition, called *alignment*, for the existence of the Néron model of the generic Jacobian of a nodal curve X over any regular base scheme S. He also proves that alignment is sufficient under additional assumptions of semifactoriality on X. Alignment is a rather strong condition that relates the local structure of X around all singularities appearing in the same cycle of a dual graph of X.

In [27], Giulio Orecchia introduces the *toric-additivity* criterion. Consider an abelian scheme A/U with semi-abelian reduction $\mathcal{A}/S$, where S is a regular base and U the complement in S of a strict normal crossings divisor. Toric-additivity is a condition on the Tate module of A. When A is the generic Jacobian of an S-curve with a nodal model, toric-additivity is sufficient for a Néron model of A to exist. It is also necessary up to some restrictions on the base characteristic. For general abelian varieties, it is proven in [28] that toric-additivity is still sufficient when S is of equicharacteristic zero, and a partial converse holds, i.e. existence of a Néron model implies a weaker version of toric-additivity.

To the author's knowledge, little has been said about Néron models of schemes with no group structure (e.g. relative curves) over bases of higher dimension. The construction of [23] for curves over Dedekind schemes consists in embedding a curve X_K into its Jacobian J_K, resolving the singularities of the scheme-theoretical closure of X_K into the Néron model of J_K, and taking the smooth locus. This does not generalize well to relative curves over bigger base schemes: in this setting, resolution of singularities is not known and regular models are not even known to exist.

Nodal curves, stable curves and log smoothness

Nodal curves can be thought of as curves that are not necessarily smooth, but only allow the simplest type of singularities: the completed localization at a singularity of a nodal curve X_k over an algebraically closed field k is a union of two lines meeting transversally. This simple description of nodal singularities in their fibers permits to define an important combinatorial invariant, the *dual graph*: its vertices are the irreducible components of X_k, and its edges are the nodal singularities, which are attached to the irreducible components they belong to. A great deal of information on a nodal curve can be recovered from just this graph (or these graphs, in the relative case) and the genera of the vertices.

In the relative case, one can nicely describe the local structure of the whole relative curve around a singularity in terms of a certain ideal of an étale local ring of the base, the *singular ideal.* Adjoining to the dual graphs the data of these singular ideals makes them an even better tool to summarize the properties of a nodal curve in a simple combinatorial object.

Nodal curves also arise naturally in the context of logarithmic geometry, i.e. algebraic geometry on the category of log schemes. A log scheme is the data of a scheme X and a map of sheaves of monoids $M \to \mathcal{O}_X$ inducing an isomorphism $M^\times \to \mathcal{O}_X^\times$, where we give $\mathcal{O}_X$ its multiplicative monoid structure. The logarithmic version of smoothness is less restrictive than scheme-theoretic smoothness, and only forces log-smooth curves to have a nodal curve as an underlying scheme.

Working with the whole category of nodal curves can be tricky, as they do not possess a well-behaved moduli stack: for example, if k is an algebraically closed field, the automorphism group of the nodal curve $\mathbb{P}^1_k$ is infinite. Its action on $\mathbb{P}^1_k$ is even 3-transitive, which implies that any nodal k-curve with a component isomorphic to $\mathbb{P}^1$, meeting the rest in only one or two points, also has infinite automorphism group. This forbids the existence of a Deligne-Mumford stack for nodal curves, even of fixed genus. Likewise, a relative nodal curve X/S smooth over a scheme-theoretically dense open subscheme $U \subset S$ can admit non-trivial blowups supported outside of X_U that are still nodal curves (see [2], Proposition 3.6, or the first part of this thesis), which forbids the existence of a separated algebraic stack for nodal curves. To obtain a Deligne-Mumford stack, smooth and proper over $\operatorname{Spec}\mathbb{Z}$, one should work instead with *n-pointed stable curves*, which are the data of a nodal curve X/S and n marked sections on it guaranteeing that the automorphism groups of the geometric fibers are finite étale.

The thesis

This thesis is divided in three parts:

- **Part I**: Nodal curves, dual graphs and resolutions;
- **Part II**: Néron models of nodal curves and their Jacobians;
- **Part III**: Base change of Néron models along finite tamely ramified maps.

Part II heavily relies on part I, while part III only depends on the first section of part II (namely, it only makes use of the definition of Néron models and of their elementary base change properties).

Part I

In part I, we will discuss a high-dimensional variant of the classic *smoothening process*. Typically, when S is the spectrum of a discrete valuation ring, a key step in constructing the Néron model of the generic fiber of a proper S-scheme X/S consists in blowing-up repeatedly in subschemes supported outside of the generic fiber X_K, so that the smooth locus "gets bigger". For example, if S is strictly local, eventually, we want to obtain a new model $X' \to X$ of X_K so that the map from the smooth locus of X' to X is surjective on S-points.

For higher-dimensional S, we can do something similar, but arbitrary choices have to be made in the process: we end up with several "partial smoothenings", whose smooth loci jointly satisfy the surjectivity property we expect. A construction is made by A.J. De Jong in [2], Proposition 3.6, for split curves over a regular base, smooth over the complement of a strict normal crossings divisor. De Jong repeatedly blows up the split curve in irreducible components of its non-smooth locus. We will construct the "partial smoothenings", that we will call *resolutions*, without hypotheses of splitness and without asking for the discriminant locus to be a normal crossings divisor.

