Néron models in high dimension: Nodal curves, Jacobians and tame base change

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Par Thibault POIRET

Modèles de Néron en dimension supérieure: courbes nodales et leurs Jacobiennes, changement de base modérément ramifié

Sous la direction de Bas Edixhoven et Qing LIU

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General introduction

Néron models

Given an integral scheme S with generic point η, a lot of proper and smooth schemes over $k(\eta)$ have no proper and smooth model over S. However, they can sometimes still have a canonical smooth S-model, the Néron model. Néron models were first introduced in 1964 by André Néron in his article [25] for abelian varieties over a Dedekind base scheme. The Néron model of X_η/η is defined as a smooth, separated scheme N/S restricting to X_η over η, satisfying the following universal property, called the *Néron mapping property*: for every smooth scheme $T \to S$ and morphism $\phi_K \colon T_\eta \to X_\eta$, there exists a unique morphism $\phi \colon T \to N$ extending ϕ_K.

The mapping property has a lot of nice consequences, making the Néron model the "best possible smooth model": among other things, it ensures that Néron models are unique up to a unique isomorphism, and inherit a group structure from X_η when the latter has one.

Néron models are always unique, but their existence is not trivially guaranteed. Néron proved in the original article [25] that abelian varieties over the fraction field of an integral Dedekind scheme always have Néron models. Recently, people have taken interest in constructing Néron models in different settings. For example, it was proved in 2013 by Qing Liu and Jilong Tong in [23] that smooth and proper curves of positive genus over a Dedekind scheme always have Néron models. This does not apply to genus 0: Proposition 4.12 of [23] shows that if S is the spectrum of a discrete valuation ring with field of fractions K, then $\mathbb{P}^1_K$ does not have a Néron model over S.

In the cases mentioned above, the Néron model is of finite type. This condition is even often included in the definition of Néron models, in which case authors refer to smooth, separated schemes satisfying the mapping property as *Néron-lft models* (where "*lft*" stands for "locally of finite type"), to emphasize the absence of quasi-compactness hypothesis. The use of this terminology is not systematic anymore in the litterature, so we will use a more flexible definition, essentially equivalent to the one given in the first paragraph. If R is an excellent and strictly henselian discrete valuation ring with field of fractions K, the simplest example of a K-group scheme with a R-Néron(-lft) model that is not of finite type is the multiplicative group $\mathbb{G}_m$ over K. In fact, it is shown in [1] (Theorem 10.2.1 and Theorem 10.2.2) that a smooth commutative K-group G_K has a Néron model if and only if it has no subgroup isomorphic to the additive group $\mathbb{G}_a$, in which case the Néron model is of finite type if and only if G_K has no subgroup isomorphic to the multiplicative group $\mathbb{G}_m$.

Among the concrete applications of the theory of Néron models, we can cite the semi-stable reduction theorem (an abelian variety over the fraction field of a discrete valuation ring acquires semi-abelian reduction after a finite extension of the base field); the Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties; the computation of canonical heights on Jacobians; as well as the linear and quadratic Chabauty methods to determine whether or not a list of rational points on a curve is exhaustive. For a geometric description of the quadratic Chabauty method, see [5]. Parallels can also be drawn to some problems in which Néron models do not explicitly intervene, such as extending the double ramification cycle on the moduli stack of smooth curves to the whole moduli stack of stable curves as in [20]. Here, one is interested in models in which one given section extends, instead of all sections simultaneously, but the two problems are closely related.

Existence in higher dimension

As we can see from the synthesis above, we understand reasonably well when Néron models exist over regular one-dimensional bases, especially for group schemes. Finding criteria for existence of Néron models over higher-dimensional bases and constructing them explicitly when they exist is a more recent and open area of research. In [21], David Holmes exhibits a necessary condition, called *alignment*, for the existence of the Néron model of the generic Jacobian of a nodal curve X over any regular base scheme S. He also proves that alignment is sufficient under additional assumptions of semifactoriality on X. Alignment is a rather strong condition that relates the local structure of X around all singularities appearing in the same cycle of a dual graph of X.

In [27], Giulio Orecchia introduces the *toric-additivity* criterion. Consider an abelian scheme A/U with semi-abelian reduction $\mathcal{A}/S$, where S is a regular base and U the complement in S of a strict normal crossings divisor. Toric-additivity is a condition on the Tate module of A. When A is the generic Jacobian of an S-curve with a nodal model, toric-additivity is sufficient for a Néron model of A to exist. It is also necessary up to some restrictions on the base characteristic. For general abelian varieties, it is proven in [28] that toric-additivity is still sufficient when S is of equicharacteristic zero, and a partial converse holds, i.e. existence of a Néron model implies a weaker version of toric-additivity.

To the author's knowledge, little has been said about Néron models of schemes with no group structure (e.g. relative curves) over bases of higher dimension. The construction of [23] for curves over Dedekind schemes consists in embedding a curve X_K into its Jacobian J_K, resolving the singularities of the scheme-theoretical closure of X_K into the Néron model of J_K, and taking the smooth locus. This does not generalize well to relative curves over bigger base schemes: in this setting, resolution of singularities is not known and regular models are not even known to exist.

Nodal curves, stable curves and log smoothness

Nodal curves can be thought of as curves that are not necessarily smooth, but only allow the simplest type of singularities: the completed localization at a singularity of a nodal curve X_k over an algebraically closed field k is a union of two lines meeting transversally. This simple description of nodal singularities in their fibers permits to define an important combinatorial invariant, the *dual graph*: its vertices are the irreducible components of X_k, and its edges are the nodal singularities, which are attached to the irreducible components they belong to. A great deal of information on a nodal curve can be recovered from just this graph (or these graphs, in the relative case) and the genera of the vertices.

In the relative case, one can nicely describe the local structure of the whole relative curve around a singularity in terms of a certain ideal of an étale local ring of the base, the *singular ideal*. Adjoining to the dual graphs the data of these singular ideals makes them an even better tool to summarize the properties of a nodal curve in a simple combinatorial object.

Nodal curves also arise naturally in the context of logarithmic geometry, i.e. algebraic geometry on the category of log schemes. A log scheme is the data of a scheme X and a map of sheaves of monoids $M \to \mathcal{O}_X$ inducing an isomorphism $M^\times \to \mathcal{O}_X^\times$, where we give $\mathcal{O}_X$ its multiplicative monoid structure. The logarithmic version of smoothness is less restrictive than scheme-theoretic smoothness, and only forces log-smooth curves to have a nodal curve as an underlying scheme.

Working with the whole category of nodal curves can be tricky, as they do not possess a well-behaved moduli stack: for example, if k is an algebraically closed field, the automorphism group of the nodal curve $\mathbb{P}^1_k$ is infinite. Its action on $\mathbb{P}^1_k$ is even 3-transitive, which implies that any nodal k-curve with a component isomorphic to $\mathbb{P}^1$, meeting the rest in only one or two points, also has infinite automorphism group. This forbids the existence of a Deligne-Mumford stack for nodal curves, even of fixed genus. Likewise, a relative nodal curve X/S smooth over a scheme-theoretically dense open subscheme $U \subset S$ can admit non-trivial blowups supported outside of X_U that are still nodal curves (see [2], Proposition 3.6, or the first part of this thesis), which forbids the existence of a separated algebraic stack for nodal curves. To obtain a Deligne-Mumford stack, smooth and proper over $\operatorname{Spec}\mathbb{Z}$, one should work instead with *n-pointed stable curves*, which are the data of a nodal curve X/S and n marked sections on it guaranteeing that the automorphism groups of the geometric fibers are finite étale.

The thesis

This thesis is divided in three parts:

- **Part I**: Nodal curves, dual graphs and resolutions;
- **Part II**: Néron models of nodal curves and their Jacobians;
- **Part III**: Base change of Néron models along finite tamely ramified maps.

Part II heavily relies on part I, while part III only depends on the first section of part II (namely, it only makes use of the definition of Néron models and of their elementary base change properties).

Part I

In part I, we will discuss a high-dimensional variant of the classic *smoothening process*. Typically, when S is the spectrum of a discrete valuation ring, a key step in constructing the Néron model of the generic fiber of a proper S-scheme X/S consists in blowing-up repeatedly in subschemes supported outside of the generic fiber X_K, so that the smooth locus "gets bigger". For example, if S is strictly local, eventually, we want to obtain a new model $X' \to X$ of X_K so that the map from the smooth locus of X' to X is surjective on S-points.

For higher-dimensional S, we can do something similar, but arbitrary choices have to be made in the process: we end up with several "partial smoothenings", whose smooth loci jointly satisfy the surjectivity property we expect. A construction is made by A.J. De Jong in [2], Proposition 3.6, for split curves over a regular base, smooth over the complement of a strict normal crossings divisor. De Jong repeatedly blows up the split curve in irreducible components of its non-smooth locus. We will construct the "partial smoothenings", that we will call *resolutions*, without hypotheses of splitness and without asking for the discriminant locus to be a normal crossings divisor.

The approach of De Jong is not convenient when one wishes to work with non-split curves. It appeared from a discussion with G. Orecchia that this problem could be solved by blowing-up in ideal sheaves of sections through singular points instead of irreducible components of the non-smooth locus. This forces us to work étale-locally on the base, which poses no problem when dealing with Néron models, as they descend even along smooth covers.

Part II

In part II, we consider a regular base scheme S and a nodal curve X/S, smooth over a dense open $U \subset S$, and we exhibit criteria for the existence of Néron models for X_U and for its Jacobian, both times in terms of the (labelled) dual graphs of X.

For Jacobians, we try to investigate the situations that are not covered by the main theorem of [21], i.e. what happens when X satisfies the alignment condition of [21], but is not semifactorial after every smooth base change. We introduce a new condition on the dual graphs of X, *strict alignment*, and we show it is equivalent to alignment of all resolutions of smooth base changes of X. In particular, strict alignment is necessary for a Néron model of the generic Jacobian to exist. We will show it is also sufficient, and we will describe explicitly the Néron model of generic Jacobians of strictly aligned nodal curves.

This thesis is not written in the language of log geometry, but the reader familiar with it can establish a parallel between strict alignment and finiteness of the tropical Jacobian as described in [24], as well as between our blowups and logarithmic modifications of a log curve inducing a given subdivision of the tropicalization. In a future work with David Holmes, Giulio Orecchia and Samouil Molcho, we intend to develop further these correspondences and show that Néron models of Jacobians can be better understood in terms of the logarithmic Picard functor.

Regarding nodal curves themselves, we will make use of a result from [8] about extending rational points of a curve into sections, when the curve has no rational components in any geometric fiber. This naturally leads us to consider the two following questions:

- When does a curve admit a nodal model with no such rational components?
- Can we use the resolutions of such a model to construct a Néron model, following the usual smoothening principles?

After base change to an étale cover, we provide an exhaustive answer to the first question. In [7], it is shown that, under certain restrictions on the genera of the curves considered, there are canonical contraction morphisms from the stack of $n+1$-pointed stable curves to that of n-pointed stable curves. A nice consequence is that if a geometric fiber of a nodal curve X has infinitely many automorphisms, after an étale extension of the base, one can always blow down X so that certain rational components of the geometric fibers are contracted. Repeating the process, we can find a unique stable curve (without markings) birational to X. Among all nodal models, this is the one with the least rational components in its geometric fibers, and we provide criteria for it to have none.

As for the second question, we will see that the answer is almost always negative: most nodal curves admit several non-isomorphic resolutions, and the smooth loci of them all cannot fit in a Néron model without creating separatedness issues. We will show, however, that a "not-necessarily-separated Néron model", i.e. a smooth object with the Néron mapping property (uniqueness included) always exists, and we will give an explicit construction. A Néron model exists if and only if this object is separated, and we will show this is equivalent to the singular locus of the stable model X^{stable} being irreducible, étale-locally on X^{stable}.

Part III

In part III, we study the base change behavior of Néron models under finite, locally free morphisms between regular schemes. We are mostly interested in descent: for example, given a regular base S, a dense open $U \subset S$, and a smooth proper curve X_U over U, if one uses some version of the semistable reduction theorem to find a nodal model of X_U over a finite extension S'/S, can we use the results of part II to recover information about the Néron model of X_U?

In [4], Bas Edixhoven investigates the base change morphism between Néron models of an abelian variety under a finite, tamely ramified extension of discrete valuation rings R'/R. More precisely, if N, N' are the Néron models over R and

R' respectively, he exhibits a filtration of N indexed by integers up to the ramification index of R'/R, and describes explicitly the successive quotients in terms of N'. Several aspects of this filtration (interpretations, applications) are developed in the papers [16] and [6] and in the book [17].

Consider a regular scheme S, a dense open $U \subset S$ and a smooth U-algebraic space X_U. The first obstruction to generalize Edixhoven's results is the following: if $S' \to S$ is a finite, locally free, tamely ramified cover and if the base change of X_U has a Néron model N'/S', it is not obvious a priori that X_U/U has a Néron model N/S. We will prove that this is actually the case and, following the ideas of [4], we will define a finite filtration of N and describe explicitly its successive quotients in terms of twisted Lie algebras of N'.

Notations

Throughout this thesis, we will adopt the following conventions:

- If $f\colon X \to S$ is a morphism of algebraic spaces locally of finite type, we call *smooth locus* of f, and write $(X/S)^{sm}$ (or X^{sm} if there is no ambiguity) the open subspace of X at which f is smooth.
- If $f\colon X \to S$ is a morphism of schemes, locally of finite presentation, with fibers of pure dimension 1, we call *singular locus* of f, and write $\mathrm{Sing}(X/S)$, the closed subscheme of X cut out by the first Fitting ideal of the sheaf of relative 1-forms of X/S. The complement of $\mathrm{Sing}(X/S)$ in X is precisely $(X/S)^{sm}$.
- Unless specified otherwise, if A is a local ring, we write $\mathfrak{m}_A$ for its maximal ideal, k_A for the its residue field and $\widehat{A}$ for its $\mathfrak{m}_A$-adic completion.

Contents

Part I

Nodal curves, dual graphs and resolutions

1 Local structure of nodal curves and their dual graphs

1.1 First definitions

The results of this subsection are mostly either well-known facts about nodal curves, or come from [21]. When the proofs are short enough, we reproduce them for convenience.

Definition 1.1. A *graph* G is a pair of finite sets (V, E), together with a map $f : E \to (V \times V)/\mathcal{S}_2$. We call V the set of vertices of G and E its set of edges. We think of f as the map sending an edge to its endpoints. We call *loop* any edge in the preimage of the diagonal of $(V \times V)/\mathcal{S}_2$. We will often omit f in the notations and write $G = (V, E)$.

Let v, v' be two vertices of G. A *path between v and v' in G* is a finite sequence $(e_1, ..., e_n)$ of edges, such that there are vertices $v_0 = v, v_1, ..., v_n = v'$ satisfying $f(e_i) = (v_{i-1}, v_i)$ for all $1 \leq i \leq n$. We call n the *length* of the path. A *chain* is a path as above, with positive n, where the only repetition allowed in the vertices $(v_i)_{0 \leq i \leq n}$ is $v_0 = v_n$. A *cycle* is a chain from a vertex to itself. The cycles of length 1 of G are its loops.

Let M be a semigroup. A *labelled graph over M* (or *labelled graph* if there is no ambiguity) is the data of a graph $G = (V, E)$ and a map $l\colon E \to M$, called *edge-labelling*. The image of an edge by this map is called the *label* of that edge.

Definition 1.2. Let X be an algebraic space. We call *geometric point* of X a morphism $\operatorname{Spec} \bar{k} \to X$ where the image of $\operatorname{Spec} \bar{k}$ is a point with residue field k, and $\bar{k}$ is a separable closure of k (notice the "separable" instead of "algebraic"). Given a geometric point x' over a point x of X, we will call *étale local ring of X at x'*, and note $\mathcal{O}_{X,x'}$, the strict henselization of $\mathcal{O}_{X,x}$ determined by the residue extension $k(x')/k(x)$. Given two geometric points s, t of X, we say that t is an *étale generization* of s (or that s is an *étale specialization* of t) when the morphism $t \to X$ factors through $\operatorname{Spec} \mathcal{O}_{X,s}$. We will often omit the word "étale" and just call them specializations and generizations.

Definition 1.3. A *curve* over a separably closed field k is a proper morphism $X \to \operatorname{Spec} k$ with X of pure dimension 1. It is called *nodal* if it is connected, and for every point x of X, either X/k is smooth at x, or x is an ordinary double point (i.e. the completed local ring of X at x is isomorphic to $k[[u, v]]/(uv)$).

A *curve* (resp. a *nodal curve*) over a scheme S is a proper, flat, finitely presented morphism $X \to S$ such that all its geometric fibers are curves (resp. nodal curves).

Remark 1.3.1. By [22], Proposition 10.3.7, our definition of nodal curves is unchanged if one defines geometric points with the standard algebraic closures instead of separable closures.

Definition 1.4. Let S be a scheme, s a point of S, and $\bar{s}$ a geometric point mapping to s. We will call *étale neighbourhood of $\bar{s}$ in S* the data of an étale morphism of schemes $V \to S$, a point v of V, and a factorization $\bar{s} \to v \to s$ of $\bar{s} \to s$. Etale neighbourhoods naturally form a codirected system, and we call *étale stalk of S at s* the limit of this system. The étale stalk of S at s is an affine scheme, and we call *étale local ring at s*, and note $\mathcal{O}^{et}_{S,s}$, its ring of global sections. We will sometimes keep the choice of geometric point $\bar{s}$ implicit and abusively call (V, v), or even V, an étale neighbourhood of s in S.

Remark 1.4.1. The étale local ring of S at s is a strict henselization of the Zariski local ring $\mathcal{O}_{S,s}$. The étale local ring of S at $\bar{s}$ is the strict henselization determined by the separable closure $k(s) \to k(\bar{s})$.

1.2 The local structure

Proposition 1.5. *Let S be a locally noetherian scheme and X/S be a nodal curve. Let s be a geometric point of S and x be a non-smooth point of X_s. There exists a unique principal ideal (Δ) of the étale local ring $\mathcal{O}^{et}_{S,s}$, called the* singular ideal of x*, such that*

$$\widehat{\mathcal{O}^{et}_{X,x}} \simeq \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta)$$

Proof. This is [21], Proposition 2.5. □

Remark 1.5.1. The singular ideal of x is generated by a nonzerodivisor if and only if X/S is generically smooth in a neighbourhood of x.

1.3 The dual graph at a geometric point

Definition 1.6. Let X, S be as above and s be a geometric point of S. We define the *dual graph* of X at s to be the graph whose vertices are the irreducible components of X_s, and whose edges are the singular points of X_s: the two vertices an edge connects are the two (not necessarily distinct) irreducible components the singular point belongs to. We also make it a labelled graph over the commutative semigroup of nontrivial principal ideals of $\mathcal{O}^{et}_{S,s}$: the label of an edge is the singular ideal of the corresponding singular point.

When S is strictly local, we will sometimes refer to the dual graph of X at the closed point as simply "the dual graph of X".

Definition 1.7. A nodal curve X over a field k is said to be *split* if its singular points are rational, and all its irreducible components are geometrically irreducible and smooth. A nodal curve is *split* when all its fibers are split. This implies that there is no geometric point of the base over which the dual graph of the curve has loops.

Proposition 1.8. *Let $S' \to S$ be a morphism of locally noetherian schemes, X/S a nodal curve, s a geometric point of S, and s' a geometric point of S' such that $s' \to S$ is a generization of $s \to S$. Let X' be the base change of X to S', Γ and Γ' be the dual graphs, respectively of X at s and of X' at s'.*

Let $R := \mathcal{O}^{et}_{S,s}$; $R' := \mathcal{O}^{et}_{S',s'}$, and ϕ be the natural map $\operatorname{Spec} R' \to \operatorname{Spec} R$. Then Γ' is obtained from Γ by contracting all edges whose label becomes invertible in R', and pulling back the labels of the other edges by ϕ.

In particular, if s' has image s, Γ and Γ' are isomorphic as non-labelled graphs, and the labels of Γ' are obtained by pulling back those of Γ.

Proof. This is [21], Remark 2.12. We reprove it here.

We can reduce to $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ affine and strictly local (i.e. isomorphic to spectra of strictly henselian local rings), of respective closed points s and s'.

Let x be a singular point of X of image s, and Δ be a generator of its (principal) singular ideal. Then we can choose an isomorphism $\widehat{\mathcal{O}_{X,x}} = \widehat{R}[[u,v]]/(uv - \Delta)$.

This yields $\widehat{\mathcal{O}_{X,x}} \otimes_R R' = \widehat{R} \otimes_R R'[[u,v]]/(uv - \Delta)$. The ring $\widehat{R} \otimes_R R'$ is local, with completion $\widehat{R'}$ with respect to the maximal ideal: as desired, if Δ is invertible in R', then X' is smooth above a neighbourhood of x, and otherwise, X' has exactly one singular closed point of image x, with singular ideal $\Delta R'$. □

Example 1.9. With notations as above, in the case $S = S'$, we have defined the *specialization maps* of dual graphs: take s, ξ geometric points of S with s specializing ξ, we have a canonical map from the dual graph at s to the dual graph at ξ, contracting an edge if and only if its label becomes the trivial ideal in $\mathcal{O}^{et}_{S,\xi}$.

It can be somewhat inconvenient to always have to look at geometric points. We can often avoid it as in [18], by reducing to a case in which the dual graphs already make sense without working étale-locally on the base.

1.4 Quasisplitness, dual graphs at non-geometric points

Definition 1.10 (see [18], Definition 4.1)**.** We say a nodal curve $X \to S$ is *quasisplit* if the two following conditions are met:

1. for any point $s \in S$ and any irreducible component E of X_s, there is a smooth section $S \to (X/S)^{sm}$ intersecting E;
2. the singular locus $\operatorname{Sing}(X/S) \to S$ is of the form

$$\coprod_{i \in I} F_i \to S,$$

where the $F_i \to S$ are closed immersions.

Example 1.11. Consider the real conic

$$X = \mathrm{Proj}(\mathbb{R}[x, y, z]/(x^2 + y^2)).$$

It is an irreducible nodal curve over $\mathrm{Spec}\,\mathbb{R}$, but the base change $X_{\mathbb{C}}$ has two irreducible components: X is not quasisplit over $\mathrm{Spec}\,\mathbb{R}$.

On the other hand, consider the real projective curve

$$Y = \mathrm{Proj}(\mathbb{R}[x, y, z]/(x^3 + xy^2 + xz^2)).$$

It has two irreducible components (respectively cut out by x and by $x^2+y^2+z^2$), both geometrically irreducible. The singular locus of $Y/\mathbb{R}$ consists of two $\mathbb{C}$-rational points, with projective coordinates $(0 : i : 1)$ and $(0 : -i : 1)$, at which $Y_{\mathbb{C}}$ is nodal. Since $\mathrm{Sing}(Y/\mathbb{R})$ is not a disjoint union of $\mathbb{R}$-rational points, Y is not quasisplit over $\mathrm{Spec}\,\mathbb{R}$. However, both X and Y become quasisplit after base change to $\mathrm{Spec}\,\mathbb{C}$.

Remark 1.11.1. Our definition of quasisplitness is similar to that of [18], but more restrictive.

Remark 1.11.2. For a quasisplit nodal curve X/S and a point $s \in S$, for any geometric point s' of S of image s, the irreducible components of $X_{s'}$ are in canonical bijection with those of X_s by the first condition defining quasisplitness, and the singular ideals of X at s' come from principal ideals of the Zariski local ring $\mathcal{O}_{S,s}$ by the second condition. Thus we can define without ambiguity the dual graph of X at s: its vertices are the irreducible components of X_s, and it has an edge for every non-smooth point $x \in X_s$, with endpoints the two components x meets, labelled by the principal ideal of $\mathcal{O}_{S,s}$ that gives rise to the singular ideal when we base change to a strict henselization.

From now on, we will call the latter the singular ideal of X at x, and talk freely about the dual graphs of quasisplit curves at (not necessarily geometric) field-valued points of S. This can clash with Definition 1.6 when x is a singular point of a geometric fiber of X/S. Unless specified otherwise, when there is an ambiguity, we always privilege Definition 1.6.

Lemma 1.12. *Quasisplit curves are stable under arbitrary base change.*

Proof. Both conditions forming the definition of quasisplitness are stable under base change. □

Lemma 1.13. *Let S be a strictly local scheme and X/S a nodal curve. Then X/S is quasisplit.*

Proof. There is a section through every closed point in the smooth locus of X/S, so in particular there is a smooth section through every irreducible component of every fiber. Proposition 1.5 implies the map $\mathrm{Sing}(X/S) \to S$ is a disjoint union of closed immersions. □

Lemma 1.14. *Let S be a locally noetherian scheme and X/S a nodal curve. Then every point $s \in S$ has an étale neighbourhood (V, v) such that X_V/V is quasisplit.*

Proof. By Lemma 1.13, and using the fact X/S is of finite presentation, there are étale neighbourhoods V_1 and V_2 of s such that the singular locus of X_{V_1}/V_1 is a disjoint union of closed immersions and there are smooth sections through all irreducible components of all fibers of X_{V_2}/V_2. Both conditions are stable under further étale localization, so the base change of X/S to $V_1 \times_S V_2$ is quasisplit. $\square$

Corollary 1.15. *Let S be a locally noetherian scheme and X/S a nodal curve. There is an étale covering morphism $V \to S$ such that X_V/V is quasisplit.*

Proof. By the preceding lemma, every point $s \in S$ admits an étale neighbourhood V_s such that X_{V_s}/V_s is quasisplit, and we can pick $V = \coprod_{s \in X} V_s$. $\square$

Lemma 1.16. *Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Quasisplitness of X/S gives a factorization*

$$x \to F \to \mathrm{Sing}(X/S) \to X \to S,$$

where $F \to S$ is a closed immersion and $F \to \mathrm{Sing}(X/S)$ is the connected component containing x. Then, there is an étale neighbourhood (V, y) of x in X; two effective Cartier divisors C, D on V; and an isomorphism $V \times_X F = C \times_V D$ such that $V \times_S F$ is the union of C and D.

Proof. Let $\bar{s}$ be a geometric point of S mapping to s, and $\bar{x} = x \times_s \bar{s}$. By Proposition 1.5, we have an isomorphism $\widehat{\mathcal{O}^{et}_{X,\bar{x}}} = \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta)$, where Δ is a generator of the singular ideal of x in $\mathcal{O}^{et}_{S,\bar{s}}$. The base change of F/S to $\mathrm{Spec}\, \mathcal{O}^{et}_{S,\bar{s}}$ is cut out by Δ, and the zero loci C_u of u and C_v of v are effective Cartier divisors on $\widehat{\mathcal{O}^{et}_{X,\bar{x}}}$, intersecting in $\widehat{\mathcal{O}^{et}_{X,\bar{x}}}/(u,v) = F \times_X \mathrm{Spec}\, \widehat{\mathcal{O}^{et}_{X,\bar{x}}}$. The union of C_u and C_v is $\widehat{\mathcal{O}^{et}_{S,\bar{s}}}[[u,v]]/(\Delta, uv)$, so the proposition follows by a limit argument. $\square$

2 Primality and base change

In this section, we will discuss questions of permanence of primality (of an element of an integral regular ring) under étale maps, smooth maps and completions. One of the reasons these considerations are important to talk about Néron models of nodal curves is that a nodal curve over an excellent and regular base admits local sections through a singular point if and only if the label of this singular point is reducible (Lemma 4.3).

An element Δ of a regular local ring R is prime in R^{sh} when the quotient $R/(\Delta)$ is *unibranch* (i.e. has integral strict henselization), so we are interested in questions of permanence of unibranch rings under tensor product. For a more detailed discussion on unibranch rings or counting geometric branches in general, see [29], chapitre IX. We also refer to [12], 23.2.

In [31], Sweedler gives a necessary and sufficient condition for the tensor product of two local algebras over a field to be local. Since we are interested in how étale stalks behave under base-change, what we would like is a suitable sufficient condition for algebras over a strictly local ring. Sweedler's proof adapts well to this situation: this is the object of the following lemma.

Lemma 2.1. *Let R be a strictly henselian local ring, $R \to A$ an integral morphism of local rings with purely inseparable residue extension, and $R \to B$ any morphism of local rings. Then $A \otimes_R B$ is local, and its residue field is purely inseparable over that of B.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $A \otimes_R B$. The map $B \to A \otimes_R B$ is integral so it has the going-up property ([30, Tag 00GU]), therefore the inverse image of $\mathfrak{m}$ in B is a maximal ideal: it must be $\mathfrak{m}_B$. Thus $\mathfrak{m}$ contains $A \otimes_R \mathfrak{m}_B$.

In particular, $\mathfrak{m}$ also contains the image of $\mathfrak{m}_R$ in $A \otimes_R B$: it corresponds to a maximal ideal of $A \otimes_R B/(\mathfrak{m}_R A \otimes_R B)$, that we will still call $\mathfrak{m}$. We have a commutative diagram

$$\begin{array}{ccc} k_R & \longrightarrow & B/\mathfrak{m}_R B \\ \downarrow & & \downarrow \\ A/\mathfrak{m}_R A & \longrightarrow & A \otimes_R B/(\mathfrak{m}_R A \otimes_R B). \end{array}$$

Since $A/\mathfrak{m}_R A$ is local and integral over the field k_R, its maximal ideal m_A is nilpotent and is its only prime. The inverse image of $\mathfrak{m}$ in $A/\mathfrak{m}_R A$ is a prime ideal, so it can only be $\mathfrak{m}_A$. This shows that, as an ideal of $A \otimes_R B$, $\mathfrak{m}$ also contains $\mathfrak{m}_A \otimes_R B$.

Every maximal ideal of $A \otimes_R B$ contains both $\mathfrak{m}_A \otimes_R B$ and $A \otimes_R \mathfrak{m}_B$, so the maximal ideals of $A \otimes_R B$ are in bijective correspondence with those of $k_A \otimes_{k_R} k_B = A \otimes_R B/(\mathfrak{m}_A \otimes_R B + A \otimes_R \mathfrak{m}_B)$. We will now show the latter is local, with purely inseparable residue extension over k_B.

