Expansions of quantum group invariants

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Summary

The subject of this thesis is a certain set of quantum groups, and the knot invariants arising from these quantum groups. A knot is an embedding of the circle S^1 into the three-dimensional space $\mathbb{R}^3$. Two knots are equivalent if they can be transformed into each other in a continuous way. Such a transformation is called an isotopy. A knot invariant is a map that maps a knot to a set S such that the image is invariant under isotopies of knots. S can be any set. In our case S will be the space of polynomials in two variables.

A quantum group is, contrary to what the name sugests, not a group. A quantum group is a Hopf algebra that originates from the functions on a Lie group. A Hopf algebra is a vector space equipped with a (co)product and a (co)unit and an antipode. In particular, a Hopf algebra is an algebra with unit. The multiplication in an algebra A can be seen as a map $\mu : A \otimes A \to A$. The dual space of an algebra is a coalgebra (ignoring infinite dimensionality issues). A coalgebra is a vector space equipped with a coproduct and a counit. The dual map of the multiplication map is a map $\mu^* : A^* \to A^* \otimes A^*$, where the tensor product is completed in the appropriate sense. This construction can be applied to the infinite dimensional case by appropriately defining the dual space of A.

A Hopf algebra is both an algebra and a coalgebra which has an antipode S. S plays the role of the inverse, but is only a convolution inverse of the coproduct. This means that when $Id \otimes S$ (or $S \otimes id$) is applied to the coproduct, and both tensor factors are multiplied, this should yield zero. Like the inverse, S is an anti-homomorphism. Some Hopf algebras can be equipped with a quasitriangular structure. These are the Hopf algebras that will be considered in this thesis. A quasitriangular Hopf algebra enables us to define a knot invariant from the Hopf algebra.

A quasitriangular structure $\mathcal{R}$ on a Hopf algebra H is called an R-matrix. An R-matrix satisfies the Yang-Baxter equation. When we write $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$, and denote $\mathcal{R}_{ij}$ for an element in $H^{\otimes n}, i, j \leq n, i \neq j$, where $\mathcal{R}_1$ is in the i-th factor and $\mathcal{R}_2$ is in the j-th factor. The Yang-Baxter equation is then written as

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

A way to construct a quasitriangular Hopf algebra is given by the Drinfel'd double construction. A standard example is given by the quantum group $U_q(sl_2)$. From this example famous invariants such as the Jones polynomial and the Alexander polynomial can be cosntructed. This algebra originates from functions on the Lie group SL(2). The $U_q(sl_2)$ Hopf algebra can be considered as the quotient of

the Drinfel'd double of a Hopf subalgebra $U_q(b^-)$ of $U_q(sl_2)$ and its dual $U_q(b^+)$. It is possible to define a variation of the $U_q(sl_2)$ quantum group by deforming the comultiplication on $U_q(b^-)$ with a parameter ϵ such that $\epsilon^2 = 0$. This deformation is equivalent to multiplying the Lie bialgebra cobracket of the Lie bialgebra b^- with ϵ, and quantizing this Lie bi algebra to obtain $U_q(b_\epsilon^-)$. A quasitriangular Hopf algebra can then be obtained by applying the Drinfel'd double construction to $U_q(b_\epsilon^-)$.
In this thesis, this construction is applied to the Lie bialgebra sl_3 to obtain the corresponding quantum group $U_q(sl_3^\epsilon)$. The knot invariant that is obtained from this quasitriangular structure is studied and calculated for a few knots. For these calculations, a new formalism is needed, and this formalism is introduced in this thesis, along with the proof of convergence. In particular it is proven that the knot invariant corresponding to $U_q(sl_3^\epsilon)$ can be calculated in polynomial time. An attempt is made to generalize this construction to sl_n.