Expansions of quantum group invariants

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A. Appendices

A.1. Mathematica

In this section we present the implementation of the zipping formalism used to do calculations with the Hopf algebra $U_q(sl_3^\epsilon)$, and to calculate the quantum double explicitly. The proof of the zipping theorem can be found in chapter 2. The program labeled *sl3invariant.nb* is an implementation of this theorem. The program *sl3invariant.nb* is based on the program *sl2invariant.nb* developed by Bar-Natan and Van der Veen. This program can be found on

http://drorbn.net/AcademicPensieve/Projects/SL2Invariant/index.html.

In *sl2invariant.nb* one can find an implementation of the invariant based on the quantum group $U_q(sl_2^\epsilon)$. The knot invariant presented in this thesis is based on the $U_q(sl_2^\epsilon)$ construction by Bar-Natan and Van der Veen.
The difference between *sl2invariant.nb* and *sl3invariant.nb* is the use of the three-stage zip. This is an essential difference, since it provides a convergent implementation of the zipping theorem for the $U_q(sl_3^\epsilon)$ Hopf algebra. The proof that this implementation is convergent can be found in chapter 2.
In this program we implement the quantum group $U_q(sl_3)$ constructed in chapter 1. We check (co)associativity, if Δ is a homomorphism, the pairing axioms, the antipode axioms, associativity and the Turaev moves. The knot invariant is computed for the Trefoil, the mirror Trefoil, the figure eight and the 6-3 knot in the Rolfson knot table.

The full sl_3 invariant using the Drinfel'd double. For compatibility reasons, we use XX instead of the generator X. This program continues sl2invariant.nb by Dror Bar-Natan and Roland van der Veen.

Profiling

```
(*BeginProfile[];*)
```

External Utilities

```
In[3]:= HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

```
In[4]:= MaxBy[list_, fun_, n_] := list[[Ordering[fun /@ list, -n]]];
```

Canonical Form:

Program

```
In[5]:= CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
          Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CCF[x]]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := PP_CF@Module[
            {vs = Cases[ℰ, (XX | Y | Z | A | B | b | s | t | a | x | y | z | XX* | Y* | Z* |
                   A* | B* | s* | t* | b* | a* | x* | y* | z*)_, ∞] ⋃ {XX, Y, Z, A, B, b,
                 s, t, a, x, y, z, XX*, Y*, Z*, A*, B*, s*, t*, b*, a*, x*, y*, z*}},
            Total[CoefficientRules[Expand[ℰ], vs] /.
              (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]
           ];
```

Program

The Kronecker δ:

Program

```
In[9]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

```
In[10]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
            CF[L1 == L2] ⋀ CF[Q1 == Q2] ⋀ CF[Normal[P1 - P2] == 0];
         𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
         𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

In[13]:=
```
𝔼3@𝔼sp__[ω_, L_, Q_, P_] := Module[
   {NP = Normal[P]},
   𝔼sp[L, ω^-1 Q, (ω^-1 NP /. ϵ → ω^-4 ϵ) + O[ϵ]^($k+1)] // CF
  ];
𝔼4@𝔼sp__[L_, Q_, P_] := Module[
   {NP = Normal[P], ω},
   ω = (NP /. ϵ → 0)^-1;
   𝔼sp[ω, L, ω Q, (ω NP /. ϵ → ω^4 ϵ) + O[ϵ]^($k+1)] // CF
  ];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

In[15]:=
```
(u_i_)* := (u*)i;
((u_i_)*)* := ui;
(((u_)*)i_)* := ui;
((u_)*)* := u;
```

Program

Finite Zips:

Program

In[19]:=
```
collect[sd_SeriesData, ξ_] := MapAt[collect[#, ξ] &, sd, 3];
collect[ℰ_, ξ_] := PPCollect@Collect[ℰ, ξ];
Zip{}[P_] := P; Zip{ξ_,ξs___}[P_] := PPZip[
  (collect[P // Zip{ξs}, ξ] /. f_. ξ^d_. ↦ ∂{ξ*,d} f) /. ξ* → 0]
```

Program

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars. $\mathbb{E}[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a,b's, Q is a combination of $x_i X_j$ (possibly starred and/or mixed with other variables), and P is a perturbation polynomial. It should be interpreted via $\mathbb{O}[\mathbb{E}[\ldots], \{X_1, Y_1, Z_1, A_1, B_1, b_1, a_1, z_1, y_1, x_1\}_i, \ldots]$, with an assumed standard ordering on the generators for an interpretation of the tensor as an expression in $U_q(\mathrm{sl}_3{}^{\epsilon})$.

Program

```
QZipξs_List@𝔼[L_, Q_, P_] := PPQZip@Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule},
    zs = Table[ξ*, {ξ, ξs}];
    c = CF[Q /. Alternatives @@ (ξs ⋃ zs) → 0];
    ys = CF@Table[∂ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}];
    ηs = CF@Table[∂z (Q /. Alternatives @@ ξs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδz,ξ* - ∂z,ξ Q, {ξ, ξs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ξrule = Thread[ξs → ξs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zipξs[P /. (zrule ⋃ ξrule)]] ];
```

Program

Upper to lower and lower to Upper:

Program

In[23]:=
```
U21 = {𝕓_i^p_· → 𝕖^(-p b_i), 𝕓^p_· → 𝕖^(-p b), 𝕒_i^p_· → 𝕖^(-p a_i), 𝕒^p_· → 𝕖^(-p a),
   T_i^p_· → 𝕖^(p t_i), T^p_· → 𝕖^(p t), S_i^p_· → 𝕖^(p s_i), S^p_· → 𝕖^(p s), 𝒶_i^p_· → 𝕖^(p a*_i), 𝒶^p_· → 𝕖^(p a*),
   𝒷_i^p_· → 𝕖^(p b*_i), 𝒷^p_· → 𝕖^(p b*), 𝔹_i^p_· → 𝕖^(-p B_i), 𝔹^p_· → 𝕖^(-p B), 𝔸_i^p_· → 𝕖^(-p A_i),
   𝔸^p_· → 𝕖^(-p A), 𝒜_i^p_· → 𝕖^(p A*_i), 𝒜^p_· → 𝕖^(p A*), ℬ_i^p_· → 𝕖^(p B*_i), ℬ^p_· → 𝕖^(p B*)};
l2U = {𝕖^(c_· b_i_+d_·) :→ 𝕓_i^-c 𝕖^d, 𝕖^(c_· b+d_·) :→ 𝕓^-c 𝕖^d, 𝕖^(c_· a_i_+d_·) :→ 𝕒_i^-c 𝕖^d, 𝕖^(c_· a+d_·) :→ 𝕒^-c 𝕖^d,
   𝕖^(c_· t_i_+d_·) :→ T_i^c 𝕖^d, 𝕖^(c_· t+d_·) :→ T^c 𝕖^d, 𝕖^(c_· s_i_+d_·) :→ S_i^c 𝕖^d, 𝕖^(c_· s+d_·) :→ S^c 𝕖^d,
   𝕖^(c_· a*_i_+d_·) :→ 𝒶_i^c 𝕖^d, 𝕖^(c_· a*+d_·) :→ 𝒶^c 𝕖^d, 𝕖^(c_· b*_i_+d_·) :→ 𝒷_i^c 𝕖^d, 𝕖^(c_· b*+d_·) :→ 𝒷^c 𝕖^d,
   𝕖^(c_· B_i_+d_·) :→ 𝔹_i^-c 𝕖^d, 𝕖^(c_· B+d_·) :→ 𝔹^-c 𝕖^d, 𝕖^(c_· A_i_+d_·) :→ 𝔸_i^-c 𝕖^d, 𝕖^(c_· A+d_·) :→ 𝔸^-c 𝕖^d,
   𝕖^(c_· A*_i_+d_·) :→ 𝒜_i^c 𝕖^d, 𝕖^(c_· A*+d_·) :→ 𝒜^c 𝕖^d, 𝕖^(c_· B*_i_+d_·) :→ ℬ_i^c 𝕖^d, 𝕖^(c_· B*+d_·) :→ ℬ^c 𝕖^d,
   𝕖^ℰ_ :→ 𝕖^(Expand@ℰ)};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = \mathrm{P}e^{L+Q}$. Such zips regard all of $\mathrm{P}e^{Q}$ as a single"P". Here the z's are A, B and a^*, b^* and the ζ's are A^*, B^* and a, b. s and t are not regarded as scalars for zip-technicalities. DB and STB are variations of B with a different choice for ζs in LZip, to speed up the zipping of tensors with s and t instead of A and B.

Program

```
LZip_ζs_List@𝔼[L_, Q_, P_] :=
  PP_LZip@Module[{ζ, z, zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_(z,ζ*) - ∂_(z,ζ) L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. U2l /. (zrule ⋃ ζrule);
    EEQ[ps___] :=
     EEQ[ps] = PP_"EEQ"@(CF[𝕖^(-Q1) D[𝕖^Q1, Sequence @@ Thread[{zs, {ps}}]]] /.
         Alternatives @@ zs → 0 //. l2U);
    CF /@ ((*CF/@*)𝔼[
       c + ηs.lt.ys, Q1 /. Alternatives @@ zs → 0,
       Det[lt] (Zip_ζs[(EQ @@ zs) (P /. U2l /. (zrule ⋃ ζrule))] /.
           Derivative[ps___][EQ][___] :→ EEQ[ps] /. _EQ → 1)
     ] //. l2U)
   ];
```

Program

In[26]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_Bind@Module[{n},
    Times[
      L /. Table[(v : XX | Y | Z | A | 𝔸 | B | 𝔹 | s | t | S | T |
             b | (*𝕓|*)a(*|𝕒*) | x | y | z)_i → v_n@i, {i, {is}}],
      R /. Table[(v : XX* | Y* | Z* | A*(*|𝒜*) | B* | (*ℬ| *)b* | s* |
             t* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_n@i, {i, {is}}]
     ] // LZip_Flatten@Table[{A*_n@i,B*_n@i,(s*)_n@i,(t*)_n@i,b_n@i,a_n@i},{i,{is}}] //
    QZip_Flatten@Table[{XX_n@i,Y*_n@i,y_n@i,x*_n@i},{i,{is}}] // QZip_Flatten@Table[{Z*_n@i,z_n@i},{i,{is}}]];
B_is___[L_, R_] := B_{is}[
   L,
   R];
```

```
DB{} [L_, R_] := L R;
DB{is__} [L_E, R_E] := PP_DBind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | s | t | b | 𝕓 | a | 𝕒 | x | y | z)_i → v_n@i, {i, {is}}],
        R /.
         Table[(v : XX* | Y* | Z* | s* | t* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_n@i, {i, {is}}]
      ] // LZip_Flatten@Table[{(s*)_n@i,(t*)_n@i,b_n@i,a_n@i},{i,{is}}] //
    QZip_Flatten@Table[{XX_n@i,Y*_n@i,y_n@i,x*_n@i},{i,{is}}] // QZip_Flatten@Table[{Z*_n@i,z_n@i},{i,{is}}]];
DB_is__ [L_, R_] := DB{is} [L, R];
```

```
In[29]:= STB{} [L_, R_] := L R;
STB{is__} [L_E, R_E] := PP_STBind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | b | 𝕓 | a | 𝕒 | x | y | z)_i → v_n@i, {i, {is}}],
        R /. Table[(v : XX* | Y* | Z* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)_i → v_n@i, {i, {is}}]
      ] // LZip_Flatten@Table[{b_n@i,a_n@i},{i,{is}}] //
    QZip_Flatten@Table[{XX_n@i,Y*_n@i,y_n@i,x*_n@i},{i,{is}}] // QZip_Flatten@Table[{Z*_n@i,z_n@i},{i,{is}}]];
STB_is__ [L_, R_] := STB{is} [L, R];
```

Program

𝔼 morphisms with domain and range.