By hypothesis, the extension k_A/k_R is purely inseparable. If k_R has characteristic 0, then $k_A = k_R$ and we are done. Suppose k_R has characteristic $p > 0$. For any $x \in k_A \otimes_{k_R} k_B$, we can write x as a finite sum $\sum\limits_{i \in I} \lambda_i \otimes \mu_i$ with the λ_i, μ_i in k_A, k_B respectively. There is an integer $N > 0$ such that for all i, $\lambda_i^{p^N}$ is in k_R. Therefore $x^{p^N} = \sum\limits_{i \in I} \lambda_i^{p^N} \mu_i^{p^N}$ is in k_B, and x is either nilpotent or invertible. It follows that $k_A \otimes_{k_R} k_B$ is local, with maximal ideal its nilradical, and that its residue field is purely inseparable over k_B as claimed. ☐

Lemma 2.2. *Let $(R, \mathfrak{m})$ be an integral and strictly local noetherian ring, and $R \to R'$ a smooth ring map. Let $\mathfrak{p}$ be a prime ideal of R' containing $\mathfrak{m}R'$. Call $\tilde{R}'$ a strict henselization of $R'_{\mathfrak{p}}$. Then $R'_{\mathfrak{p}}$ is geometrically unibranch, i.e. $\tilde{R}'$ is an integral domain.*

Proof. We know $\tilde{R}'$ is reduced since it is a filtered colimit of smooth R-algebras. Let B, B' be the integral closures of $R, R'_{\mathfrak{p}}$ in their respective fraction fields. The ring $R'_{\mathfrak{p}}$ is integral so by [29], chapitre IX, corollaire 1, $\tilde{R}'$ is an integral domain if and only if B' is local and the extension of residue fields of $R'_{\mathfrak{p}} \to B'$

is purely inseparable. But any smooth base change of B/R remains normal (see [22], Corollary 8.2.25), so $B \otimes_R R'_{\mathfrak{p}}$ is normal as a filtered colimit of normal B-algebras. Moreover, any normal algebra over $R'_{\mathfrak{p}}$ must factor through $B \otimes_R R'_{\mathfrak{p}}$, so we have $B' = B \otimes_R R'_{\mathfrak{p}}$. Applying Lemma 2.1, we find that B' is local and the extension of residue fields of $R'_{\mathfrak{p}} \to B'$ is purely inseparable, which concludes the proof. □

Corollary 2.3. *Let S be a regular scheme, $Y \to S$ a smooth morphism and $y \to Y$ a geometric point. Then for any prime element Δ of $\mathcal{O}^{et}_{S,y}$, the image of Δ in $\mathcal{O}^{et}_{Y,y}$ is prime.*

Proof. Base change to $\operatorname{Spec} \mathcal{O}^{et}_{S,y}/(\Delta)$, replace Y by an affine neighbourhood of y in Y, and apply Lemma 2.2. □

Lemma 2.4. *Let R be a strictly henselian excellent local ring and $\widehat{R}$ its completion with respect to the maximal ideal. Then an element Δ of R is prime in R if and only if it is in $\widehat{R}$.*

Proof. The nontrivial implication is the direct sense. Suppose Δ is prime in R. Since R is excellent, the morphism $R \to \widehat{R}$ is regular. Then by Popescu's thorem ([30, Tag 07GB]), $\widehat{R}$ is a directed colimit of smooth R-algebras. We conclude by writing $R/(\Delta) \to \widehat{R}/(\Delta)$ as a colimit of smooth $R/(\Delta)$-algebras and applying Lemma 2.2. □

Lemma 2.5. *Let S be an excellent and regular scheme, X/S an S-scheme of finite presentation, $\bar{s}$ a geometric point of S, and x a closed point of $X_{\bar{s}}$ with an isomorphism $\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{\mathcal{O}^{et}_{S,\bar{s}}}[[u,v]]/(uv - \Delta)$ for some $\Delta \in \mathfrak{m}_{\bar{s}} \subset \mathcal{O}^{et}_{S,\bar{s}}$. For every $t_1, t_2 \in \mathfrak{m}_{\bar{s}}$ such that $t_1 t_2 = \Delta$, there exists an étale neighbourhood $S' \to S$ of $\bar{s}$ and a section $S' \to X$ through x such that the induced map $\widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{\mathcal{O}^{et}_{S',\bar{s}}}$ sends u, v respectively to a generator of (t_1) and a generator of (t_2).*

Proof. Put $R = \mathcal{O}^{et}_{S,\bar{s}}$ and consider the map $\widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{R}$ that sends u, v to t_1, t_2 respectively. Compose it with $\mathcal{O}^{et}_{X,x} \to \widehat{\mathcal{O}^{et}_{X,x}}$ to get a map $f_0 : \mathcal{O}^{et}_{X,x} \to \widehat{R}$.

For noetherian local rings, quotients commute with completion with respect to the maximal ideal, so two distinct ideals are already distinct modulo some power of the maximal ideal. Let $\prod_{i=1}^{n} \Delta_i^{\nu_i}$ be the prime factor decomposition of Δ in R. Principal ideals of R of the form $(\Delta_i^{\mu_i})$ with $0 \leq \mu_i \leq \nu_i$ are pairwise distinct and in finite number, so there exists some $N \in \mathbb{N}$ such that their images in $R/\mathfrak{m}_R^N$ are pairwise distinct. Since R is henselian and excellent, it has the Artin approximation property ([30, Tag 07QY]), so there exists a map $f \colon \mathcal{O}^{et}_{X,x} \to R$ that coincides with f_0 modulo m_R^N. This f induces a map $\widehat{f} : \widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{R}$. Call a, b the respective images of u, v by $\widehat{f}$, we have $a = t_1$ and $b = t_2$ in $R/\mathfrak{m}_R^N$. But $ab = \Delta$ in $\widehat{R}$ and, by Lemma 2.4, Δ has the same prime factor decomposition in R and $\widehat{R}$, so the only principal ideals of $\widehat{R}$ containing Δ are of the form $(\Delta_i^{\mu_i})$ with $0 \leq \mu_i \leq \nu_i$. By definition of N we get $a\widehat{R} = t_1\widehat{R}$ and $b\widehat{R} = t_2\widehat{R}$. Since

X/S is of finite presentation, f comes from an S-morphism $S' \to X$ where S' is an étale neighbourhood of $\bar{s}$ in S. $\square$

3 Sections of nodal curves

We present a few technicalities regarding sections of a quasisplit nodal curve X/S. There is a double interest in this. First, blow-ups of nodal curves in the ideal sheaves of sections will be an important tool to construct ns-Néron models of smooth curves with nodal reduction. Second, we can view the Néron mapping property as a condition of extension of rational points into sections after smooth base change. Thus, we will be interested in conditions under which a section of a nodal curve factors through the blowing-up in the ideal sheaf of another section.

The basis for this formalism was thought of together with Giulio Orecchia, and these notions should appear again in a future joint work describing the sheaf of regular models of a nodal curve, smooth over the complement of a normal crossings divisor.

3.1 Type of a section

We will define a combinatorial invariant, the *type* of a section, summarizing information about the behavior of said section around the singular locus of a nodal curve X/S. Later on, we will show that sections of all types exist étale-locally on the base (Proposition 3.9) and that the type of a section locally characterizes the blowing-up of X in the ideal sheaf of that section (Corollary 4.7).

Definition 3.1. Let S be a regular scheme and $s \to S$ a geometric point. Let X/S be a nodal curve, smooth over a dense open subscheme U of S. Let x be a singular point of X_s, we call *thickness* of x (in X/S) the image of the singular ideal of x in the monoid $\mathcal{O}_{S,s}^{et}/(\mathcal{O}_{S,s}^{et})^*$ of principal ideals of $\mathcal{O}_{S,s}^{et}$.

Definition 3.2. Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let F be the connected component of $\mathrm{Sing}(X/S)$ containing x. Then the set of connected components of $(X\backslash F) \times_X \operatorname{Spec} \mathcal{O}_{X,x}^{et} \times_S F$ is a pair $\{C, D\}$ (see Proposition 1.5 or Lemma 1.16), on which the Galois group $\mathrm{Aut}_{O_S}(\mathcal{O}_{S,s}^{et}) = \mathrm{Gal}(k(s)^{sep}/k(s))$ acts. If this action is trivial, we say X/S is *orientable at* x, and we call *orientations of* X/S *at* x the ordered pairs (C, D) and (D, C). The scheme-theoretical closures of C and D in $\operatorname{Spec} \mathcal{O}_{X,x}^{et}$ are effective Cartier divisors, and we will often also call them C and D.

Remark 3.2.1. The curve X/S is orientable at x if and only if the preimage of x in the normalization of X_s consists of two $k(s)$-rational points, in which case an orientation is the choice of one of these points. Roughly speaking, this also corresponds to picking an orientation of the edge corresponding to x in the dual graph of X at s. The "roughly speaking" is due to the case of loops: there is an ambiguity on how to orient them. We could get rid of this ambiguity by using a heavier notion of dual graphs (such as the tropical curves often used in log geometry), but this work does not require it.

Remark 3.2.2. Since the base change of X/S to $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ is orientable at x, there is an étale neighbourhood (V,v) of s in S such that X_V/V is orientable at $x \times_s v$.

Lemma 3.3. *Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let x' be a singular point of X/S specializing to x and s' the image of x' in S. Suppose X/S is orientable at x, then it is orientable at x'. Moreover, in that case, there is a canonical bijection between orientations at x and orientations at x'.*

Proof. Suppose X/S is orientable at x and let (C_1, C_2) be an orientation at x. Let F be the connected component of $\operatorname{Sing}(X/S)$ containing x and x'. Pick a (non-canonical) isomorphism

$$\mathcal{O}_{X,x}^{et} = \mathcal{O}_{S,s}^{et}[[u,v]]/(uv - \Delta). \tag{1}$$

Permuting u and v if necessary, we can assume C_1 and C_2 come from the zero loci of u and v respectively. Pick a (canonical up to the residue extension $k(s)^{sep} \to k(s')^{sep}$) factorization

$$\operatorname{Spec} \mathcal{O}_{S,s'}^{et} \to \operatorname{Spec} \mathcal{O}_{S,s}^{et} \to S.$$

Then, tensoring by $\mathcal{O}_{S,s'}^{et}$ in equation (1), we get an isomorphism

$$\mathcal{O}_{X,x'}^{et} = \mathcal{O}_{S,s'}^{et}[[u,v]]/(uv - \Delta),$$

and the two connected components C_1' and C_2' of $(X \backslash F) \times_X \operatorname{Spec} \mathcal{O}_{X,x'}^{et} \times_S F$ come from the zero loci of u and v. Therefore, in order to prove that X/S is orientable at x' and that the order (C_1', C_2') is canonically induced by the order (C_1, C_2), we only need to know that the action of $G = \operatorname{Aut}_{\mathcal{O}_{S,s}^{et}}(\mathcal{O}_{S,s'}^{et})$ on $\{C_1', C_2'\}$ is trivial, which is true since the action of G on $\mathcal{O}_{S,s'}^{et}[[u,v]]/(uv - \Delta)$ preserves the zero loci of u and v. ☐

From now on, given an orientation (C_1, C_2) at a singular point x, we will also write (C_1, C_2) for the induced orientation at a singular generization of x.

Definition 3.4 (type of a section)**.** Let X/S be a quasisplit nodal curve with S regular. Suppose X is smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s at which X/S is orientable. Pick an orientation (C_1, C_2) at x and an isomorphism

$$\widehat{\mathcal{O}_{X,x}^{et}} = \widehat{\mathcal{O}_{S,s}^{et}}[[u,v]]/(uv - \Delta_x),$$

where C_1 corresponds to $u = 0$ and Δ_x is a generator of the singular ideal of x in $\mathcal{O}_{S,s}^{et}$. We call *type at x* any element of the monoid $\mathcal{O}_{S,s}^{et}/(\mathcal{O}_{S,s}^{et})^*$ strictly comprised between 1 and the thickness of x (for the order induced by divisibility). There are only finitely many types at x, given by the association classes of the strict factors of Δ_x in $\mathcal{O}_{S,s}^{et}$ (which is a unique factorization domain).

Let σ be a section of X/S through x. It induces a morphism

$$\widehat{\sigma\#} : \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta_x) \to \widehat{\mathcal{O}^{et}_{S,s}}$$

By Lemma 2.4, Δ_x has the same prime factor decomposition in $\mathcal{O}^{et}_{S,s}$ and in $\widehat{\mathcal{O}^{et}_{S,s}}$, so there is a canonical embedding of the submonoid of $\widehat{\mathcal{O}^{et}_{S,s}}/(\widehat{\mathcal{O}^{et}_{S,s}})^*$ generated by the factors of Δ_x into $\mathcal{O}^{et}_{S,s}/(\mathcal{O}^{et}_{S,s})^*$. We call *type of σ at x relatively to (C_1, C_2)* the image of u in in $\mathcal{O}^{et}_{S,s}/(\mathcal{O}^{et}_{S,s})^*$. It is a type at x, and since X/S is orientable at x, it does not depend on our choice of isomorphism $\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{\mathcal{O}^{et}_{S,s}}[[u,v]]/(uv - \Delta_x)$ as long as C_1 is given by $u = 0$. When they are clear from context, we will omit x and (C_1, C_2) from the notation and just call it the *type of* σ. In general, given a type T at x, there need not exist a section of type T.

If the type of a section σ at x relatively to (C_1, C_2) is equal to the type of a section σ' at x relatively to (C_2, C_1), we say σ and σ' are of *opposite type* at x.

Lemma 3.5. *Let X/S, s, x, (C_1, C_2) and U be as in Definition 3.4, and σ be a section $S \to X$ of type T at x. Let s' be a generization of s. Then there is a singular point of $X_{s'}$ specializing to x if and only if the thickness of x does not map to 1 in $\mathcal{O}^{et}_{S,s'}/(\mathcal{O}^{et}_{S,s'})^*$. Suppose it is the case and write x' this singular point, then*

- *if the image of T in $\mathcal{O}^{et}_{S,s'}/(\mathcal{O}^{et}_{S,s'})^*$ is either 1 or the thickness of x', then $\sigma(s')$ is a smooth point of $X_{s'}$;*
- *otherwise, the image of T is a type at s', that we still write T, and σ is of type T at x' relatively to (C_1, C_2).*

Proof. By Proposition 1.5, if the thickness of x maps to 1 in $\mathcal{O}^{et}_{S,s'}/(\mathcal{O}^{et}_{S,s'})^*$, then all points of X mapping to s' and specializing to x are S-smooth, and otherwise there is a unique singular point x' of $X_{s'}$ specializing to s. Suppose the latter holds, then by Lemma 3.3, (C_1, C_2) induces an orientation of X/S at x', and the lemma follows from the definition of types. $\square$

Remark 3.5.1. One can think of the thickness of x as the relative version of a length, and of the type of a section σ relatively to an orientation (C_1, C_2) as a measure of the intersection of σ with C_1, seen as an effective Cartier divisor locally around x as in Lemma 1.16. In other words, the type is a measure of "how close to C_1" the section is.

Proposition 3.6. *Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let σ and σ' be two S-sections of X. Then the union of $(X/S)^{sm}$ with the set of singular points x of X/S at which σ and σ' have the same type (resp. opposite types) is an open subscheme of X.*

Proof. Since the singular locus $\mathrm{Sing}(X/S)$ is finite over S, every singular point x of X/S has a Zariski-open neighbourhood $V \subset X$ containing only S-smooth points and singular generizations of x. Thus, since the smooth locus of X/S is open in X, the proposition reduces to the following claim: if σ and σ' have the same type (resp. opposite types) at a singular geometric point $x \to X$, then they have the same type (resp. opposite types) at every singular generization of x. This claim is true by Lemma 3.5. □

Definition 3.7. The open subschemes of X described in Proposition 3.6 above are called respectively the *same type locus* and the *opposite type locus* of σ and σ'.

3.2 Admissible neighbourhoods

Here we will show that, when one works étale-locally on the base (in a sense that we will make precise), one can always assume sections of all types exist.

Definition 3.8. Let S be a regular scheme and X/S a nodal curve, smooth over a dense open U of S. Let s be a point of S and (V, v) an étale neighbourhood of s in S. We say (V, v) is an *admissible neighbourhood of* s (relatively to X/S) when the following conditions are met:

1. X_V/V is quasisplit;
2. X_V/V is orientable at all singular points of X_v;
3. for any singular point x of X_v with singular ideal $(\Delta_x) \subset \mathcal{O}^{et}_{S,s}$, all prime factors of Δ_x in $\mathcal{O}^{et}_{S,s}$ come from global sections of $\mathcal{O}_V$;
4. for every singular point x of X_v (however oriented), there are sections $V \to X_V$ of all types at x.

When $\bar{s} \to S$ is a geometric point with image s and (V, v) an admissible neighbourhood of s with a factorization $\bar{s} \to v$, we will also sometimes call V an *admissible neighbourhood of* $\bar{s}$.

Remark 3.8.1. In the situation of Definition 3.8, if S is strictly local, then it is an admissible neighbourhood of its closed point.

Proposition 3.9. *Let X/S be a nodal curve, where S is a regular and excellent scheme. Then any point $s \in S$ has an admissible neighbourhood.*

Proof. Replacing S by an étale neighbourhood, we can assume X/S is quasisplit (for example using Lemma 1.13 and the fact X/S is of finite presentation). By Remark 3.2.2 and since X_s has finitely many singular points $x_1, ..., x_n$, we can assume X/S is orientable at all the x_i. Each x_i has only finitely many prime factors in its singular ideal in $\mathcal{O}^{et}_{S,s}$, so we can shrink S again into a neighbourhood satisfying condition 3. of the definition of admissibility. The fact this V can be shrinked again until it meets all four conditions follows from Lemma 2.5. □

Remark 3.9.1. If (V, v) is an admissible neighbourhood of a point s of S, then V need not be an admissible neighbourhood of all of its points (even if V is strictly local with closed point v, condition 3. of the definition may fail, see Example 6.10). Thus, it is not easy a priori to find a good global notion of admissible cover.

The next proposition states that admissible neighbourhoods are compatible with smooth (and not just étale) morphisms.

Proposition 3.10. *Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let $Y \to S$ be a smooth morphism and $y \to Y$ a geometric point. Let V be an admissible neighbourhood of y in S, then $V \times_S Y$ is an admissible neighbourhood of y in Y.*

Proof. This follows from Corollary 2.3 and the definition of admissible neighbourhoods. □

4 Refinements and resolutions

This section is dedicated to techniques aiming at constructing inductively nodal models of a smooth curve with prime singular ideals, starting from an arbitrary nodal model. When Néron models are concerned, the interest of nodal models with prime labels lies in two facts: they are locally factorial, which is a crucial hypothesis in the existence result of [21] for Néron models of Jacobians, and all their sections factor through their smooth locus, which will allow us to construct Néron models of curves as gluings of smooth loci of nodal models. We will now make these two statements precise and prove them, and the consequences will be developped in part II.

4.1 Arithmetic complexity and motivation for refinements

We start by defining what will be our recursion parameter, the *arithmetic complexity* of a nodal curve.

4.1.1 Arithmetic complexity

Definition 4.1. Let M be the free commutative semigroup over a set of generators G. We call *word length* of $m \in M$, and note $wl(m)$, the (unique) $n \in \mathbb{N}^*$ such that we can write $m = \prod_{i=1}^{n} g_i$ with all the g_i in G.

Given a labelled graph $\Gamma = (V, E, l)$ over M and an edge $e \in E$, we call *arithmetic complexity* of e and note n_e the natural integer $wl(l(e)) - 1$. We call *arithmetic complexity* of Γ and note n_Γ the sum of the arithmetic complexities of all its edges.

Given a nodal curve X/S where $S = \operatorname{Spec} R$ is a local unique factorization domain, the semigroup of nontrivial principal ideals of R is the free commutative semigroup over the set of primes of height 1. From now on, we will talk freely about arithmetic complexities of edges of dual graphs, always implicitly referring to this set of generators.

Thus, when X/S is quasisplit, we define the *arithmetic complexity* of a closed singular point x, noted n_x, as the arithmetic complexity of the corresponding edge of the dual graph: If S is local, we define the *arithmetic complexity* of X, noted n_X, to be that of its dual graph.

Note that X is of arithmetic complexity 0 if and only if every singular ideal is prime: it is an integer measuring "how far away from being prime" the singular ideals are. Arithmetic complexity is not stable even under étale base change.

4.1.2 Factoriality of completed étale local rings

Lemma 4.2. *Let R be a regular complete local ring, and Δ be an element of $\mathfrak{m}_R$. Let $\widehat{A} = R[[u,v]]/(uv - \Delta)$, then $\widehat{A}$ is a unique factorization domain if and only if Δ is prime in R.*

Proof. Suppose that $\widehat{A}$ is a unique factorization domain, and let d be a prime factor of Δ in R. Call S the complement of the prime ideal (u,d) in $\widehat{A}$. Let $\mathfrak{p}$ be a nonzero prime ideal of $S^{-1}\widehat{A}$. Then, p contains a nonzero element $x = ux_u + x_v$, with x_u and x_v in $R[[u]]$ and $R[[v]]$ respectively. Since $\mathfrak{p} \neq S^{-1}\widehat{A}$, we have $d|x_v$. Call n and m respectively the maximal elements of $\mathbb{N}^* \cup \{+\infty\}$ such that $u^n|ux_u$ and $d^n|x_v$. Since x is nonzero, we know either n or m is finite. If $n \leq m$, then $v^n x = \Delta^n \frac{x_u}{u^{n-1}} + d^n \frac{v^n x_v}{d^n}$ is in $\mathfrak{p}$, and is associated to d^n in $S^{-1}\widehat{A}$, so we obtain $d \in \mathfrak{p}$, from which it follows that $\mathfrak{p} = (u,d)$. If $m < n$, a similar argument shows that $\mathfrak{p}$ contains u^m and thus equals (u,d). Therefore, $S^{-1}\widehat{A}$ has Krull dimension one, i.e. (u,d) has height 1 in $\widehat{A}$. Since $\widehat{A}$ is a unique factorization domain, it follows that (u,d) is principal in it, from which we deduce that Δ and d are associated in $\widehat{A}$. In particular, Δ is prime in R.

The interesting part is the converse: let us assume that Δ is prime in R. We want to show that $\widehat{A}$ is a unique factorization domain. We first prove that $A := R[u,v]/(uv - \Delta)$ is a unique factorization domain: let p be a prime ideal of A of height 1, we have to show p is principal in A. We observe that u is a prime element of A, since the quotient $A/(u) = R/(\Delta)[v]$ is an integral domain. Therefore, if p contains u, then $p = (u)$ is principal. Otherwise, p gives rise to a prime ideal of height 1 in $A_u := A[u^{-1}]$, which is principal since $A[u^{-1}] = R[u, u^{-1}]$ is a unique factorization domain. In that case, write $pA_u = fA_u$ for some $f \in A_u$. Multiplying by a power of the invertible u of A_u, we can choose the generator f to be in $A \backslash uA$. Since p is a prime ideal of A not containing u, we know p contains f and thus fA. We will now prove the reverse inclusion. Let x be an element of p. The localization $pA_u = fA_u$ contains x, so x satisfies a relation of the form $u^n x = fy$ for some $n \in \mathbb{N}$ and some $y \in A$. But since u is prime in A, we know u^n divides y and x is in fA.

Now, we will deduce the factoriality of $\widehat{A}$ from that of A. The author would like to thank Ofer Gabber for providing the following proof. Let q be a prime ideal of $\widehat{A}$ of height 1, we will show q is principal. We put $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, $\widehat{X} = \operatorname{Spec} \widehat{A}$, and $Z = \operatorname{Spec}(\widehat{A}/q)$, so that Z is a prime Weil divisor on $\widehat{X}$. Let η, η' be the generic points of the respective zero loci of u, v in the closed fiber $\operatorname{Spec} k_R[[u,v]]/(uv)$ of $\widehat{X} \to S$. Since u and v are prime elements of $\widehat{A}$, we can once again assume Z contains neither η nor η'. It follows that the closed fiber of $Z \to S$ is of dimension 0: the morphism $Z \to S$ is quasi-finite, hence finite by [10], chapter 0, 7.4. A fortiori, $\widehat{A}/q$ is finite over A, so by Nakayama's lemma, the morphism $A \to \widehat{A}/q$ is surjective. Call p its kernel. Then $A/p = \widehat{A}/q$ is $\mathfrak{m}_A$-adically complete and separated, so it maps isomorphically to its completion $\widehat{A}/p\widehat{A}$. The prime ideal p of A is of height 1 since $\widehat{X} \to X$ is a flat map of normal noetherian schemes. Therefore, p is principal in A, and $q = p\widehat{A}$ is principal in $\widehat{A}$. □

4.1.3 Factoring sections through the smooth locus

Lemma 4.3. *Let X/S be a quasisplit nodal curve, where $S = \operatorname{Spec} R$ is a regular, strictly local and excellent scheme. Let σ be a section of X/S. Let $s \in S$ be the closed point and $x = \sigma(s)$, then x is either smooth over S, or singular of arithmetic complexity ≥ 1.*

Proof. Suppose by contradiction that x is singular of arithmetic complexity 0. The section σ factors through $\operatorname{Spec} \mathcal{O}_{X,x}$, and gives rise to a $\operatorname{Spec} \widehat{R}$-section of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$. As $\widehat{\mathcal{O}_{X,x}}$ is of the form $\widehat{R}[[u,v]]/(uv-\Delta)$ for some prime $\Delta \in R$, this section is given by a morphism $\widehat{R}[[u,v]]/(uv-\Delta) \to \widehat{R}$, which is fully described by the images of u and v. But Δ is prime in $\widehat{R}$ by Lemma 2.4, so either u or v has invertible image in $\widehat{R}$. Thus the image of s factors through the complement of one of the two irreducible components of $\operatorname{Spec} \widehat{X}_{s,x}$, a contradiction. □

4.2 Refinements of graphs

Now, to reap the benefits of the properties of nodal curves with prime labels, all we need is an algorithm that takes a generically smooth nodal curve as an input, and returns a nodal curve, birational to the first, with strictly lower arithmetic complexity.

Definition 4.4. As in [21], Definition 3.2, for a graph $\Gamma = (V, E, l)$ with edges labelled by elements of a semigroup M, we call *refinement* of Γ the data of another labelled graph $\Gamma' = (V', E', l')$ labelled by M and two maps

$$E' \to E$$
$$V' \to E \coprod V$$

such that:

- every vertex v in V has a unique preimage v' in V';
- for every edge $e \in E$ with endpoints $v_1, v_2 \in V$, there is a chain $C(e)$ from v_1', v_2' in Γ' such that the preimage of $\{e\}$ in $V' \coprod E'$ consists of all edges and intermediate vertices of $C(e)$;
- for all $e \in E$, the length of e is the sum of the lengths of all edges of $C(e)$.

We will often keep the maps implicit in the notation, in which case we call Γ' a *refinement* of Γ and write $\Gamma' \preceq \Gamma$. We say Γ' is a *strict refinement* of Γ and note $\Gamma' \prec \Gamma$, if in addition the map $E' \to E$ is not bijective.

Remark 4.4.1. Informally, a refinement of a graph is obtained by "replacing every edge by a chain of edges of the same total length". Suppose $\Gamma' \preceq \Gamma$, then $\Gamma' \prec \Gamma$ if and only if at least one of the chains $C(e)$ is of length ≥ 2, i.e. if and only if Γ' has strictly more edges than Γ.

Now we want to blow up X in a way that does not affect X_U, but refines the dual graph. We will define *refinements* of curves (Definition 4.5). We can obtain any refinement of a dual graph of X by iterating these refinements of curves, but they exist only étale-locally on the base.

4.3 Refinements of curves

Definition 4.5. Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s at which X/S is orientable, and (C, D) an orientation of X/S at x. Let T be a type at x. We will call *T-refinement* of X (at x, relatively to (C, D)) the blowing-up of X in the sheaf of ideals of a section $S \to X$ through x, of type T. We will often omit x and (C, D) in the notation and just call these T-refinements of X.

A map $X' \to X$ is called a *refinement* if it is a T-refinement for some such $X, x, (C, D), T$.

Remark 4.5.1.

- If S is excellent, then any geometric point $s \in S$ has an admissible neighbourhood V by Proposition 3.9, so X_V/V has a T-refinement for any type T at any singular point of X_s.
- Consider any morphism $S' \to S$ where S' is still regular (e.g. any smooth map $S' \to S$). Let x be a singular point of X and x' a singular point of X' of image x. Then any orientation of X/S at x pulls back to an orientation of $X_{S'}/S'$ at x'; any type T at x pulls back to a type T' at x'; and any T-refinement pulls back to a T'-refinement.
- Let $x \in X$ be a singular point at which X/S is orientable, and y a generization of x. By Lemma 3.5, for any type T at x and any T-refinement $X' \to X$, either T corresponds to a type (still noted T) at y, in which case $X' \to X$ is a T-refinement at y, or T becomes trivial at y, in which case $X' \to X$ restricts to an isomorphism above a Zariski neighbourhood of y.

Lemma 4.6. *Let $f : X \to S$ be a quasisplit nodal curve with S regular and excellent. Suppose X is smooth over some dense open $U \subset S$. Let $\sigma\colon S \to X$ be a section and $\phi\colon X' \to X$ the blow-up in the ideal sheaf of σ.*

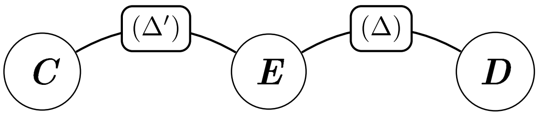
Then ϕ is an isomorphism above the complement in X of the intersection of $\mathrm{Sing}(X/S)$ *with the image of σ. In particular, it is an isomorphism above the smooth locus of X/S, which contains X_U, so X' is a model of X_U.*

Moreover, X' is a nodal curve, and its dual graphs are refinements of those of X. More precisely, let s be a point of S and suppose $\sigma(s)$ is a singular point x of X_s. Choose an orientation (C,D) of $X_{\mathcal{O}^{et}_{S,s}}$ at x, and call T the type of σ at x relatively to (C,D). Then, the singular ideal of x in $\mathcal{O}^{et}_{S,s}$ is generated by

$$\Delta_x = \Delta\Delta',$$

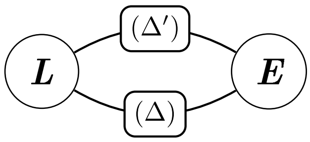
where Δ and Δ' are lifts to $\mathcal{O}^{et}_{S,s}$ of T and of the opposite type of T respectively. Let Γ, Γ' be the respective dual graphs of X and X' at s, and let e be the edge of Γ corresponding to x. Then e has label generated by $\Delta_x = \Delta\Delta'$, and one obtains Γ' from Γ as follows:

- *if e is not a loop, then C and D come from two distinct irreducible components of X_s (that we still call C and D). In that case, Γ' is obtained from Γ by replacing e by a chain*

  where the strict transforms of C and D are still called C and D, and E is the inverse image of x.