The approach of De Jong is not convenient when one wishes to work with non-split curves. It appeared from a discussion with G. Orecchia that this problem could be solved by blowing-up in ideal sheaves of sections through singular points instead of irreducible components of the non-smooth locus. This forces us to work étale-locally on the base, which poses no problem when dealing with Néron models, as they descend even along smooth covers.

Part II

In part II, we consider a regular base scheme S and a nodal curve X/S, smooth over a dense open $U \subset S$, and we exhibit criteria for the existence of Néron models for X_U and for its Jacobian, both times in terms of the (labelled) dual graphs of X.

For Jacobians, we try to investigate the situations that are not covered by the main theorem of [21], i.e. what happens when X satisfies the alignment condition of [21], but is not semifactorial after every smooth base change. We introduce a new condition on the dual graphs of X, *strict alignment*, and we show it is equivalent to alignment of all resolutions of smooth base changes of X. In particular, strict alignment is necessary for a Néron model of the generic Jacobian to exist. We will show it is also sufficient, and we will describe explicitly the Néron model of generic Jacobians of strictly aligned nodal curves.

This thesis is not written in the language of log geometry, but the reader familiar with it can establish a parallel between strict alignment and finiteness of the tropical Jacobian as described in [24], as well as between our blowups and logarithmic modifications of a log curve inducing a given subdivision of the tropicalization. In a future work with David Holmes, Giulio Orecchia and Samouil Molcho, we intend to develop further these correspondences and show that Néron models of Jacobians can be better understood in terms of the logarithmic Picard functor.

Regarding nodal curves themselves, we will make use of a result from [8] about extending rational points of a curve into sections, when the curve has no rational components in any geometric fiber. This naturally leads us to consider the two following questions:

- When does a curve admit a nodal model with no such rational components?
- Can we use the resolutions of such a model to construct a Néron model, following the usual smoothening principles?

After base change to an étale cover, we provide an exhaustive answer to the first question. In [7], it is shown that, under certain restrictions on the genera of the curves considered, there are canonical contraction morphisms from the stack of $n+1$-pointed stable curves to that of n-pointed stable curves. A nice consequence is that if a geometric fiber of a nodal curve X has infinitely many automorphisms, after an étale extension of the base, one can always blow down X so that certain rational components of the geometric fibers are contracted. Repeating the process, we can find a unique stable curve (without markings) birational to X. Among all nodal models, this is the one with the least rational components in its geometric fibers, and we provide criteria for it to have none.

As for the second question, we will see that the answer is almost always negative: most nodal curves admit several non-isomorphic resolutions, and the smooth loci of them all cannot fit in a Néron model without creating separatedness issues. We will show, however, that a "not-necessarily-separated Néron model", i.e. a smooth object with the Néron mapping property (uniqueness included) always exists, and we will give an explicit construction. A Néron model exists if and only if this object is separated, and we will show this is equivalent to the singular locus of the stable model X^{stable} being irreducible, étale-locally on X^{stable}.

Part III

In part III, we study the base change behavior of Néron models under finite, locally free morphisms between regular schemes. We are mostly interested in descent: for example, given a regular base S, a dense open $U \subset S$, and a smooth proper curve X_U over U, if one uses some version of the semistable reduction theorem to find a nodal model of X_U over a finite extension S'/S, can we use the results of part II to recover information about the Néron model of X_U?

In [4], Bas Edixhoven investigates the base change morphism between Néron models of an abelian variety under a finite, tamely ramified extension of discrete valuation rings R'/R. More precisely, if N, N' are the Néron models over R and

R' respectively, he exhibits a filtration of N indexed by integers up to the ramification index of R'/R, and describes explicitly the successive quotients in terms of N'. Several aspects of this filtration (interpretations, applications) are developed in the papers [16] and [6] and in the book [17].

Consider a regular scheme S, a dense open $U \subset S$ and a smooth U-algebraic space X_U. The first obstruction to generalize Edixhoven's results is the following: if $S' \to S$ is a finite, locally free, tamely ramified cover and if the base change of X_U has a Néron model N'/S', it is not obvious a priori that X_U/U has a Néron model N/S. We will prove that this is actually the case and, following the ideas of [4], we will define a finite filtration of N and describe explicitly its successive quotients in terms of twisted Lie algebras of N'.

Notations

Throughout this thesis, we will adopt the following conventions:

- If $f\colon X \to S$ is a morphism of algebraic spaces locally of finite type, we call *smooth locus* of f, and write $(X/S)^{sm}$ (or X^{sm} if there is no ambiguity) the open subspace of X at which f is smooth.
- If $f\colon X \to S$ is a morphism of schemes, locally of finite presentation, with fibers of pure dimension 1, we call *singular locus* of f, and write $\mathrm{Sing}(X/S)$, the closed subscheme of X cut out by the first Fitting ideal of the sheaf of relative 1-forms of X/S. The complement of $\mathrm{Sing}(X/S)$ in X is precisely $(X/S)^{sm}$.
- Unless specified otherwise, if A is a local ring, we write $\mathfrak{m}_A$ for its maximal ideal, k_A for the its residue field and $\widehat{A}$ for its $\mathfrak{m}_A$-adic completion.