Program

```
In[32]:= B_is_List[𝔼_d1_→r1_ [L1_, Q1_, P1_], 𝔼_d2_→r2_ [L2_, Q2_, P2_]] :=
   𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ B_is [𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
STB_is_List[𝔼_d1_→r1_ [L1_, Q1_, P1_], 𝔼_d2_→r2_ [L2_, Q2_, P2_]] :=
   𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ STB_is [𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_d1_→r1_ [L1_, Q1_, P1_] // 𝔼_d2_→r2_ [L2_, Q2_, P2_] :=
   B_r1⋂d2 [𝔼_d1→r1 [L1, Q1, P1], 𝔼_d2→r2 [L2, Q2, P2]];
𝔼_d1_→r1_ [L1_, Q1_, P1_] ≡ 𝔼_d2_→r2_ [L2_, Q2_, P2_] ^:=
   (d1 == d2) ⋀ (r1 == r2) ⋀ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_d1_→r1_ [L1_, Q1_, P1_] 𝔼_d2_→r2_ [L2_, Q2_, P2_] ^:=
   𝔼_(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_d_→r_ [L_, Q_, P_]_$k_ := 𝔼_d→r @@ 𝔼[L, Q, P]_$k;
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

Program

“Define” code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

In[39]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is_ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

Program

Booting Up

Program

```
$k = 1;
```

The multiplication tensors on both halves of the double are defined here. $k indicates the degree $\epsilon^{\$k+1} = 0$ we are working over. The am and bm tensors given here only work for $k=1 and $k=0.

In[43]:=
```
Define[am_{i,j→k} =
  𝔼_{{i,j}→{k}}[(a*_i + a*_j) a_k + (b*_i + b*_j) b_k, (e^{(b*)_j+(a*)_j} z*_i + z*_j + e^{-(b*)_j+2 (a*)_j} x*_i (y*)_j) z_k +
     (e^{-(a*)_j+2 (b*)_j} y*_i + y*_j) y_k + (e^{-(b*)_j+2 (a*)_j} x*_i + x*_j) x_k,
    1 + ϵ (ℏ (z*)_j (x*)_i e^{-(b*)_j+2 (a*)_j} z_k x_k + ℏ (z*)_j (x*)_i (y*)_j e^{-(b*)_j+2 (a*)_j} z_k z_k -
        ℏ (z*)_j (y*)_i e^{2 (b*)_j-(a*)_j} z_k y_k - ℏ (y*)_i (x*)_i (y*)_j e^{(a*)_j+(b*)_j} z_k y_k - ℏ (x*)_i
         e^{-(b*)_j+2 (a*)_j} (y*)_j (y*)_j z_k y_k - ℏ (x*)_i e^{-(b*)_j+2 (a*)_j} (y*)_j y_k x_k) + O[ϵ]^2]_{$k},
 bm_{i,j→k} = 𝔼_{{i,j}→{k}}[A_k A*_i + A_k A*_j + B_k B*_i + B_k B*_j, XX_k XX*_i + XX_k XX*_j + Y_k Y*_i +
     Y_k Y*_j + Z_k Z*_i + Z_k Z*_j, 1 + (-XX_k (A*)_i (XX*)_j + ℏ XX_k Y_k (XX*)_j (Y*)_i -
        Y_k (B*)_i (Y*)_j + 2 Z_k (XX*)_i (Y*)_j - ℏ XX_k Z_k (XX*)_j (Z*)_i +
        ℏ Y_k Z_k (Y*)_j (Z*)_i - Z_k (A*)_i (Z*)_j - Z_k (B*)_i (Z*)_j) ϵ + O[ϵ]^2]_{$k}];
```

The R-matrix is defined with the Faddeev-Quesne formula.

Program

$$\text{Define}\Big[R_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar A_i a_j + \hbar B_i b_j,\ \hbar XX_i x_j + \hbar Y_i y_j + \hbar Z_i z_j,\ e\wedge\Big(\sum_{k=2}^{\$k+1} \frac{(e^{-2\epsilon\hbar}-1)^k (\hbar XX_i x_j)^k}{k(1-e^{-2k\epsilon\hbar})}\Big)\ e\wedge\Big(\sum_{l=2}^{\$k+1} \frac{(e^{-2\epsilon\hbar}-1)^l (\hbar Y_i y_j)^l}{l(1-e^{-2l\epsilon\hbar})}\Big)\ e\wedge\Big(\sum_{m=2}^{\$k+1} \frac{(e^{-2\epsilon\hbar}-1)^m (\hbar Z_i z_j)^m}{m(1-e^{-2m\epsilon\hbar})}\Big)\Big]_{\$k},$$

$$\overline{R}_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[-\hbar a_j A_i - \hbar b_j B_i,\ -\hbar x_j XX_i \mathbb{A}_i^{2\hbar} \mathbb{B}_i^{-\hbar} + \hbar^2 XX_i Y_i z_j \mathbb{A}_i^{\hbar} \mathbb{B}_i^{\hbar} - \hbar z_j Z_i \mathbb{A}_i^{\hbar} \mathbb{B}_i^{\hbar} - \hbar y_j Y_i \mathbb{A}_i^{-\hbar} \mathbb{B}_i^{2\hbar},\ 1 + \text{If}\big[\$k == 0,\ 0,\ (\overline{R}_{\{i,j\},\$k-1})_{\$k}[3] - (((\overline{R}_{\{i,j\},0})_{\$k}\, R_{1,2}\, (\overline{R}_{\{3,4\},\$k-1})_{\$k}) \,//\, (bm_{i,1\to i}\, am_{j,2\to j}) \,//\, (bm_{i,3\to i}\, am_{j,4\to j}))[3]\big]\Big],$$

$$P_{i,j} = \mathbb{E}_{\{i,j\}\to\{\}}\Big[\frac{1}{\hbar} A^*{}_i\, a^*{}_j + \frac{1}{\hbar} B^*{}_i\, (b^*)_j,\ \frac{1}{\hbar} XX^*{}_i\, x^*{}_j + \frac{1}{\hbar} Y^*{}_i\, y^*{}_j + \frac{1}{\hbar} Z^*{}_i\, z^*{}_j,\ 1 + \text{If}\big[\$k == 0,\ 0,\ (P_{\{i,j\},\$k-1})_{\$k}[3] - (R_{1,2} \,//\, ((P_{\{1,j\},0})_{\$k}\, (P_{\{i,2\},\$k-1})_{\$k}))[3]\big]\Big]\Big]$$

Program

In[45]:=

$$\text{Define}\Big[aS_j = \overline{R}_{i,j} \sim B_i \sim P_{i,j},\ \overline{aS}_i = \mathbb{E}_{\{i\}\to\{i\}}\Big[-a^*{}_i\, a_i - b^*{}_i\, b_i,\ -e^{-(b^*)_i-(a^*)_i} z^*{}_i\, z_i + e^{-(b^*)_i-(a^*)_i} y^*{}_i\, (x^*)_i\, z_i - e^{-2(b^*)_i+(a^*)_i} y^*{}_i\, y_i - e^{(b^*)_i-2(a^*)_i} x^*{}_i\, x_i,\ 1 + \text{If}\big[\$k == 0,\ 0,\ (\overline{aS}_{\{i\},\$k-1})_{\$k}[3] - ((\overline{aS}_{\{i\},0})_{\$k} \sim B_i \sim aS_i \sim B_i \sim (\overline{aS}_{\{i\},\$k-1})_{\$k})[3]\big]\Big]\Big]$$

Program

In[46]:=

$$\text{Define}\Big[bS_i = R_{i,1} \sim B_1 \sim aS_1 \sim B_1 \sim P_{i,1},\ \overline{bS}_i = R_{i,1} \sim B_1 \sim \overline{aS}_1 \sim B_1 \sim P_{i,1},\ a\Delta_{i\to j,k} = (R_{1,j}\, R_{2,k}) \,//\, bm_{1,2\to 3} \,//\, P_{3,i},\ b\Delta_{i\to j,k} = (R_{j,1}\, R_{k,2}) \,//\, am_{1,2\to 3} \,//\, P_{i,3}\Big]$$

Program

In[47]:=

$$\text{Define}\Big[dm_{i,j\to k} = \Big(\mathbb{E}_{\{i,j\}\to\{i,j\}}\big[(A^*)_i A_i + (B^*)_i B_i + b^*{}_j b_j + a^*{}_j a_j,\ y^*{}_j y_j + x^*{}_j x_j + z^*{}_j z_j + (XX^*)_i XX_i + (Y^*)_i Y_i + Z^*{}_i Z_i,\ 1\big]\ (a\Delta_{i\to 1,2} \,//\, a\Delta_{2\to 2,3} \,//\, \overline{aS}_3)\ (b\Delta_{j\to -1,-2} \,//\, b\Delta_{-2\to -2,-3})\Big) \,//\, (P_{-1,3}\, P_{-3,1}\, am_{2,j\to k}\, bm_{i,-2\to k}),$$

$$dS_i = \mathbb{E}_{\{i\}\to\{1,2\}}\big[(A^*)_i A_1 + (B^*)_i B_1 + b^*{}_i b_2 + a^*{}_i a_2,\ y^*{}_i y_2 + x^*{}_i x_2 + z^*{}_i z_2 + (XX^*)_i XX_1 + (Y^*)_i Y_1 + Z^*{}_i Z_1,\ 1\big] \,//\, (\overline{bS}_1\, aS_2) \,//\, dm_{2,1\to i},$$

$$d\Delta_{i\to j,k} = (b\Delta_{i\to 3,1}\, a\Delta_{i\to 2,4}) \,//\, (dm_{3,4\to k}\, dm_{1,2\to j}),$$

$$\overline{dS}_i = \mathbb{E}_{\{i\}\to\{1,2\}}\big[(A^*)_i A_1 + (B^*)_i B_1 + b^*{}_i b_2 + a^*{}_i a_2,\ y^*{}_i y_2 + x^*{}_i x_2 + z^*{}_i z_2 + (XX^*)_i XX_1 + (Y^*)_i Y_1 + Z^*{}_i Z_1,\ 1\big] \,//\, (bS_1\, \overline{aS}_2) \,//\, dm_{2,1\to i}\Big]$$

Program

In[48]:=

$$\text{Define}\Big[C_i = \mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ \frac{1}{\mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar}} - \frac{\hbar(a_i+b_i)\epsilon}{\mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar}} + O[\epsilon]^2\Big]_{\$k},\ \overline{C}_i = \mathbb{E}_{\{\}\to\{i\}}\big[0,\ 0,\ (\mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + (\hbar a_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + \hbar b_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar})\epsilon) + O[\epsilon]^2\big]_{\$k},\ \text{Kink}_i = (R_{1,3}\, \overline{C}_2) \,//\, dm_{1,2\to 1} \,//\, dm_{1,3\to i},\ \overline{\text{Kink}}_i = (\overline{R}_{1,3}\, C_2) \,//\, dm_{1,2\to 1} \,//\, dm_{1,3\to i}\Big]$$

Program

Note: s=2A-B+ϵa, t=2B-A+ϵb. This substitution is implemented in the following tensors.

Program

```
In[49]:= Define[AB2st_i = E_{i}→{i}[b*_i b_i + a*_i a_i + (A*)_i (2/3 (s_i + t_i/2)) + (B*)_i (2/3 (t_i + s_i/2)),
      (XX*)_i XX_i + (Y*)_i Y_i + Z*_i Z_i + y*_i y_i + x*_i x_i + z*_i z_i,
      1 - ϵ (A*)_i 1/3 (2 a_i + b_i) - ϵ (B*)_i 1/3 (2 b_i + a_i) + O[ϵ]^2]_$k,
    st2AB_i = E_{i}→{i}[b*_i b_i + a*_i a_i + (s*)_i (2 A_i - B_i) + (t*)_i (2 B_i - A_i), (XX*)_i XX_i +
      (Y*)_i Y_i + Z*_i Z_i + y*_i y_i + x*_i x_i + z*_i z_i, 1 + ϵ (s*)_i a_i + ϵ (t*)_i b_i + O[ϵ]^2]_$k]
```

The following definitions are used for a slightly faster implementation of the quantum-double that leaves out the s and the t from the zip. Since s and t are central, this is well defined and yields the same result. These tensors should be zipped using the STB and DB zip function. As such, we also check the axioms for these tensors.