- *if e is a loop, i.e. x belongs to only one component L of X_s, then Γ' is obtained from Γ by replacing e by a cycle*

  where the strict transform of L is still called L and E is the inverse image of x.

Proof. The ideal sheaf of σ is already Cartier above the smooth locus of X/S and outside the image of σ, so by the universal property of blow-ups ([30, Tag 0806]), we only need to describe ϕ above the étale localizations $\operatorname{Spec} \mathcal{O}^{et}_{X,x}$, where $x, s, (C,D)$ are as in the statement of the lemma. We can assume $S = \operatorname{Spec} R$ is strictly local, with closed point s. Pick an isomorphism

$$\widehat{\mathcal{O}^{et}_{X,x}} = \widehat{R}[[u,v]]/(uv - \Delta\Delta')$$

such that C is locally given by $u = 0$. The map

$$\widehat{\sigma} \colon \widehat{\mathcal{O}^{et}_{X,x}} \to \widehat{R}$$

yielded by σ sends u to a generator of $\Delta\widehat{R}$ and v to a generator of $\Delta'\widehat{R}$. Scaling u and v by a unit of $\widehat{R}$ if necessary, we can assume $\widehat{\sigma}(u) = \Delta$ and $\widehat{\sigma}(v) = \Delta'$.

The completed local rings of $\operatorname{Spec}\mathcal{O}^{et}_{X,x} \times_X X'$ can be computed using the blowing-up of the algebra $B := R[u,v]/(uv - \Delta\Delta')$ in the ideal $(u-\Delta, v-\Delta')$ (since the completion of B at $(u,v,\mathfrak{m}_R)$ is $\widehat{\mathcal{O}_{X,x}}$).

The latter is covered by two affine patches:

- the patch where $u-\Delta$ is a generator, given by the spectrum of
$$R[u,v,\alpha]/((v-\Delta') - \alpha(u-\Delta), u\alpha + \Delta') \simeq R[u,\alpha]/(u\alpha + \Delta')$$
since, in the ring $R[u,v,\alpha]/((v-\Delta') - \alpha(u-\Delta))$, the element $uv - \Delta\Delta'$ is equal to $(u-\Delta)(u\alpha + \Delta')$
- and the patch where $v - \Delta'$ is a generator, where we obtain symmetrically the spectrum of
$$R[v,\beta]/(v\beta + \Delta)$$

with the obvious gluing maps. Thus we see that X' remains nodal, and that the edge e of Γ (of label $(\Delta\Delta')$) is replaced in Γ' by a chain of two edges, one labelled (Δ) and one labelled (Δ'). It also follows from this description that the strict transform of C (resp. D) in $X' \times_X \operatorname{Spec}\widehat{\mathcal{O}^{et}_{X,x}}$ contains the singularity of label (Δ') (resp. (Δ)). □

Corollary 4.7. *With the same hypotheses and notations as in Lemma 4.6, for any two sections σ, σ' of X/S, the blow-ups $Y \to X$ and $Y' \to X$ in the respective ideal sheaves of σ and σ' are canonically isomorphic above the same type locus of σ and σ' in X.*

Proof. It suffices to exhibit, for any point $s \to S$ and any singular point x of X_s such that $\sigma(s) = \sigma'(s) = x$ and σ, σ' have the same type T at x, a Zariski neighbourhood V of x in X and an isomorphism $Y \times_X V \to Y' \times_X V$ compatible with the canonical identifications $Y \times_X X^{sm} = X^{sm} = Y' \times_X X^{sm}$. Since X, Y, Y' are of finite presentation over S, this can be done assuming $S = \operatorname{Spec} R$ is strictly local, with closed point s. Using the universal property of blow-ups ([30, Tag 0806]), we reduce to proving that the pull-back of the ideal sheaf of σ' (resp. σ) to Y (resp. Y') is Cartier. The proofs are symmetric, so we will only show that the pull-back to Y of the ideal sheaf of σ' is Cartier. This, in turn, reduces to proving that the ideal sheaf of σ' in $\operatorname{Spec}\widehat{\mathcal{O}^{et}_{X,x}}$ becomes Cartier in $Y \times_X \operatorname{Spec}\widehat{\mathcal{O}^{et}_{X,x}}$. Pick an isomorphism

$$\widehat{A} := \widehat{R}[[u,v]]/(uv - \Delta_x) = \widehat{\mathcal{O}^{et}_{X,x}},$$

where $\Delta_x \in R$ is a generator of the singular ideal of x. The map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ sends u, v to elements Δ, Δ' of $\widehat{R}$ with $\Delta\Delta' = \Delta_x$. Since σ and σ' have the same type at x, there is a unit $\lambda \in \widehat{R}^\times$ such that the map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ' sends u and v to $\lambda\Delta$ and $\lambda^{-1}\Delta'$ respectively. We have reduced to proving that the sheaf given by the ideal $(u-\lambda\Delta, v-\lambda^{-1}\Delta')$ of $\widehat{A}$ becomes Cartier in the blow-up of $\widehat{A}$ in $(u-\Delta, v-\Delta')$. Put

$$A = \widehat{R}[u,v]/(uv-\Delta\Delta'),$$

then it is enough to prove that the ideal $I = (u-\lambda\Delta, v-\lambda^{-1}\Delta')$ of A becomes invertible in the two affine patches (as described in the proof of Lemma 4.6) forming the blowing-up of A in $(u-\Delta, v-\Delta')$. By symmetry, we only check it in the patch generated by $u-\Delta$, which is the spectrum of

$$A_1 = \widehat{R}[u,\alpha]/(u\alpha+\Delta'),$$

where v maps to $\Delta'+\alpha(u-\Delta)$. We have $I = (u-\lambda\Delta, \lambda v-\Delta')$, and in A_1 we can write

$$\begin{aligned}\lambda v-\Delta' &= \lambda(\Delta'+\alpha(u-\Delta))+u\alpha\\ &= -\lambda\alpha\Delta+u\alpha\\ &= \alpha(u-\lambda\Delta).\end{aligned}$$

Thus, the preimage of I in A_1 is the invertible ideal $(u-\lambda\Delta)$, and we are done. □

4.4 Resolutions of nodal curves

Lemma 4.8. *Let Γ, Γ' be two labelled graphs over a free commutative semigroup. If $\Gamma' \preceq \Gamma$ (Definition 4.4), then $n_{\Gamma'} \leq n_\Gamma$. If $\Gamma' \prec \Gamma$, then $n_{\Gamma'} < n_\Gamma$.*

Proof. Suppose $\Gamma' \preceq \Gamma$. Then, by definition, the sum of lengths of edges of Γ is equal to the sum of lengths of edges of Γ' and Γ' has at least as many edges as Γ, so $n_{\Gamma'} \leq n_\Gamma$.

If equality holds in the latter, then Γ' and Γ have the same number of edges, which, combined with the fact $\Gamma' \preceq \Gamma$, implies they are isomorphic. □

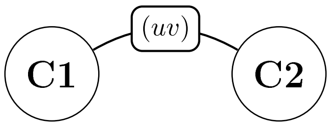
Now we want, starting from X, to find a model of X_U "refining Γ_s as much as possible", in the sense that it will be of arithmetic complexity 0 and its dual graph will be a refinement of Γ_s. This model's alignment will then determine the existence of a Néron model for the Jacobian of X_U. We do it following the ideas of [2], proposition 3.6, as follows:

Definition 4.9. Let X be a nodal curve over a regular and strictly local scheme S with closed point s, smooth over a dense open subscheme $U \subset S$. We call *resolution of* X any nodal S-model X' of X_U, obtained by a finite sequence of refinements, and of arithmetic complexity 0.

Proposition 4.10. *With notations and hypotheses as above, X admits a resolution.*

Proof. The base S is strictly local so it is an admissible neighbourhood of s. Take $X' \to X$ a finite sequence of refinements minimizing $n_{X'}$, then $X' \to X$ is a resolution. Indeed, suppose it was not, then there would be a closed singular point x of X' of arithmetic complexity ≥ 1. There would exist a type T at x, and a T-refinement $X'' \to X$. We would have $n_{X''} < n_{X'}$, a contradiction. □

Remark 4.10.1. Resolutions are not unique in general. For example, consider $S = \operatorname{Spec} \mathbb{C}[[u, v]]$, and suppose X is a nodal curve over S with dual graph



There are two types at the closed singular point x of X/S with respect to (C_1, C_2), namely the class T of u and the class T' of v. The T-refinement and the T'-refinement of X are both resolutions, but they are not isomorphic as models of X_U (they are not even isomorphic as schemes as soon as $C1, C2$ are not isomorphic, e.g. of distinct genera).

Definition 4.11. Let X be a quasisplit nodal curve over a regular scheme S, smooth over a dense open subscheme $U \subset S$. Let s be a point of S. We say X/S is *square-free at s* when all labels of edges of the dual graph of X' at s are square-free. We say X is *square-free* if it is square-free at every point of S.

Remark 4.11.1. In the definition above, we consider the dual graphs at points of S labelled by ideals of the Zariski local rings, see Remark 1.11.2. However, if $R \to R^{sh}$ is a strict henselization of a regular local ring, any square-free element $\Delta \in R$ has square-free image in R^{sh}, so the definition would be unchanged if we asked for the labels of the dual graphs at geometric points (which are ideals of the étale local rings of S) to be square-free.

Definition 4.12. Let X be a quasisplit nodal curve over a regular scheme S, smooth over a dense open subscheme $U \subset S$. Let s be a point of S. We call *partial resolution of X at s* any map $X' \to X$, composition of a finite number of refinements, such that X' is square-free at s. We will call *partial resolution of X over S*, or just *partial resolution of X* if there is no ambiguity, a map $X' \to X$ that is a partial resolution at every point of S.

Remark 4.12.1. The property "being a square-free principal ideal of the regular local ring R" is preserved by tensor product with R' for any morphism of local rings $R \to R'$ that is a directed colimit of étale morphisms. Therefore, a square-free quasisplit nodal curve remains square-free after base change to any codirected limit of étale maps, e.g. a localization or a strict localization. In particular, for any point $s \in S$ and any geometric point $\bar{s}$ above s, $X' \to X$ is a partial resolution at s if and only if $X' \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,\bar{s}} \to X \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,\bar{s}}$ is a partial resolution at the closed point.

Lemma 4.13. *Let X/S be a quasisplit nodal curve, smooth over a dense open $U \subset S$, with S regular. Then, the set of points $s \in S$ at which X/S is square-free is open in S.*

Proof. By quasisplitness, the singular locus of X/S is a disjoint union of closed immersions cut out by locally principal ideals $I_1, ..., I_n$ of $\mathcal{O}_S$. Thus, X is square-free at a point t if and only if the quotient $\mathcal{O}_S/I_i$ is reduced at t for every i, which is an open condition on S. □

Proposition 4.14. *Let X/S be a nodal curve, smooth over a dense open $U \subset S$, with S excellent and regular. Then, every point $s \in S$ admits an étale neighbourhood $V \subset S$ such that X_V/V has a partial resolution.*

Proof. Let s be a point of S, we will show it has such a neighbourhood. Shrinking S if necessary, we assume S is an admissible neighbourhood of s. If the arithmetic complexity of X at s is not 0, then there is always a refinement $X' \to X$ such that the arithmetic complexity of X' at s is strictly lower than that of X, so by induction we may assume all labels appearing in the dual graph of X at s are prime. In particular, X is square-free at s. The proposition now follows immediately from Lemma 4.13. □

Part II

Néron models of nodal curves and their Jacobians

5 Generalities about Néron models

5.1 Definitions

Definition 5.1. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a U-algebraic space. An *S-model* of Z (or just *model* if there is no ambiguity) is an S-algebraic space X together with an isomorphism $X_U = Z$. A *morphism of S-models* between two models X and Y of Z is an S-morphism $X \to Y$ that commutes over U with the given isomorphisms $X_U = Z$ and $Y_U = Z$.

Definition 5.2. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a smooth separated U-scheme. An *ns-S-Néron model* of Z (or just *ns-Néron model* if there is no ambiguity) is a smooth S-model N satisfying the following universal property, called the *Néron mapping property*:

For each smooth S-algebraic space Y, the restriction map

$$\mathrm{Hom}_S(Y, N) \to \mathrm{Hom}_U(Y_U, Z)$$

is bijective.

If N is separated, we call it a *S-Néron model*, or just *Néron model*, of X_U.

Remark 5.2.1. In the litterature, Néron models are often required to be of finite type over the base, and what we called Néron model here is referred to as a Néron-lft model, where "lft" stands for "locally of finite type". The use of this terminology is not systematical anymore, so we prefer the more flexible definition above. To the author's knowledge, however, the separatedness hypothesis is usually never omitted, so we use the prefix "ns" (not necessarily separated) to avoid generating unnecessary confusion.

Remark 5.2.2. As an immediate consequence of the universal property, a ns-Néron model, when it exists, is unique up to a unique isomorphism. A fortiori, the same holds for Néron models.

Remark 5.2.3. Let S, U, Z be as above, and N be a smooth, separated S-model of Z. Consider a smooth S-algebraic space Y/S and two morphisms $f_1, f_2 \colon Y \to N$ that coincide over U. The separatedness of N/S implies that the equalizer of f_1 and f_2 is a closed subspace of Y containing Y_U, and flatness of Y/S implies that the open subscheme Y_U of Y is scheme-theoretically dense (see [13], théorème 11.10.5). Thus, we automatically have *uniqueness in the Néron mapping property*, i.e. injectivity of the restriction map

$$\mathrm{Hom}_S(Y, N) \to \mathrm{Hom}_U(Y_U, Z).$$

Therefore, we can try to construct Néron models as separated S-spaces satisfying existence in the Néron mapping property (i.e. surjectivity of the restriction map).

5.2 Base change and descent properties

Proposition 5.3. *The formation of ns-Néron models (resp. Néron models) is compatible with smooth base change, i.e. given a smooth morphism $S' \to S$, a scheme-theoretically dense open $U \subset S$ and an S-algebraic space X which is a ns-Néron model (resp. Néron model) of X_U, the base change $X_{S'}$ is a ns-Néron model (resp. Néron model) of $X_{U'}$.*

Proof. First, note that $X_{S'}/S'$ is smooth since X/S is, separated if X/S is, and that U' is scheme-theoretically dense in S' by [13], théorème 11.10.5. Thus, we only need to check that X'/S' has the Néron mapping property.

Let Y' be a smooth S'-scheme and $u' : Y'_{U'} \to X_{U'}$ a U'-morphism. Composing with the projection: $X_{U'} \to X_U$, we get a U-morphism $Y'_{U'} \to X_U$, which extends to a unique S-morphism $Y' \to X$ by the Néron mapping property since Y'/S is smooth. Then the induced morphism $Y' \to X'$ extends u', and this extension is unique since a morphism $Y' \to X'$ is uniquely determined by the two composites $Y' \to X$ and $Y' \to S'$. □

Corollary 5.4. *If S'/S is a cofiltered limit of smooth S-schemes (indexed by a cofiltered partially ordered set, e.g. a localization, a henselization when S is local...), and X is the (ns-)S-Néron model of X_U, then $X_{S'}$ is the (ns-)S'-Néron model of $X_{U'}$.*

Lemma 5.5 (Néron models are compatible with disjoint unions on the base)**.** *Let I be a set, $(S_i)_{i \in I}$ a family of schemes, and $(N_i \to S_i)_{i \in I}$ a family of* *morphisms of algebraic spaces. Write* $S = \coprod_{i \in I} S_i$ *and* $N = \coprod_{i \in I} N_i$*. Let* U *be a scheme-theoretically dense open of* S*, and write* $U_i = U \times_S S_i$ *for every* i*. Then* N *is the* S*-ns-Néron model of* N_U *(resp. the* S*-Néron model of* N_U*) if and only if for all* i *in* I*,* N_i *is the* S_i*-ns-Néron model of* $N_i \times_{S_i} U_i$ *(resp. the* S_i*-Néron model of* $N_i \times_{S_i} U_i$*).*

Proof. Suppose N is the ns-Néron model of N_U. Then, by Proposition 5.3, for all i in I, N_i is the ns-Néron model of $N_i \times_{S_i} U_i$. Conversely, suppose that for every i in I, N_i/S_i is the ns-Néron model of its restriction to U, and consider a smooth S-algebraic space Y with a morphism $f_u \colon Y_U \to N_U$. For each i, we write $Y_i = Y \times_S S_i$. We have

$$\begin{aligned}\mathrm{Hom}_S(Y, N) &= \prod_{i \in I} \mathrm{Hom}_{S_i}(Y_i, N_i) \\ &= \prod_{i \in I} \mathrm{Hom}_{U_i}(Y_i \times_{S_i} U_i, N_i \times_{S_i} U_i) \\ &= \mathrm{Hom}_U(Y_U, N_U),\end{aligned}$$

where the first and third equalities hold since Y is the disjoint union of the Y_i, and the second one because each Y_i/S_i is smooth. Since N/S is smooth (resp. smooth and separated) if and only if all N_i/S_i are smooth (resp. smooth and separated), we are done. □

Proposition 5.6 (Néron models descend along smooth covers)**.** *Let* S *be a scheme and* U *a scheme-theoretically dense open of* S*. Let* $S' \to S$ *be a smooth surjective morphism and* $U' = U \times_S S'$*. Let* X_U *be a smooth* U*-algebraic space, and suppose* $X_{U'}$ *has a (ns-)*S'*-Néron model* X'*. Then* X_U *has a (ns-)*S*-Néron model* X *satisfying* $X' = X \times_S S'$*.*

Proof. We first show X' comes via base change from an S-algebraic space X. Call p_1, p_2 the two projections $S'' := S' \times_S S' \to S'$. They are smooth morphisms, so by Proposition 5.3 and uniqueness of the Néron model, we know $p_1^* X' = p_2^* X'$ is the S''-Néron model of $X_{U''}$ with $U'' = U \times_S S''$. It follows from effectiveness of fppf descent for algebraic spaces ([30, Tag 0ADV]) that X' comes from an S-algebraic space X/S.

The morphism $X \to S$ is smooth since X'/S' is, and separated if X'/S' is (both properties are even fpqc local on the base, see [30, Tag 02KU] and [30, Tag 02VL]). Therefore, we only need to show X/S has the Néron mapping property. Take Y a smooth S-algebraic space with a generic morphism $f_U \colon Y_U \to X_U$, and write Y' (resp. f'_U) for the pullbacks of Y (resp. f_U) under $S' \to S$. Then Y'/S' is smooth so f'_U extends to a unique $f' \colon Y' \to X'$. We have a cartesian diagram

$$\begin{array}{ccccc}
Y'' & \rightrightarrows & Y' & \longrightarrow & Y \\
\downdownarrows & & \downarrow & & \\
X'' & \rightrightarrows & X' & \longrightarrow & X \\
\downarrow & & \downarrow & & \downdownarrows \\
S'' & \rightrightarrows & S' & \longrightarrow & S
\end{array}$$

$$Y \to S$$

where $S'' := S' \times_S S'$ and the arrows $S'' \to S'$ are the two projections p_1, p_2, so that all horizontal rows are equalizers. We only need to show that $p_1^* f' = p_2^* f'$, which follows from uniqueness in the Néron mapping property of X''/S'' since they coincide over U''. □

Proposition 5.7. *Let S be a scheme, U a scheme-theoretically dense open subscheme of S, X_U/U a smooth U-scheme and N/S a model of X_U of finite type. Then N is the (ns-)Néron model of X_U if and only if for all $s \in S$, $N \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ is a (ns-)$\operatorname{Spec} \mathcal{O}_{S,s}^{et}$-Néron model of its restriction to U.*

Proof. If all the $N \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{et} / \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ are separated, then N/S is also separated, and the "only if" part is a special case of Corollary 5.4. All that remains to do is prove N/S is the ns-Néron model of N_U/U, assuming that for all $s \in S$, N is the ns-Néron model over $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ of its restriction to U. Let Y/S be a smooth S-algebraic space and $f_u \colon Y_U \to X_U$ a U-morphism. Since Y/S is locally of finite presentation, by [13], théorème 8.8.2, every point $s \in S$ has an étale neighbourhood $V_s \to S$ such that f_u extends uniquely to a morphism $Y \times_S V_s \to X \times_S V_s$. By [13], théorème 11.10.5, U remains scheme-theoretically dense in every V_s, so these maps glue as in the proof of Proposition 5.6 and f_U extends to a morphism $Y \to X$. □

5.3 Schemes vs algebraic spaces

Here we introduce the (perhaps more standard) definition of a Néron model as a scheme and not an algebraic space, and go for a little sanity check by showing both notions coincide under conditions of existence.

Definition 5.8. Let S be a scheme and U a dense open subscheme of S. Let Z/U be a smooth separated U-scheme. A *ns-S-Sch-Néron model* of Z is a smooth S-scheme N, with an identification $N_U = Z$, satisfying the *Sch-Néron mapping property*:

For each scheme Y with a smooth morphism $Y \to S$, the restriction map

$$\operatorname{Hom}_S(Y, N) \to \operatorname{Hom}_U(Y_U, Z)$$

is bijective.

Remark 5.8.1. • The ns-Sch-Néron model, if it exists, is unique up to a unique isomorphism.

- Again, when U is scheme-theoretically dense, given a smooth separated S-scheme N with $N_U = Z$, it is the ns-Sch-Néron model if and only if it satisfies existence in the mapping property.
- When a ns-Néron model is a scheme, it is automatically the ns-Sch-Néron model since it satisfies the Sch-Néron mapping property.

Proposition 5.9. *Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a smooth U-scheme. Suppose Z admits a ns-Sch-Néron model N. Then N is also a ns-Néron model of Z.*

Proof. We show that N has the Néron mapping property. Let Y be a smooth S-algebraic space, together with a morphism $Y_U \to Z$ of algebraic spaces. We can choose a presentation of Y as a quotient of a scheme by an étale equivalence relation ([30, Tag 0262]), i.e. S-schemes R and V with an étale covering map $V \to Y$ and an equivalence relation $R \to V \times_S V$ such that the two induced maps $R \to V$ are étale, and such that the diagram

$$R \rightrightarrows V \to Y$$

is a coequalizer of sheaves of sets on $(Sch/S)_{fppf}$. This presentation is compatible with the base change $U \to S$ ([30, Tag 03I4]), so we get a coequalizer

$$R_U \rightrightarrows V_U \to Y_U$$

in the category of sheaves of sets on $(Sch/U)_{fppf}$. Thus $Y_U \to Z$ can be seen as a map $V_U \to Z$ such that both composites $R_U \to Z$ coincide. Then, since V and R are smooth over S by composition, applying the Sch-Néron mapping property, we can extend uniquely $V_U \to Z$ to an S-map $V \to N$. The two composites $R \to N$ both extend the same $R_U \to Z$, so they are equal by uniqueness in the Sch-Néron mapping property. So we have a unique morphism $Y \to N$ of algebraic spaces extending $Y_U \to Z$, as required. □

Corollary 5.10. *If Z admits a ns-Néron model N and a ns-Sch-Néron model N', then $N = N'$ is a scheme.*

6 Néron models of Jacobians

6.1 Alignment and its relation to the Picard space

This subsection summarizes the main results of [21] and introduces a few definitions to adapt them to our context. From now on, given a local ring R, we will note R^{sh} for a strict henselization of R.

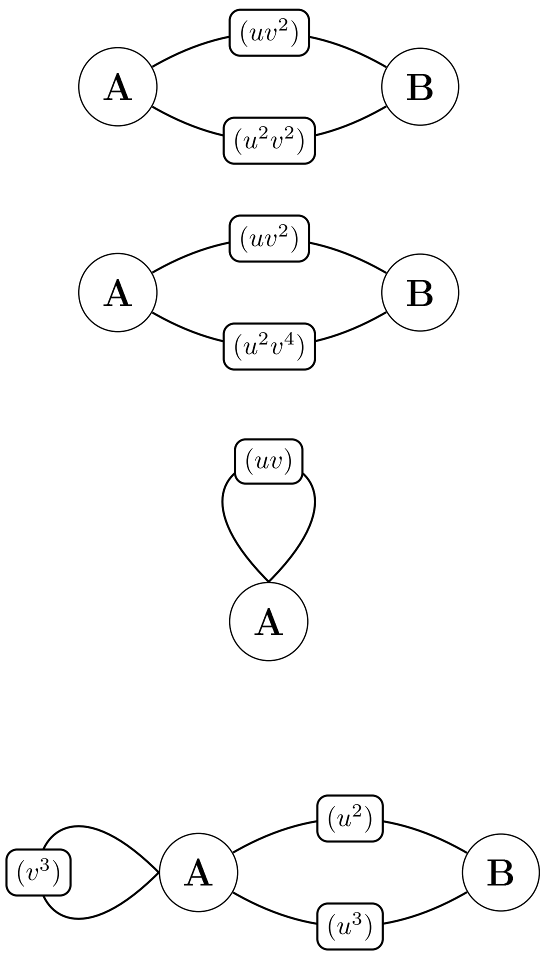
6.1.1 Definition and examples

Definition 6.1. Suppose S is a regular scheme. Let s be a geometric point of S and $R = \mathcal{O}^{et}_{S,s}$. Following [21], Definition 2.11, we say that a labelled graph Γ

is *aligned* when for every cycle Γ^0 in Γ, all the labels figuring in Γ^0 are positive powers of the same principal ideal; and that a nodal curve X/S is *aligned at s* when its dual graph Γ_s at s is aligned. We say X/S is *aligned* if it is aligned at every geometric point of S.

We define Γ_s to be *strictly aligned*, or X to be *strictly aligned at s*, when it satisfies the following condition: for any cycle $\Gamma^0 \subset \Gamma_s$, there exists a *prime* element $\Delta \in R$ such that all the labels of Γ^0 are powers of the principal ideal (Δ) of R. We say that X is *strictly aligned* if it is strictly aligned at every geometric point of S.

Example 6.2. Over $S = \operatorname{Spec} \mathbb{C}[[u, v]]$, at the closed point, among the 4 following dual graphs, the first is non-aligned; the second and the third are aligned but not strictly, and the last one is strictly aligned.



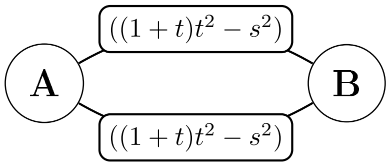
Remark 6.2.1. Strict alignment implies alignment, and is equivalent to strict alignment in the sense of [21], Definition 3.4. In particular, when S is regular, excellent and separated, and X is split and smooth over the complement of a strict normal crossings divisor (i.e. the singular ideals are generated by products of the elements of some regular system of parameters), using [21], Proposition 3.6, we see strict alignment is equivalent to the existence of a Néron model for the Jacobian. We want to investigate what happens when X is a generically smooth nodal curve, but not necessarily smooth over the complement of a normal crossings divisor.

We have to be a little careful about the fact that alignment and strict alignment both deal with étale neighbourhoods. Let us consider two examples.

Example 6.3. The curve over $R = \mathbb{C}[s,t]_{(s,t)}$, given in the weighted projective space $\mathbb{P}_S(1,2,1)$ (in affine coordinates (x,y)) by

$$y^2 = \left((x-1)^2 - (1+t)t^2 + s^2\right)\left((x+1)^2 + (1+t)t^2 - s^2\right)$$

is quasisplit, and its dual graph at the closed point is the following 2-gon:

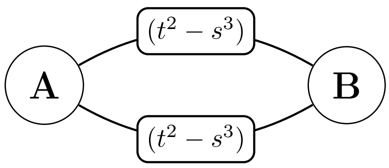


but it is not strictly aligned, even though $(1+t)t^2 - s^2$ is a prime element of R. Indeed, $(1+t)t^2 - s^2$ is prime in R but has two distinct prime factors in a strict henselization, since $(1+t)$ becomes a square, and it is the prime factor decomposition in the étale local rings that counts in the definition of strict alignment.

Example 6.4. On the other hand, the equation

$$y^2 = \left((x-1)^2 - t^2 + s^3\right)\left((x+1)^2 + t^2 - s^3\right)$$

defines a nodal curve over $\operatorname{Spec} R$ with dual graph



which is strictly aligned at the closed point, because $t^2 - s^3$ remains prime in R^{sh}.

6.1.2 Alignment and Néron models

A classical way of obtaining a Néron model for the Jacobian of a proper smooth curve X_U/U with a nodal model X/S, when X is "nice enough", is to consider the biggest separated quotient of the subspace $\mathrm{Pic}^{[0]}_{X/S}$ of $\mathrm{Pic}_{X/S}$ consisting of line bundles of total degree 0 (see for example [1]). In other words, a good "candidate Néron model" is the quotient of $\mathrm{Pic}^{[0]}_{X/S}$ by the closure of its unit section. This works well when three conditions are met: $\mathrm{Pic}^{[0]}_{X/S}$ is representable by an S-algebraic space; the closure of its unit section is flat over S (so that the quotient is also representable); and $\mathrm{Pic}^{[0]}_{X/S}$ satisfies existence in the Néron mapping property (i.e. X is semifactorial after every smooth base change). These are the ideas behind the main result of [21], that we will recall here, and behind the notion of alignment.

Proposition 6.5. *Let S be a regular scheme, $U \subset S$ a dense open, and X/S a nodal curve, smooth over U. Let $P = \mathrm{Pic}^{[0]}_{X/S}$ be the subsheaf of $\mathrm{Pic}_{X/S}$ consisting of line bundles of total degree 0. It is representable by a smooth quasi-separated algebraic space, that we call P again ([1], 8.3.1 and 9.4.1). Let E be the scheme-theoretical closure in P of its unit section. Then the following conditions are equivalent:*

1. *E/S is flat.*
2. *E/S is étale.*
3. *X/S is aligned.*

Proof. This is [21], Theorem 5.17. □

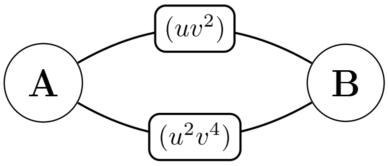
Proposition 6.6. *With the same hypotheses and notations as in Proposition 6.5 above, let J be the Jacobian of X_U. If a Néron model N for J exists, then E/S is flat. Conversely, if $X \times_S S'$ is locally factorial for every smooth base change $S' \to S$ and E/S is flat, then P/E is an S-Néron model for J.*

Proof. This is [21], Theorem 6.2 and Remark 6.3. The idea is that, when the regularity condition we give on X is satisfied, we can use the correspondence between Weil divisors and Cartier divisors to show that line bundles over U extend to the whole base, so P satisfies existence in the Néron mapping property. It follows that its biggest separated quotient P/E (which exists as an algebraic space if and only if E/S is flat) also does. □

We want to investigate the in-between zone, i.e. what happens if we are given an aligned nodal curve that is not locally factorial.