```
In[50]:= Define[stR_{i,j} = (R_{i,j} ~ B_{i,j} ~ (AB2st_i AB2st_j)) // Simplify,
    stR̄_{i,j} = (R̄_{i,j} ~ B_{i,j} ~ (AB2st_i AB2st_j)) // Simplify,
    stC̄_i = (C̄_i ~ B_{i} ~ (AB2st_i)) // Simplify,
    stC_i = (C_i ~ B_{i} ~ (AB2st_i)) // Simplify,
    stKink̄_i = (Kink̄_i ~ B_{i} ~ (AB2st_i)) // Simplify,
    stKink_i = (Kink_i ~ B_{i} ~ (AB2st_i)) // Simplify,
    stdm_{i,j→k} = ((st2AB_i st2AB_j) ~ B_{i,j} ~ dm_{i,j→k} ~ B_k ~ AB2st_k),
    stdΔ_{i→j,k} = (st2AB_i // dΔ_{i→j,k} // (AB2st_j AB2st_k)),
    stdS_i = (st2AB_i // dS_i // AB2st_i),
    stP_{i,j} = ((st2AB_i st2AB_j) // P_{i,j}),
    ddΔ_{i→j,k} = (st2AB_i // dΔ_{i→j,k} // (AB2st_j AB2st_k)) /.
      {S_{j|k} → S, T_{j|k} → T, t_{j|k} → t, s_{j|k} → s, (s*)_i → 0, (t*)_i → 0},
    ddS_i = (st2AB_i // dS_i // AB2st_i) /. {S_i → S, T_i → T,
       s_i → s, t_i → t, (s*)_i → 0, (t*)_i → 0},
    PP_{i,j} = ((st2AB_i st2AB_j) // P_{i,j}) /. {(s*)_{i|j} → 0, (t*)_{i|j} → 0},
    RR_{i,j} = (R_{i,j} // (AB2st_i AB2st_j)) /. {t_{i|j} → t, s_{i|j} → s},
    RR̄_{i,j} = (R̄_{i,j} // (AB2st_i AB2st_j)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T},
    CC̄_i = (C̄_i ~ B_{i} ~ (AB2st_i)) /. {S_{i|j} → S, T_{i|j} → T} // Simplify,
    CC_i = (C_i ~ B_{i} ~ (AB2st_i)) /. {S_{i|j} → S, T_{i|j} → T} // Simplify,
    KKink̄_i =
     (Kink̄_i ~ B_{i} ~ (AB2st_i)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T} // Simplify,
    KKink_i = (Kink_i ~ B_{i} ~ (AB2st_i)) /. {t_{i|j} → t, s_{i|j} → s, S_{i|j} → S, T_{i|j} → T} //
      Simplify,
    ddm_{i,j→k} = ((st2AB_i st2AB_j) // dm_{i,j→k} // AB2st_k) /.
      {S_k → S, T_k → T, t_k → t, s_k → s, (s*)_{i|j} → 0, (t*)_{i|j} → 0}];

Define[BB_{i,j→k} = ((dΔ_{i→1,r1} dΔ_{j→2,r2}) // dS_{r1} // dS̄_{r2} // dm_{r1,r2→k} // dm_{k,1→k} // dm_{k,2→k})]
```

Testing

```
Block[{$k = 1}, {
   am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j}, aS → aS_i,
   aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k}, dΔ → dΔ_{i→j,k},
   C → C_i, C̄ → C̄_i, Kink → Kink_i, Kink̄ → Kink̄_i, AB2st → AB2st_i, st2AB → st2AB_i
  }] //
 Column
```

Check that on the generators this agrees with our conventions in the handout:

```
Timing@{{"[x,a]" →
     ((𝔼_{{}→{1,2}}[0, 0, a_2 x_1] // am_{1,2→1})[3] - (𝔼_{{}→{1,2}}[0, 0, a_1 x_2] // am_{1,2→1})[3]),
    "[A,X]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 A_1] // bm_{1,2→1})[3] -
       (𝔼_{{}→{1,2}}[0, 0, XX_1 A_2] // bm_{1,2→1})[3]), "[x,y]" →
     ((𝔼_{{}→{1,2}}[0, 0, y_2 x_1] // am_{1,2→1})[3] - (𝔼_{{}→{1,2}}[0, 0, y_1 x_2] // am_{1,2→1})[3]),
    "[Y,X]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 Y_1] // bm_{1,2→1})[3] -
       (𝔼_{{}→{1,2}}[0, 0, XX_1 Y_2] // bm_{1,2→1})[3])} /. z_1 → z,
  {"Δ[X]" → Last[𝔼_{{}→{1}}[0, 0, XX_1] ~B_1~ bΔ_{1→1,2}],
   "Δ[A]" → Last[𝔼_{{}→{1}}[0, 0, A_1] ~B_1~ bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼_{{}→{1}}[0, 0, a_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[z]" → Last[𝔼_{{}→{1}}[0, 0, z_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼_{{}→{1}}[0, 0, x_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[Z]" → Last[𝔼_{{}→{1}}[0, 0, Z_1] ~B_1~ bΔ_{1→1,2}]},
  {
    "S(a)" → ((𝔼_{{}→{1}}[0, 0, a_1] ~B_1~ aS_1)[3]),
    "S(z)" → ((𝔼_{{}→{1}}[0, 0, z_1] ~B_1~ aS_1)[3]),
    "S(x)" → ((𝔼_{{}→{1}}[0, 0, x_1] ~B_1~ aS_1)[3]),
    "S(A)" → ((𝔼_{{}→{1}}[0, 0, A_1] ~B_1~ bS_1)[3]),
    "S(X)" → ((𝔼_{{}→{1}}[0, 0, XX_1] ~B_1~ bS_1)[3]),
    "S(Z)" → ((𝔼_{{}→{1}}[0, 0, Z_1] ~B_1~ bS_1)[3])
   } /. z_1 → z}
```

$\{1.265625, \{\{[x,a] \to 2\,x + O[\epsilon]^2,\ [A,X] \to -XX\,\epsilon + O[\epsilon]^2,$
$[x,y] \to z - x\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ [Y,X] \to (-2\,Z + XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2\},$
$\{\Delta[X] \to \left(XX_2 + XX_1\,\mathbb{A}_2^{-2\hbar}\,\mathbb{B}_2^{\hbar}\right) + O[\epsilon]^2,\ \Delta[A] \to (A_1 + A_2) + O[\epsilon]^2,$
$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^2,\ \Delta[z] \to (z_1 + z_2) + (2\,\hbar\,x_1\,y_2 - \hbar\,a_1\,z_2 - \hbar\,b_1\,z_2)\,\epsilon + O[\epsilon]^2,$
$\Delta[x] \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + O[\epsilon]^2,\ \Delta[Z] \to \left(Z_2 + Z_1\,\mathbb{A}_2^{-\hbar}\,\mathbb{B}_2^{-\hbar} + \hbar\,XX_1\,Y_2\,\mathbb{A}_2^{-2\hbar}\,\mathbb{B}_2^{\hbar}\right) + O[\epsilon]^2\},$
$\{S(a) \to -a + O[\epsilon]^2,\ S(z) \to -z + (2\,x\,y\,\hbar + 2\,z\,\hbar - a\,z\,\hbar - b\,z\,\hbar)\,\epsilon + O[\epsilon]^2,$
$S(x) \to -x - a\,x\,\hbar\,\epsilon + O[\epsilon]^2,\ S(A) \to -A + O[\epsilon]^2,\ S(X) \to -XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar} + O[\epsilon]^2,$
$S(Z) \to \left(-Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar} + XX\,Y\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\right) + \left(2\,Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar - XX\,Y\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar^2\right)\,\epsilon + O[\epsilon]^2\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
Timing@Block[{$k = 1},
  HL /@
   {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
 ]
```

```
{0.437500, {True, True}}
```

R and *P* are inverses:

```
Timing@
 Block[{$k = 1}, {HL[(R_{i,j} // P_{i,k}) ≡ 𝔼_{{k}→{j}}[a_j a*_k + b_j b*_k, x_j x*_k + y_j y*_k + z_j z*_k, 1]]}]
{0.031250, {True}}
```

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
Timing[HL /@ {(aS̄_1 // aS_1) ≡ 𝔼_{{1}→{1}}[a_1 a*_1 + b_1 b*_1, x_1 x*_1 + y_1 y*_1 + z_1 z*_1, 1],
   (bS̄_1 // bS_1) ≡ 𝔼_{{1}→{1}}[A_1 A*_1 + B_1 B*_1, XX_1 XX*_1 + Y_1 Y*_1 + Z_1 Z*_1, 1]}]
{0.406250, {True, True}}
```

(co)-associativity on both sides

```
Timing[HL /@
  {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
   (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
{1.078125, {True, True, True, True}}
```

Δ is an algebra morphism

```
Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
   (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
{1.312500, {True, True}}
```

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@ {
   (aΔ_{1→1,2} ~B_1~ aS_1) ~B_{1,2}~ am_{1,2→1}, (aΔ_{1→1,2} ~B_2~ aS_2) ~B_{1,2}~ am_{1,2→1},
   (bΔ_{1→1,2} ~B_1~ bS_1) ~B_{1,2}~ bm_{1,2→1}, (bΔ_{1→1,2} ~B_2~ bS_2) ~B_{1,2}~ bm_{1,2→1}}]
{1.015625, {True, True, True, True}}
```

S is an algebra anti-(co)morphism

```
Timing[HL /@
  {am_{1,2→1} ~B_1~ aS_1 ≡ (aS_1 aS_2) ~B_{1,2}~ am_{2,1→1}, bm_{1,2→1} ~B_1~ bS_1 ≡ (bS_1 bS_2) ~B_{1,2}~ bm_{2,1→1},
   aS_1 ~B_1~ aΔ_{1→1,2} ≡ aΔ_{1→2,1} ~B_{1,2}~ (aS_1 aS_2), bS_1 ~B_1~ bΔ_{1→1,2} ≡ bΔ_{1→2,1} ~B_{1,2}~ (bS_1 bS_2)}]
{2.500000, {True, True, True, True}}
```

R-matrix and antipode

```
R_{1,2} ~B_1~ (bS_1) ≡ R̄_{1,2}
True
```

Pairing axioms

```
Timing[HL /@ {(bm_{1,2→1} 𝔼_{{3}→{3}}[b*_3 b_3 + a*_3 a_3, y*_3 y_3 + x*_3 x_3 + z*_3 z_3, 1]) ~B_{1,3}~ P_{1,3} ≡
     (𝔼_{{1}→{1}}[(A*)_1 A_1 + (B*)_1 B_1, (XX*)_1 XX_1 + (Y*)_1 Y_1 + Z*_1 Z_1, 1]
        𝔼_{{2}→{2}}[(A*)_2 A_2 + (B*)_2 B_2, (XX*)_2 XX_2 + (Y*)_2 Y_2 + Z*_2 Z_2, 1] aΔ_{3→4,5}) ~B_{1,4}~
      P_{1,4} ~B_{2,5}~ P_{2,5}, (bΔ_{1→1,2} 𝔼_{{3}→{3}}[b*_3 b_3 + a*_3 a_3, y*_3 y_3 + x*_3 x_3 + z*_3 z_3, 1]
        𝔼_{{4}→{4}}[b*_4 b_4 + a*_4 a_4, y*_4 y_4 + x*_4 x_4 + z*_4 z_4, 1]) ~B_{1,3}~ P_{1,3} ~B_{2,4}~ P_{2,4} ≡
     (𝔼_{{1}→{1}}[(A*)_1 A_1 + (B*)_1 B_1, (XX*)_1 XX_1 + (Y*)_1 Y_1 + Z*_1 Z_1, 1] am_{3,4→3}) ~B_{1,3}~ P_{1,3}}]
{0.34375, {True, True}}
```

```
Timing[HL /@ {((bS₁ 𝔼{2}→{2}[b*₂ b₂ + a*₂ a₂, y*₂ y₂ + x*₂ x₂ + z*₂ z₂, 1]) // P₁,₂) ≡
    ((𝔼{1}→{1}[(A*)₁ A₁ + (B*)₁ B₁, (XX*)₁ XX₁ + (Y*)₁ Y₁ + Z*₁ Z₁, 1] aS₂) // P₁,₂),
   (bS̄₁ 𝔼{2}→{2}[b*₂ b₂ + a*₂ a₂, y*₂ y₂ + x*₂ x₂ + z*₂ z₂, 1]) ~B₁,₂~P₁,₂ ≡
    (𝔼{1}→{1}[(A*)₁ A₁ + (B*)₁ B₁, (XX*)₁ XX₁ + (Y*)₁ Y₁ + Z*₁ Z₁, 1] aS̄₂) ~B₁,₂~P₁,₂}]
{0.28125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
{
  "[a,y]" → ((𝔼{}→{1,2}[0, 0, y₂ a₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, y₁ a₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[b,x]"  → ((𝔼{}→{1,2}[0, 0, x₂ b₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, x₁ b₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[b,y]" → ((𝔼{}→{1,2}[0, 0, y₂ b₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, y₁ b₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[a,x]" → ((𝔼{}→{1,2}[0, 0, x₂ a₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, x₁ a₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[a,z]" → ((𝔼{}→{1,2}[0, 0, z₂ a₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, z₁ a₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[b,z]" → ((𝔼{}→{1,2}[0, 0, z₂ b₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, z₁ b₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[x,z]" → ((𝔼{}→{1,2}[0, 0, z₂ x₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, z₁ x₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[y,z]" → ((𝔼{}→{1,2}[0, 0, z₂ y₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, z₁ y₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[x,y]" → ((𝔼{}→{1,2}[0, 0, y₂ x₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, y₁ x₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[Y,y]" → ((𝔼{}→{1,2}[0, 0, y₂ Y₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, y₁ Y₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[Y,x]" → ((𝔼{}→{1,2}[0, 0, x₂ Y₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, x₁ Y₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[X,y]" → ((𝔼{}→{1,2}[0, 0, y₂ XX₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, y₁ XX₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[XX,x]" → ((𝔼{}→{1,2}[0, 0, x₂ XX₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, x₁ XX₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[Z,z]" → ((𝔼{}→{1,2}[0, 0, z₂ Z₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, z₁ Z₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[Z,y]" → ((𝔼{}→{1,2}[0, 0, y₂ Z₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, y₁ Z₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[Z,x]" → ((𝔼{}→{1,2}[0, 0, x₂ Z₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, x₁ Z₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[XX,z]" → ((𝔼{}→{1,2}[0, 0, z₂ XX₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, z₁ XX₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[Y,z]" → ((𝔼{}→{1,2}[0, 0, z₂ Y₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
       (𝔼{}→{1,2}[0, 0, z₁ Y₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧) // Expand // Simplify,
  "[a,Z]" → ((𝔼{}→{1,2}[0, 0, Z₂ a₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, Z₁ a₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[a,XX]" → ((𝔼{}→{1,2}[0, 0, XX₂ a₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
     (𝔼{}→{1,2}[0, 0, XX₁ a₂] ~B₁,₂~dm₁,₂→₁)⟦3⟧),
  "[a,Y]" → ((𝔼{}→{1,2}[0, 0, Y₂ a₁] ~B₁,₂~dm₁,₂→₁)⟦3⟧ -
```

```
        (𝔼_{{}→{1,2}}[0, 0, Y_1 a_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[b,Z]" → ((𝔼_{{}→{1,2}}[0, 0, Z_2 b_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Z_1 b_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[b,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 b_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, XX_1 b_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[b,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 b_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Y_1 b_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[A,Z]" → ((𝔼_{{}→{1,2}}[0, 0, Z_2 A_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Z_1 A_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[A,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 A_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, XX_1 A_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[A,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 A_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Y_1 A_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[B,Z]" → ((𝔼_{{}→{1,2}}[0, 0, Z_2 B_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Z_1 B_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[B,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 B_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, XX_1 B_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[B,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 B_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Y_1 B_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[XX,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 XX_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Y_1 XX_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[Z,Y]" → ((𝔼_{{}→{1,2}}[0, 0, Y_2 Z_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, Y_1 Z_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[Z,XX]" → ((𝔼_{{}→{1,2}}[0, 0, XX_2 Z_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, XX_1 Z_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[A,x]" → ((𝔼_{{}→{1,2}}[0, 0, x_2 A_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, x_1 A_2] ~B_{1,2}~dm_{1,2→1})[[3]]),
      "[A,y]" → ((𝔼_{{}→{1,2}}[0, 0, y_2 A_1] ~B_{1,2}~dm_{1,2→1})[[3]] -
        (𝔼_{{}→{1,2}}[0, 0, y_1 A_2] ~B_{1,2}~dm_{1,2→1})[[3]])
    } /. {v__1 → v} // Expand // Factor
{
   "Δ(a)" → ((𝔼_{{1,2}→{}}[0, 0, a_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(x)" → ((𝔼_{{1,2}→{}}[0, 0, x_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(b)" → ((𝔼_{{1,2}→{}}[0, 0, b_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(y)" → ((𝔼_{{1,2}→{}}[0, 0, y_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(z)" → ((𝔼_{{1,2}→{}}[0, 0, z_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(XX)" → ((𝔼_{{1,2}→{}}[0, 0, XX_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(Y)" → ((𝔼_{{1,2}→{}}[0, 0, Y_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(Z)" → ((𝔼_{{1,2}→{}}[0, 0, Z_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(A)" → ((𝔼_{{1,2}→{}}[0, 0, A_1] ~B_1~dΔ_{1→1,2})[[3]]),
   "Δ(B)" → ((𝔼_{{1,2}→{}}[0, 0, B_1] ~B_1~dΔ_{1→1,2})[[3]])} // Simplify
{
    "S(a)" → ((𝔼_{{1}→{1}}[0, 0, a_1] ~B_1~dS_1)[[3]]),
    "S(x)" → ((𝔼_{{1}→{1}}[0, 0, x_1] ~B_1~dS_1)[[3]]),
    "S(b)" → ((𝔼_{{1}→{1}}[0, 0, b_1] ~B_1~dS_1)[[3]]),
    "S(y)" → ((𝔼_{{1}→{1}}[0, 0, y_1] ~B_1~dS_1)[[3]]),
    "S(z)" → ((𝔼_{{1}→{1}}[0, 0, z_1] ~B_1~dS_1)[[3]]),
    "S(XX)" → ((𝔼_{{1}→{1}}[0, 0, XX_1] ~B_1~dS_1)[[3]]),
    "S(Y)" → ((𝔼_{{1}→{1}}[0, 0, Y_1] ~B_1~dS_1)[[3]]),
    "S(Z)" → ((𝔼_{{1}→{1}}[0, 0, Z_1] ~B_1~dS_1)[[3]]),
    "S(A)" → ((𝔼_{{1}→{1}}[0, 0, A_1] ~B_1~dS_1)[[3]]),
    "S(B)" → ((𝔼_{{1}→{1}}[0, 0, B_1] ~B_1~dS_1)[[3]])
```

$$\{[a,y] \to y + O[\epsilon]^2,\ [b,x] \to x + O[\epsilon]^2,\ [b,y] \to -2\,y + O[\epsilon]^2,$$
$$[a,x] \to -2\,x + O[\epsilon]^2,\ [a,z] \to -z + O[\epsilon]^2,\ [b,z] \to -z + O[\epsilon]^2,$$
$$[x,z] \to x\,z\,\hbar\,\epsilon + O[\epsilon]^2,\ [y,z] \to -y\,z\,\hbar\,\epsilon + O[\epsilon]^2,\ [x,y] \to z - x\,y\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[Y,y] \to \frac{-1 + \mathbb{A}^{\hbar}\,\mathbb{B}^{-2\hbar}}{\hbar} + \left(-b\,\mathbb{A}^{\hbar}\,\mathbb{B}^{-2\hbar} + 2\,y\,Y\,\hbar\right)\epsilon + O[\epsilon]^2,\ [Y,x] \to -x\,Y\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[X,y] \to -XX\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ [XX,x] \to \frac{-1 + \mathbb{A}^{-2\hbar}\,\mathbb{B}^{\hbar}}{\hbar} + \left(-a\,\mathbb{A}^{-2\hbar}\,\mathbb{B}^{\hbar} + 2\,x\,XX\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$[Z,z] \to \frac{-1 + \mathbb{A}^{-\hbar}\,\mathbb{B}^{-\hbar}}{\hbar} + \mathbb{A}^{-\hbar}\,\mathbb{B}^{-\hbar}\left(-a - b + 2\,z\,Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$[Z,y] \to -XX + y\,Z\,\hbar\,\epsilon + O[\epsilon]^2,\ [Z,x] \to Y\,\mathbb{A}^{-2\hbar}\,\mathbb{B}^{\hbar} + \mathbb{A}^{-2\hbar}\left(x\,Z\,\mathbb{A}^{2\hbar} - (-1+a)\,Y\,\mathbb{B}^{\hbar}\right)\hbar\,\epsilon + O[\epsilon]^2,$$
$$[XX,z] \to (-2\,y + XX\,z\,\hbar)\,\epsilon + O[\epsilon]^2,\ [Y,z] \to \left(2\,x\,\mathbb{A}^{\hbar}\,\mathbb{B}^{-2\hbar} + Y\,z\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$[a,Z] \to Z + O[\epsilon]^2,\ [a,XX] \to 2\,XX + O[\epsilon]^2,\ [a,Y] \to -Y + O[\epsilon]^2,\ [b,Z] \to Z + O[\epsilon]^2,$$
$$[b,XX] \to -XX + O[\epsilon]^2,\ [b,Y] \to 2\,Y + O[\epsilon]^2,\ [A,Z] \to -Z\,\epsilon + O[\epsilon]^2,$$
$$[A,XX] \to -XX\,\epsilon + O[\epsilon]^2,\ [A,Y] \to O[\epsilon]^2,\ [B,Z] \to -Z\,\epsilon + O[\epsilon]^2,\ [B,XX] \to O[\epsilon]^2,$$
$$[B,Y] \to -Y\,\epsilon + O[\epsilon]^2,\ [XX,Y] \to (2\,Z - XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2,\ [Z,Y] \to Y\,Z\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[Z,XX] \to -XX\,Z\,\hbar\,\epsilon + O[\epsilon]^2,\ [A,x] \to x\,\epsilon + O[\epsilon]^2,\ [A,y] \to O[\epsilon]^2\}$$

$$\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^2,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + O[\epsilon]^2,$$
$$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^2,\ \Delta(y) \to (y_1 + y_2) - \hbar\,b_1\,y_2\,\epsilon + O[\epsilon]^2,$$
$$\Delta(z) \to (z_1 + z_2) + \hbar\,(2\,x_1\,y_2 - (a_1 + b_1)\,z_2)\,\epsilon + O[\epsilon]^2,\ \Delta(XX) \to \left(XX_1 + XX_2\,\mathbb{A}_1^{-2\hbar}\,\mathbb{B}_1^{\hbar}\right) + O[\epsilon]^2,$$
$$\Delta(Y) \to \left(Y_1 + Y_2\,\mathbb{A}_1^{\hbar}\,\mathbb{B}_1^{-2\hbar}\right) + O[\epsilon]^2,\ \Delta(Z) \to \left(Z_1 + Z_2\,\mathbb{A}_1^{-\hbar}\,\mathbb{B}_1^{-\hbar} + \hbar\,XX_2\,Y_1\,\mathbb{A}_1^{-2\hbar}\,\mathbb{B}_1^{\hbar}\right) + O[\epsilon]^2,$$
$$\Delta(A) \to (A_1 + A_2) + O[\epsilon]^2,\ \Delta(B) \to (B_1 + B_2) + O[\epsilon]^2\}$$

$$\{S(a) \to -a + O[\epsilon]^2,\ S(x) \to -x - a\,x\,\hbar\,\epsilon + O[\epsilon]^2,\ S(b) \to -b + O[\epsilon]^2,$$
$$S(y) \to -y - b\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ S(z) \to -z + (2\,x\,y - (-2 + a + b)\,z)\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$S(XX) \to -XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar} - 2\left(XX\,\mathbb{A}^{2\hbar}\,\mathbb{B}^{-\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$S(Y) \to -Y\,\mathbb{A}^{-\hbar}\,\mathbb{B}^{2\hbar} - 2\left(Y\,\mathbb{A}^{-\hbar}\,\mathbb{B}^{2\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$S(Z) \to \mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,(-Z + XX\,Y\,\hbar) + \mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\,(-2\,Z + 3\,XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2,$$
$$S(A) \to -A + O[\epsilon]^2,\ S(B) \to -B + O[\epsilon]^2\}$$

(co)-associativity

$$\mathbf{Timing[HL\ /@\ \{(d\Delta_{1\to1,2}\ //\ d\Delta_{2\to2,3}) \equiv (d\Delta_{1\to1,3}\ //\ d\Delta_{1\to1,2}),\ (dm_{1,2\to1}\ //\ dm_{1,3\to1}) \equiv (dm_{2,3\to2}\ //\ dm_{1,2\text{->}1})\}]}$$

{14.218750, {True, True}}

$$\mathbf{Timing[HL\ /@\ \{(std\Delta_{1\to1,2}\ //\ std\Delta_{2\to2,3}) \equiv (std\Delta_{1\to1,3}\ //\ std\Delta_{1\to1,2}),\ (stdm_{1,2\to1}\ //\ stdm_{1,3\to1}) \equiv (stdm_{2,3\to2}\ //\ stdm_{1,2\text{->}1})\}]}$$

{7.531250, {True, True}}

$$\mathbf{Timing[HL\ /@\ \{(ddm_{1,2\to1} \sim STB_1 \sim ddm_{1,3\to1}) \equiv (ddm_{2,3\to2} \sim STB_2 \sim ddm_{1,2\text{->}1})\}]}$$

{2.125000, {True}}

Δ is an algebra morphism

$$\mathbf{Timing@HL[dm_{1,2\to1} \sim B_1 \sim d\Delta_{1\to1,2} \equiv (d\Delta_{1\to1,3}\ d\Delta_{2\to2,4}) \sim B_{1,2,3,4} \sim (dm_{3,4\to2}\ dm_{1,2\to1})]}$$