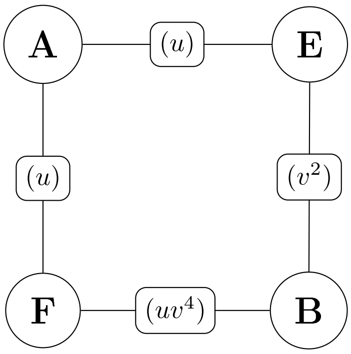
A consequence of Proposition 6.6 is that if J has a Néron model, then *every* nodal model of X_U must be aligned. This is stronger than just alignment of X since alignment is not stable under modifications of nodal curves (see the example below).

Example 6.7. Consider a nodal curve X over $S = \mathrm{Spec}\,\mathbb{C}[[u,v]]$ having the following dual graph at the closed point:

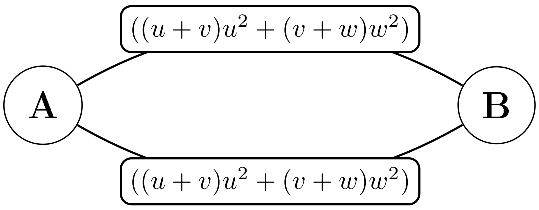


This graph is aligned, and X is smooth over the complement U in S of $\mathrm{Div}(uv)$, but X is not locally factorial (see 4.2), so Proposition 6.5 and Proposition 6.6 do not allow us to immediately conclude to either existence or nonexistence of a Néron model.

The zero locus of u in X has two irreducible components, one containing A and one containing B, and explicit computation shows that if we blow up X in the one containing A, the result is still a nodal curve, with dual graph



This new curve coincides with X over U, and it is not aligned: the Jacobian of X_U cannot have a Néron model. Similarly, if $S = \operatorname{Spec} R$ with $R = \mathbb{C}[[u, v, w]]$ and X has dual graph



then, again, we cannot immediately apply 6.5 and 6.6: the curve X is aligned, but its (smooth) base change to $R' = \mathbb{C}[[u, v, w]][v^{-1}, \sqrt{u + v}, \sqrt{v + w}]$ fails to be locally factorial, since in R' the labels become sums of two squares and factor into a product of two primes.

However, we can observe that if the Jacobian of X_U/U had a Néron model, it would still be a Néron model over R', and after that base change, we are now in a case similar to that of the previous example! The curve $X \times_R R'$ can be blown up into a non-aligned nodal curve over R', which means its Jacobian does not have a Néron model.

In conclusion, asking for X to remain aligned after smooth base changes and birational morphisms of nodal curves is a strictly stronger condition than just asking for X to be aligned, yet it remains necessary for a Néron model of the Jacobian to exist. We will see that strict alignment is precisely the closure of alignment under those operations, and that it is the right notion to talk about Néron models of Jacobians in terms of dual graphs.

6.2 Étale-universally prime elements

Here, we will elaborate on a phenomenon illustrated in Example 6.7, namely the fact that some prime elements of a regular, strictly henselian local ring can have several prime factors in a further étale localization. If such an element labels a cycle of an aligned nodal curve, then there is an étale base change after which this curve has a non-aligned refinement (which forbids the existence of a Néron model for the generic Jacobian). We will also show this is actually the *only* possible reason for the smooth base change of an aligned curve to have non-aligned refinements.

Definition 6.8. Let R be a regular local ring and Δ be a non-invertible element of R. We say Δ is *étale-universally prime* when, for all prime ideals $\mathfrak{p} \subset R$ containing Δ, the image of Δ in a strict henselization of $R_\mathfrak{p}$ is prime.

Example 6.9. Take $R = \mathbb{C}[s,t]_{(s,t)}$ and $\Delta = (t^2 - s^3)$. Then Δ is étale-universally prime, since Δ remains prime in both R^{sh} and $(R_{(\Delta)})^{sh}$, and the maximal ideal and (Δ) are the only primes of R containing Δ.

Example 6.10. On the other hand, some prime elements are not étale-universally prime even if R is strictly henselian: let $R = (\mathbb{C}[u,v,w]_{(u,v,w)})^{sh}$ and $\Delta = u^2(v+w) - v^2(v-w)$. Then Δ is prime in R (it is even prime in $\mathbb{C}[[u,v,w]]$), but if we consider the prime ideal $\mathfrak{p} = (u,v)$ of R, which contains Δ, we see that Δ has a nontrivial factorization in $R_\mathfrak{p}^{sh}$ since the units $v+w$ and $v-w$ of $R_\mathfrak{p}$ become squares in $R_\mathfrak{p}^{sh}$.

Remark 6.10.1. An element $\Delta \in R$ is étale-universally prime if and only if all localizations of $R/(\Delta)$ are geometrically unibranch in the sense of [12], 23.2.1 or [29], IX, Définition 2.

We are interested in those étale-universal primes to study Néron models because they behave well with respect to the smooth topology. Their key property is Lemma 6.11.

Lemma 6.11. *Let $S = \operatorname{Spec} R$ be an affine regular scheme and Δ be an element of R. Then Δ is étale-universally prime in R if and only if for every smooth morphism $Y \to \operatorname{Spec} R$ and every geometric point $y \in Y$, the image of Δ in $\mathcal{O}^{et}_{Y,y}$ is either invertible or prime.*

Proof. The "if" sense is immediate since the identity $\operatorname{Spec} R \to \operatorname{Spec} R$ is smooth. For the converse, suppose Δ is étale-universally prime. Since smoothness is a Zariski-local property, it is enough to prove that for each smooth map of affines $\operatorname{Spec} A \to \operatorname{Spec} R$, Δ is étale-universally prime in A. Let $\mathfrak{p} \subset A$ be a prime ideal containing Δ and $\mathfrak{m}$ the preimage of $\mathfrak{p}$ in R, the map $R \to (A_\mathfrak{p})^{sh}$

factors as $R \to (R_\mathfrak{m})^{sh} \to A \otimes_R (R_\mathfrak{m})^{sh} \to (A_\mathfrak{p})^{sh}$. Since the middle arrow $(R_\mathfrak{m})^{sh} \to A \otimes_R (R_\mathfrak{m})^{sh}$ is smooth, and since $A \otimes_R (R_\mathfrak{m})^{sh} \to (A_\mathfrak{p})^{sh}$ is a strict localization at a prime containing the image of $\mathfrak{m}$, we can conclude by quotienting by (Δ) and applying Lemma 2.2. $\square$

Proposition 6.12. *Take X/S a generically smooth nodal curve with S regular. Take s a geometric point of S such that X is strictly aligned at s. Then X is strictly aligned at every étale generization t of s if and only if for every cycle Γ_0 of Γ_s, the only prime of $\mathcal{O}^{et}_{S,s}$ appearing as a factor of the labels of Γ_0 is étale-universally prime.*

Proof. It follows from Proposition 1.8 applied to the specialization morphism: $\operatorname{Spec} \mathcal{O}^{et}_{S,t} \to \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ and the definitions. $\square$

This allows us to detect strict alignment, only looking at the dual graph of the closed fiber:

Definition 6.13. Let X/S be a generically smooth nodal curve with S regular.

We say that Γ_s is *étale-strictly aligned*, or that X is *étale-strictly aligned at s*, when it satisfies the following condition: for any cycle $\Gamma^0 \subset \Gamma_s$, including loops, there exists an étale-universally prime element $\Delta \in R$ such that all the labels of Γ^0 are powers of the principal ideal (Δ) of R. We say that X is *étale-strictly aligned* if it is étale-strictly aligned at every geometric point of S.

Proposition 6.14. *If X/S is a nodal curve with S regular, the following conditions are equivalent:*

1. *X is strictly aligned.*
2. *X is étale-strictly aligned at the closed geometric points of S.*
3. *X is étale-strictly aligned.*

Proof. (3) $\implies$ (2) and (2) $\implies$ (1) are clear. (2) $\implies$ (3) follows from observing that in a locally noetherian scheme, every point specializes to a closed point, and if R is a local ring, Δ an étale-universally prime element of R, and $\mathfrak{p}$ a prime ideal of R containing Δ, then Δ is also étale-universally prime in $R_\mathfrak{p}$. We will show (1) $\implies$ (2).

Take X/S a strictly aligned generically smooth nodal curve with S regular. We will show it is étale-strictly aligned at the closed geometric points of S. We can assume $S = \operatorname{Spec} R$ is local and strictly henselian, with closed point s. Let Γ be the dual graph of X at s, and Γ^0 be a cycle of Γ. There is a prime $\Delta \in R$ such that all labels of Γ_0 are powers of Δ, and we have to show Δ is étale-universally prime in R.

Let $\mathfrak{p}$ be a prime ideal of R containing Δ, and choose a strict henselization $R^{sh}_\mathfrak{p}$ of R_p. It gives an étale generization t of s, at which X is strictly aligned, so the cycle pulled back from Γ^0 in the dual graph of X at t has all its labels generated by powers of some prime of $R_{\mathfrak{p}}^{sh}$. Thus, the image of Δ in $R_{\mathfrak{p}}^{sh}$ is a power of a prime. Therefore it is enough to show $R_{\mathfrak{p}}^{sh}/(\Delta)$ is reduced.

But $R_{\mathfrak{p}}^{sh}/(\Delta)$ is a strict henselization of $R_{\mathfrak{p}}/(\Delta)$ since the quotient of a henselian ring is henselian, so it is reduced as a directed colimit of reduced $R_{\mathfrak{p}}/(\Delta)$-algebras. □

6.3 Strict alignment is necessary and sufficient for Néron models to exist

The goal of this subsection is to prove Theorem 6.20. It is to be noted that a variant of the theorem probably holds under a weaker assumption than regularity of S: what we care about is extending generic line bundles on X after étale base change, so having S parafactorial along the complement of the discriminant locus after every smooth base change, plus some other minor assumptions, should suffice. Of course, strict alignment would then have to be defined in that context, since the étale local rings of S would not be unique factorization domains anymore. In addition, if one were in need of such generality, they would need to verify that the material we use (e.g. in [21]) also works after weakening the hypotheses.

6.3.1 The necessity of strict alignment

We start with the easy implication: we will show that if a nodal curve is not strictly aligned, then over some étale local ring of the base, we can find a non-aligned refinement of it (which means there can be no Néron model for the generic Jacobian).

Proposition 6.15. *Let S be a regular scheme and X/S a nodal curve, smooth over a dense open $U \subset S$. If the Jacobian of X_U/U has a Néron model over S, then X/S is strictly aligned.*

Proof. We will work by contradiction, assuming there is a geometric point $s \in S$ at which X is not strictly aligned. Using Corollary 5.4, we can assume S is a strictly local scheme $\operatorname{Spec} R$, with closed point s. Remember that R is regular, thus a unique factorization domain. Note Γ the dual graph of X at s and l the edge-labelling of Γ. By assumption, there is a cycle Γ^0 in Γ and two (not necessarily distinct) edges e and e' of Γ^0 such that $l(e)l(e')$ has at least two distinct prime factors.

We know X is aligned by Proposition 6.5 and Proposition 6.6: there is an element $\Delta \in R$ such that all edges of Γ^0 are labelled by positive powers of ΔR. This applies in particular to e and e', so we can write Δ as a product $(\Delta_1 \Delta_2)$, where Δ_1, Δ_2 are non-invertible elements of R with no common factor.

Let x be the singular point of X corresponding to e. Since S is strictly local with closed point s, we can pick an orientation (C, D) of X/S at x. Call $X' \to X$ the (Δ_1)-refinement of X at x relatively to (C, D). By Lemma 4.6, the dual graph of X' at s contains a cycle, refining Γ^0, such that the edge corresponding to x has been replaced by a chain of two edges, one of label (Δ_1) and one of label (Δ_2). In particular, X' is not aligned at s. However, $X'_U = X_U$, so the jacobian of X'_U has a Néron model: we get a contradiction by virtue of Proposition 6.5 and Proposition 6.6. □

6.3.2 Fiberwise-disconnecting locus of nodal curves and closure of the unit section of the Picard scheme

As strict alignment is only a condition on the cycles of the dual graphs, we have to show that "labels of disconnecting points do not matter", in a sense that will made precise by Proposition 6.19. The idea is that when one blows up a nodal curve over a strictly local base in a section through a disconnecting singular point, all "new" line bundles are killed by the growth of the closure of the unit section, and the quotient P/E does not change. We start with a few technical lemmas.

Lemma 6.16. *Let $S = \operatorname{Spec} R$ be a trait (i.e. the spectrum of a discrete valuation ring) and $f : X \to S$ a generically smooth quasisplit nodal curve. Let $\pi : X' \to X$ be the blowing-up in a closed non-smooth point x of X/S. Let $\mathfrak{L}$ be a line bundle on X', trivial over the exceptional fiber of π. Then $\pi_*\mathfrak{L}$ is a line bundle on X.*

Proof. This is [26], Proposition 4.2. □

Lemma 6.17. *Let $S = \operatorname{Spec} R$ be a regular and strictly local scheme. Let $f \colon X \to S$ be a quasisplit nodal curve, smooth over some dense open $U \subset S$. Let $\pi \colon X' \to X$ be a refinement and $\mathfrak{L}$ be a line bundle on X'. Let $Y \subset X'$ be the exceptional locus of π and suppose $\mathfrak{L}|_Y \simeq \mathcal{O}_Y$. Then $\pi_*\mathfrak{L}$ is a line bundle on X.*

Proof. $\pi_*\mathfrak{L}$ is a coherent $\mathcal{O}_X$-module and X is reduced, so it is enough to check that, for all $y \in X$, we have $\dim_{k(y)} \pi_*\mathfrak{L} \otimes_{\mathcal{O}_X} k(y) = 1$. It is obvious for all y such that π is a local isomorphism at y, so we only need to check it when y is in the image of the exceptional locus of π.

Take a section $\sigma \colon S \to X$ such that π is the blowing-up in the sheaf of ideals of σ. Let x be in the image of the exceptional locus of π and s its image in S: we have $x = \sigma(s)$. The base change of π to $\operatorname{Spec} \mathcal{O}^{sh}_{S,s}$ is still the blowing-up in the sheaf of ideals of σ, and the condition $\dim_{k(x)} \pi_*\mathfrak{L} \otimes_{\mathcal{O}_X} k(x) = 1$ can also be checked after base change to $\operatorname{Spec} \mathcal{O}^{sh}_{S,s}$, so we can assume s is the closed point of S. Iterating the prime avoidance lemma, we see that $\mathcal{O}_S(S)$ admits a quotient D, that is a discrete valuation ring, such that the generic point of $T = \operatorname{Spec} D$ lands in U. Pick a uniformizer t of D. We have $\Delta_x D = t^n D$ for some $n \geq 1$, where Δ_x is a generator of the singular ideal of x.

The base change X_T/T is a nodal curve. The point corresponding to x, that we still call x, has singular ideal $t^n D$. The sheaf of ideals $\mathcal{I}$ of σ in X_T is trivial away from x, and given at the completed étale local ring $\widehat{\mathcal{O}^{et}_{X_T,x}} = \widehat{D^{sh}}[[u,v]]/(uv - t^n)$

by the ideal $(u - t^k, v - t^l)$ with $k + l = n$ for a good choice of isomorphism. If $n = 1$, then $\mathcal{I}$ is trivial (and a fortiori Cartier) so $X'_T = X_T$.

Suppose $n \geq 2$ and pick $d = \lfloor \frac{n}{2} \rfloor$. There is a sequence

$$X'' = X_d \to X_{d-1} \to ... \to X_0 = X_T$$

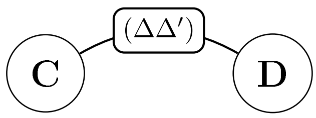
where each $X_{i+1} \to X_i$ is a blowing-up in a closed point of image x, namely the only closed non-T-smooth point of X_i of image x and of singular ideal $\neq tD$, and the preimage of x in X_d is a chain of $n-1$ copies of $\mathbb{P}^1_{k(x)}$, intersecting in $n-2$ non-smooth points of ideal tD. The sheaf of ideals $\mathcal{I}$ on X_T is Cartier in X'', which, by the universal property of blowing-ups, implies that $X'' \to X_T$ factors through $X'_T \to X_T$.

Now, the restriction $\mathfrak{L}|_{X''}$ is a line bundle on X'', trivial on the exceptional locus of $X'' \to X_T$. Thus, using the preceding lemma, we see inductively that its pushforward to every X_i, and in particular to $X_0 = X_T$, is a line bundle. This, in turn, gives us $\dim_{k(x)} \pi_*\mathfrak{L} \otimes_{\mathcal{O}_X} k(x) = 1$: $\pi_*\mathfrak{L}$ is a line bundle on X. □

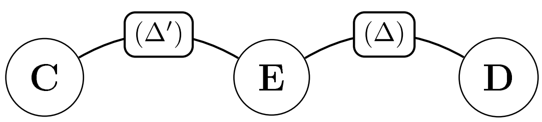
Lemma 6.18. *Let $S = \operatorname{Spec} R$ be a regular and strictly local scheme with closed point s. Let $f: X \to S$ be a nodal curve, smooth over some dense open $U \subset S$. Let $\pi: X' \to X$ be a refinement such that the exceptional locus of π is disconnecting in the closed fiber. Let $\mathfrak{L}$ be a line bundle of total degree 0 on X'. There exists a line bundle $\mathfrak{L}'$ on X', trivial over U, such that $(\mathfrak{L} \otimes \mathfrak{L}')|_Z \simeq \mathcal{O}_Z$, where Z is the exceptional locus of $X' \to X$.*

Proof. The morphism $X' \to X$ is the blowup in a section $\sigma: S \to X$. Set $x = \sigma(s)$, and call Γ, Γ' the respective dual graphs of X and X' at s. By hypothesis, x is a singular point of X, disconnecting in its fiber.

We only look at the local picture at x, since π is an isomorphism away from x. Pick an isomorphism $\widehat{\mathcal{O}_{X,x}} \simeq \widehat{R}[[u,v]]/uv - \Delta_x$ where Δ_x is a generator of the singular ideal of x. The map $\widehat{\mathcal{O}_{X,x}} \to \widehat{R}$ given by σ sends u, v to elements Δ, Δ' of R with $\Delta\Delta' = \Delta_x$. Lemma 4.6 shows that the edge



corresponding to x in Γ (where C, D are necessarily distinct since x is disconnecting) is replaced in Γ' by a chain



Where we still write C, D for the respective strict transforms of C and D in X'.

The new nodal curve $f' \colon X' \to S$ is quasisplit since S is strictly local. Call z the singular point $E \cap D$ of X' (of ideal (Δ)) and Y the connected component of the non-smooth locus of X'/S containing z. The map $Y \to S$ is a closed immersion, cut out by Δ.

Now, since $Y \times_S Y$ is disconnecting in $X \times_S Y$, we know $(X \backslash Y) \times_S Y$ has two distinct connected components Y_1^0 and Y_2^0, respectively containing the images of C and D in $(X' \backslash Y) \times_S Y$. Call Y_2 the scheme-theoretical closure of Y_2^0 in X'. We will show it is a Cartier divisor on X'.

The sheaf of ideals $\mathcal{J}$ defining Y_2 in X' is locally principal away from Y, cut out by Δ in Y_2^0 and by 1 in $X \backslash (Y \cup Y_2^0)$: we only need to check it is invertible on $\mathcal{O}_{X',z}$, which is a consequence of Lemma 1.16 (or can be seen explicitly in $\operatorname{Spec} \widehat{\mathcal{O}^{et}_{X',z}}$). Thus $\mathcal{J}$ is Cartier, and $\mathfrak{L}'' := \mathcal{O}(Y_2)$ is a line bundle on X', trivial over U.

Let V be the closed subscheme of S cut out by the ideal $(\Delta, \Delta') \subset R$. The exceptional locus Z of π is a $\mathbb{P}^1$-bundle on V. But Z and Y_2 intersect transversally at one double point in each fiber over V, so $\deg \mathfrak{L}''|_Z = 1$. Let d be the degree of $\mathfrak{L}$ on Z and $\mathfrak{L}' = \mathfrak{L}''^{\otimes -d}$, then $\mathfrak{L} \otimes_{\mathcal{O}_{X'}} \mathfrak{L}'$ has degree zero on Z, hence is trivial on Z since $Z \simeq \mathbb{P}^1_V$. □

Proposition 6.19. *Let $f : X \to S$ be a nodal curve with $S = \operatorname{Spec} R$ regular and strictly local. Let $\pi : X' \to X$ be a refinement, such that its exceptional locus is disconnecting in the closed fiber. Set $P = \operatorname{Pic}^{[0]}_{X/S}$ and $P' = \operatorname{Pic}^{[0]}_{X'/S}$ and call E and E' the scheme-theoretical closures of the unit sections of P and P' respectively. Then the canonical morphism of algebraic spaces $P \to P'$ induces an isomorphism $P/E \to P'/E'$.*

Proof. First, we show $P \to P'$ is an open immersion.

The fiberwise-connected component of unity $\operatorname{Pic}^0_{X/S}$ is an open neighbourhood of the unit section in P, and same goes for $\operatorname{Pic}^0_{X'/S} \to P'$. We have a commutative diagram of S-spaces

$$\begin{array}{ccccc} \operatorname{Pic}^0_{X/S} & \hookrightarrow & P & \twoheadrightarrow & \Phi_{X/S} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Pic}^0_{X'/S} & \hookrightarrow & P' & \twoheadrightarrow & \Phi_{X'/S} \end{array}$$

where both horizontal rows are exact, and $\operatorname{Pic}^0_{X/S} \to \operatorname{Pic}^0_{X'/S}$ is an isomorphism by Lemma 6.17, so $P \to P'$ is locally on the source an open immersion: to deduce it is an actual open immersion, we only need to show it is set-theoretically injective, which can be checked on its fibers over S. Let s be a point of S and k its residue field. The smooth locus of X_s^{sm} has a k-rational point in every irreducible component by quasisplitness of X (which follows from the fact S is strictly local), and π is an isomorphism above X_s^{sm}, so $\Phi_s \to \Phi'_s$ is set-theoretically injective. It follows $P \to P'$ is set-theoretically injective, so it is an open immersion.

Now this implies the scheme-theoretical closure in P' of the unit section of P is E', so $E = E' \times_{P'} P$. Thus $P/E = P'/E' \times_{P'} P$, and $P/E \to P'/E'$ is an open immersion as a base change of the open immersion $P \hookrightarrow P'$. Moreover, the formation of P and P' commutes with base change, so $P/E \to P'/E'$ will be surjective (thus an isomorphism) if it is surjective on S-points: take a section $\sigma\colon S \to P'/E'$, Lemma 6.18 shows that σ can be represented by a line bundle $\mathfrak{L}$ on X', trivial over the exceptional locus of π. But then by Lemma 6.17, $\pi_*\mathfrak{L}$ is a line bundle on X, so it gives a section $S \to P/E$. Composing with $P/E \to P'/E'$, we obtain the S-point of P'/E' corresponding to the line bundle $\pi^*\pi_*\mathfrak{L}$, which is none other than σ since $\pi^*\pi_*\mathfrak{L} \otimes_{\mathcal{O}_{X'}} \mathfrak{L}^{\otimes -1}$ is trivial over U. Thus σ comes from an S-point of P/E and we are done. □

6.3.3 The main theorem

Theorem 6.20. *Let S be an excellent regular scheme, $U \subset S$ a dense open subscheme, and X a nodal S-curve, smooth over U. The following conditions are equivalent:*

(i) The Jacobian J of X_U admits a Néron model over S.

(ii) X is strictly aligned.

(iii) X is étale-strictly aligned.

(iv) X is étale-strictly aligned at all closed points of S.

If these conditions are met, the Néron model is of finite type. If in addition X/S has a partial resolution $X' \to X$, the Néron model of J is P/E, where $P = \mathrm{Pic}^{[0]}_{X'/S}$ and E is the scheme-theoretical closure of the unit section in P.

Proof. The Néron model is of finite type if it exists by [19], Theorem 2.1.

Conditions (ii), (iii) and (iv) are equivalent by Proposition 6.14, and $(i) \to (ii)$ is Proposition 6.15.

We will show $(iii) \to (i)$. As both (iii) and (i) can be checked after base change to an étale neighbourhood of an arbitrary point s of S, we can assume X/S is quasisplit (using Lemma 1.14). Base-changing to a further étale cover, we can assume X/S has a partial resolution (see Proposition 4.14). Replacing X by this partial resolution, we can assume X is square-free and étale-strictly aligned. Thus, every non-smooth point of X/S that is not disconnecting in its fiber has étale-universally prime label. Take $P = \mathrm{Pic}^{[0]}_{X/S}$, we will show P/E is a Néron model for J.

The base change of X to any étale local ring of S still has étale-universally prime labels in all cycles of all dual graphs, so our claim that P/E is a Néron model for J can be checked over the étale local rings of S using 5.7 and the fact that the formation of P and E commutes with base change to the étale local rings of S. Thus we also assume S is strictly local, with closed point s.

The quotient P/E is a smooth and separated model of J (since $J = \mathrm{Pic}^0_{X_U/U}$) so we only have to show it satisfies existence in the Néron mapping property. Let Y be a smooth S-algebraic space, together with a generic morphism $f_u : Y_U \to J$. We want to show f_u extends to a morphism $Y \to P/E$. Using the uniqueness in the Néron mapping property and the effectiveness of fppf descent for algebraic spaces ([30, Tag 0ADV]), we can work étale-locally on Y: it is enough to extend f_u to $\mathcal{O}_{Y,y}$ for every geometric point y of Y. Thus we can and will work assuming $Y = \mathrm{Spec}\, A$ is a strictly local scheme, and replacing the hypothesis that Y/S is smooth by the hypothesis that A is a filtered colimit of smooth R-algebras.

The map $Y_U \to J$ corresponds to a line bundle $\mathfrak{L}_U$ on $(X \times_S Y)_U$. We only have to show $\mathfrak{L}_U$ comes from a line bundle on $X \times_S Y$: indeed, such a line bundle would have total degree zero and give a morphism $Y \to P$. Composing with $P \to P/E$, we would get a map extending f_U as desired.

The base change $X \times_S Y$ is still a nodal curve whose non-disconnecting singular points have étale-universally prime label by Lemma 6.11. Take a Y-resolution $X_0 \to X \times_S Y$. There is a Cartier divisor D_U on $(X \times_S Y)_U = (X_0)_U$ such that $\mathfrak{L}_U = \mathcal{O}(D_U)$. Write it as a finite sum $D_U = \sum_{i=1}^{k} n_i D_i$ where the n_i are integers and the D_i are primitive Weil divisors on $(X_0)_U$, and take $D = \sum_{i=1}^{k} n_i \overline{D_i}$, where $\overline{D_i}$ is the scheme-theoretical closure of D_i in X_0. By definition D is only a Weil divisor on X_0, but, by Lemma 4.2, X_0 is locally factorial, so D is automatically Cartier and the line bundle $\mathfrak{L} = \mathcal{O}(D)$ on X_0 restricts to $\mathfrak{L}_U$. Moreover, $E \times_S Y$ is still the closure of the unit section in $P \times_S Y$, so the quotient of $\mathrm{Pic}^{[0]}_{X_0/Y}$ by the closure of its unit section is equal to $P/E \times_S Y$ by Proposition 6.19, and we get the desired line bundle on $X \times_S Y$ extending $\mathfrak{L}_U$. □

7 Néron models of curves with nodal models

Let S be a regular base scheme, U a dense open subscheme of S, and X/S a nodal relative curve, smooth over U. In what follows, we are interested in the existence of a Néron model over S for the curve X_U/U.

We will end up getting a very restrictive condition on the local structure of singularities for an actual Néron model to exist. When X/S is quasisplit, almost all connected components of its singular locus need to be irreducible. However, we will also see one can often exhibit a smooth (bot not necessarily separated) S-algebraic space with the Néron mapping property. Our condition of local irreducibility of the singular locus of X/S then becomes a condition for separability of this object, i.e. a condition for it to be a true Néron model. More precisely, the main results of this section are:

Theorem 7.1 (Theorem 7.40). *Let S be a regular excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Suppose X has no rational loops, and suppose no geometric fiber of X contains a rational component meeting the non-exceptional other irreducible components in three points or more. Then X_U/U has a ns-Néron model N/S. If in addition*

X/S is quasisplit, then N is the smooth aggregate (Construction 7.5) of the stable model X^{stable} of X_U (Definition 7.26).

Theorem 7.2 (Theorem 7.48). *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. If X_U has a Néron model over S, then the two following conditions are met:*

- *The singular locus $\mathrm{Sing}(X/S)$ is irreducible around every non-exceptional singular geometric point of X/S.*
- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}_{S,s}^{et}$ have the same radical.*

Conversely, suppose these conditions are met. Suppose in addition that no geometric fiber of X/S contains either a rational cycle or a rational component meeting the non-exceptional other components in at least three points. Then X_U/U has a Néron model, i.e. the ns-Néron model of X_U/U exhibited in Theorem 7.40 is separated over S.

7.1 Factoring sections through refinements

A first question, easier to tackle than existence of a Néron model, is "given a U-point of X_U, can we extend it to a section of a smooth S-model of X_U".

We answer with a two-step strategy: first, when X has no rational component in its geometric fibers, all U-points of X extend to sections by the following result from [8]:

Proposition 7.3 ([8], Proposition 6.2). *Let X/S be a proper morphism of schemes, where S is noetherian, regular and integral. Let K be the function field of S, and suppose that no geometric fiber of X/S contains a rational curve. Then every K-rational point of X_K extends to a section $S \to X$.*

The S-section of X we obtain might meet the singular locus. Our second step consists in finding a refinement of X such that the section comes (at least locally on S) from a smooth section of this refinement.