{7.296875, True}

S_2 inverts R, but not S_1:

In[65]:=
```
Timing@{HL[R1,2 ~ B2 ~ dS2 ≡ R̄1,2]}
```
Out[65]=
```
{0.796875, {True}}
```

S is convolution inverse of id

```
Timing[HL[# ≡ E{1}→{1}[0, 0, 1]] & /@
  {(dΔ1→1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2→1, (dΔ1→1,2 ~ B2 ~ dS2) // dm1,2→1}]
```
```
{4.703125, {True, True}}
```

S is a (co)-algebra anti-morphism

```
Timing[HL /@ Expand /@
   {dm1,2→1 ~ B1 ~ dS1 ≡ (dS1 dS2) ~ B1,2 ~ dm2,1→1, dS1 ~ B1 ~ dΔ1→1,2 ≡ dΔ1→2,1 ~ B1,2 ~ (dS1 dS2)}]
```
```
{22.718750, {True, True}}
```

Quasi-triangular axiom 1:

```
Timing@HL[R1,2 ~ B1 ~ dΔ1->1,3 ≡ (R1,4 R3,2) ~ B2,4 ~ dm2,4→2]
```
```
{0.375000, True}
```

Quasi-triangular axiom 2:

```
Timing@HL[
  ((dΔ1->1,2 R3,4) ~ B1,2,3,4 ~ (dm1,3→1 dm2,4→2)) ≡ ((dΔ1→2,1 R3,4) ~ B1,2,3,4 ~ (dm3,1→1 dm4,2→2))]
```
```
{2.359375, True}
```

The Drinfel'd element inverse property, $(u_1 \bar{u}_2) \sim B_{1,2} \sim \mathrm{dm}_{1,2\to 1} \equiv \mathbb{E}[0, 0, 1]$:

```
Timing@
 HL[((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1→i) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1→j)) ~ Bi,j ~ dmi,j→i ≡
   E{}→{i}[0, 0, 1]]
```
```
{2.453125, True}
```

The ribbon element v satisfies $v^2 = S(u)\, u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\, u^{-1}$ which is something easy. Taking the square root of z and multiplying it with S(u) yields the ribbon element v.

```
Timing@Block[{$k = 1},
  (((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1→i) ~ Bi ~ dSi) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1→j)) ~
   Bi,j ~ dmi,j→i]
```
$$\{8.062500,\ \mathbb{E}_{\{\}\to\{i\}}[0,\ 0,\ \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} + (2\hbar a_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} + 2\hbar b_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar})\,\epsilon + O[\epsilon]^2]\}$$

Turaev moves are checked here.

T-4:

```
Timing@Block[{$k = 1},
  HL /@ {(C̄1 C̄2 R3,4 C5 C6) ~ B1,3 ~ dm1,3→1 ~ B1,5 ~ dm1,5→1 ~ B2,4 ~ dm2,4→2 ~ B2,6 ~ dm2,6→2 ≡ R1,2}]
```
```
{3.796875, {True}}
```

T-5, T-6

In[63]:=

$$\text{Timing@Block}\Big[\{\$k = 1\},\ \text{HL} /@ \Big\{ \big(C_i\, \overline{C}_j\big) \sim B_{i,j} \sim dm_{i,j\to i} \equiv \mathbb{E}_{\{\}\to\{i\}}[0,0,1],\ \big(C_i\, \overline{C}_j\big) \sim B_{i,j} \sim dm_{j,i\to i} \equiv \mathbb{E}_{\{\}\to\{i\}}[0,0,1],\ \big(\overline{C}_i\, \overline{C}_j\big) \sim B_{i,j} \sim dm_{i,j\to i} \equiv \Big(\big((R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1\to i}) \sim B_i \sim dS_i\big)\ \big(R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1\to j}\big)\Big) \sim B_{i,j} \sim dm_{i,j\to i}\Big\}\Big]$$

Out[63]= {16.296875, {True, True, True}}

Reidemeister 2 or T-3:

$$\text{Timing}\Big[\text{HL}[\# \equiv \mathbb{E}_{\{\}\to\{1,2\}}[0,0,1]]\ \&/@\ \Big\{\big(\overline{R}_{1,2}\, R_{3,4}\big) \sim B_{1,2,3,4} \sim (dm_{1,3\to 1}\, dm_{2,4\to 2}),\ \big(R_{1,2}\, \overline{R}_{3,4}\big) \sim B_{1,2,3,4} \sim (dm_{1,3\to 1}\, dm_{2,4\to 2})\Big\}\Big]$$

{2.500000, {True, True}}

$$\text{Timing}\Big[\text{HL}[\# \equiv \mathbb{E}_{\{\}\to\{1,2\}}[0,0,1]]\ \&/@\ \Big\{\Big(\big(\big(\overline{RR}_{1,2}\, RR_{3,4}\big)\big) \sim B_{1,2,3,4} \sim ((ddm_{1,3\to 1}\, ddm_{2,4\to 2}))\Big),\ \Big(\big(RR_{1,2}\, \overline{RR}_{3,4}\big) \sim STB_{1,2,3,4} \sim (ddm_{1,3\to 1}\, ddm_{2,4\to 2})\Big)\Big\}\Big]$$

{2.718750, {True, True}}

Cyclic Reidemeister 2 or T-2:

$$\text{Timing@HL}\Big[\big(R_{1,4}\, \overline{R}_{5,2}\, \overline{C}_3\big) \sim B_{2,4} \sim dm_{2,4\to 2} \sim B_{1,3} \sim dm_{1,3\to 1} \sim B_{1,5} \sim dm_{1,5\to 1} \equiv \overline{C}_1\, \mathbb{E}_{\{\}\to\{2\}}[0,0,1]\Big]$$

{7.765625, True}

Reidemeister 3 or T-1:

$$\text{Timing@HL}[((R_{1,2}\, R_{4,3}\, R_{5,6}) \sim B_{1,4} \sim dm_{1,4\to 1} \sim B_{2,5} \sim dm_{2,5\to 2} \sim B_{3,6} \sim dm_{3,6\to 3}) \equiv ((R_{1,6}\, R_{2,3}\, R_{4,5}) \sim B_{1,4} \sim dm_{1,4\to 1} \sim B_{2,5} \sim dm_{2,5\to 2} \sim B_{3,6} \sim dm_{3,6\to 3})]$$

{5.343750, True}

$$\text{Timing@HL}[((RR_{1,2}\, RR_{4,3}\, RR_{5,6}) \sim STB_{1,4} \sim ddm_{1,4\to 1} \sim STB_{2,5} \sim ddm_{2,5\to 2} \sim STB_{3,6} \sim ddm_{3,6\to 3}) \equiv ((RR_{1,6}\, RR_{2,3}\, RR_{4,5}) \sim STB_{1,4} \sim ddm_{1,4\to 1} \sim STB_{2,5} \sim ddm_{2,5\to 2} \sim STB_{3,6} \sim ddm_{3,6\to 3})]$$

{1.656250, True}

Relations between the four kinks or T-7

In[64]:=

$$\text{Timing}\Big[\text{HL} /@ \Big\{\text{Kink}_i \equiv (R_{3,1}\, C_2) \sim B_{1,2} \sim dm_{1,2\to 1} \sim B_{1,3} \sim dm_{1,3\to i},\ \overline{\text{Kink}}_j \equiv \big(\overline{R}_{3,1}\, \overline{C}_2\big) \sim B_{1,2} \sim dm_{1,2\to 1} \sim B_{1,3} \sim dm_{1,3\to j},\ \big(\text{Kink}_i\, \overline{\text{Kink}}_j\big) \sim B_{i,j} \sim dm_{i,j\to 1} \equiv \mathbb{E}_{\{\}\to\{1\}}[0,0,1],\ \big(\text{Kink}_i\, \overline{\text{Kink}}_j\big) \sim B_{i,j} \sim dm_{j,i\to 1} \equiv \mathbb{E}_{\{\}\to\{1\}}[0,0,1]\Big\}\Big]$$

Out[64]= {9.187500, {True, True, True, True}}

The Trefoil

$$\text{Timing@Block}\Big[\{\$k = 1\},\ ZZ = RR_{1,5}\, RR_{6,2}\, RR_{3,7}\, \overline{CC}_4\, \overline{KKink}_8\, \overline{KKink}_9\, \overline{KKink}_{10};\ \text{Do}[ZZ = (ZZ /. \{\epsilon \to 0\}) \sim B_{1,r} \sim (ddm_{1,r\to 1} /. \{\epsilon \to 0\}), \{r, 2, 10\}];\ \{\text{Simplify} /@ (ZZ /. \{\epsilon \to 0\})\}\Big]$$

$$\Big\{1.250000,\ \Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{S^2\, T^2}{(1 - S + S^2)\ (1 - T + T^2)\ (1 - S\,T + S^2\, T^2)}\Big]\Big\}\Big\}$$

$$\text{Timing@Block}\Big[\{\$k = 1\},\ ZZ = \big(RR_{1,5}\, RR_{6,2}\, RR_{3,7}\, \overline{CC}_4\, \overline{KKink}_8\, \overline{KKink}_9\, \overline{KKink}_{10}\big);\ \text{Do}[\text{Print}[\text{"Doing "}, r];\ ZZ = (ZZ) \sim B_{1,r} \sim (ddm_{1,r\to 1}), \{r, 2, 10\}];\ \{\text{Simplify@}ZZ\}\Big]$$

$$\{28.890625,$$

$$\{\mathbb{E}_{\{\}\to\{1\}}[0,\ 0,\ \frac{S^2 T^2}{(1-S+S^2)\ (1-T+T^2)\ (1-S T+S^2 T^2)} + (2 S^2 T^2 (S - 2 S^2 + 3 S^3 - 2 S^4 + T -$$

$$2 S T + S^2 T + 2 S^3 T - 5 S^4 T + 6 S^5 T - 2 T^2 + S T^2 - S^3 T^2 - 4 S^4 T^2 + 5 S^5 T^2 - 11 S^6 T^2 +$$

$$3 T^3 + 2 S T^3 - S^2 T^3 + 4 S^3 T^3 + 6 S^4 T^3 - S^5 T^3 + 7 S^6 T^3 + 10 S^7 T^3 - 2 T^4 - 5 S T^4 -$$

$$4 S^2 T^4 + 6 S^3 T^4 - 24 S^4 T^4 + 10 S^5 T^4 - 12 S^6 T^4 - 11 S^7 T^4 - 6 S^8 T^4 + 6 S T^5 +$$

$$5 S^2 T^5 - S^3 T^5 + 10 S^4 T^5 + 12 S^5 T^5 - S^6 T^5 + 10 S^7 T^5 + 13 S^8 T^5 - 11 S^2 T^6 + 7 S^3 T^6 -$$

$$12 S^4 T^6 - S^5 T^6 - 16 S^6 T^6 + 11 S^7 T^6 - 20 S^8 T^6 + 10 S^3 T^7 - 11 S^4 T^7 + 10 S^5 T^7 +$$

$$11 S^6 T^7 - 14 S^7 T^7 + 15 S^8 T^7 - 6 S^4 T^8 + 13 S^5 T^8 - 20 S^6 T^8 + 15 S^7 T^8 - 8 S^8 T^8 +$$

$$(1 - S + S^2)\ (1 - T + T^2)^2\ (-2 + S + 3 S T + 2 S^6 T^4 + S^4 T^2\ (3 + T) - S^5 T^3\ (3 + T) -$$

$$S^2 T\ (1 + 3 T) + S^3 T\ (-1 + T^2))\ a_1 + (1 - S + S^2)^2\ (1 - T + T^2)\ (-2 + T + 3 S T -$$

$$S\ (1 + 3 S)\ T^2 + S\ (-1 + S^2)\ T^3 + S^2\ (3 + S)\ T^4 - S^3\ (3 + S)\ T^5 + 2 S^4 T^6)\ b_1 +$$

$$2 x_1 XX_1 + 2 S^3 x_1 XX_1 - 5 T x_1 XX_1 - 2 S T x_1 XX_1 - 3 S^2 T x_1 XX_1 - 2 S^3 T x_1 XX_1 -$$

$$5 S^4 T x_1 XX_1 + 9 T^2 x_1 XX_1 + 5 S T^2 x_1 XX_1 + 9 S^2 T^2 x_1 XX_1 + 9 S^3 T^2 x_1 XX_1 +$$

$$5 S^4 T^2 x_1 XX_1 + 9 S^5 T^2 x_1 XX_1 - 7 T^3 x_1 XX_1 - 12 S T^3 x_1 XX_1 - 12 S^2 T^3 x_1 XX_1 -$$

$$14 S^3 T^3 x_1 XX_1 - 12 S^4 T^3 x_1 XX_1 - 12 S^5 T^3 x_1 XX_1 - 7 S^6 T^3 x_1 XX_1 + 4 T^4 x_1 XX_1 +$$

$$10 S T^4 x_1 XX_1 + 18 S^2 T^4 x_1 XX_1 + 14 S^3 T^4 x_1 XX_1 + 14 S^4 T^4 x_1 XX_1 + 18 S^5 T^4 x_1 XX_1 +$$

$$10 S^6 T^4 x_1 XX_1 + 4 S^7 T^4 x_1 XX_1 - 7 S T^5 x_1 XX_1 - 12 S^2 T^5 x_1 XX_1 - 12 S^3 T^5 x_1 XX_1 -$$

$$14 S^4 T^5 x_1 XX_1 - 12 S^5 T^5 x_1 XX_1 - 12 S^6 T^5 x_1 XX_1 - 7 S^7 T^5 x_1 XX_1 + 9 S^2 T^6 x_1 XX_1 +$$

$$5 S^3 T^6 x_1 XX_1 + 9 S^4 T^6 x_1 XX_1 + 9 S^5 T^6 x_1 XX_1 + 5 S^6 T^6 x_1 XX_1 + 9 S^7 T^6 x_1 XX_1 -$$

$$5 S^3 T^7 x_1 XX_1 - 2 S^4 T^7 x_1 XX_1 - 3 S^5 T^7 x_1 XX_1 - 2 S^6 T^7 x_1 XX_1 - 5 S^7 T^7 x_1 XX_1 +$$

$$2 S^4 T^8 x_1 XX_1 + 2 S^7 T^8 x_1 XX_1 + 2 y_1 Y_1 - 5 S y_1 Y_1 + 9 S^2 y_1 Y_1 - 7 S^3 y_1 Y_1 +$$

$$4 S^4 y_1 Y_1 - 2 S T y_1 Y_1 + 5 S^2 T y_1 Y_1 - 12 S^3 T y_1 Y_1 + 10 S^4 T y_1 Y_1 - 7 S^5 T y_1 Y_1 -$$

$$3 S T^2 y_1 Y_1 + 9 S^2 T^2 y_1 Y_1 - 12 S^3 T^2 y_1 Y_1 + 18 S^4 T^2 y_1 Y_1 - 12 S^5 T^2 y_1 Y_1 +$$

$$9 S^6 T^2 y_1 Y_1 + 2 T^3 y_1 Y_1 - 2 S T^3 y_1 Y_1 + 9 S^2 T^3 y_1 Y_1 - 14 S^3 T^3 y_1 Y_1 +$$

$$14 S^4 T^3 y_1 Y_1 - 12 S^5 T^3 y_1 Y_1 + 5 S^6 T^3 y_1 Y_1 - 5 S^7 T^3 y_1 Y_1 - 5 S T^4 y_1 Y_1 +$$

$$5 S^2 T^4 y_1 Y_1 - 12 S^3 T^4 y_1 Y_1 + 14 S^4 T^4 y_1 Y_1 - 14 S^5 T^4 y_1 Y_1 + 9 S^6 T^4 y_1 Y_1 -$$

$$2 S^7 T^4 y_1 Y_1 + 2 S^8 T^4 y_1 Y_1 + 9 S^2 T^5 y_1 Y_1 - 12 S^3 T^5 y_1 Y_1 + 18 S^4 T^5 y_1 Y_1 -$$

$$12 S^5 T^5 y_1 Y_1 + 9 S^6 T^5 y_1 Y_1 - 3 S^7 T^5 y_1 Y_1 - 7 S^3 T^6 y_1 Y_1 + 10 S^4 T^6 y_1 Y_1 -$$

$$12 S^5 T^6 y_1 Y_1 + 5 S^6 T^6 y_1 Y_1 - 2 S^7 T^6 y_1 Y_1 + 4 S^4 T^7 y_1 Y_1 - 7 S^5 T^7 y_1 Y_1 +$$

$$9 S^6 T^7 y_1 Y_1 - 5 S^7 T^7 y_1 Y_1 + 2 S^8 T^7 y_1 Y_1 - 2 XX_1 Y_1 z_1 + 7 S XX_1 Y_1 z_1 -$$

$$9 S^2 XX_1 Y_1 z_1 + 7 S^3 XX_1 Y_1 z_1 - 2 S^4 XX_1 Y_1 z_1 - 5 S T XX_1 Y_1 z_1 - 5 S^4 T XX_1 Y_1 z_1 +$$

$$12 S T^2 XX_1 Y_1 z_1 - 9 S^2 T^2 XX_1 Y_1 z_1 + 21 S^3 T^2 XX_1 Y_1 z_1 - 9 S^4 T^2 XX_1 Y_1 z_1 +$$

$$12 S^5 T^2 XX_1 Y_1 z_1 - 2 T^3 XX_1 Y_1 z_1 - 5 S T^3 XX_1 Y_1 z_1 - 9 S^2 T^3 XX_1 Y_1 z_1 -$$

$$7 S^3 T^3 XX_1 Y_1 z_1 - 7 S^4 T^3 XX_1 Y_1 z_1 - 9 S^5 T^3 XX_1 Y_1 z_1 - 5 S^6 T^3 XX_1 Y_1 z_1 -$$

$$2 S^7 T^3 XX_1 Y_1 z_1 + 7 S T^4 XX_1 Y_1 z_1 + 21 S^3 T^4 XX_1 Y_1 z_1 - 7 S^4 T^4 XX_1 Y_1 z_1 +$$

$$21 S^5 T^4 XX_1 Y_1 z_1 + 7 S^7 T^4 XX_1 Y_1 z_1 - 9 S^2 T^5 XX_1 Y_1 z_1 - 9 S^4 T^5 XX_1 Y_1 z_1 -$$

$$9 S^5 T^5 XX_1 Y_1 z_1 - 9 S^7 T^5 XX_1 Y_1 z_1 + 7 S^3 T^6 XX_1 Y_1 z_1 - 5 S^4 T^6 XX_1 Y_1 z_1 +$$

$$12 S^5 T^6 XX_1 Y_1 z_1 - 5 S^6 T^6 XX_1 Y_1 z_1 + 7 S^7 T^6 XX_1 Y_1 z_1 - 2 S^4 T^7 XX_1 Y_1 z_1 -$$

$$2 S^7 T^7 XX_1 Y_1 z_1 + 4 z_1 Z_1 - 7 S z_1 Z_1 + 9 S^2 z_1 Z_1 - 5 S^3 z_1 Z_1 + 2 S^4 z_1 Z_1 - 7 T z_1 Z_1 +$$

$$10 S T z_1 Z_1 - 12 S^2 T z_1 Z_1 + 5 S^3 T z_1 Z_1 - 2 S^4 T z_1 Z_1 + 9 T^2 z_1 Z_1 - 12 S T^2 z_1 Z_1 +$$

$$18 S^2 T^2 z_1 Z_1 - 12 S^3 T^2 z_1 Z_1 + 9 S^4 T^2 z_1 Z_1 - 3 S^5 T^2 z_1 Z_1 - 5 T^3 z_1 Z_1 +$$

$$5 S T^3 z_1 Z_1 - 12 S^2 T^3 z_1 Z_1 + 14 S^3 T^3 z_1 Z_1 - 14 S^4 T^3 z_1 Z_1 + 9 S^5 T^3 z_1 Z_1 -$$

$$2 S^6 T^3 z_1 Z_1 + 2 S^7 T^3 z_1 Z_1 + 2 T^4 z_1 Z_1 - 2 S T^4 z_1 Z_1 + 9 S^2 T^4 z_1 Z_1 - 14 S^3 T^4 z_1 Z_1 +$$

$$14 S^4 T^4 z_1 Z_1 - 12 S^5 T^4 z_1 Z_1 + 5 S^6 T^4 z_1 Z_1 - 5 S^7 T^4 z_1 Z_1 - 3 S^2 T^5 z_1 Z_1 +$$

$$9 S^3 T^5 z_1 Z_1 - 12 S^4 T^5 z_1 Z_1 + 18 S^5 T^5 z_1 Z_1 - 12 S^6 T^5 z_1 Z_1 + 9 S^7 T^5 z_1 Z_1 -$$

$$2 S^3 T^6 z_1 Z_1 + 5 S^4 T^6 z_1 Z_1 - 12 S^5 T^6 z_1 Z_1 + 10 S^6 T^6 z_1 Z_1 - 7 S^7 T^6 z_1 Z_1 +$$

$$2 S^3 T^7 z_1 Z_1 - 5 S^4 T^7 z_1 Z_1 + 9 S^5 T^7 z_1 Z_1 - 7 S^6 T^7 z_1 Z_1 + 4 S^7 T^7 z_1 Z_1)\ \in)\ /$$

$$\left(\left(1-S+S^2\right)^3\left(1-T+T^2\right)^3\left(1-S\,T+S^2\,T^2\right)^3\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

The Figure Eight knot

Timing@Block[{$k = 1},
ZZ = ($\overline{\mathrm{RR}}_{8,1}\,\overline{\mathrm{RR}}_{2,6}\,\mathrm{RR}_{5,9}\,\mathrm{RR}_{10,3}\,\overline{\mathrm{CC}}_{7}\,\mathrm{CC}_{4}$);
Do[Print["Doing ", r];
ZZ = (ZZ /. {$\epsilon \to 0$}) ~ $\mathrm{STB}_{1,r}$ ~ ($\mathrm{ddm}_{1,r\to 1}$ /. {$\epsilon \to 0$}), {r, 2, 10}];
{Simplify@ZZ}]

$$\left\{2.125000,\left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\left(1-3\,T+T^2\right)\left(1-3\,S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

Timing@Block[{$k = 1},
ZZ = ($\overline{\mathrm{RR}}_{8,1}\,\overline{\mathrm{RR}}_{2,6}\,\mathrm{RR}_{5,9}\,\mathrm{RR}_{10,3}\,\overline{\mathrm{CC}}_{7}\,\mathrm{CC}_{4}$);
Do[Print["Doing ", r]; ZZ = (ZZ) ~ $\mathrm{STB}_{1,r}$ ~ ($\mathrm{ddm}_{1,r\to 1}$), {r, 2, 10}];
{Simplify@ZZ}]

$$\Big\{532.593750,$$

$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\,0,\,-\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\left(1-3\,T+T^2\right)\left(1-3\,S\,T+S^2\,T^2\right)}-\Big(2\,\Big(S^2\,T^2\,\Big(4-9\,S+2\,S^2-9\,T+$$
$$12\,S\,T+6\,S^2\,T+2\,T^2+6\,S\,T^2-6\,S^3\,T^2-2\,S^4\,T^2-6\,S^2\,T^3-12\,S^3\,T^3+9\,S^4\,T^3-$$
$$2\,S^2\,T^4+9\,S^3\,T^4-4\,S^4\,T^4+\left(1-3\,T+T^2\right)\left(-2+2\,S^4\,T^2+3\,S\,(1+T)-3\,S^3\,T\,(1+T)\right)$$
$$a_1+\left(1-3\,S+S^2\right)\left(-2+3\,(1+S)\,T-3\,S\,(1+S)\,T^3+2\,S^2\,T^4\right)b_1+$$
$$2\,x_1\,XX_1+2\,S\,x_1\,XX_1-9\,T\,x_1\,XX_1-3\,S\,T\,x_1\,XX_1-9\,S^2\,T\,x_1\,XX_1+4\,T^2\,x_1\,XX_1+$$
$$16\,S\,T^2\,x_1\,XX_1+16\,S^2\,T^2\,x_1\,XX_1+4\,S^3\,T^2\,x_1\,XX_1-9\,S\,T^3\,x_1\,XX_1-3\,S^2\,T^3\,x_1\,XX_1-$$
$$9\,S^3\,T^3\,x_1\,XX_1+2\,S^2\,T^4\,x_1\,XX_1+2\,S^3\,T^4\,x_1\,XX_1+2\,y_1\,Y_1-9\,S\,y_1\,Y_1+4\,S^2\,y_1\,Y_1+$$
$$2\,T\,y_1\,Y_1-3\,S\,T\,y_1\,Y_1+16\,S^2\,T\,y_1\,Y_1-9\,S^3\,T\,y_1\,Y_1-9\,S\,T^2\,y_1\,Y_1+16\,S^2\,T^2\,y_1\,Y_1-$$
$$3\,S^3\,T^2\,y_1\,Y_1+2\,S^4\,T^2\,y_1\,Y_1+4\,S^2\,T^3\,y_1\,Y_1-9\,S^3\,T^3\,y_1\,Y_1+2\,S^4\,T^3\,y_1\,Y_1-$$
$$2\,XX_1\,Y_1\,z_1+11\,S\,XX_1\,Y_1\,z_1-2\,S^2\,XX_1\,Y_1\,z_1-2\,T\,XX_1\,Y_1\,z_1-8\,S\,T\,XX_1\,Y_1\,z_1-$$
$$8\,S^2\,T\,XX_1\,Y_1\,z_1-2\,S^3\,T\,XX_1\,Y_1\,z_1+11\,S\,T^2\,XX_1\,Y_1\,z_1-8\,S^2\,T^2\,XX_1\,Y_1\,z_1+11\,S^3\,T^2$$
$$XX_1\,Y_1\,z_1-2\,S^2\,T^3\,XX_1\,Y_1\,z_1-2\,S^3\,T^3\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-9\,S\,z_1\,Z_1+2\,S^2\,z_1\,Z_1-$$
$$9\,T\,z_1\,Z_1+16\,S\,T\,z_1\,Z_1-3\,S^2\,T\,z_1\,Z_1+2\,S^3\,T\,z_1\,Z_1+2\,T^2\,z_1\,Z_1-3\,S\,T^2\,z_1\,Z_1+$$
$$16\,S^2\,T^2\,z_1\,Z_1-9\,S^3\,T^2\,z_1\,Z_1+2\,S\,T^3\,z_1\,Z_1-9\,S^2\,T^3\,z_1\,Z_1+4\,S^3\,T^3\,z_1\,Z_1\Big)\Big)\,\epsilon\Big)\Big/$$
$$\left(\left(1-3\,S+S^2\right)^2\left(1-3\,T+T^2\right)^2\left(1-3\,S\,T+S^2\,T^2\right)^2\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