Lemma 7.4. *Let X be a quasisplit nodal curve over a regular scheme S. Suppose X is smooth over a scheme-theoretically dense open subscheme $U \subset S$. Let σ, τ be sections of X/S, and $\phi \colon X' \to X$ the blowing-up in the ideal sheaf of τ. Let s be a point of S and suppose $\tau(s)$ is a singular point x of X_s at which X/S is orientable. Then the three following conditions are equivalent:*

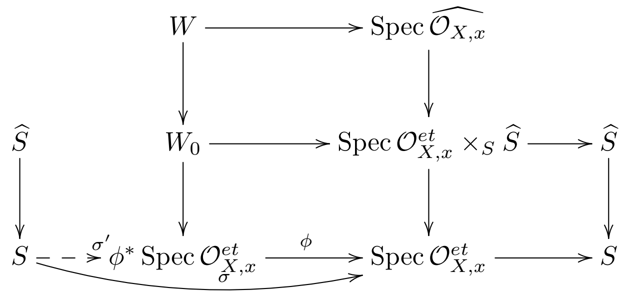
1. *The restriction of σ to the étale local ring $\mathrm{Spec}\,\mathcal{O}_{S,s}^{et}$ factors through a smooth section of $X' \times_S \mathrm{Spec}\,\mathcal{O}_{S,s}^{et} / \mathrm{Spec}\,\mathcal{O}_{S,s}^{et}$.*
2. *There exists an étale neighbourhood V of s such that the restriction of σ to V factors through a smooth section of $X' \times_S V/V$.*

3. *Either $\sigma(s)$ is a smooth point of X_s, or $\sigma(s) = x$ and σ and τ are of opposite types at x.*

Proof. Conditions (1) and (2) are equivalent since nodal curves are of finite presentation. We will now prove that (1) and (3) are equivalent. This can be done assuming $S = \operatorname{Spec} R$ is strictly local, with closed point s. We can also assume $\sigma(s) = x$ since otherwise, the equivalence of (1) and (3) follows from the fact ϕ is an isomorphism away from x.

Under these additional hypotheses, let us assume (3) and prove (1). We know σ factors uniquely through $\operatorname{Spec} \mathcal{O}^{et}_{X,x}$.

Let us note $\widehat{S} = \operatorname{Spec} \widehat{R}$. We have the following commutative diagram:



where $W_0 = \phi^* \operatorname{Spec} \mathcal{O}^{et}_{X,x} \times_S \widehat{S}$ and $W = W_0 \times_{X \times_S \widehat{S}} \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$, so that all squares are pullbacks, and σ' is the strict transform of σ in X'. Then σ' is a rational map (defined at least over U) and our goal is to prove that it is defined everywhere.

There are sections $\widehat{\sigma}$ and $\widehat{\tau}$ of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}} / \widehat{S}$ induced by σ and τ respectively. Pick an isomorphism

$$\operatorname{Spec} \widehat{\mathcal{O}_{X,x}} \simeq \widehat{R}[[u, v]]/(uv - \Delta\Delta'),$$

where the comorphism of $\widehat{\tau}$ sends u, v to Δ, Δ' respectively. The section $\widehat{\sigma}$ is fully described by the images t_1 of u and t_2 of v in $\widehat{R}$ by its comorphism. Since σ and τ have opposite types at x, there is a unit λ such that $t_1 = \lambda\Delta'$ and $t_2 = \lambda^{-1}\Delta$.

We claim that $\widehat{\sigma}$ factors through $W \to \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$. Since $W \to \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$ is the blow-up in the ideal $I_{\widehat{\tau}} = (u - \Delta, v - \Delta')$ defining $\widehat{\tau}$, by the universal property of blow-ups ([30, Tag 085U]), it suffices to show that the pull-back of $I_{\widehat{\tau}}$ to $\widehat{S}$ by $\widehat{\sigma}$ is Cartier. Blow-ups commute with completions, so our claim reduces to proving that the ideal $(u - \Delta, v - \Delta)$ of

$$A := \widehat{R}[u, v]/(uv - \Delta\Delta')$$

becomes invertible in $\widehat{R}$ when we map A to $\widehat{R}$ via

$$\begin{aligned} A &\to \widehat{R} \\ u &\mapsto \lambda\Delta' \\ v &\mapsto \lambda^{-1}\Delta. \end{aligned}$$

The image of $I_{\widehat{\tau}}$ under this map is the ideal $I = (\lambda\Delta' - \Delta)$. If $\lambda\Delta' \neq \Delta$, then I is invertible and the claim holds. Otherwise, we reduce to this case by observing that the blow-up of A in $(u-\Delta, v-\Delta')$ is canonically isomorphic to the blow-up of A in $(u - \mu\Delta, v - \mu^{-1}\Delta')$ for any unit μ of $\widehat{R}$, as can be seen in the proof of Corollary 4.7.

Now, let us check that σ factors through X' if and only if $\widehat{\sigma}$ factors through W. Looking at the diagram above, we see that a factorization of $\widehat{\sigma}$ through W yields a factorization of $\sigma \times_S \widehat{S}$ through W_0, which means the (faithfully flat) base change to $\widehat{S}$ of the rational map σ' is defined everywhere, so σ' itself is defined everywhere. Conversely, if σ' is an actual S-section, it yields a section from $\widehat{S}$ to a completed local ring of X', and all completed local rings of X' at points above x factor through W.

We have proven σ factors through a section $\sigma' \colon S \to X'$. We need to show this section is smooth. It suffices to show $\sigma'(s)$ is a smooth point of X'/S. Call E the preimage of x in X'_s. The point $\sigma'(s)$ must be in E since $\sigma(s) = x$. Looking at the local description of X' in the proof of Lemma 4.6, we see E contains exactly two non-smooth points y and y' of X_s, and there is an isomorphism $\widehat{\mathcal{O}_{X',y}} = \widehat{R}[[\beta, v]]/(\beta v + \Delta)$ such that the natural map $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{X',y}}$ sends u, v to $\beta(v - \Delta') + \Delta$ and v respectively. It follows that $\sigma'(s) = y$ if and only if t_2 strictly divides Δ, i.e. if and only if Δ' strictly divides t_1. Symmetrically, $\sigma'(s) = y'$ if and only if Δ strictly divides t_2. Thus, $\sigma'(x)$ is in the smooth locus of X'/S as claimed, and we have proven (3) implies (1).

For the converse, suppose σ comes from a section $\sigma' \colon (X'/S)^{sm}$. By our additional hypothesis that $\sigma(s) = x$, we know $\sigma'(s)$ is a point of E that is neither y nor y', and it follows from the discussion in the paragraph above that σ and τ are of opposite types at x. □

7.2 First construction of the ns-Néron model

In the previous subsection, we have seen how to factor (at least locally) one section of X to the smooth locus of some refinement of X. If we want to approach the Néron mapping property, we would rather have a smooth model of X_U, mapping to X, through which *all* sections will simultaneously factor. Intuitively speaking, we need this model to contain the smooth loci of all possible refinements of X, at all singular points and of all types, after any smooth base change. We will now present the formal construction.

Construction 7.5. Let S be a regular and excellent scheme and X/S a quasisplit nodal curve, smooth over a dense open U of S. For each point s of S, pick an admissible neighbourhood $V^{(s)}$ of s in S as in Definition 3.8. We will write $V^{(s,s')}$ the fiber product $V^{(s)} \times_S V^{(s')}$. For each s and each singular point x of

X_s, pick an orientation of $X_{V^{(s)}}$ at x. For each type T at x, pick a $V^{(s)}$-section $\tau^{(x,T)}$ of $X_{V^{(s)}}$ of type T at x, and write $X^{(x,T)} \to X_{V^{(s)}}$ the blowing-up in that section. Write

$$X^{tot} = \coprod_{(s,x,T)} (X^{(x,T)}/V^{(s)})^{sm}.$$

Then X^{tot} is a X-scheme, smooth over S. Consider two index triples (s, x, T) and (s', x', T') and call R' the same type locus of $\tau^{(x,T)}|_{V^{(s,s')}}$ and $\tau^{(x',T')}|_{V^{(s,s')}}$. Then R' is an open subscheme of $X_{V^{(s,s')}}$ by Proposition 3.6, and the pull-back $R^{(x,T,x',T')}$ of R' to $(X^{(x,T)}_{V^{(s,s')}}/V^{(s,s')})^{sm}$ is canonically isomorphic to the pull-back of R' to $(X^{(x',T')}_{V^{(s,s')}}/V^{(s,s')})^{sm}$ by Corollary 4.7. Therefore, we have étale maps

$$R^{(x,T,x',T')} \to (X^{(x,T)}/V^{(s)})^{sm}$$
$$R^{(x,T,x',T')} \to (X^{(x',T')}/V^{(s')})^{sm}.$$

These maps define an étale equivalence relation on X^{tot}. We write N the quotient algebraic space (see [30, Tag 02WW]), and call it the *smooth aggregate* of X.

Proposition 7.6. *With the same hypotheses and notations as in Construction 7.5, N is well-defined, smooth over S, and depends only on X (i.e. if one makes different choices of admissible neighbourhoods $V^{(s)}$ and of sections $\tau^{(x,T)}$, the resulting smooth aggregate N' is canonically isomorphic to N). The map $N \to X$ is an isomorphism above the smooth locus of X/S.*

Proof. First, let us prove that N is well-defined, i.e. that we have indeed given an étale equivalence relation on X^{tot}. For any pair of index triples (s, x, T) and (s', x', T'), the maps

$$R^{(x,T,x',T')} \to (X^{(x,T)}/V^{(s)})^{sm},$$
$$R^{(x,T,x',T')} \to (X^{(x',T')}/V^{(s')})^{sm}$$

are étale since $V^{(s)} \to S$ and $V^{(s')} \to S$ are. These maps jointly form an étale equivalence relation since for any quasisplit nodal curve Y/R with R regular, and any singular point y at which Y/R is orientable, "having the same type at y" is an equivalence relation on the set of sections $R \to Y$. Since X^{tot} is S-smooth, N is also S-smooth. The fact that $N \to X$ is an isomorphism above the smooth locus of X/S follows from observing that all $X^{(x,T)} \to X_{V^{(s)}}$ are isomorphisms above said smooth locus, and that the $V^{(s)}$ form an étale cover of S.

Now, we have to show N only depends on X. For every (s, x, T), consider another admissible neighbourhood $W^{(s)}$ of s and a section $\sigma^{(x,T)}$ of $X_{W^{(s)}}$ of type T at x. This gives rise to another smooth aggregate N', and we will prove N and N' are canonically isomorphic. We can assume $V^{(s)}$ and $W^{(s)}$ are admissible neighbourhoods of the same geometric point $\bar{s} \to S$ mapping to s.

First, we will do so assuming that the $W^{(s)}$ are smaller than the $V^{(s)}$ and that the $\sigma^{(x,T)}$ are obtained from the $\tau^{(x,T)}$ via pullback. In that case, there is a canonical map from $X'^{tot} := \coprod_{(s,x,T)} (X^{(x,T)}/V^{(s)})^{sm}$ to X^{tot}, compatible with the étale equivalence relations defining N and N', so we get a canonical map $N' \to N$ of S-algebraic spaces. This map restricts to an isomorphism over the étale stalks of all geometric points $\bar{s} \to S$, so it is an isomorphism.

Now, let us drop the assumption that the $\sigma^{(x,T)}$ are obtained from the $\tau^{(x,T)}$ via pullback. For all (s,x,T), by the special case proven above, we can assume $V^{(s)} = W^{(s)}$. Using Proposition 3.6 and the special case proven above, we can assume (shrinking $V^{(s)}$ if necessary) that $\sigma^{(x,T)}$ and $\tau^{(x,T)}$ have the same type everywhere. It follows from Corollary 4.7 that the blowing-ups in the sheaves of ideals of $\sigma^{(x,T)}$ and $\tau^{(x,T)}$ are canonically isomorphic, and this holds for all (s,x,T), so $N = N'$ by construction.

Finally, we also drop the assumption that there exist maps of étale neighbourhoods $W^{(s)} \to V^{(s)}$. Then, N and N' are still canonically isomorphic by the special cases above since $W^{(s)} \times_S V^{(s)}$ is an admissible neighbourhood of s that factors through both $V^{(s)}$ and $W^{(s)}$. □

Proposition 7.7. *The formation of smooth aggregates commutes with smooth base change, i.e. if S is a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, N the smooth aggregate of X/S, and Y/S a smooth morphism of schemes, then $N \times_S Y$ is the smooth aggregate of X_Y/Y.*

Proof. Immediate from Proposition 3.10 and Proposition 7.6. □

Corollary 7.8. *If S is a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, N the smooth aggregate of X/S, and Y/S a cofiltered limit of smooth morphisms, then $N \times_S Y$ is the smooth aggregate of X_Y/Y.*

Proposition 7.9. *Let S be a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, and N the smooth aggregate of X/S. Then every S-section of X/S factors uniquely through N.*

Proof. First, we prove uniqueness: suppose σ comes from two sections σ_0, σ_1 of N/S an let us show $\sigma_0 = \sigma_1$. Since σ_0 and σ_1 coincide on $N_U = X_U$, it is enough to show that for any $t \in S$ we have $\sigma_0(t) = \sigma_1(t)$. This can be done assuming S is strictly local with closed point t. Describe N as in Construction 7.5 using admissible neighbourhoods $V^{(s)}$ of every point s of S and sections $\tau^{(x,T)}$ of $X_{V^{(s)}}$ of type T at x for every singular point x of X_s and every type T at x. By Proposition 7.6, we can assume none of the $V^{(s)}$ contains t except $V^{(t)}$ and $V^{(t)} = S$. Put $y = \sigma(t)$. If y is a smooth point of X/S, then σ factors through the smooth locus of X/S, above which $N \to X$ is an isomorphism, so we are done. Otherwise, by Lemma 7.4, we see that σ_0 and σ_1 must both factor through the Zariski-open subscheme $X^{(t,y,T)}$ of N, where T is the type at y opposite to that of σ. Since $X^{(t,y,T)}$ is a nodal curve over S, it is S-separated, and we conclude using the fact σ_0 and σ_1 coincide over U.

Next, we have to show existence. We recycle the notations of Construction 7.5. By descent, using the uniqueness part we have already proven, it is enough to show that for all $s \in S$, the section $\sigma^{(s)} := (\sigma, \mathrm{Id})$ of $X_{V^{(s)}}/V^{(s)}$ comes from a map $V^{(s)} \to N$. Put $y = \sigma(s)$. By Lemma 7.4 and Proposition 7.6, we can assume (shrinking $V^{(s)}$ if necessary) that $\sigma^{(s)}$ factors through $X^{(y,T)}$, where T is the type at y opposite to that of σ, so we are done. □

The properties of smooth aggregates proven above allow us to see them as the solutions of a universal problem:

Proposition 7.10. *Let S be a regular, excellent scheme and X/S a quasisplit nodal curve. Then for any smooth S-algebraic space Y together with a morphism $f\colon Y \to X$, f factors uniquely through the canonical map $N \to X$.*

Proof. The section (f, Id) of X_Y/Y factors uniquely through N_Y by Proposition 7.9 since the latter is the smooth aggregate of X_Y/Y by Proposition 7.7. Projecting onto N, we get the unique map $Y \to N$ through which f factors. □

Corollary 7.11. *Let S be a regular and excellent scheme and $X' \to X$ a morphism between two quasisplit nodal curves over S. Let N be the smooth aggregate of X, then $N \times_X X'$ is the smooth aggregate of X'.*

Now, we are equipped to prove the following result, which is a weak version of our main theorem of existence for ns-Néron models of nodal curves:

Proposition 7.12. *Let S be a regular and excellent scheme and X/S a nodal curve, smooth over a dense open subscheme U of S, with no rational component in any geometric fiber. Then X_U has a ns-Néron model N/S, and there is a canonical morphism $N \to X$ of models of X_U. When X/S is quasisplit, N is the smooth aggregate of X.*

Proof. By Lemma 1.14, Proposition 5.6 and Lemma 5.5, we can assume X/S is quasisplit and S is integral. Let N be the smooth aggregate of X/S. Then N is a smooth S-model of X_U with a canonical S-map $N \to X$. Consider a smooth S-scheme Y, then we have

$$\begin{aligned}\mathrm{Hom}_U(Y_U, X_U) &= \mathrm{Hom}_S(Y, X)\\ &= \mathrm{Hom}_S(Y, N),\end{aligned}$$

where the first equality holds by Proposition 7.3 applied to the connected components of X_Y/Y, and the second by Proposition 7.7. Thus, N/S has the Néron mapping property. □

The remainder of this section will be dedicated to improving Proposition 7.12 by weakening the hypothesis that the geometric fibers of X/S have no rational components, and determining conditions under which N/S is separated, i.e. a Néron(-lft) model in the classical sense.

7.3 Exceptional components and minimal proper regular models

It is known that an elliptic curve over the fraction field of discrete valuation ring has a Néron model, given by the smooth locus of its minimal proper regular model. It is proven in [23] that the same holds for any smooth curve of positive genus. In particular, rational components of the special fiber that can be contracted to smooth points have a "special status": they must map to a mere point of the Néron model. Therefore, if one wants to weaken the hypotheses of Proposition 7.12 to allow for rational components, one must take this phenomenon into account. Over a discrete valuation ring, these components of the special fiber that can be contracted to smooth points, the so-called *exceptional components*, are characterized by Castelnuovo's criterion ([22], Theorem 9.3.8). This criterion uses intersection theory on fibered surfaces, so is not easy to generalize to higher-dimensional bases for arbitrary relative curves, but the nodal case is much simpler. We will discuss the analogue of the notion of exceptional components for nodal curves over arbitrary regular base schemes.

7.3.1 Definition

Definition 7.13. Let k be a separably closed field and X_k/k a nodal curve. Define a sequence of subsets of the (finite) set I of irreducible components of X_k by $J_0 = \emptyset$, and for all $n \in \mathbb{N}$, J_{n+1} is the subset of I consisting of components C meeting one of the following conditions:

- C is in J_n;
- C is rational and k-smooth, and intersects $\left(\bigcup_{D \in I - J_n - \{C\}} D\right)$ in exactly one point.

The sequence $(J_n)_{n \in N}$ is increasing, so it is stationary at some subset J of I, which we call the set of *exceptional components* of X.

We call *exceptional trees* the connected components of $\bigcup_{C \in J} C$.

A non-smooth point of X/k is called *exceptional* if it belongs to at least one exceptional component.

When X/S is a nodal relative curve, smooth over a schematic dense open $U \subset S$, we call *exceptional point* of X a singular point, exceptional in a fiber of X over a separably closed field-valued point of S.

If X/S is quasisplit, for any $s \in S$, we define the exceptional components (resp. exceptional points, resp. exceptional trees) of X_s as those giving rise to the exceptional components (resp. points, resp. trees) of $X_{\bar{s}}$ for some geometric point $\bar{s} \to s$.

Remark 7.13.1. Neither the components of X lying in a cycle of the dual graph, nor its components of genus ≥ 1 are exceptional. In particular, the exceptional trees correspond to actual trees of the dual graph, and they are not covering as soon as X is of genus ≥ 1.

7.3.2 The minimal proper regular model

Here, we discuss briefly the case of one-dimensional bases, where there is a canonical *minimal proper regular model* of which the Néron model is the smooth locus.

Proposition 7.14. *Let R be a discrete valuation ring, with field of fractions K and residue field k, and X_K a smooth K-curve of genus ≥ 1. Then X_K admits a unique minimal proper regular model X_{min} over S (i.e. X_{min} is a terminal object in the category of proper regular S-models of X_K).*

Moreover, if X_K has a regular nodal model X, then X_{min} is nodal and the map $X \to X_{min}$ is just a contraction of every exceptional tree of the special fiber of X into a smooth point (i.e. the image of an exceptional tree of the special fiber of X is a smooth point of X_{min}/S, and $X \to X_{min}$ restricts to an isomorphism over the rest of X_{min}).

Proof. The existence of the minimal proper regular model is [22], Theorem 9.3.21.

For the second part of the proposition, suppose X_K has a nodal regular model X/S. It follows from [22], Definition 3.1 and Theorem 3.8, that there exists a regular proper model X'/S of X_K and a map $X \to X'$ that is just a contraction of every exceptional tree into a smooth point. In particular, X'/S is nodal. But then X' is relatively minimal in the sense of [22], Definition 3.12, so it is X_{min} and we are done. □

Theorem 7.15 ([23], Theorem 4.1.)**.** *Let S be a connected Dedekind scheme (i.e. a regular scheme of dimension 1) with field of functions K. Let X_K/K be a proper regular connected curve of genus ≥ 1. Suppose either S is excellent, or X_K/K is smooth, and let X_{min} be the minimal proper regular S-model of X_K. Then $(X_{min}/S)^{sm}$ is the Néron model of $(X_K/K)^{sm}$.*

7.3.3 Van der Waerden's purity theorem

We will define the exceptional locus of a birational morphism, and cite a result of purity of this exceptional locus when the target is factorial. This will allow us to describe explicitly some open subsets of the ns-Néron model (when it exists) of a curve with a nodal model.

Definition 7.16. Let $f : X \to Y$ be a morphism locally of finite type between two locally noetherian algebraic spaces. We say f is a *local isomorphism* at some $x \in X$ when f induces an isomorphism $\mathcal{O}_{Y,f(x)} = \mathcal{O}_{X,x}$ (or, equivalently, if x has a Zariski open neighbourhood $V \subset X$ such that f induces an isomorphism from V onto its image in Y). The set of all points at which f is a local isomorphism is an open subscheme W of X, and we call its complement the *exceptional locus* of f. If $W = X$, we say f is a *local isomorphism*.

Example 7.17. Let Y be a noetherian integral scheme and $X \to Y$ be the blowing-up along a closed subscheme $Z \to Y$ of codimension ≥ 1. Then the exceptional locus of $X \to Y$ is the preimage of the set of all $z \in Z$ around which Z is not a Cartier divisor.

Example 7.18. Let k be a field, and glue two copies of the identity $\mathbb{A}^1_k \to \mathbb{A}^1_k$ along the complement of the origin. The resulting map $A \to \mathbb{A}^1_k$, where A is the affine line with double origin, is a local isomorphism.

In Example 7.18, the birational map $f\colon A \to \mathbb{A}^1_k$ has empty exceptional locus, but it is not separated, so in particular not an open immersion. In the following lemma, we will see that non-separatedness is essentially the only possible obstruction preventing such maps from being open immersions.

Lemma 7.19. *Let $f\colon X \to Y$ be a separated local isomorphism between two locally noetherian integral algebraic spaces. Then f is an open immersion.*

Proof. We need to show f is injective. Call η the generic point of X. Since f is a local isomorphism at η, we know $f(\eta)$ is the generic point of Y. Consider two points x, x' of X with the same image y in Y. There are Zariski-open neighbourhoods U, U' of x and x' respectively, such that $U \to Y$ and $U' \to Y$ are open immersions. By separatedness of f, the canonical map $U \times_X U' \to U \times_Y U'$ is a closed immersion. But it follows from the fact f is a local isomorphism that $U \times_Y U'$ is integral, with generic point (η, η). Since this point is in the image of $U \times_X U'$, the map $U \times_X U' \to U \times_Y U'$ is an isomorphism, so the point $(x, x') \to X \times_Y X'$ factors through $U \times_X U'$, i.e. $x = x'$. □

Theorem 7.20 (Van Der Waerden)**.** *Let X, Y be locally noetherian integral schemes with Y locally factorial and $f: X \to Y$ a birational morphism of finite type. Then the exceptional locus of f is of pure codimension one in X.*

Proof. This is [14], Theorem 21.12.12. □

Lemma 7.21. *Let S be a regular scheme, U a dense open subscheme of S, and X/S a quasisplit nodal curve, smooth over U. Let E be the union in X of the exceptional components of all fibers X_s (which are well-defined by quasisplitness). Suppose that X_U admits a ns-Néron model N/S, then the map $(X \backslash E)^{sm} \to N$ extending the identity over U is an open immersion.*

Proof. The scheme $(X\backslash E)^{sm}$ is separated over S, hence separated over N. Therefore, using Lemma 7.19, we only need to prove the exceptional locus of $(X\backslash E)^{sm} \to N$ is empty. The subset E is Zariski-closed in X by 1.8. The unique morphism of algebraic spaces $g: X^{sm} \to N$ extending the identity over U is birational and of finite type, and the domain and codomain are S-smooth, hence regular. We only have to show that its exceptional locus E_0 is contained in E. Take an étale cover $V_0 \to N$ where V_0 is a scheme, it is enough to show $E_0 \times_N V_0 \subset E \times_N V_0$. Therefore, it is enough to prove that for any integral scheme V and any étale map $V \to N$, we have $E_0 \times_N V \subset E \times_N V$. The scheme V is smooth over S so it is regular, and $g_V : X^{sm} \times_N V \to V$ is birational and of finite type since g is. Furthermore, since the property "being an isomorphism" is local on the target for the fpqc topology, the exceptional locus of g_V

is precisely $E_0 \times_N V$. Thus $E_0 \times_N V$ is either empty or pure of codimension one in $X^{sm} \times_N V$ by Theorem 7.20. If it is empty, we are done. Otherwise, since $E \times_X X^{sm} \times_N V$ is closed in $X^{sm} \times_N V$, it is enough to prove that every point of $E_0 \times_N V$ of codimension 1 in $X^{sm} \times_N V$ is contained in $E \times_N V$. Since $V \to N$ is an étale cover, this is true if and only if every point of E_0 of codimension 1 in X^{sm} is contained in E.

Let x be a point of E_0 of codimension 1. Let ξ be the image of x in S, we have $\operatorname{codim}(x, X) = \operatorname{codim}(x, X_\xi) + \operatorname{codim}(\xi, S)$, so ξ has codimension ≤ 1 in S. Since X/S is smooth over U, ξ cannot be of codimension 0 in S, so it must be of codimension 1: $\mathcal{O}_S, \xi$ is a discrete valuation ring.

But then $N \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ is the $\operatorname{Spec} \mathcal{O}_{S,\xi}$-Néron model of its generic fiber by Proposition 5.4, so it is the smooth locus of the minimal proper regular model of $X \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ by Theorem 7.15. Now, by Proposition 7.14, the minimal proper regular model of $X \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ is the contraction of the exceptional trees of its special fiber into smooth points, and in particular contains $(X \backslash E) \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ as an open subscheme. This implies g is an isomorphism at every point of $(X \backslash E)^{sm} \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$, so x must be in E. □

Remark 7.21.1. With hypotheses and notations as in Lemma 7.21, if E is empty, it follows that the canonical morphism $N_0 \to N$, where N_0 is the smooth aggregate of X, is an open immersion. We will see in the next subsection that one can always reduce to this situation: if E is not empty, one can always contract X into a new nodal model of X_U with no exceptional components. However, we will also see that nodal models with no rational components at all do not always exist, so ns-Néron models cannot always be easily described in terms of smooth aggregates.

7.4 Contractions and stable models

So far, we have met two features of a nodal curve X/S, smooth over a dense open $U \subset S$, that can cause problems for us: one is the complexity of its singularities (for example because there can be sections through a singular point of positive arithmetic complexity, meaning we lose relevant information if we take the smooth locus and forget this point), and the other one is the presence of rational components in its geometric fibers (in the absence of such components, we can construct explicitly a ns-Néron model, see 7.2). Refinements allow us to "turn the first problem into the second": we get a new model of X_U with less complex singularities, but more rational components. Looking at Proposition 7.12, it is clear that we also have an interest in the inverse problem: if X/S has rational components, is it possible to blow them down and obtain a new nodal model with more complex singularities, but less rational components?

This question finds its answer in [7], in which the author introduces and studies contraction morphisms for the moduli stacks of n-pointed stable curves. In this subsection, we will see how this translates into the algorithm we need.

7.4.1 The stack of n-pointed stable curves and the contraction morphism

Definition 7.22 ([7], Definition 1.1.)**.** Let n, g be natural integers such that $2g-2+n > 0$. A *n-pointed stable curve of genus g over S* is a nodal relative curve X/S of genus g, together with n pairwise disjoint sections $\sigma_1, ..., \sigma_n \colon S \to X^{sm}$, such that for every geometric fiber X_s and every nonsingular rational component C of X_s, the sum of the number of intersection points between C and the union of all other irreducible components of X_s, and of the number of σ_i passing through C, is at least 3. When the sections are clear from context, we will sometimes omit them in the notation.

We define a *morphism* between two n-pointed stable curves $(X'/S', \sigma'_1, ..., \sigma'_n)$ and $(X/S, \sigma_1, ..., \sigma_n)$ as a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{g} & S \end{array}$$

such that $f\sigma'_i = \sigma_i g$ for all i.

Remark 7.22.1. The condition on the number of special points appearing on a rational component aims to guarantee that the S-automorphism group of X is finite.

Theorem 7.23. *Call $\mathcal{M}_{g,n}$ the category of stable n-pointed curves of genus g. As a category fibered in groupoids over schemes, it is a separated Deligne-Mumford stack, smooth and proper over* $\operatorname{Spec} \mathbb{Z}$*.*

Proof. This is [7], Theorem 2.7. □

Definition 7.24 ([7], Definition 1.3.)**.** Let S be a scheme, g a natural integer, and $f : X \to X'$ a morphism of S-schemes between stable pointed S-curves of genus g. It is called a *contraction*, or *contraction of X*, if:

- X is $n+1$-pointed and X' is n-pointed, with $2g-2+n > 0$, and their respective sections $(\sigma_i)_{1 \le i \le n+1}, (\sigma'_i)_{1 \le i \le n}$ satisfy $f \circ \sigma(i) = \sigma'(i)$ for all $0 \le i \le n$.
- For any geometric point $s \in S$, either $X_s \to X'_s$ is an isomorphism, or $\sigma_{n+1}(s)$ is in a rational component C of X_s such that $f(C)$ is a point $x \in X'_s$, and $X_s \backslash C \to X'_s \backslash \{x\}$ is an isomorphism.

Remark 7.24.1. We do not use the same notion of geometric point as [7], but the two subsequent definitions of contractions are equivalent by [22], Proposition 10.3.7.

Theorem 7.25. *Let S be a scheme and X/S a $n+1$-pointed stable curve of genus g with $2g-2+n > 0$. Then X admits a contraction, unique up to a canonical isomorphism.*

Proof. This is [7], Proposition 2.1. □

7.4.2 The stable model

Definition 7.26. Let S be a scheme and $U \subset S$ a scheme-theoretically dense open. Let X_U/U be a smooth curve of genus $g \geq 2$. We call *stable S-model* of X_U a 0-pointed stable curve X of genus g with an isomorphism $X \times_S U = X_U$.

Lemma 7.27. *Let S be a normal, noetherian and strictly local scheme, $U \subset S$ a scheme-theoretically dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Then X_U has a unique stable model X^{stable}, and there is a unique morphism of models $X \to X^{stable}$.*

Proof. Let $s \in S$ be the closed point. The fiber X_s has finitely many rational components, and there are infinitely many disjoint smooth sections of X/S through each of those. If a rational component E of X_s contains only one singular point, consider two disjoint smooth sections through E, and if it contains two singular points, consider one smooth section through E. This gives a finite number $\sigma_1, ..., \sigma_n$ of sections through X^{sm}. Applying Proposition 1.8, and using the fact that the irreducible components of X are geometrically irreducible by quasisplitness, we see that for any geometric fiber X_t of X/S, any rational component of X_t intersecting the other components in two points contains $\sigma_i(t)$ for some i, and any rational component of X_t intersecting the other components in one point contains $\sigma_i(t)$ and $\sigma_j(t)$ for some $i \neq j$.

Thus X/S endowed with the $(\sigma_i)_{0 \leq i \leq n}$ becomes a n-pointed stable curve of genus g, restricting over U to the data of X_U and the n U-sections $\sigma_i|_U : U \to X_U$, and we can apply repeatedly Theorem 7.25 to get a stable 0-pointed curve $X' \to S$ with a map $X \to X'$. Since X_U/U is smooth, the restriction of $X \to X'$ to U just forgets the sections (and induces an isomorphism on the curves), so $X^{stable} := X'$ is a stable model of X_U.

Consider two stable models of X_U, corresponding to two maps $a, b \colon S \rightrightarrows \mathcal{M}_{g,0}$ extending $U \to \mathcal{M}_{g,0}$. Call Z the equalizer of f and g, we have a cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle (a,b)} \\ \mathcal{M}_{g,0} & \longrightarrow & \mathcal{M}_{g,0} \times \mathcal{M}_{g,0} \end{array}$$

where the bottom arrow is the diagonal. Since $\mathcal{M}_{g,0}$ is Deligne-Mumford and separated, its diagonal is finite. Thus, $Z \to S$ is a finite and birational morphism of algebraic spaces, hence an isomorphism by Zariski's main theorem: the stable model X^{stable} is unique up to a unique isomorphism. As for uniqueness of the morphism $X \to X^{stable}$ of models of X_U, let f, g be two such morphisms, then their equalizer is a closed subscheme of X (by separatedness of X^{stable}/S), which contains X_U. But X_U is scheme-theoretically dense in X since U is scheme-theoretically dense in S and X/S is flat, so f and g must be equal. □

Proposition 7.28. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$. Then*

1. *X_U has a stable S-model X^{stable}, unique up to a unique isomorphism, and there is a canonical map $X \to X^{stable}$.*

2. *The formation of X^{stable} commutes with any base change $S' \to S$ such that S' is normal and locally noetherian and $U \times_S S'$ is scheme-theoretically dense in S'.*

Proof. (2) is a consequence of Proposition 1.8. In (1), uniqueness of X^{stable} and of the map $X \to X^{stable}$ holds by the same argument as in the proof of Lemma 7.27 above. We will now prove their existence. Let s be a point of S. By Lemma 7.27, $X_U \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s}$ admits a stable model $X^{0,s}$ over $\operatorname{Spec} \mathcal{O}^{et}_{S,s}$. But then $X^{0,s}/\operatorname{Spec} \mathcal{O}^{et}_{S,s}$ is of finite presentation, thus comes via base change from a morphism $X^s \to V^s$, where V^s is an étale neighbourhood of s in S. Moreover, by [13], Proposition 8.14.2, restricting V^s if necessary, the map $X \times_S \operatorname{Spec} \mathcal{O}^{et}_{S,s} \to X^{0,s}$ extends to a V^s-map $X \times_S V^s \to X^s$. Restricting V^s once again if necessary, we take this map to be an isomorphism over U. Now, the locus on X^s where X^s/V^s is at-worst nodal is open in X^s and contains X_s so, restricting V_s again if necessary, we can assume X^s/V^s is a nodal curve. Finally, the union of all nonsingular rational components of fibers of X^s/V^s meeting the other components of their fiber in at most two points is closed in X^s, and does not meet X_s, so, restricting V_s one last time, we can assume X^s/V^s is stable. In particular, it corresponds to a morphism $V^s \to \mathcal{M}_{g,0}$.

For any $s, s' \in S$, the diagram of stacks

$$\begin{array}{ccc} V^s \times_S V^{s'} & \longrightarrow & V^s \\ \downarrow & & \downarrow \\ V^{s'} & \longrightarrow & \mathcal{M}_{g,0} \end{array}$$

commutes by uniqueness of the stable model of $X_{U \times_S V^s \times_S V^{s'}}$.

Applying a similar argument to the triple fibered products, we see the maps $V^s \to \mathcal{M}_{g,0}$ and their gluing isomorphisms satisfy the cocycle condition with respect to the (étale) covering of S by the V^s. Therefore, they come via base change from a map $S \to \mathcal{M}_{g,0}$ i.e. the X^s are obtained via base change from a stable curve X^{stable}/S (which is a model of X_U as desired). Likewise, the local maps $X \times_S V^s \to X^s$ glue to a morphism $X \to X^{stable}$. □

7.4.3 Rational components of the stable model

We will now determine conditions on X guaranteeing that X^{stable} has no rational components in any geometric fiber. When said conditions are met, this allows us to use Proposition 7.12 to describe explicitly the ns-Néron model of X_U.

Definition 7.29. Let k be a separably closed field and X/k a nodal curve. We say X *has rational cycles* if there is a union of rational components of X that is 2-connected, and *no rational cycles* otherwise. If S is a scheme and X/S a nodal curve, we say X/S *has rational cycles* if a fiber over some geometric point of S does.

Remark 7.29.1. The curve $X/\operatorname{Spec} k$ has rational cycles if and only if there is a cycle of its dual graph in which each vertex corresponds to a rational component. We call such cycles the *rational cycles* of the dual graph.

Remark 7.29.2. If $X/\operatorname{Spec} k$ is of genus $g \geq 2$ and has rational cycles, then every rational cycle of the dual graph either is a loop, or contains a rational component meeting the non-exceptional other components in at least three points.

Definition 7.30. Let k be a separably closed field and X/k a nodal curve. We call *rational loop* of X any singular rational irreducible component of X. If S is a scheme and X/S a nodal curve, we say *X/S has rational loops* if a fiber over some geometric point of S does.

Lemma 7.31. *Let k be a separably closed field, $(Y/k, y_1, ..., y_{n+1})$ a stable $n+1$-pointed curve of genus g over k with $2g-2+n > 0$, and $Y \to Z$ the contraction. Then Z has rational cycles if and only if Y does.*

Proof. If $Y \to Z$ is an isomorphism of schemes, it is obvious. Otherwise, there is a rational component C of Y whose image is a point $z \in Z$, and $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism. If C does not belong to a 2-connected union of rational components of Y, we are done. Otherwise, let Γ be said union, the image of Γ in Z is still 2-connected and still contains only rational components. □

Corollary 7.32. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$. Then X^{stable} has rational cycles if and only if X does.*

Rational cycles are an example of rational components we cannot get rid of by contracting. There is another family of such "problematic components": suppose for example that a rational component intersects three other non-rational components, then no contraction will get rid of it. The ones we *can* get rid of are described in the two following lemmas:

Lemma 7.33. *Let k be a separably closed field and $(Y/k, y_1, ..., y_{n+1})$ a stable $(n+1)$-pointed curve of genus g with $2g-2+n > 0$. Suppose Y/k has no rational cycles, and $Y \to Z$ is the contraction. Let F_Y, F_Z denote the union of all rational components meeting the non-exceptional other components in at most two points, in Y and Z respectively. Then $Y \backslash F_Y$ is isomorphic to its image in Z, and the image of F_Y in Z is either F_Z or the union of F_Z and a point.*

Proof. If $Y \to Z$ is an isomorphism of schemes, this is obvious. Otherwise, there is a rational component C of Y whose image is a point $z \in Z$, and $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism. We will prove that the image of F_Y in Z is $F_Z \cup \{x\}$.

The contracted component C meets the other irreducible components of Y in at most two points so C is in F_Y. There is a canonical bijection between irreducible components of Y that are not C and irreducible components of Z (namely, it sends a component of Y to its image in Z), so we just need to prove that a component $D \neq C$ of Y is in F_Y if and only if it is sent to a component of F_Z.

If D is not rational or does not meet C, this is true. Suppose D is rational and meets C. Then $D \cap C$ is exactly one point c_1 (otherwise $D \cup C$ would be 2-connected and Y would have rational cycles).

It is enough to show that D has as many intersection points with non-exceptional other irreducible components of Y than its image D' has in Z. Since the map $Y\backslash C \to Z\backslash\{z\}$ is an isomorphism, it comes down to saying that if C is not exceptional, then the other irreducible component of Y that C meets is not exceptional either, which follows from the definition of exceptional components. □

Lemma 7.34. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$, with no rational cycles. Let s be a field-valued point of S, F the union of all rational components of X_s intersecting the non-exceptional other components in at most two points, and F' its image in X^{stable}. Then*

1. *$X \to X^{stable}$ induces an isomorphism between $X_s\backslash F$ and $X_s^{stable}\backslash F'$.*
2. *F' is a disjoint union of points, one for each connected component of F.*

Proof. By quasisplitness we can assume $k(s)$ is separably closed, and base-changing to $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ (which preserves X^{stable} by Proposition 7.28), we can assume S is strictly local, with closed point s. Then there are S-sections $\sigma_1, ..., \sigma_n$ of X making it a n-pointed stable curve. Consider the sequence $X = X_n \to X_{n-1} \to ... \to X_0 = X^{stable}$, where $X_i \to X_{i-1}$ is the contraction of σ_i. Call F_i the image of F in $(X_i)_s$ for all i, and call G_i the union of all rational components of $(X_i)_s$ meeting the non-exceptional other ones in at most two points. It follows from the preceding lemma that F_i is the union of G_i and a finite number of points. In particular, since $(X_i)_s \to (X_{i-1})_s$ induces an isomorphism from $(X_i)_s\backslash G_i$ to its image, we have $(X_i)_s\backslash F_i = (X_{i-1})_s\backslash F_{i-1}$, and (1) follows inductively. Then, (2) follows from observing that G_0 is empty: indeed, X_s^{stable} is a stable 0-pointed curve over $k(s)$, so it has no exceptional components, thus a component in G_0 would be rational, unmarked, and meet the other irreducible components in at most two points, which is forbidden by the definition of stable curves. □

Proposition 7.35. *Let S be a normal and locally noetherian scheme and X/S a quasisplit nodal curve of genus $g \geq 2$, smooth over a scheme-theoretically dense open $U \subset S$. Let X^{stable} be the stable model of X_U. Then the two following conditions are equivalent:*

1. *There is a geometric point $s \to S$ such that X_s^{stable} has a rational component.*
2. *One of the fibers of X over S contains either a rational loop, or a rational component meeting the non-exceptional other irreducible components in at least three points.*

Proof. If X has rational loops, then X^{stable} has a geometric fiber with a rational component by Corollary 7.32. If a fiber X_s has a rational component E meeting the other non-exceptional other components in at least three points, then any contraction of X_s is an isomorphism over the open subscheme $E \cap (X/S)^{sm}$ of

X_s, so X_s^{stable} has a rational component (thus some geometric fiber of X^{stable} also does).

Conversely, suppose X has no rational loops, and every rational component of a fiber X_s meets the other non-exceptional irreducible components of X_s in at most two points. By Remark 7.29.2, X has no rational cycles, so we conclude with Lemma 7.34. □

7.4.4 Singular ideals of the stable model

Now we want to understand precisely what the stable model looks like as a nodal curve, i.e. we want to compute its singular ideals. This will be made clear by Lemma 7.39. We build up to it with a few technicalities.

Lemma 7.36. *Let $S = \operatorname{Spec} R$ be a trait with generic point η. Let X be a nodal S-model of a smooth η-curve X_η. Then the sum of thicknesses of non-exceptional singular points of X is the number of singular points of the special fiber of X_{min}, where X_{min} is the minimal proper regular model of X_η.*

Proof. The sum of thicknesses of non-exceptional singular points does not change when one blows up in a singular point of the closed fiber, and after a finite sequence of such blow-ups, we obtain a regular model $X_{reg} \to X$ of X_η. Therefore, we can assume X is regular. But then, Proposition 7.14 allows us to conclude. □

Corollary 7.37. *The sum of thicknesses of non-exceptional singular points is a birational invariant for generically smooth nodal curves over a discrete valuation ring.*

Proposition 7.38. *Let $S = \operatorname{Spec} R$ be a regular local scheme with closed point s, U a dense open subscheme of S and X/S, Y/S two quasisplit nodal curves, with $X_U = Y_U$ smooth over U. Then the product of singular ideals of non-exceptional points of X_s is equal to that of Y_s.*

Remark 7.38.1. If we call *thickness* of a singular point its singular ideal, and note additively the monoid of principal prime ideals of R, we can rephrase this "the sum of thicknesses of non-exceptional points in the closed fiber is a birational invariant for generically smooth quasisplit nodal curves over a regular local ring".

Proof. Since R is regular, it is a unique factorization domain. Let $\Delta_1, ..., \Delta_k$ be the prime elements of R such that the generic point of $\{\Delta_i = 0\}$ is not in U. Every singular point of X_s has singular ideal of the form $\left(\prod_{i=1}^{k} \Delta_i^{\nu_i}\right)$ for some integers ν_i, not all zero, and the same goes for Y. Therefore, if we call λ, μ the products of all singular ideals of non-exceptional points of X_s and Y_s respectively, we have integers $n_1, .., n_k, m_1, ..., m_k$ with $\lambda = \left(\prod_{i=1}^{k} \Delta_i^{n_i}\right)$ and $\mu = \left(\prod_{i=1}^{k} \Delta_i^{m_i}\right)$, and we only need to show $n_i = m_i$ for all i.

Pick some $1 \leq i \leq k$, let t be the generic point of the zero locus of Δ_i in S and set $T = \operatorname{Spec} \mathcal{O}_{S,t}$. Base-changing to T, we get nodal curves X_T, Y_T, with the same smooth generic fiber. Proposition 1.8 implies that the sum of thicknesses of their non-exceptional singular points are respectively n_i and m_i, but they must be equal by Corollary 7.37, so $n_i = m_i$ for all i and we are done. □

The next lemma describes how to compute the singular ideals of X^{stable} from the singular ideals of X.

Lemma 7.39. *Let X/S be a quasisplit nodal curve with no rational cycles, of genus ≥ 2, with S regular. Suppose X is smooth over a dense open $U \subset S$. Let s be a point of S and $F \subset X_s$ be the union of all rational components of X_s intersecting the non-exceptional other irreducible components in at most two points. Call Z the set of non-exceptional singular points of X_s. The image in X^{stable} of a connected component G of F is a smooth point if all singular points in G are exceptional, and a singular point of label $\prod_{y \in Z \cap G} l(y)$ otherwise, where we note $l(y)$ the label of y.*

Remark 7.39.1. To put this in simpler words, in the formalism of Remark 7.38.1, with the additional convention that the points of thickness 0 are the S-smooth points, Lemma 7.39 says that the thickness of a point of X^{stable} is the sum of thicknesses of all non-exceptional singular points of X above it.

Proof. We can assume S is local, with closed point s. Let $\sigma_1, ..., \sigma_n$ be such that X/S endowed with the σ_i is in $\mathcal{M}_{g,n}$ (they exist by quasisplitness). Permuting the σ_i if necessary, we assume there is an index $0 \leq m \leq n$ such that for all $1 \leq i \leq m$, $\sigma_i(s)$ is in G, and for all $m < i \leq n$, $\sigma_i(s)$ is not in G. Consider the sequence $X_n \to ... \to X_0$, where X_n is X endowed with the σ_i, and each $X_{i+1} \to X_i$ is the contraction of σ_{i+1}: we have $X_0 = X^{stable}$. Call F_i, Z_i the images of F, Z in $(X_i)_s$. For all $y \in Z_i$, we call $l_i(y)$ the singular ideal of y in X_i.

Call G_i the image of G in X_i for every i. Since none of the $\sigma_i(s)$ with $1 \leq i \leq m$ are in G_i, we know that $G_m \to G_0$ is an isomorphism. But G_0 is a point, since it is connected, X_0 is stable, and all components of G are rational and meet the others in at most two points. Thus, G_m is a $k(s)$-point x of X_m.

Now, observe that for all $m < i \leq n$, $\sigma_i(s)$ is in G_i. Therefore, $X \to X_m$ induces an isomorphism $X \backslash G \to X_m \backslash G_m$. In particular, the product of all singular ideals of non-exceptional points of X outside of G is equal to the product of all singular ideals of non-exceptional points of X_m distinct from x. But we also know by Proposition 7.38 that the product of singular ideals of non-exceptional singular points of X and X_m are the same: it follows that if G consists only of exceptional components, then x is smooth over S, and otherwise, x is singular of label $\prod_{y \in Z \cap G} l(y)$. □

7.4.5 The main theorem

We can now generalize Proposition 7.12 by applying it to the stable model: since the latter is less likely to have rational components than the original nodal model, it is more likely to fall under the hypotheses of the proposition.

Theorem 7.40. *Let S be a regular excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Suppose X has no rational loops, and suppose no geometric fiber of X contains a rational component meeting the non-exceptional other irreducible components in three points or more. Then X_U/U has a ns-Néron model N/S. If in addition X/S is quasisplit, then N is the smooth aggregate of the stable model X^{stable} of X_U.*

Proof. We can assume X/S is quasisplit by Lemma 1.14, Proposition 5.6 and Lemma 5.5. Then X/S has a stable model X^{stable}, which has no rational components in any geometric fiber by Proposition 7.35. We conclude by applying Proposition 7.12 to X^{stable}/S. □

Remark 7.40.1. Our hypotheses on the rational components of the geometric fibers of X/S are quite unnatural, and merely come from the fact the rational components we allow are the only ones that can be contracted *while staying in the realm of nodal curves*. One could maybe get rid of these hypotheses in the following way:

1. Locally on the base if necessary, obtain a (not necessarily nodal) model of X_U with no rational components in any geometric fiber.

2. Try to see if the models obtained this way always fit into a category in which we can solve the universal problem 7.10.

Over one-dimensional bases, the standard way to contract a set E of rational components is to consider the projectivisation of the symmetric graded algebra of a very ample divisor that does not meet E. In higher dimension, however, it is not obvious that the resulting scheme would even remain flat over the base. In the case of nodal curves, this was proven for us in [7], so these subtleties are hidden behind Theorem 7.25, but in order to go beyond nodal curves, one would have to be careful about such matters.

7.5 Separatedness of the ns-Néron model

Ns-Néron models of nodal curves are often non-separated, which can make them a little difficult to work with. Here, we discuss (quite restrictive) criteria under which they *are* separated, i.e. under which a Néron model exists. Roughly speaking, the defect of separatedness of the ns-Néron model comes from the existence of non-isomorphic locally factorial models. In fact, we will show that a Néron model can only exist when there is a canonical "minimal étale-locally factorial model": this is quite similar to the case of one-dimensional bases studied in [23].

Definition 7.41. Let S be a regular scheme and $X \to S$ a nodal curve, smooth over a dense open $U \subset S$. Let s be a geometric point of S and x a singular point of X_s. We say $\operatorname{Sing}(X/S)$ is *irreducible around* x when the connected component of the singular locus of $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{et} / \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ containing x is irreducible. We say $\operatorname{Sing}(X/S)$ is *étale-locally irreducible* if it is irreducible around every singular geometric point of X. We will sometimes omit the "étale" and just say $\operatorname{Sing}(X/S)$ is *locally irreducible.*

Lemma 7.42. *With hypotheses and notations as in Definition 7.41,* $\operatorname{Sing}(X/S)$ *is irreducible around x if and only if the singular ideal of x is of the form $(\Delta)^n$, where n is a positive integer and Δ a prime element of* $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$.

Proof. The base change of X/S to $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ is quasisplit, so if we write Y the connected component of the singular locus of $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{et} / \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ containing x, then the structural morphism $Y \to \operatorname{Spec} \mathcal{O}_{S,s}^{et}$ is a closed immersion. Therefore, the irreducible components of Y are in bijection with the distinct irreducible factors of the singular ideal of x. □

Corollary 7.43. *Keeping the same hypotheses and notations,* $\operatorname{Sing}(X/S)$ *is irreducible around every étale generization of x if and only if the singular ideal of x is generated by an étale-universally prime element of* $\operatorname{Spec} \mathcal{O}_{S,s}$. *In particular, X/S has étale-locally irreducible singular locus if and only if all of its singular geometric points have a power of an étale-universally prime element as a label.*

Remark 7.43.1. With notations as above, $\operatorname{Sing}(X/S)$ is irreducible around x if and only if the radical of the singular ideal of x is generated by a prime element of $\mathcal{O}_{S,s}^{et}$: if the singular ideal of x is of the form (Δ^n) with Δ prime in $\mathcal{O}_{S,s}^{et}$ and $n > 0$, its radical is precisely (Δ).

Proposition 7.44. *Let $S = \operatorname{Spec} R$ be a strictly local unique factorization domain, $U \subset S$ a dense open subscheme, and take two non-units Δ_1, Δ_2 of R with no common prime factor. There exists a trait $T = \operatorname{Spec} A \to S$ such that*

- *The generic point of T is sent to a point of U.*
- *The special point of T is sent to the closed point of S*
- *The images in A of Δ_1 and Δ_2 are equal, nonzero, and not units.*

Proof. Take $\pi : S' \to S$ the blowing-up in the (non-invertible) ideal (Δ_1, Δ_2) of R. Call D_1 and D_2 the strict transforms of the divisors cut out in S by Δ_1 and Δ_2 respectively, and E the exceptional divisor. Let s be a closed point of S' in the zero locus of $\frac{\Delta_1}{\Delta_2} - 1$ (which is contained in $E \backslash (D_1 \cup D_2)$. By [11], Proposition 7.1.9, there exists a trait $T \to S'$ such that the closed point is mapped to s and the generic point to a point of U. The map $T \to S'$ factors through $\mathcal{O}_{S',s}$ and $\Delta_1 = \Delta_2$ in $\mathcal{O}_{S',s}$, so $T \to S$ satisfies all the desired properties. □

Remark 7.44.1. Though over one-dimensional bases, uniqueness in the Néron mapping property is already a weaker condition than separatedness, Proposition 7.44 illustrates the fact that the gap between these two conditions becomes much greater in higher base dimension. Indeed, as the base gets bigger, smooth morphisms of base change remain pretty rare and tame, while the quantity (and array of potentially wild behavior) of traits on the base gets much bigger. This is why non-separated ns-Néron models are so prevalent in higher dimension, even though, to the author's knowledge, no examples are known over a Dedekind scheme.

Lemma 7.45. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. Suppose X_U has a Néron model over S. Then $\mathrm{Sing}(X/S)$ is irreducible around every non-exceptional (Definition 7.13) singular geometric point x of X/S.*

Proof. We will work by contradiction: suppose there is some geometric point $s \in S$ and a singular point $x \in X_s$ around which $\mathrm{Sing}(X/S)$ is not irreducible. Since $X_U \times_S \mathrm{Spec}\, \mathcal{O}^{et}_{S,s}$ has a Néron model by 5.4, we can assume x is a closed point of X and $S = \mathrm{Spec}\, R$ is strictly local with closed point s. In particular, S is an admissible neighbourhood of s relatively to X/S. Let $\prod_{i=1}^{r} \Delta_i^{\nu_i}$ be the decomposition in prime factors of a generator Δ_x of the singular ideal of x in R. By hypothesis, we have $r \geq 2$.

Let (C_1, C_2) be an orientation of X/S at x. Define T_1 and T_2 to be the images in the (multiplicative) monoid $R/R^\times$ of $\prod_{i=1}^{r-1} \Delta_i^{\nu_i}$ and $\Delta_r^{\nu_r}$ respectively. They are opposite types at x. For $j \in \{1, 2\}$, consider a section $\sigma_j \colon S \to X$ of type T_j at x relatively to (C_1, C_2). We define X_j as the blowing-up of X in the ideal sheaf of σ_{1-j} (note the index). The X_j are refinements of X, in particular models of X_U, so by hypothesis there is a Néron model N for $(X_1)_U = (X_2)_U = X_U$. Call F the union of all exceptional components of X, and F_1, F_2 the preimages of F in X_1 and X_2 respectively. Since S is strictly local, X/S is quasisplit so F, F_1, F_2 are well-defined closed subsets of X, X_1, X_2 respectively. Using Lemma 4.6, we know F_i is the union of all exceptional components of X_i for $i = 1, 2$. By Lemma 7.21, the canonical maps

$$(X_i \backslash F_i)^{sm} \to N$$

are open immersions. Thus, they induce isomorphisms on open subspaces V_1, V_2 of N. Call V the open subspace $V_1 \cup V_2$ of N. Then V is isomorphic to the gluing of $(X_1 \backslash F_1)^{sm}$ and $(X_2 \backslash F_2)^{sm}$ along the preimages of $V_1 \cap V_2$ in each of them. We will conclude by proving V is not separated, which is absurd since it is an open subspace of N.

Using Lemma 7.4, we see that σ_1 factors through X_1^{sm}. Since x is not in F, σ_1 even factors through a section $\sigma_1' \colon S \to (X_1 \backslash F_1)^{sm}$. However, $X_2 \to X$ is precisely the blowing-up in the ideal sheaf of σ_1, so σ_1 does not factor through X_2. Symmetrically, σ_2 factors through a section $\sigma_2' \colon X_2^{sm} \backslash F_2$, but not through X_1. Let x_j be the image of $\sigma_j'(s)$ in N, the fact σ_j does not factor through X_{1-j} implies x_j is not in $V_1 \cap V_2$, so $x_1 \neq x_2$.

Proposition 7.44 gives a trait $T = \mathrm{Spec}\, A \to S$, with generic point η and closed point t, such that η is sent to a point of U and t to s, and such that the images in A of $\left(\prod_{i=1}^{r-1} \Delta_i^{\nu_i}\right)$ and $\Delta_r^{\nu_r}$ are equal, nonzero, and not units.

In particular, the T-sections $(\sigma_1)_T$ and $(\sigma_2)_T$ of X_T induced by σ_1 and σ_2 are equal (because $\left(\prod_{i=1}^{r-1} \Delta_i^{\nu_i}\right)$ and $\Delta_r^{\nu_r}$ are equal in A). Since η is sent to a point of U, it follows that $(\sigma'_1)_T(\eta) = (\sigma'_2)_T(\eta)$ in V_T. However, $(\sigma'_1)_T(t) = \sigma'_1(s) = x_1 \neq x_2 = \sigma'_2(s) = (\sigma'_2)_T(t)$ since x_1 is in $V_1 \backslash V_2$ and x_2 in $V_2 \backslash V_1$: V does not satisfy the valuative criterion for separatedness, a contradiction. □

Example 7.46.

- Local irreducibility of the singular locus is only necessary around non-exceptional points: for example, consider $S = \operatorname{Spec} \mathbb{C}[[a,b]]$ and $U = D(ab)$, and take the elliptic (so a fortiori nodal) curve X/S cut out in the weighted projective space $\mathbb{P}^2(2,3,2)$ by $y^2 = x(x^2 - z^2)$, so that X is the S-Néron model of X_U. Consider the blowing-up $X' \to X$ of X in the sheaf of ideals $\mathcal{I}$ given by (y, abz). Since $\mathcal{I}$ is Cartier outside of the zero locus of ab, we have $X'_U = X_U$: in particular, X'_U has a Néron model over S (namely X). However, computing the blowup explicitly, we find that X' is nodal and that its closed fiber consists of two irreducible components, intersecting in a point p of label (ab): the singular locus is not irreducible around p, but p is exceptional.
- Local irreducibility of the singular locus around non-exceptional points is not sufficient either: take X to be a nodal curve over $S = \operatorname{Spec} \mathbb{C}[[a,b]]$, whose closed fiber has two irreducible components C_1 and C_2 of genus 1, intersecting in a singular point of label (ab). In this case, X is smooth over the dense open $U = D(ab)$ of S and the singular point is not exceptional, so by (2), X_U has no Néron model over S. Take $X' \to X$ to be the (C_1, a)-refinement, this means X'_U has no Néron model over S, even though X' has étale-locally irreducible singular locus. However, we will see in Theorem 7.48 that if X has no rational components in any geometric fiber, the condition (which then just becomes "X has étale-locally irreducible singular locus") is necessary and sufficient.

As seen in Example 7.46, there are situations in which we cannot conclude to nonexistence of a Néron model by applying directly Lemma 7.45, but we can if we apply it to a different nodal model. This argument can be made systematic, and gives the following (more restrictive) necessary condition:

Lemma 7.47. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. Suppose X_U has a Néron model over S. Then the following two conditions are met:*

- *The singular locus* $\operatorname{Sing}(X/S)$ *is irreducible around every non-exceptional singular geometric point of X/S.*
- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}^{et}_{S,s}$ have the same radical.*

Proof. By Corollary 5.4, we can assume S is strictly local. In particular, X/S is quasisplit. If X/S is of genus 1, this is a special case of Proposition 6.15. Otherwise, we can apply Lemma 7.45 to the stable model X^{stable} and conclude using Lemma 7.39. □

Remark 7.47.1. In the genus 1 case, there is no 0-pointed stable model, but we could use a 1-pointed stable model instead of referring to our work on Jacobians.

One could wonder if the necessary conditions of Lemma 7.47 are still too weak. We will now show that, when we know the ns-Néron model exists, its separatedness (and therefore the existence of a Néron model) is equivalent to these conditions.

Theorem 7.48. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. If X_U has a Néron model over S, then the two following conditions are met:*

- *The singular locus $\mathrm{Sing}(X/S)$ is irreducible around every non-exceptional singular geometric point of X/S.*
- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}_{S,s}^{et}$ have the same radical.*

Conversely, suppose these conditions are met. Suppose in addition that no geometric fiber of X/S contains either a rational cycle or a rational component meeting the non-exceptional other components in at least three points. Then X_U/U has a Néron model, i.e. the ns-Néron model of X_U/U exhibited in Theorem 7.40 is separated over S.

Proof. The first part of the theorem is Lemma 7.47. We will now prove the "conversely" part. Let N/S be the ns-Néron model of X_U exhibited in Theorem 7.40. Separatedness of N/S can be checked over the étale stalks of S: we can and will assume $S = \operatorname{Spec} R$ is strictly local, and we call s its closed point. In particular, X/S is quasisplit. By Proposition 7.35, we know X^{stable} has no rational components in any geometric fiber, and by Lemma 7.39, we can assume $X = X^{stable}$ while preserving all hypotheses made on X. Then, N is the smooth aggregate of X/S. By Corollary 7.43, all prime factors of all singular ideals of X/S are étale-universally prime, so S is an admissible neighbourhood of *all* its geometric points (and not just of s). Therefore, if for every pair (x, T) where x is a singular point of X_s and T a type at x, we write $X^{(x,T)}$ the blowing-up of X in a section of type T at x, N is the gluing of the $(X^{(x,T)}/S)^{sm}$ along the strict transforms of $(X/S)^{sm}$ in each of them.