The 6-3 knot

Timing@Block[{$k = 1},
ZZ = ($\overline{\mathrm{RR}}_{6,12}\,\overline{\mathrm{RR}}_{10,14}\,\overline{\mathrm{RR}}_{13,7}\,\mathrm{RR}_{1,5}\,\mathrm{RR}_{8,2}\,\mathrm{RR}_{3,9}\,\overline{\mathrm{CC}}_{4}\,\mathrm{CC}_{11}$);
Do[Print["Doing ", r]; ZZ = (ZZ) ~ $\mathrm{STB}_{1,r}$ ~ ($\mathrm{ddm}_{1,r\to 1}$), {r, 2, 14}];
{Simplify@ZZ}]

$$\Big\{337.828125,\,\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\,0,\,\left(S^4\,T^4\right)\Big/\Big(\left(1-3\,S+5\,S^2-3\,S^3+S^4\right)$$
$$\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)\left(1-3\,S\,T+5\,S^2\,T^2-3\,S^3\,T^3+S^4\,T^4\right)\Big)+$$
$$\Big(2\,S^4\,T^4\,\Big(8-21\,S+30\,S^2-15\,S^3+4\,S^4-21\,T+36\,S\,T-30\,S^2\,T-24\,S^3\,T+18\,S^4\,T-$$
$$6\,S^5\,T+30\,T^2-30\,S\,T^2+12\,S^2\,T^2+45\,S^3\,T^2+6\,S^4\,T^2-6\,S^5\,T^2-15\,T^3-24\,S\,T^3+$$
$$45\,S^2\,T^3-48\,S^3\,T^3-24\,S^4\,T^3+6\,S^6\,T^3+6\,S^7\,T^3+4\,T^4+18\,S\,T^4+6\,S^2\,T^4-$$
$$24\,S^3\,T^4+24\,S^5\,T^4-6\,S^6\,T^4-18\,S^7\,T^4-4\,S^8\,T^4-6\,S\,T^5-6\,S^2\,T^5+24\,S^4\,T^5+$$
$$48\,S^5\,T^5-45\,S^6\,T^5+24\,S^7\,T^5+15\,S^8\,T^5+6\,S^3\,T^6-6\,S^4\,T^6-45\,S^5\,T^6-12\,S^6\,T^6+$$

$$30 S^7 T^6 - 30 S^8 T^6 + 6 S^3 T^7 - 18 S^4 T^7 + 24 S^5 T^7 + 30 S^6 T^7 - 36 S^7 T^7 + 21 S^8 T^7 -$$
$$4 S^4 T^8 + 15 S^5 T^8 - 30 S^6 T^8 + 21 S^7 T^8 - 8 S^8 T^8 + \left(1 - 3 T + 5 T^2 - 3 T^3 + T^4\right)$$
$$\left(-4 + 4 S^8 T^4 + 9 S (1 + T) - 9 S^7 T^3 (1 + T) - 2 S^2 \left(5 + 9 T + 5 T^2\right) +\right.$$
$$\left.2 S^6 T^2 \left(5 + 9 T + 5 T^2\right) + 3 S^3 \left(1 + 5 T + 5 T^2 + T^3\right) - 3 S^5 T \left(1 + 5 T + 5 T^2 + T^3\right)\right) a_1 +$$
$$\left(1 - 3 S + 5 S^2 - 3 S^3 + S^4\right) \left(-4 + 9 (1 + S) T - 2 \left(5 + 9 S + 5 S^2\right) T^2 +\right.$$
$$3 \left(1 + 5 S + 5 S^2 + S^3\right) T^3 - 3 S \left(1 + 5 S + 5 S^2 + S^3\right) T^5 + 2 S^2 \left(5 + 9 S + 5 S^2\right) T^6 -$$
$$\left.9 S^3 (1 + S) T^7 + 4 S^4 T^8\right) b_1 + 4 x_1 XX_1 - 2 S x_1 XX_1 - 2 S^2 x_1 XX_1 + 4 S^3 x_1 XX_1 -$$
$$15 T x_1 XX_1 + 3 S T x_1 XX_1 - 3 S^2 T x_1 XX_1 + 3 S^3 T x_1 XX_1 - 15 S^4 T x_1 XX_1 +$$
$$30 T^2 x_1 XX_1 + 6 S T^2 x_1 XX_1 + 12 S^2 T^2 x_1 XX_1 + 12 S^3 T^2 x_1 XX_1 + 6 S^4 T^2 x_1 XX_1 +$$
$$30 S^5 T^2 x_1 XX_1 - 21 T^3 x_1 XX_1 - 51 S T^3 x_1 XX_1 - 6 S^2 T^3 x_1 XX_1 - 30 S^3 T^3 x_1 XX_1 -$$
$$6 S^4 T^3 x_1 XX_1 - 51 S^5 T^3 x_1 XX_1 - 21 S^6 T^3 x_1 XX_1 + 8 T^4 x_1 XX_1 + 44 S T^4 x_1 XX_1 +$$
$$56 S^2 T^4 x_1 XX_1 + 8 S^3 T^4 x_1 XX_1 + 8 S^4 T^4 x_1 XX_1 + 56 S^5 T^4 x_1 XX_1 + 44 S^6 T^4 x_1 XX_1 +$$
$$8 S^7 T^4 x_1 XX_1 - 21 S T^5 x_1 XX_1 - 51 S^2 T^5 x_1 XX_1 - 6 S^3 T^5 x_1 XX_1 - 30 S^4 T^5 x_1 XX_1 -$$
$$6 S^5 T^5 x_1 XX_1 - 51 S^6 T^5 x_1 XX_1 - 21 S^7 T^5 x_1 XX_1 + 30 S^2 T^6 x_1 XX_1 + 6 S^3 T^6 x_1 XX_1 +$$
$$12 S^4 T^6 x_1 XX_1 + 12 S^5 T^6 x_1 XX_1 + 6 S^6 T^6 x_1 XX_1 + 30 S^7 T^6 x_1 XX_1 - 15 S^3 T^7 x_1 XX_1 +$$
$$3 S^4 T^7 x_1 XX_1 - 3 S^5 T^7 x_1 XX_1 + 3 S^6 T^7 x_1 XX_1 - 15 S^7 T^7 x_1 XX_1 + 4 S^4 T^8 x_1 XX_1 -$$
$$2 S^5 T^8 x_1 XX_1 - 2 S^6 T^8 x_1 XX_1 + 4 S^7 T^8 x_1 XX_1 + 4 y_1 Y_1 - 15 S y_1 Y_1 + 30 S^2 y_1 Y_1 -$$
$$21 S^3 y_1 Y_1 + 8 S^4 y_1 Y_1 - 2 T y_1 Y_1 + 3 S T y_1 Y_1 + 6 S^2 T y_1 Y_1 - 51 S^3 T y_1 Y_1 +$$
$$44 S^4 T y_1 Y_1 - 21 S^5 T y_1 Y_1 - 2 T^2 y_1 Y_1 - 3 S T^2 y_1 Y_1 + 12 S^2 T^2 y_1 Y_1 -$$
$$6 S^3 T^2 y_1 Y_1 + 56 S^4 T^2 y_1 Y_1 - 51 S^5 T^2 y_1 Y_1 + 30 S^6 T^2 y_1 Y_1 + 4 T^3 y_1 Y_1 +$$
$$3 S T^3 y_1 Y_1 + 12 S^2 T^3 y_1 Y_1 - 30 S^3 T^3 y_1 Y_1 + 8 S^4 T^3 y_1 Y_1 - 6 S^5 T^3 y_1 Y_1 +$$
$$6 S^6 T^3 y_1 Y_1 - 15 S^7 T^3 y_1 Y_1 - 15 S T^4 y_1 Y_1 + 6 S^2 T^4 y_1 Y_1 - 6 S^3 T^4 y_1 Y_1 +$$
$$8 S^4 T^4 y_1 Y_1 - 30 S^5 T^4 y_1 Y_1 + 12 S^6 T^4 y_1 Y_1 + 3 S^7 T^4 y_1 Y_1 + 4 S^8 T^4 y_1 Y_1 +$$
$$30 S^2 T^5 y_1 Y_1 - 51 S^3 T^5 y_1 Y_1 + 56 S^4 T^5 y_1 Y_1 - 6 S^5 T^5 y_1 Y_1 + 12 S^6 T^5 y_1 Y_1 -$$
$$3 S^7 T^5 y_1 Y_1 - 2 S^8 T^5 y_1 Y_1 - 21 S^3 T^6 y_1 Y_1 + 44 S^4 T^6 y_1 Y_1 - 51 S^5 T^6 y_1 Y_1 +$$
$$6 S^6 T^6 y_1 Y_1 + 3 S^7 T^6 y_1 Y_1 - 2 S^8 T^6 y_1 Y_1 + 8 S^4 T^7 y_1 Y_1 - 21 S^5 T^7 y_1 Y_1 +$$
$$30 S^6 T^7 y_1 Y_1 - 15 S^7 T^7 y_1 Y_1 + 4 S^8 T^7 y_1 Y_1 - 4 XX_1 Y_1 z_1 + 19 S XX_1 Y_1 z_1 -$$
$$32 S^2 XX_1 Y_1 z_1 + 19 S^3 XX_1 Y_1 z_1 - 4 S^4 XX_1 Y_1 z_1 + 2 T XX_1 Y_1 z_1 - 22 S T XX_1 Y_1 z_1 +$$
$$16 S^2 T XX_1 Y_1 z_1 + 16 S^3 T XX_1 Y_1 z_1 - 22 S^4 T XX_1 Y_1 z_1 + 2 S^5 T XX_1 Y_1 z_1 +$$
$$2 T^2 XX_1 Y_1 z_1 + 35 S T^2 XX_1 Y_1 z_1 - 28 S^2 T^2 XX_1 Y_1 z_1 + 34 S^3 T^2 XX_1 Y_1 z_1 -$$
$$28 S^4 T^2 XX_1 Y_1 z_1 + 35 S^5 T^2 XX_1 Y_1 z_1 + 2 S^6 T^2 XX_1 Y_1 z_1 - 4 T^3 XX_1 Y_1 z_1 -$$
$$22 S T^3 XX_1 Y_1 z_1 - 28 S^2 T^3 XX_1 Y_1 z_1 - 4 S^3 T^3 XX_1 Y_1 z_1 - 4 S^4 T^3 XX_1 Y_1 z_1 -$$
$$28 S^5 T^3 XX_1 Y_1 z_1 - 22 S^6 T^3 XX_1 Y_1 z_1 - 4 S^7 T^3 XX_1 Y_1 z_1 + 19 S T^4 XX_1 Y_1 z_1 +$$
$$16 S^2 T^4 XX_1 Y_1 z_1 + 34 S^3 T^4 XX_1 Y_1 z_1 - 4 S^4 T^4 XX_1 Y_1 z_1 + 34 S^5 T^4 XX_1 Y_1 z_1 +$$
$$16 S^6 T^4 XX_1 Y_1 z_1 + 19 S^7 T^4 XX_1 Y_1 z_1 - 32 S^2 T^5 XX_1 Y_1 z_1 + 16 S^3 T^5 XX_1 Y_1 z_1 -$$
$$28 S^4 T^5 XX_1 Y_1 z_1 - 28 S^5 T^5 XX_1 Y_1 z_1 + 16 S^6 T^5 XX_1 Y_1 z_1 - 32 S^7 T^5 XX_1 Y_1 z_1 +$$
$$19 S^3 T^6 XX_1 Y_1 z_1 - 22 S^4 T^6 XX_1 Y_1 z_1 + 35 S^5 T^6 XX_1 Y_1 z_1 - 22 S^6 T^6 XX_1 Y_1 z_1 +$$
$$19 S^7 T^6 XX_1 Y_1 z_1 - 4 S^4 T^7 XX_1 Y_1 z_1 + 2 S^5 T^7 XX_1 Y_1 z_1 + 2 S^6 T^7 XX_1 Y_1 z_1 -$$
$$4 S^7 T^7 XX_1 Y_1 z_1 + 8 z_1 Z_1 - 21 S z_1 Z_1 + 30 S^2 z_1 Z_1 - 15 S^3 z_1 Z_1 + 4 S^4 z_1 Z_1 -$$
$$21 T z_1 Z_1 + 44 S T z_1 Z_1 - 51 S^2 T z_1 Z_1 + 6 S^3 T z_1 Z_1 + 3 S^4 T z_1 Z_1 - 2 S^5 T z_1 Z_1 +$$
$$30 T^2 z_1 Z_1 - 51 S T^2 z_1 Z_1 + 56 S^2 T^2 z_1 Z_1 - 6 S^3 T^2 z_1 Z_1 + 12 S^4 T^2 z_1 Z_1 -$$
$$3 S^5 T^2 z_1 Z_1 - 2 S^6 T^2 z_1 Z_1 - 15 T^3 z_1 Z_1 + 6 S T^3 z_1 Z_1 - 6 S^2 T^3 z_1 Z_1 + 8 S^3 T^3 z_1 Z_1 -$$
$$30 S^4 T^3 z_1 Z_1 + 12 S^5 T^3 z_1 Z_1 + 3 S^6 T^3 z_1 Z_1 + 4 S^7 T^3 z_1 Z_1 + 4 T^4 z_1 Z_1 +$$
$$3 S T^4 z_1 Z_1 + 12 S^2 T^4 z_1 Z_1 - 30 S^3 T^4 z_1 Z_1 + 8 S^4 T^4 z_1 Z_1 - 6 S^5 T^4 z_1 Z_1 +$$
$$6 S^6 T^4 z_1 Z_1 - 15 S^7 T^4 z_1 Z_1 - 2 S T^5 z_1 Z_1 - 3 S^2 T^5 z_1 Z_1 + 12 S^3 T^5 z_1 Z_1 -$$
$$6 S^4 T^5 z_1 Z_1 + 56 S^5 T^5 z_1 Z_1 - 51 S^6 T^5 z_1 Z_1 + 30 S^7 T^5 z_1 Z_1 - 2 S^2 T^6 z_1 Z_1 +$$
$$3 S^3 T^6 z_1 Z_1 + 6 S^4 T^6 z_1 Z_1 - 51 S^5 T^6 z_1 Z_1 + 44 S^6 T^6 z_1 Z_1 - 21 S^7 T^6 z_1 Z_1 +$$
$$4 S^3 T^7 z_1 Z_1 - 15 S^4 T^7 z_1 Z_1 + 30 S^5 T^7 z_1 Z_1 - 21 S^6 T^7 z_1 Z_1 + 8 S^7 T^7 z_1 Z_1\Big) \epsilon\Big) \Big/$$