Now, for any singular point y of X_s, the singular ideal of y in R is of the form $(\Delta_y^{\nu_y})$, where Δ_y is an étale-universally prime element of R and ν_y a positive integer. Consider the morphism $X' \to X$ obtained as a composition of $\nu_y - 1$ blowing-ups in sections through points of positive arithmetic complexity above y, it follows that all the $X^{y,T}$ factor uniquely through X'. Repeating the process for every x, we find a nodal model X_{min} of X_U of arithmetic complexity 0, with a map $X_{min} \to X$, such that N is the smooth locus of X_{min}. In particular, N is separated. □

Remark 7.48.1. Here, we only constructed X^{min} locally, but when X/S is quasisplit and the hypotheses of the "conversely" part of Theorem 7.48 are met, these local models always flue into a canonical "minimal étale-locally factorial model" of X_U, of which the Néron model is the smooth locus.

Part III

Base change of Néron models along finite tamely ramified maps

8 Motivation

Given an open immersion $U \subset S$ and a smooth algebraic space over U, we can sometimes get informations about the Néron model of this space (existence, nonexistence, explicit construction if it exists) only after base change to some finite, locally free extension S'/S. Examples include smooth curves acquiring nodal reduction over S', Jacobians of smooth curves acquiring nodal reduction, and to some extent abelian varieties acquiring semi-abelian reduction (see Theorem 6.20, Theorem 7.40 and Theorem 7.48, as well as [28]). Therefore, it is interesting to have tools to turn this into information on the Néron model over S. In [4], one studies the base change behaviour of Néron models of abelian varieties over discrete valuation rings along finite tamely ramified extensions. We are interested in what happens when the base is higher-dimensional. The first complication appearing in this setting is that the existence of Néron models is no longer guaranteed. We address this problem by proving

Theorem 8.1 (Theorem 10.5). *Let $S' \to S$ be a finite, locally free, tamely ramified map between regular schemes and U the complement of a strict normal crossings divisor $D \to S$. Suppose $U' := U \times_S S'$ is étale over U.*

Let X_U be a smooth U-algebraic space, such that $X_U \times_U U'$ has a Néron model X' over S'. Then the scheme-theoretical closure of X_U in $\prod_{S'/S} X'$ is the Néron model of X_U over S (where X_U maps to $\prod_{S'/S} X'$ as in Example 9.3).

Moreover, if $S' \to S$ is a quotient for the right-action of a finite group G, then G acts on X' via its action on S': it follows that G acts on $\prod_{S'/S} X'$ as in Remark 9.4.1. In that case, the Néron model of X_U is $(\prod_{S'/S} X')^G$.

Then, when the Néron models N and N' exist, we introduce filtrations of certain strata of N (quite similar to the filtration of the closed fiber described in [4]), and describe explicitly the successive quotients of these filtrations in terms of N'. Namely, after reducing to its hypotheses by working étale-locally on the base (see Lemma 10.12), we prove the following result:

Theorem 8.2 (Theorem 10.13). *Let $S = \operatorname{Spec} R$ be a regular affine scheme,*

$f_1, ..., f_r$ regular parameters of R, $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where the n_j are invertible on S, and let $S' = \operatorname{Spec} R'$. Suppose R contains the group μ_{n_j} of n_j-th roots of unity for all j. Let U be the locus in S where all f_j are invertible, and Z the locus where all f_j vanish. Let X_U be a proper smooth U-group algebraic space with a Néron model N' over S'. Then X_U has a Néron model N over S, and we have sub-Z-group spaces $(F^d N_Z)_{d\in\mathbb{N}}$ of N_Z (see Definition 10.7 and Definition 10.10) such that:

- *For all $d \in \mathbb{N}$, $F^{d+1}N_Z \subset F^d N_Z$.*
- *$F^0 N_Z = N_Z$.*
- *If $d > \prod_{j=1}^{r} (n_j - 1)$ then $F^d N_Z = 0$.*
- *$F^0 N_Z / F^1 N_Z$ is the subspace of N'_Z invariant under the action of $G = \prod_{j=1}^{r} \mu_{n_j}$, where $(\xi_j)_{1\leq j\leq r}$ acts by multiplying T_j by ξ_j.*
- *If $d > 0$, $F^d N_Z / F^{d+1} N_Z$ is isomorphic to the fiber product over Z of the $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ where $\mathbf{k}$ ranges through all r-uples of integers $(k_1, ..., k_r)$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j; $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ is the subspace of $\mathrm{Lie}_{N'_Z/Z}$ where all $(\xi_j)_{1\leq j\leq r}$ in G act by multiplication by $\prod_{j=1}^{r} \xi_j^{k_j}$; and the map $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}] \to Z$ is given by identifying $\mathrm{Lie}_{N'_Z/Z}$ with $\mathrm{Lie}_{N'_Z/Z}(P.R)$, where $P = \prod_{i=1}^{r} T_i^{k_i}$.*

9 Prerequisites

9.1 Weil restrictions

As in [4], we introduce the Weil restriction functor and give some well-known representability properties.

Definition 9.1. Let S'/S be a morphism of schemes, and X' a contravariant functor from (Sch/S') to (Set). We call *Weil restriction* of X'/S' to S, and we note $\prod_{S'/S} X'$, the contravariant functor from (Sch/S) to (Set) sending $T \to S$ to $X'(T \times_S S')$. If $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ are affine, we will sometimes write $\prod_{R'/R} X'$.

Remark 9.1.1. The Weil restriction of a presheaf $X' : (\mathrm{Sch}/S')^{op} \to \mathrm{Set}$ to S is its pushforward to $\mathrm{Hom}((\mathrm{Sch}/S)^{op}, \mathrm{Set})$.

Proposition 9.2. *If $S' \to S$ is a flat and proper morphism between locally noetherian schemes, and X' is a quasi-projective S-algebraic space with X'/S*

factoring through S', then $\prod_{S'/S} X'$ is (representable by) an algebraic space. If in addition X' is a scheme, then $\prod_{S'/S} X'$ is a scheme.

Proof. The general statement follows from the case where X' is a scheme, which is treated in [9], 4.c. □

Example 9.3. Let $S' \to S$ be a morphism of schemes and Y/S be an S-algebraic space. Let Y' be the S'-space $Y \times_S S'$. Then for all T/S we have $\mathrm{Hom}_S(T, \prod_{S'/S} Y') = \mathrm{Hom}_{S'}(T \times_S S', Y') = \mathrm{Hom}_S(T \times_S S', Y)$. In particular, there is a natural map $Y \to \prod_{S'/S} Y'$.

Proposition 9.4. *If $S' \to S$ is a flat and finite morphism of schemes, and X' is a smooth quasi-projective S'-algebraic space, then $\prod_{S'/S} X'$ is smooth over S.*

Proof. The formation of $\prod_{S'/S} X'$ is étale-local on both X' and S, so we can assume $X' = \mathrm{Spec}\, A'$ and $S = \mathrm{Spec}\, R$ are affine schemes. Then S' is a disjoint union of affine schemes of the form $\mathrm{Spec}\, R'$, with R'/R finite and flat, and $\prod_{S'/S} X'$ is a scheme by Proposition 9.2. By hypothesis, each $R' \to A'$ is formally smooth and locally of finite presentation. But then $\prod_{S'/S} X'/S$ is also formally smooth, and it follows from [13], Proposition 8.14.2, that it is locally of finite presentation as well. □

Remark 9.4.1. Suppose given equivariant right-actions of a group G on a morphism of schemes $S' \to S$ and on a morphism from an algebraic space X' to S'. Suppose moreover that G acts trivially on S. Then $\prod_{S'/S} X' \to S$ carries a natural G-action, defined as follows: for any S-scheme T, define $T' := T \times_S S'$, every $g \in G$ induces an automorphism $\rho_{X'}(g)$ of X' and an automorphism $\rho_{T'}(g)$ of T' (obtained by extending the automorphism on S' by the identity on T). The action takes $f \in \mathrm{Hom}(T, \prod_{S'/S} X') = \mathrm{Hom}(T', X')$ to

$$f.g = \rho_{X'}(g) \circ f \circ \rho_T(g)^{-1}.$$

When $\prod_{S'/S} X'$ is representable, this action is equivariant.

9.2 Fixed points

We will show later that under certain hypotheses, we can construct a Néron model by considering the Weil restriction of a Néron model over a bigger base, and looking at its subspace of fixed points under a Galois action: here we define the functor of fixed points and talk about its representability and possible smoothness. This is all contained in [4], to which we refer for the proofs unless they are short enough.

Definition 9.5. Let $\pi\colon S' \to S$ be a morphism of schemes and G a finite group, acting on the right on S'. We say that π is a *quotient* for this action if it is affine, and for every affine open subscheme $\operatorname{Spec} A \subset S$ of pullback $\operatorname{Spec} A' \subset S'$ by π, A is the subring A'^G of G-invariants of A'.

Definition 9.6. Let S be a scheme and X an algebraic space. Suppose a group G acts equivariantly on $X \to S$ with the trivial action on S. We define the *subfunctor of fixed points* $X^G\colon (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ by $X^G(T) = (X(T))^G$.

Proposition 9.7. *With notations as above, X^G is an algebraic space, and its formation commutes with base change. If $X \to S$ is locally separated (resp. separated), then X^G is a subspace (resp. a closed subspace) of X.*

Proof. Compatibility with base change is immediate. Each $g \in G$ gives an automorphism $\rho_X(g)$ of X, thus a graph $\Gamma_g : X \to X \times_S X$. We can write Γ_g as the composition

$$X \xrightarrow{\Delta} X \times_S X \xrightarrow{(p_1, \rho_X(g)\circ p_2)} X \times_S X$$

where Δ is the diagonal map, and p_1, p_2 are the two projections from $X \times_S X$ onto X. Since $\rho_X(g)$ is an automorphism of X, $(p_1, \rho_X(g) \circ p_2)$ is an automorphism of $X \times_S X$. Write $Z \to X \times_S X$ for the fiber product of all Γ_g. Then Z is an algebraic space, which represents X^G. Suppose X is locally separated (resp. separated) over S. Then, $\Delta = \Gamma_0$ is an immersion (resp. a closed immersion), so Z is a subspace (resp. closed subspace) of X. □

Proposition 9.8. *With the same hypotheses and notations as in Definition 9.6, if $f\colon X \to S$ is smooth and $n := \#G$ is invertible on X, then $X^G \to S$ is smooth.*

Proof. This follows from [4], Proposition 3.4 and Proposition 3.5. □

Corollary 9.9. *Let G be a finite group acting equivariantly on a smooth morphism of algebraic spaces $X \to S$. If $\#G$ is invertible on X, then $X^G \to S^G$ is smooth.*

9.3 Twisted Lie algebras

We will make use of a slightly broader than usual notion of tangent space and Lie algebra of a group algebraic space over a base scheme, so we present the definition and a few properties here. These objects are studied in much more detail in [3].

Definition 9.10. Let S be a scheme and X an S-group algebraic space. Let $\mathcal{M}$ be a free $\mathcal{O}_S$-module of finite type. We write $\mathrm{T}_{X/S}(\mathcal{M})$ for the functor $(\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ taking T/S to $\mathrm{Hom}_S(\operatorname{Spec}(\mathcal{O}_T \oplus \mathcal{M}_T), X)$, where the $\mathcal{O}_T$-module $\mathcal{O}_T \oplus \mathcal{M}_T$ is endowed with the $\mathcal{O}_T$-algebra structure making $\mathcal{M}_T$ a square-zero ideal. The morphism of $\mathcal{O}_S$-modules $\mathcal{M} \to 0$ induces a morphism

$$\mathrm{T}_{X/S}(\mathcal{M}) \to X = \mathrm{T}_{X/S}(0),$$

and we write $\mathrm{Lie}_{X/S}(\mathcal{M})$ for the pullback of $\mathrm{T}_{X/S}(\mathcal{M})$ by the unit section $S \to X$. In particular, when $\mathcal{M} = \mathcal{O}_S$, they are the usual tangent bundle and Lie algebra of X over S, that we write $\mathrm{T}_{X/S}$ and $\mathrm{Lie}_{X/S}$.

Proposition 9.11. *With hypotheses and notations as above, $\mathrm{T}_{X/S}(\mathcal{M})$ and $\mathrm{Lie}_{X/S}(\mathcal{M})$ are representable by group S-algebraic spaces, and the canonical morphisms*

$$\mathrm{Lie}_{X/S}(\mathcal{M}) \to \mathrm{T}_{X/S}(\mathcal{M}) \to X$$

are morphisms of S-groups.

Proof. Representability when X is a scheme is [3], exposé 2, Proposition 3.3. The case of algebraic spaces is similar. Existence of the group structure, and the fact the canonical maps respect them, is [3], exposé 2, Corollaire 3.8.1. □

Proposition 9.12. *Let $(e_1, ..., e_n)$ be a basis for the free $\mathcal{O}_S$-module $\mathcal{M}$. Then, we have a natural isomorphism between $T_{X/S}(\mathcal{M})$ and the fiber product over X of the $T_{X/S}(\mathcal{O}_S.e_i)$, which we write $\prod_{1\leq i\leq n,X} T_{X/S}(\mathcal{O}_S.e_i)$; as well as between $\mathrm{Lie}_{X/S}(\mathcal{M})$ and the fiber product $\prod_{1\leq i\leq n,S} \mathrm{Lie}_{X/S}(\mathcal{O}_S.e_i)$.*

Proof. When X is a scheme, this is [3], exposé 2, Proposition 2.2. The case of algebraic spaces is similar. □

10 The morphism of base change for tame extensions

10.1 Compatibility with Weil restrictions

In this section, $S' \to S$ will be a finite locally free morphism between regular schemes, X'/S' an algebraic space, U a scheme-theoretically dense open subscheme of S, and $U' = U \times_S S'$. Note that U' is scheme-theoretically dense in S' by [13], théorème 11.10.5.

Definition 10.1. Let t be a point of $S\backslash U$ of codimension 1 in S, so that $\mathcal{O}_{S,t}$ is a discrete valuation ring. We call *ramification index of S'/S at t* the lcm of ramification indexes of all valuation ring maps $\mathcal{O}_{S,t} \to \mathcal{O}_{S',t'}$ with $t' \in S'$ of image t. Let $\mathbf{t} = (t_1, ..., t_r)$ be a r-uple of generic points of $S\backslash U$, we call *ramification index of S'/S at* $\mathbf{t}$ the r-uple $(n_1, ..., n_r)$, where n_i is the ramification index of S'/S at t_i for all i.

Proposition 10.2 (see [4], Proposition 4.1)**.** *Let X' be a smooth quasi-projective S'-algebraic space. Suppose X' is the S'-Néron model of $X'_{U'}$. Then $\prod_{S'/S}(X')$ is the S-Néron model of $\prod_{U'/U}(X'_{U'})$.*

Proof. The restriction $\prod_{S'/S}(X')$ is smooth and separated over S by Proposition 9.4 and, if Y is a smooth S-algebraic space, then

$$\begin{aligned}\mathrm{Hom}\left(Y, \prod_{S'/S}(X')\right) &= \mathrm{Hom}(Y', X') \\ &= \mathrm{Hom}(Y'_{U'}, X'_{U'}) \\ &= \mathrm{Hom}\left(Y_U, \prod_{U'/U}(X'_{U'})\right)\end{aligned}$$

where Y' denotes the base change $Y \times_S S'$. □

Proposition 10.3 (Abhyankar's lemma)**.** *Let $S = \operatorname{Spec} R$ be a regular local scheme and $S' \to S$ a finite locally free tamely ramified morphism of schemes, étale over the complement U of a strict normal crossings divisor D of S. Let $(f_1, ..., f_r)$ be part of a regular system of parameters of R such that $D = \mathrm{Div}(f_1...f_r)$. Then there are integers $n_1, ..., n_r$, prime to the residue characteristic p of R, such that if $\tilde{S} = \operatorname{Spec} R[T_1, ..., T_r]/(T_i^{n_i} - f_i)_{1\leq i\leq r}$, the normalization of $\tilde{S}$ in the total ring of fractions of $S' \times_S \tilde{S}$ is étale over $\tilde{S}$.*

Proof. This is [15], exposé XIII, Proposition 5.2. □

Proposition 10.4. *Under the hypotheses of Proposition 10.3, suppose that S is strictly local, and that S' is connected and regular. Then there are integers $n_1, ..., n_r$, prime to p, such that $S' = \operatorname{Spec} R[T_1..., T_r]/(T_i^{n_i} - f_i)$.*

Proof. By Proposition 10.3, there are integers $m_1, ..., m_r$ prime to p such that if we call $\tilde{S}$ the spectrum of

$$\tilde{R} = R[T_1, ..., T_r]/(T_i^{m_i} - f_i)_{1\leq i\leq r},$$

then the normalization Y of $\tilde{S}$ in the fraction field of $S' \times_S \tilde{S}$ is finite étale over $\tilde{S}$. Since $\tilde{S}$ is strictly local and S' is connected, $Y \to \tilde{S}$ must be an isomorphism. Since $S' \to S$ is integral, we get a factorization $\tilde{S} \to S' \to S$. Let G be the group of S'-automorphisms of $\tilde{S}$. As both S' and $\tilde{S}$ are spectra of free R-algebras, the map $\tilde{S} \to S'$ is a quotient for the G-action. We can see G as a subgoup of the group of S-automorphisms of $\tilde{S}$, which is the product $\prod_{i=1}^{r} \mu_{m_i}$ of the groups μ_{m_i} of m_i-th roots of unity of R (where $\xi \in \mu_{m_i}$ acts by sending T_i to ξT_i). We know that $R' = \tilde{R}^G$ is generated as a R-module by all of the G-invariant monomials in $(T_1, ..., T_r)$. Let n_i be the minimal integer such that T_i is G-invariant. The $T_i^{n_i}$ are irreducible - hence prime - elements of the regular local ring $R' = \tilde{R}^G$, and we have $n_i|m_i$ for each i. We claim that G is of the form $< \xi_1, \xi_2, ..., \xi_r >$, where $\xi_i \in \mu_{m_i}$ is a primitive $d_i := \frac{m_i}{n_i}$-th root of unity. We will now prove the claim. Let $M = \prod_{i=1}^{r} T_i^{k_i}$ be a G-invariant unitary monomial. It suffices to show that for all i, we have $n_i|k_i$. But M divides some positive power of $\prod_{i=1}^{r} f_i = \prod_{i=1}^{r} T_i^{m_i}$ in

$R' = \tilde{R}^G$, and by uniqueness of the prime factor decomposition in R' it follows that $n_i | k_i$ for all i. This concludes the proof of the claim.

Therefore, $\prod_{i=1}^{r} \mu_{m_i}/G$ is a product of quotients of the μ_{m_i}, i.e. there are integers $n_1, ..., n_r$ such that n_i divides m_i for all i and S' itself is of the form $\operatorname{Spec} R[T_1, ..., T_r]/(T_i^{n_i} - f_i)_{1 \leq i \leq r}$. □

Remark 10.4.1. The integer n_i is the ramification index of S'/S at the generic point of $\{f_i = 0\}$ in S.

Remark 10.4.2. This proof means that, étale-locally on any regular base, a finite tamely ramified morphism either does nothing more than adding roots of regular parameters, or must have a scheme that is not locally factorial as a source. Since in many practical situations, the behaviour of Néron models is only well-known over (at least) locally factorial bases, we will only be considering the "adding roots" case. For the same reason, we always take D to be strict: indeed, suppose D is a (non-strict) normal crossings divisor, and suppose there is an étale morphism $\tilde{S} \to S$ and an irreducible component D_0 of D which breaks into multiple irreducible components in $\tilde{S}$, then no extension S'/S with ramification index > 1 over the generic point of D_0 can have factorial étale local rings.

Theorem 10.5 (see [4], Theorem 4.2. for the case $\dim S = 1$)**.** *Let $S' \to S$ be a finite, locally free, tamely ramified map between regular schemes and U the complement of a strict normal crossings divisor $D \to S$. Suppose $U' := U \times_S S'$ is étale over U.*

Let X_U be a smooth U-algebraic space, such that $X_U \times_U U'$ has a S'-Néron model X'. Then the scheme-theoretical closure of X_U in $\prod_{S'/S} X'$ is the S-Néron model of X_U (where X_U maps to $\prod_{S'/S} X'$ as in Example 9.3).

Moreover, if $S' \to S$ is a quotient for the right-action of a finite group G, then G acts on X' via its action on S': it follows that G acts on $\prod_{S'/S} X'$ as in Remark 9.4.1. In that case, the Néron model of X_U is $(\prod_{S'/S} X')^G$.

Proof. By Corollary 5.4 and Proposition 5.7, we can assume $S = \operatorname{Spec} R$ is strictly local. We can also assume S' is connected, in which case Proposition 10.4 shows that $S' \to S$ is a quotient for the action of a finite group G. Call Z the restriction $\prod_{S'/S} (X')$ and N the scheme-theoretical closure of X_U in Z. The theorem now reduces to the floowing claims: Z^G is a smooth S-model of X_U; there is a canonical isomorphism $N = Z^G$; and N is a separated S-space satisfying existence in the Néron mapping property. We will now prove these claims in order.

The group G acts on $Z = \prod_{S'/S} X_{S'}$ via its right-action on S', and $S' \to S$ is a quotient for the latter. As seen in Proposition 10.4, $\#G$ is prime to the residue characteristic of R, so by Proposition 9.8, Z^G is S-smooth. We have a pullback diagram of algebraic spaces

$$\begin{array}{ccc} X'_{U'} & \longrightarrow & X_U \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

where both horizontal arrows are quotients for the action of G. We will show $X_U = Z_U^G$, which can be checked Zariski-locally: let V be an affine open subscheme of U, $V' = \operatorname{Spec} A$ its pullback to U', X_0 an affine open of X_U with image contained in V and $X'_0 = \operatorname{Spec} B$ its pullback to $X'_{U'}$. We have $V = \operatorname{Spec}(A^G)$ and $X_0 = \operatorname{Spec}(B^G)$, and there is a pushout diagram of rings

$$\begin{array}{ccc} B & \longleftarrow & B^G \\ \uparrow & & \uparrow \\ A & \longleftarrow & A^G \end{array}$$

Let W be the Weil restriction of X'_0 to V. Then we see that $W = \operatorname{Spec} R$ is affine, and for any A^G-algebra C, we have $\operatorname{Hom}_{A^G}(R, C) = \operatorname{Hom}_A(B, C \otimes_{A^G} A)$. But the G-invariant A-maps from B to $C \otimes_{A^G} A$ are precisely those lying in the image of $\operatorname{Hom}_{A^G}(B^G, C)$. Therefore $W^G = \operatorname{Spec}(B^G)$, and it follows from the sheaf property that $(Z_U)^G = X_U$: Z^G is a smooth S-model of X_U as claimed.

Now, observe that $Z^G \to Z$ is a closed immersion through which X_U factors, so $N \to Z$ factors through a closed immersion $N \to Z^G$. But Z^G is S-smooth, hence S-flat, so $Z_U^G = X_U$ is scheme-theoretically dense in Z^G by [13], théorème 11.10.5, which means N is scheme-theoretically dense in Z^G: the closed immersion $N \to Z^G$ is an isomorphism.

The map $N \to S$ is separated since $Z \to S$ is. Let Y be a smooth S-algebraic space with a map $f_U \colon Y_U \to X_U$. Call Y' the base change $Y \times_S S'$. The map $Y'_{U'} \to X'_{U'}$ obtained by base change extends uniquely to a map $Y' \to X'$, which induces a map $Y \to Z$ extending f_U. By definition of the scheme-theoretical closure, $Y \to Z$ factors through a map $f \colon Y \to N$ extending f_U. We have shown $\operatorname{Hom}_S(Y, N) \to \operatorname{Hom}_U(Y_U, X_U)$ is surjective, so $N/S = Z^G/S$ is the Néron model of X_U. □

10.2 A filtration of the Néron model over the canonical stratification

By Proposition 5.3 and Proposition 5.6, Néron models can always be described over an étale covering of the base. Therefore, in this section, unless mentioned otherwise, we will work assuming that $S = \operatorname{Spec} R$ is an affine regular connected scheme, that R contains all roots of unity of order invertible on S, that D is a strict normal crossings divisor on S cut out by regular parameters $f_1, ..., f_r$ of R, and that $S' = \operatorname{Spec} R'$ with $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)_{1 \leq j \leq r}$. Note that in our previous setting (where $S' \to S$ was a finite, locally free and tamely ramified morphism between regular schemes, étale over the complement of a strict normal crossings divisor), all these assumptions hold in an étale neighbourhood of any given point of S.

We put $A = R/(f_j)_{1\leq j\leq r}$ and $A' = A\otimes_R R' = A[T_1, ..., T_r]/(T_j^{n_j})_{1\leq j\leq r}$. The closed subscheme $Z = \operatorname{Spec} A$ of S is the closed stratum of D. We let X_U be a proper and smooth U-group space with a Néron model N' over S'. It follows from Theorem 10.5 that X_U has a Néron model N/S, and that N is the subspace of G-invariants of the Weil restriction of N' to S, where the action of $G = \prod_{j=1}^{r} \mu_{n_j}$ on S' is given by multiplying T_j by the j-th coordinate of an element of G.

In [4], section 5, when S is a discrete valuation ring, one computes the successive quotients of a filtration of the closed fiber of N. We adapt this construction to our context to get a filtration of N_Z and express its successive quotients in terms of N'.

For all $d \in \mathbb{N}^*$, we write Λ_d the set of monomials of the form $\prod_{j=1}^{r} T_j^{k_j}$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j. The set $A'_d \subset A'$ of homogenous polynomials of degree d in the T_j is a finite free A-module with basis Λ_d.

Definition 10.6. For $d \in \mathbb{N}^*$, we define a sheaf

$$\operatorname{Res}^d N'_Z : (\operatorname{Sch}/Z)^{op} \to \operatorname{Set}$$

as follows: for any A-algebra C, we put $\operatorname{Res}^d N'_Z(C) = N'(C \otimes_A A'/(\Lambda_d))$.

Remark 10.6.1. The functor $\operatorname{Res}^d N'_Z$ is (representable by) the Z-algebraic space $\prod_{(A'/(\Lambda_d))/A} N'_{A'/(\Lambda_d)}$. We have $\operatorname{Res}^1 N'_Z = N'_Z$, and for any $d > \prod_{j=1}^{r}(n_j - 1)$, we have $\operatorname{Res}^d N'_Z = \left(\prod_{S'/S} N'\right) \times_S Z$ since Λ_d is empty. There are natural maps $\operatorname{Res}^{d+1} N'_Z \to \operatorname{Res}^d N'_Z$.

Definition 10.7. For $d \in \mathbb{N}^*$, we define $F^d N'_Z$ as the kernel of the canonical morphism $\left(\prod_{S'/S} N'\right) \times_S Z \to \operatorname{Res}^d N'_Z$ of Z-group spaces. We also put $F^0 N'_Z = \left(\prod_{S'/S} N'\right) \times_S Z$. The $F^d N'_Z$ form a descending filtration of $\left(\prod_{S'/S} N'\right) \times_S Z$ by Z-subgroup spaces, stationary at 0 starting from $d = 1 + \prod_{j=1}^{r}(n_j - 1)$. We call $\operatorname{Gr}^d N'_Z$ the quotient $F^d N'_Z / F^{d+1} N'_Z$.

Proposition 10.8. *We have* $\operatorname{Gr}^0 N'_Z = N'_Z$*, and for any* $d \geq 1$*,* $\operatorname{Gr}^d N'_Z$ *is canonically isomorphic to* $\operatorname{Lie}_{N'_Z/Z}(A'_d) = \prod_{P\in\Lambda_d, Z} \operatorname{Lie}_{N'_Z/Z}(PA)$.

Proof. The proof of [4], 5.1. carries over without much change: let $d \geq 1$, and let C be an A-algebra. Let $\lambda_1, ..., \lambda_k$ be parameters for the formal group of N'

over R'. An element $a \in F^d N'_Z(C)$ corresponds to a ring map

$$\phi\colon R'[[\lambda_1, ..., \lambda_k]] \to C[T_1, ..., T_r]/(T_j^{n_j})_{1 \le j \le n}$$

such that for all $1 \le i \le k$, $\phi(\lambda_i)$ is in the ideal generated by Λ_d, i.e. is of the form $\sum_P a_{i,P} P$ where the $a_{i,P}$ are in C and P runs over all nonzero monomials $\prod_{j=1}^r T^{k_j}$ with $\sum_j k_j \ge d$. Thus, we can associate to a an element of $\mathrm{Lie}_{N'_Z/Z}(A_d)(C)$ by truncature, sending λ_i to $\sum_{P \in \Lambda_d} a_{i,P} P$. This gives a surjective morphism of Z-groups $F^d N'_Z \to \mathrm{Lie}_{N'_Z/Z}(A_d)$, with kernel $F^{d+1} N'_Z$. The identification $\mathrm{Lie}_{N'_Z/Z}(A'_d) = \prod_{P \in \Lambda_d, Z} \mathrm{Lie}_{N'_Z/Z}(PA)$ is Proposition 9.12. □

Proposition 10.9. *With the same hypotheses and notations as in Proposition 10.8, the action of G on $\mathrm{Lie}_{N'_Z/Z}(A_d)$ is obtained from equivariant actions of G on each factor $\mathrm{Lie}_{N'_Z/Z}(PA) \to Z$, where P ranges through Λ_d. Moreover, for any $P = \prod_{j=1}^r T_j^{k_j}$ in Λ_d, the bijection $\mathrm{Lie}_{N'_Z/Z}(PA) = \mathrm{Lie}_{N'_Z/Z}$ induced by $P \mapsto 1$ identifies the subspace $\mathrm{Lie}_{N'_Z/Z}(PA)^G$ with the subspace of $\mathrm{Lie}_{N'_Z/Z}$ where all $\xi = (\xi_j)_{1 \le j \le r}$ in G act by multiplication by $\prod_{j=1}^r \xi_j^{k_j}$. We will write this subspace $\mathrm{Lie}_{N'_Z/Z}[k_1, ..., k_r]$ or $\mathrm{Lie}_{N'_Z/Z}[P]$.*

Proof. For any A-algebra C, the action of ξ on $\mathrm{Hom}(R'[[\lambda_1, ..., \lambda_k]], C \otimes_A A')$ makes the following diagram commute:

$$\begin{array}{ccc} R'[[\lambda_1, ..., \lambda_k]] & \xrightarrow{\xi.\psi} & C \otimes_A A' \\ \downarrow{\scriptstyle \xi} & T_j \mapsto \xi_j^{-1} T_j & \uparrow \\ R'[[\lambda_1, ..., \lambda_k]] & \xrightarrow{\psi} & C \otimes_A A' \end{array}$$

where the map $R'[[\lambda_1, ..., \lambda_k]] \xrightarrow{\xi} R'[[\lambda_1, ..., \lambda_k]]$ is given by the G-action on N'. Therefore, the G-action on $\mathrm{Lie}_{N'_Z/Z}(A_d)$ comes from G-actions on the factors $\mathrm{Lie}_{N'_Z/Z}(PA)$, given for $P = \prod_{j=1}^r T_j^{k_j}$ by

$$\begin{array}{ccc} R'[[t_1, ..., t_d]] & \xrightarrow{\xi.\psi} & C \oplus PC \\ \downarrow{\scriptstyle \xi} & P \mapsto \prod_j \xi_j^{-k_j} P & \uparrow \\ R'[[t_1, ..., t_d]] & \xrightarrow{\psi} & C \oplus PC \end{array}$$

from which the proposition follows. □

Definition 10.10. For any integer $d \in \mathbb{N}$, we define

$$F^d N_Z := (F^d N'_Z)^G$$

and

$$G^d N_Z := F^d N_Z / F^{d+1} N_Z.$$

Remark 10.10.1. The F^dN_Z form a descending filtration of sub-Z-group spaces of N_Z, with $F^0N_Z = N_Z$ and $F^dN_Z = 0$ when $d > \prod_{j=1}^{r}(n_j - 1)$.

Proposition 10.11. *Keeping the notations of Proposition 10.9, for all $d \in \mathbb{N}$, we have $G^dN_Z = (G^dN'_Z)^G$. In particular, $G^0N_Z = (N'_Z)^G$, and for all $d \geq 1$, $G^dN_Z = \prod_{P \in \Lambda_{d,Z}} \mathrm{Lie}_{N'_Z/Z}[P]$.*

Proof. (see [4], 5.2.) The Z-group spaces $F^dN'_Z$ are unipotent for $d \geq 1$ and the order of G is invertible on Z, so the exact sequence

$$0 \to F^dN'_Z \to F^{d+1}N'_Z \to G^dN'_Z \to 0$$

remains exact after taking the G-invariants. □

We summarize all this into Theorem 10.13 below, and justify its hypotheses by Lemma 10.12 and Proposition 5.6.

Lemma 10.12. *Let $S' \to S$ be a finite, locally free, morphism between regular connected schemes. Let D be a strict normal crossings divisor of S and put $U = S\backslash D$. Suppose $S' \to S$ is étale over U. Let s be a point of S, and $D_1, ..., D_r$ the irreducible components of D containing s. Then there is an affine étale neighbourhood $V = \mathrm{Spec}\, R \to S$ of s in S such that:*

- *For all $1 \leq j \leq r$, $D_j|_V$ is cut out by a regular parameter f_j of R.*
- *There is an isomorphism $V \times_S S' = \mathrm{Spec}\, R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where n_j is the ramification index of $S' \to S$ at the generic point of D_j (in particular, if $S' \to S$ is tamely ramified, then n_j is invertible on R).*
- *R contains all n_j-th roots of unity for all j.*

Proof. Immediate from Proposition 10.4. □

Theorem 10.13. *Let $S = \mathrm{Spec}\, R$ be a regular affine scheme, $f_1, ..., f_r$ regular parameters of R, $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where the n_j are invertible on S, and let $S' = \mathrm{Spec}\, R'$. Suppose R contains the group μ_{n_j} of n_j-th roots of unity for all j. Let U be the locus in S where all f_j are invertible, and Z the locus where all f_j vanish. Let X_U be a proper smooth U-group algebraic space with a Néron model N' over S'. Then X_U has a Néron model N over S, and we have sub-Z-group spaces $(F^dN_Z)_{d \in \mathbb{N}}$ of N_Z (see Definition 10.7 and Definition 10.10) such that:*

- *For all $d \in \mathbb{N}$, $F^{d+1}N_Z \subset F^dN_Z$.*
- *$F^0N_Z = N_Z$.*
- *If $d > \prod_{j=1}^{r}(n_j - 1)$ then $F^dN_Z = 0$.*

- *F^0N_Z/F^1N_Z is the subspace of N'_Z invariant under the action of $G = \prod_{j=1}^{r} \mu_{n_j}$, where $(\xi_j)_{1\leq j\leq r}$ acts by multiplying T_j by ξ_j.*
- *If $d > 0$, $F^dN_Z/F^{d+1}N_Z$ is isomorphic to the fiber product over Z of the $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ where $\mathbf{k}$ ranges through all r-uples of integers $(k_1, ..., k_r)$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j; $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ is the subspace of $\mathrm{Lie}_{N'_Z/Z}$ where all $(\xi_j)_{1\leq j\leq r}$ in G act by multiplication by $\prod_{j=1}^{r} \xi_j^{k_j}$; and the map $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}] \to Z$ is given by identifying $\mathrm{Lie}_{N'_Z/Z}$ with $\mathrm{Lie}_{N'_Z/Z}(P.R)$, where $P = \prod_{i=1}^{r} T_i^{k_i}$.*

Remark 10.13.1. Our choice of quotienting by all monomials of the same degree in Definition 10.6 is somewhat arbitrary, other choices could perhaps lead to interesting things as well.

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Abstract

In several areas of mathematics, one encounters families of objects (groups, varieties, schemes, graphs...) parametrized by one or several unknowns, that are well-behaved and easy to define except for a few specific values of these unknowns. Think, for example, of an elliptic curve over the field of rational numbers: starting with an equation with rational coefficients, one can clear denominators and get an equation with integer coefficients, and this equation reduces to the equation of an elliptic curve modulo p for all but a finite number of primes p. Even then, it is often convenient to be able to extend our family into a global, compact one, or at least satisfying some good continuity properties with respect to the parameters. A *model* of a family of objects defined for all but certain values of the parameters is a way of extending it to all possible values. Incomplete smooth families of schemes (or, more generally, of stacks) sometimes admit a "best possible smooth model", the *Néron model*. This thesis deals with questions of existence and explicit construction of Néron models. It is divided in three parts.

In the first part, we study generically smooth families of nodal curves (i.e. curves with at worst ordinary double points) over a regular base scheme. We define certain birational modifications of such nodal (relative) curves, which we call *refinements*. We prove that refinements always exist étale-locally on the base. We define invariants measuring the complexity of the singularities of a nodal curve, and explain how refinements can be used to find the nodal models of a generic smooth curve with the simplest singularities.

In the second part, we are interested in the existence of Néron models for (families of) Jacobians and curves, over a regular base with no restriction of dimension. First, we introduce a condition on nodal curves called *strict alignment*. Strict alignment can be read on the *dual graph*, a simple combinatorial invariant summarizing information about the global structure of the curve and how its singularities vary in families. We show that the generic Jacobian of a generically smooth nodal curve has a Néron model if and only if the curve is strictly aligned. Then, we prove that for a smooth curve to have a Néron model, it is necessary that the singular locus of any nodal model be locally irreducible for the étale topology. Using the contraction morphisms of $\mathcal{M}_{g,n}$ stacks, we deduce an even stronger necessary condition in terms of dual graphs (equivalent to the closure of this local irreducibility of the singular locus under étale base change and contraction), and we show that this new necessary condition is also sufficient under some technical hypotheses.

In the third part, we study the base change behavior of Néron models under finite, tamely ramified morphisms $S' \to S$ between regular schemes. We show that if an abelian variety defined generically over S has a Néron model N'/S' after such a base change, then it admits a Néron model N/S, and we make explicit the successive quotients of a certain filtration of N in terms of N'.

Keywords: Nodal curves, Néron models, Jacobians, high dimension.

Samenvatting

In verscheidene gebieden van de wiskunde treft men families van objecten (variëteiten, groepen, schemas, grafen...), geparametriseerd door een of meerdere variabelen, die zich goed gedragen en eenvoudig te definiëren zijn, afgezien voor een paar specifieke waarden van deze variabelen. Denk bijvoorbeeld aan een elliptische kromme over het lichaam van rationale getallen: beginnend met een vergelijking met rationale coëfficiënten kun je de noemers wegwerken en een vergelijking met gehele coëfficiënten krijgen, en deze vergelijking reduceert naar de vergelijking van een elliptische kromme modulo p voor alle p, op eindig veel p na. Zelfs dan is het vaak handig om de familie uit te breiden naar een globale, compacte familie, of op zijn minst een familie die goede continuïteitseigenschappen heeft met betrekking tot de variabelen. Een *model* van een familie van objecten die gedefinieerd is voor alle waarden van de variabelen, op een aantal na, is een manier om deze familie uit te breiden naar alle waarden van de variabelen. Onvolledige gladde families van schemas hebben soms een "best mogelijk glad model", het zogeheten *Néron model.* Dit proefschrift gaat over de existentie en expliciete constructie van Néron modellen. Het bestaat uit drie delen.

In het eerste deel bestuderen we generiek gladde families van nodale krommen (d.w.z. krommen met op zijn slechtst gewone dubbelpunten) over een regulier basisschema. We definiëren zekere birationale modificaties van zulke nodale (relatieve) krommen, welke we *verfijningen* noemen. We bewijzen dat verfijningen étale-lokaal op de basis bestaan. We definiëren invarianten die de complexiteit van de singulariteiten van een nodale kromme meten, en leggen uit hoe verfijningen gebruikt kunnen worden om de nodale modellen van een generieke gladde kromme te vinden die de eenvoudigste singulariteiten hebben.

In het tweede deel zijn we geïnteresseerd in het bestaan van Néron modellen voor (families van) Jacobianen en krommen, over een reguliere basis, zonder beperking op de dimensie. Als eerst introduceren we een conditie op nodale krommen, genaamd *stricte uitlijning.* Stricte uitlijning kan afgelezen worden van de duale graaf, een eenvoudige combinatorische invariant die informatie samenvat over de globale structuur van de kromme, en over hoe zijn singulariteiten veranderen in families. We tonen aan dat de generieke Jacobiaan van een generiek gladde nodale kromme een Néron model heeft dan en slechts dan als de kromme strict uitgelijnd is. Dan bewijzen we dat als een gladde kromme een Néron model heeft, dan is de singuliere locus van elk nodale model lokaal irreducibel voor de étale topologie. Door gebruik te maken van de contractiemorfismen van de $\mathcal{M}_{g,n}$ schelven, leiden we een sterkere nodige conditie in termen van duale grafen af, en we tonen aan dat deze nieuwe nodige conditie ook voldoende is.

In het derde deel bestuderen we het gedrag van Néron modellen onder basisverandering langs eindige, tam vertakte morfismen $S' \to S$ tussen reguliere schemas. We tonen aan dat als een abelse variëteit, generiek over S gedefinieerd, een Néron model N'/S' heeft, na zo'n basisverandering, dan heeft hij een Néron model N/S, en we maken de achtereenvolgende quotienten van een zekere filtratie op N expliciet in termen van N'.

Trefwoorden: Nodale krommen, Néron modellen, Jacobianen, hoge dimensie.

Résumé

Dans de nombreux domaines des mathématiques, il arrive de rencontrer des familles d'objets (groupes, graphes, variétés, schémas, champs algébriques...) paramétrisées par une ou plusieurs variables, qui admettent une définition simple et se comportent bien seulement en dehors de certaines valeurs exceptionnelles de ces variables. Par exemple, étant donné une courbe elliptique sur $\mathbb{Q}$, on peut en trouver une équation à coefficients entiers, dont la réduction modulo p sera une courbe elliptique pour tout nombre premier p à l'exception d'un nombre fini d'entre eux. On a alors une "famille continue" de courbes elliptiques paramétrée par l'ensemble des nombres premiers, privé d'un sous-ensemble fini. Il est naturel de souhaiter construire des *modèles* de ces familles incomplètes, i.e. de les étendre en familles définies sur toutes les valeurs possibles des paramètres. Les familles incomplètes de schémas (ou même champs algébriques) ont parfois un "meilleur modèle lisse", le modèle de Néron. Cette thèse traite de questions d'existence et de construction explicite de modèles de Néron. Elle comporte trois parties.

Dans la première partie, nous nous intéressons à des familles de courbes nodales génériquement lisses paramétrisées par un schéma régulier. Nous introduisons certains éclatements birationnels de telles familles, les *raffinements*. Nous montrons que, localement sur la base dans la topologie étale, les raffinements existent toujours. Nous définissons certains invariants mesurant la complexité des singularités d'une courbe nodale relative, et nous montrons que les raffinements permettent, partant d'une courbe lisse avec un modèle nodal, d'en trouver les modèles nodaux avec les singularités les plus simples.

Dans la deuxième partie, nous nous intéressons à l'existence de modèles de Néron pour des familles de Jacobiennes, puis pour des familles de courbes, sur une base régulière sans restriction de dimension. D'abord, nous introduisons une condition appelée *alignement strict* sur la structure locale d'une courbe nodale génériquement lisse X/S autour de ses singularités. Nous montrons que la Jacobienne générique de X/S a un modèle de Néron si et seulement si X/S est strictement alignée. Ensuite, nous prouvons que si une courbe lisse a un modèle de Néron, alors le lieu singulier de tout modèle nodal de cette courbe est localement irréductible pour la topologie étale. Nous utilisons les morphismes de contraction des champs $\mathcal{M}_{g,n}$ pour en déduire une condition nécessaire plus forte (équivalente à la clôture de cette irréductibilité locale du lieu singulier sous les morphismes de contraction et les changements de base lisses), et nous montrons que cette nouvelle condition est également suffisante sous quelques hypothèses techniques.

Dans la troisième partie, nous étudions le comportement des modèles de Néron sous des changements de base $S' \to S$ finis et modérément ramifiés entre schémas réguliers. Nous verrons qu'une variété abélienne définie génériquement au-dessus de S, et admettant un modèle de Néron N'/S' après un tel changement de base, doit aussi avoir un modèle de Néron N/S. Dans ce cas, nous expliciterons les quotients successifs d'une filtration de N en termes de N'.

Mots-clés: Courbes nodales, modèles de Néron, Jacobiennes, dimension supérieure.

Résumé substantiel

Étant donné un schéma S et un ouvert schématiquement dense $U \subset S$, de nombreux U-schémas propres et lisses Z/U n'admettent pas de modèle propre et lisse sur S. Par exemple, toute courbe elliptique sur $\mathbb{Q}$ s'étend en courbe elliptique sur un ouvert dense de $\operatorname{Spec}\mathbb{Z}$, mais aucune ne s'étend en courbe elliptique sur tout $\operatorname{Spec}\mathbb{Z}$. Afin de trouver le "meilleur modèle lisse possible" en l'absence de modèle propre et lisse, il peut alors être intéressant de remplacer la condition de propreté par une autre. Soit X/S un modèle de Z/U, on dit que X/S a la ***propriété de Néron*** (relativement à $U \to S$) lorsque pour tout S-schéma lisse Y/S, tout morphisme $Y_U \to Z$ s'étend de manière unique en un morphisme $Y \to X$. Un *modèle de Néron* est un modèle lisse et séparé ayant la propriété de Néron.

La propriété de Néron est universelle, et garantit donc l'unicité des modéles de Néron, à unique isomorphisme près. Cependant, le modèle de Néron n'existe pas toujours: par exemple, si S est le spectre d'un anneau de valuation discrète et $U \to S$ l'inclusion du point générique, alors $\mathbb{P}^1_U$ n'a pas de modèle de Néron sur S.

Néron a prouvé que toute variété abélienne au-dessus d'un ouvert dense d'un schéma de Dedekind S admet un modèle de Néron sur S. Plus récemment, la question s'est posée de construire des modèles de Néron dans des cadres plus généraux. Qing Liu et Jilong Tong ont montré que toute courbe de genre ≥ 1 définie sur un ouvert dense d'un schéma de Dedekind connexe a un modèle de Néron, qui est le lieu lisse de son modèle propre régulier minimal. Lorsque S est un schéma régulier, $U \subset S$ un ouvert dense et X/S est une courbe nodale[1] lisse au-dessus de U, David Holmes a décrit une condition nécessaire pour que la Jacobienne de X_U ait un modéle de Néron sur S, en termes de la structure locale de X autour de ses singularités. Cette thèse traite de questions d'existence et de comportement des modèles de Néron sur une base régulière sans restriction de dimension, avec une emphase particulière sur les modèles de Néron de courbes nodales et de leurs Jacobiennes. Elle comporte trois parties.

Partie I: Courbes nodales, graphes duaux et résolutions

Dans cette partie, nous partons d'un schéma régulier S, un ouvert dense $U \subset S$, et une courbe nodale X/S, lisse au-dessus de U. Lorsque S est un trait[2] et U son point générique, une étape clé dans la construction du modèle de Néron de X_U consiste typiquement à "lissifier" X, i.e. à éclater X en des sous-schémas dont le support est disjoint de U, pour rendre le lieu lisse "plus gros". Par exemple, lorsque S est un trait strictement local, après un nombre fini d'éclatements successifs en des points singuliers de la fibre spéciale de X/S, on obtient un nouveau modèle nodal $X' \to X$ de X_U, tel que toute section de X/S provient d'une unique section lisse de X'/S. Nous présentons un analogue de ce procédé de lissification lorsque S est un schéma régulier sans restriction de dimension.

[1] Une courbe propre et de présentation finie, dont les fibres géométriques ont au pire des points doubles ordinaires

[2] Le spectre d'un anneau de valuation discrète

D'abord, nous définissons le *graphe dual* de X/S en un point géométrique s de S. Les sommets de ce graphe (respectivement, ses arêtes) sont les composantes irréductibles (respectivement, les points singuliers) de X_s. À une arête du graphe dual, correspondant à un point singulier x de X_s, nous associons un idéal de l'anneau local étale $\mathcal{O}^{et}_{S,s}$, l'*idéal singulier* de x, représentatif de la complexité globale du lieu singulier de X/S autour de x. L'idéal singulier généralise la notion d'*épaisseur* d'un point singulier de la fibre spéciale de X/S, lorsque S est un trait strictement local. Par exemple, lorsque l'idéal singulier de x est premier, aucune section de X/S ne peut passer par x.

Nous appelons *raffinement* de X/S l'éclatement de X en le faisceau d'idéaux d'une S-section. Nous montrons que tout raffinement de X/S est un modèle nodal de X_U. Nous montrons que, lorsque S est excellent, pour tout point géométrique s de S et toute section $\sigma\colon S \to X$, il existe un voisinage étale V de s dans S et un raffinement X' de X_V/V tels que $\sigma|_V$ provient d'une section lisse de X_V/V. Nous prouvons également que, étale localement sur S, par composition d'un nombre fini de raffinements, il est toujours possible d'obtenir une *résolution* de X/S en s, i.e. un modèle nodal X' de X_U tel que tous les idéaux singuliers des points de X'_s sont premiers. Il s'ensuit que si S est strictement local et excellent, toute section σ de X/S provient d'une section du lieu lisse d'une certaine résolution de X/S. Cependant, contrairement au cas où S est de dimension 1, il peut exister plusieurs classes d'isomorphisme de résolutions de X/S, et σ ne se factorise pas nécessairement par *toute* résolution de X/S.

Partie II: Modèles de Néron de courbes nodales et de leurs Jacobiennes

Dans cette partie, nous conservons les notations X, S, U de la partie précédente, et supposons aussi que S est excellent. Nous décrivons alors des critères pour que X_U et sa Jacobienne aient des modèles de Néron au-dessus de S.

D'abord, nous cherchons à construire un modèle de Néron pour la Jacobienne J de X_U/U. Nous nous appuyons fortement sur le travail de David Holmes, qui a prouvé que si J admet un modèle de Néron, alors les graphes duaux de X/S en tout point géométrique doivent satisfaire une condition appelée *alignement*: un graphe dual est aligné lorsque tous les labels des arêtes d'un même cycle (i.e. les idéaux singuliers des points doubles correspondants) sont des puissances d'un même idéal. Une conséquence immédiate est que si J admet un modèle de Néron, alors *tout* modèle nodal de X_U doit être aligné. Nous montrons que les deux conditions suivantes sont équivalentes:

- Tout raffinement de tout changement de base lisse de X/S est aligné.
- X/S est *strictement alignée*, i.e. les labels d'un même cycle d'un même graphe dual de X/S sont toujours puissances d'un même idéal *premier*.

Puisque les modèles de Néron sont compatibles avec les changements de base lisses, il s'ensuit que si J a un modèle de Néron, alors X/S est strictement alignée. Nous montrons que cette condition est également suffisante: J a un modèle de Néron si et seulement si X/S est strictement alignée. Localement sur S, nous décrivons explicitement ce modèle de Néron N de J lorsqu'il existe: tout point géométrique s de S a un voisinage étale V tel que X_V/V ait une résolution X', et N_V est canoniquement isomorphe au quotient P/E, où P est l'espace de Picard $\mathrm{Pic}^{[0]}_{X'/V}$ paramétrisant les faisceaux inversibles de degré total 0; et E la clôture schématique de la section unité dans P.

En ce qui concerne la construction d'un modèle de Néron pour la courbe X_U/U elle-même, on s'intéresse d'abord à un problème étroitement lié, mais un peu différent: existe-t-il un morphisme de S-espaces algébriques $N \to X$, avec N/S lisse, tel que tout autre tel morphisme $Y \to X$ avec Y/S lisse se factorise par N ? Nous répondons par l'affirmative, et construisons un tel N en recollant dans la topologie étale une famille bien choisie de lieux lisses de raffinements de changements de base étales de X/S. Cet espace N, que l'on appelle *agrégat lisse* de X/S, possède des propriétés analogues à la propriété de Néron, mais il n'est en général pas séparé. En conséquence, plutôt qu'un modèle de Néron, nous allons chercher à construire un ns-modèle de Néron de X_U/U, i.e. un modèle lisse (mais pas nécessairement séparé) avec la propriété de Néron.

Par un résultat de Gabber, Liu et Lorenzini, on sait que si X/S n'a pas de courbe rationnelle dans ses fibres géométriques, alors tout morphisme $U \to X$ s'étend uniquement en une S-section. Il s'ensuit que, sous cette hypothèse additionnelle, l'agrégat lisse de X est un ns-modèle de Néron de X_U/U. Dans le cas général, cela soulève la question de construire un modèle nodal de X_U sans composante rationnelle dans ses fibres géométriques. Nous utilisons les morphismes de contraction des champs $\mathcal{M}_{g,n}$ décrits par Knudsen pour montrer que X/S peut toujours être contractée en un modèle nodal stable X^{stable} de X_U. Nous donnons une caractérisation explicite en termes de X pour que les fibres géométriques de X^{stable} n'aient pas de composantes rationnelles (i.e. pour que X_U ait un modèle nodal sans composante rationnelle dans ses fibres géométriques). Lorsque cette condition est remplie, l'agrégat lisse de X^{stable} est donc le ns-modèle de Néron de X_U/U.

Ensuite, nous cherchons des conditions sous lesquelles X_U a un modèle de Néron proprement dit, i.e. un ns-modèle de Néron séparé. Nous montrons que si X_U a un modèle de Néron, alors le lieu singulier de X/S doit être localement irréductible pour la topologie étale. En termes de graphes duaux, cela revient à demander que tous les idéaux singuliers de X/S (en tout point géométrique de S) soient des puissances d'idéaux premiers.

Une fois encore, cette condition nécessaire s'applique à tout modèle nodal de X_U, et pas seulement à X: il est possible d'obtenir une condition nécessaire plus restrictive (en termes de X) en appliquant la précédente à X^{stable}. Nous expliquons donc comment les idéaux singuliers de X^{stable} peuvent être calculés à partir de ceux de X, et nous en déduisons que si X_U a un modéle de Néron, alors X satisfait les deux conditions suivantes:

1. Tous les idéaux singuliers de X/S sont des puissances d'idéaux premiers.

2. Pour tout point géométrique s de S, et toute composante rationnelle E de

X_s rencontrant les autres composantes irréductibles de X_s en exactement deux points x et y, les idéaux singuliers de x et y sont des puissances du même idéal premier de $\mathcal{O}^{et}_{S,s}$.

Enfin, nous montrons que si X satisfait ces deux conditions et si X^{stable} n'a pas de composantes rationnelles dans ses fibres géométriques, alors X_U a bien un modèle de Néron, i.e. le ns-modèle de Néron construit précédemment est séparé.

Partie III: Changement de base modérément ramifié de modèles de Néron

Les modèles de Néron passent aux changements de base lisses, et descendent sous les recouvrements lisses. Cela est propre aux morphismes lisses. Soit $f\colon S' \to S$ un morphisme plat, $U \subset S$ un ouvert schématiquement dense et X_U/U un U-schéma propre et lisse. On pose $U' = U \times_S S'$ et $X_{U'} = X \times_S S'$. En général, lorsque X_U et $X_{U'}$ ont des modèles de Néron N et N', on a seulement un morphisme de changement de base $N \times_S S' \to N'$. Dans cette partie, nous étudions ce morphisme de changement de base lorsque S et S' sont des schémas réguliers; U est le complément d'un diviseur de S à croisements strictement normaux; f est un morphisme fini et localement libre, lisse sur U et modérément ramifié sur S; et X_U est une variété abélienne. Il s'agit d'une généralisation en dimension arbitraire d'un travail similaire réalisé par Bas Edixhoven sur la descente des modèles de Néron de variétés abéliennes le long d'extensions modérément ramifiées d'anneaux de valuation discrète.

Nous nous intèressons principalement à des questions de descente: supposons que $X_{U'}$ ait un modèle de Néron N'/S', X_U a-t-il un modèle de Néron N ? Et si oui, que peut-on dire de N ?

Nous répondons par l'affirmative à la première de ces questions. Nous rappelons que le foncteur $T/S \mapsto N'(T \times_S S')$, appelé *restriction de Weil* de N' à S et noté $\prod_{S'/S} N'$, est représentable, et qu'il existe un morphisme naturel $X_U \to \prod_{S'/S} N'$. Nous montrons que la clôture schématique de X_U dans $\prod_{S'/S} N'$ est le modèle de Néron de X_U.

Ensuite, nous tentons de rendre explicite N en termes de N'. Tout point géométrique s de S a un voisinage étale affine $V = \operatorname{Spec} R$, tel que $V \times_S S'$ soit de la forme $\operatorname{Spec} R'$ avec

$$R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)_{1 \leq j \leq r}$$

pour un système régulier de paramètres $(f_1, ..., f_r)$ de R; et tel que tous les f_i s'annulent en s. En conséquence, par la suite, nous allons supposer que $S = \operatorname{Spec} R$ et $S' = \operatorname{Spec} R'$ sont tels que ci-dessus, et nous allons nous intéresser à N_Z, où Z est le sous-schéma de S où tous les f_i s'annulent. Remarquons que Z est aussi le sous-schéma fermé de S' où tous les T_i s'annulent. Nous allons décrire une certaine filtration de N_Z.

Soit μ_{n_j} le groupe des racines n_j-èmes de l'unité dans R. Quitte à remplacer R par une extension étale, on suppose que μ_{n_j} est d'ordre n_j. Le groupe $G = \prod_{i=1}^{r} \mu_j$ agit à droite sur S' via $(\xi_i)_{1\leq i\leq r}.T_j = \xi_j T_j$. Cette action induit naturellement une action de G sur $\prod_{S'/S} N'$, et nous montrons que N représente le foncteur $\left(\prod_{S'/S} N'\right)^G$ des invariants sous cette action.

Appelons A'_d le sous-A-module de A' composé des polynômes homogènes de degré d en les T_j; et Λ_d la A-base de A'_d constituée de ses monômes unitaires. Nous exhibons une filtration descendante $(F^d N'_Z)_{d\in\mathbb{N}}$ de sous-Z-espaces algébriques en groupes de $\left(\prod_{S'/S} N'\right) \times_S Z$. Cette filtration est stationnaire et triviale à partir de $d = 1 + \prod_{j=1}^{r}(n_j - 1)$. Nous montrons que le quotient $\mathrm{Gr}^0 N'_Z := F^0 N'_Z / F^1 N'_Z$ est canoniquement isomorphe à N'_Z; et pour $d \geq 1$, nous exhibons un isomorphisme canonique entre $\mathrm{Gr}^d N'_Z := F^d N'_Z / F^{d+1} N'_Z$ et l'algèbre de Lie $\mathrm{Lie}_{N'_Z/Z}(A'_d)$. Cette dernière est elle-même canoniquement isomorphe au produit fibré $\prod_{P\in\Lambda_d, Z} \mathrm{Lie}_{N'_Z/Z}(PA)$, où chaque $\mathrm{Lie}_{N'_Z/Z}(PA)$ est (non-canoniquement) isomorphe à l'algèbre de Lie "classique" $\mathrm{Lie}_{N'_Z/Z}$ puisque PA est un A-module libre de rang 1.

L'action naturelle de G sur N' induit des actions équivariantes sur les $F^d N'_Z$, donc sur les $\mathrm{Gr}^d N'_Z$. Nous montrons que cette action naturelle $G \curvearrowright \mathrm{Gr}^d N'_Z$ est obtenue à partir d'actions de G sur chaque facteur $\mathrm{Lie}_{N'_Z/Z}(PA)$, et que si P est le monôme $\prod_{i=1}^{r} T_i^{k_i}$, alors l'espace des invariants $\mathrm{Lie}_{N'_Z/Z}(PA)^G$ s'identifie via le morphisme $P \mapsto 1$ au sous-espace de $\mathrm{Lie}_{N'_Z/Z}$ où chaque $(\xi_i)_{1\leq i\leq r}$ dans G agit par multiplication par $\prod_i \xi_i^{k_i}$. En conséquence, les espaces $F^d N_Z := (F^d N'_Z)^G$ forment une filtration descendante de sous-Z-espaces algébriques en groupes de $N_Z = (F^0 N'_Z)^G$, stationnaire et triviale à partir de $d = 1 + \prod_{j=1}^{r}(n_j - 1)$, dont nous avons explicité les quotients successifs.

Curriculum Vitae

Thibault Poiret was born on December 4, 1993, in Ermont, France. He grew up in Sartrouville, in which he attended high school at Lycée Evariste Galois.

In 2010, after finishing high school, he attended preparatory classes in Lycée Condorcet, Paris. In 2012, he entered ENS Rennes, and studied at ENS Rennes and Université de Rennes 1 until 2016. There, he obtained a Bachelor's degree in mathematics and one in computer science; prepared and obtained the Agrégation of mathematics; and pursued a Master's degree in mathematics, with a focus on algebra, geometry and number theory.

Then, he started a PhD in mathematics in cotutelle between Université de Bordeaux, where he worked between 2016 and 2018 under the supervision of Prof. Qing Liu; and Universiteit Leiden, where he worked until 2020 under the supervision of Prof. Bas Edixhoven. He will defend the thesis on October 20, 2020.

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