$$\left(\left(1-3\,S+5\,S^2-3\,S^3+S^4\right)^2\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)^2\right.$$
$$\left.\left(1-3\,S\,T+5\,S^2\,T^2-3\,S^3\,T^3+S^4\,T^4\right)^2\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

Mirror Trefoil

```
Timing@Block[{$k = 1},
  ZZ = (RR̄_{1,5} RR̄_{6,2} RR̄_{3,7} CC_4 KKink_8 KKink_9 KKink_10);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~STB_{1,r}~ (ddm_{1,r→1} /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{3.562500,\left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR̄_{1,5} RR̄_{6,2} RR̄_{3,7} CC_4 KKink_8 KKink_9 KKink_10);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB_{1,r}~ (ddm_{1,r→1}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{49.390625,$$

$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\,0,\,\frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}+\Big(2\,S^2\,T^2\,\Big(8-15\,S+20\,S^2-13\,S^3+$$
$$6\,S^4-15\,T+14\,S\,T-11\,S^2\,T-10\,S^3\,T+11\,S^4\,T-10\,S^5\,T+20\,T^2-11\,S\,T^2+16\,S^2\,T^2+$$
$$S^3\,T^2+12\,S^4\,T^2-7\,S^5\,T^2+11\,S^6\,T^2-13\,T^3-10\,S\,T^3+S^2\,T^3-12\,S^3\,T^3-10\,S^4\,T^3+$$
$$S^5\,T^3-5\,S^6\,T^3-6\,S^7\,T^3+6\,T^4+11\,S\,T^4+12\,S^2\,T^4-10\,S^3\,T^4+24\,S^4\,T^4-6\,S^5\,T^4+$$
$$4\,S^6\,T^4+5\,S^7\,T^4+2\,S^8\,T^4-10\,S\,T^5-7\,S^2\,T^5+S^3\,T^5-6\,S^4\,T^5-4\,S^5\,T^5+S^6\,T^5-$$
$$2\,S^7\,T^5-3\,S^8\,T^5+11\,S^2\,T^6-5\,S^3\,T^6+4\,S^4\,T^6+S^5\,T^6-S^7\,T^6+2\,S^8\,T^6-6\,S^3\,T^7+$$
$$5\,S^4\,T^7-2\,S^5\,T^7-S^6\,T^7+2\,S^7\,T^7-S^8\,T^7+2\,S^4\,T^8-3\,S^5\,T^8+2\,S^6\,T^8-S^7\,T^8+$$
$$\left(1-S+S^2\right)\left(1-T+T^2\right)^2\left(-2+S+3\,S\,T+2\,S^6\,T^4+S^4\,T^2\,(3+T)-S^5\,T^3\,(3+T)-\right.$$
$$\left.S^2\,T\,(1+3\,T)+S^3\,T\left(-1+T^2\right)\right)a_1+\left(1-S+S^2\right)^2\left(1-T+T^2\right)\left(-2+T+3\,S\,T-\right.$$
$$\left.S\,(1+3\,S)\,T^2+S\left(-1+S^2\right)T^3+S^2\,(3+S)\,T^4-S^3\,(3+S)\,T^5+2\,S^4\,T^6\right)b_1+$$
$$2\,x_1\,XX_1+2\,S^3\,x_1\,XX_1-5\,T\,x_1\,XX_1-2\,S\,T\,x_1\,XX_1-3\,S^2\,T\,x_1\,XX_1-2\,S^3\,T\,x_1\,XX_1-$$
$$5\,S^4\,T\,x_1\,XX_1+9\,T^2\,x_1\,XX_1+5\,S\,T^2\,x_1\,XX_1+9\,S^2\,T^2\,x_1\,XX_1+9\,S^3\,T^2\,x_1\,XX_1+$$
$$5\,S^4\,T^2\,x_1\,XX_1+9\,S^5\,T^2\,x_1\,XX_1-7\,T^3\,x_1\,XX_1-12\,S\,T^3\,x_1\,XX_1-12\,S^2\,T^3\,x_1\,XX_1-$$
$$14\,S^3\,T^3\,x_1\,XX_1-12\,S^4\,T^3\,x_1\,XX_1-12\,S^5\,T^3\,x_1\,XX_1-7\,S^6\,T^3\,x_1\,XX_1+4\,T^4\,x_1\,XX_1+$$
$$10\,S\,T^4\,x_1\,XX_1+18\,S^2\,T^4\,x_1\,XX_1+14\,S^3\,T^4\,x_1\,XX_1+14\,S^4\,T^4\,x_1\,XX_1+18\,S^5\,T^4\,x_1\,XX_1+$$
$$10\,S^6\,T^4\,x_1\,XX_1+4\,S^7\,T^4\,x_1\,XX_1-7\,S\,T^5\,x_1\,XX_1-12\,S^2\,T^5\,x_1\,XX_1-12\,S^3\,T^5\,x_1\,XX_1-$$
$$14\,S^4\,T^5\,x_1\,XX_1-12\,S^5\,T^5\,x_1\,XX_1-12\,S^6\,T^5\,x_1\,XX_1-7\,S^7\,T^5\,x_1\,XX_1+9\,S^2\,T^6\,x_1\,XX_1+$$
$$5\,S^3\,T^6\,x_1\,XX_1+9\,S^4\,T^6\,x_1\,XX_1+9\,S^5\,T^6\,x_1\,XX_1+5\,S^6\,T^6\,x_1\,XX_1+9\,S^7\,T^6\,x_1\,XX_1-$$
$$5\,S^3\,T^7\,x_1\,XX_1-2\,S^4\,T^7\,x_1\,XX_1-3\,S^5\,T^7\,x_1\,XX_1-2\,S^6\,T^7\,x_1\,XX_1-5\,S^7\,T^7\,x_1\,XX_1+$$
$$2\,S^4\,T^8\,x_1\,XX_1+2\,S^7\,T^8\,x_1\,XX_1+2\,y_1\,Y_1-5\,S\,y_1\,Y_1+9\,S^2\,y_1\,Y_1-7\,S^3\,y_1\,Y_1+$$
$$4\,S^4\,y_1\,Y_1-2\,S\,T\,y_1\,Y_1+5\,S^2\,T\,y_1\,Y_1-12\,S^3\,T\,y_1\,Y_1+10\,S^4\,T\,y_1\,Y_1-7\,S^5\,T\,y_1\,Y_1-$$
$$3\,S\,T^2\,y_1\,Y_1+9\,S^2\,T^2\,y_1\,Y_1-12\,S^3\,T^2\,y_1\,Y_1+18\,S^4\,T^2\,y_1\,Y_1-12\,S^5\,T^2\,y_1\,Y_1+$$
$$9\,S^6\,T^2\,y_1\,Y_1+2\,T^3\,y_1\,Y_1-2\,S\,T^3\,y_1\,Y_1+9\,S^2\,T^3\,y_1\,Y_1-14\,S^3\,T^3\,y_1\,Y_1+$$
$$14\,S^4\,T^3\,y_1\,Y_1-12\,S^5\,T^3\,y_1\,Y_1+5\,S^6\,T^3\,y_1\,Y_1-5\,S^7\,T^3\,y_1\,Y_1-5\,S\,T^4\,y_1\,Y_1+$$
$$5\,S^2\,T^4\,y_1\,Y_1-12\,S^3\,T^4\,y_1\,Y_1+14\,S^4\,T^4\,y_1\,Y_1-14\,S^5\,T^4\,y_1\,Y_1+9\,S^6\,T^4\,y_1\,Y_1-$$
$$2\,S^7\,T^4\,y_1\,Y_1+2\,S^8\,T^4\,y_1\,Y_1+9\,S^2\,T^5\,y_1\,Y_1-12\,S^3\,T^5\,y_1\,Y_1+18\,S^4\,T^5\,y_1\,Y_1-$$
$$12\,S^5\,T^5\,y_1\,Y_1+9\,S^6\,T^5\,y_1\,Y_1-3\,S^7\,T^5\,y_1\,Y_1-7\,S^3\,T^6\,y_1\,Y_1+10\,S^4\,T^6\,y_1\,Y_1-$$
$$12\,S^5\,T^6\,y_1\,Y_1+5\,S^6\,T^6\,y_1\,Y_1-2\,S^7\,T^6\,y_1\,Y_1+4\,S^4\,T^7\,y_1\,Y_1-7\,S^5\,T^7\,y_1\,Y_1+$$

$$9\,S^6\,T^7\,y_1\,Y_1-5\,S^7\,T^7\,y_1\,Y_1+2\,S^8\,T^7\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+7\,S\,XX_1\,Y_1\,z_1-$$
$$9\,S^2\,XX_1\,Y_1\,z_1+7\,S^3\,XX_1\,Y_1\,z_1-2\,S^4\,XX_1\,Y_1\,z_1-5\,S\,T\,XX_1\,Y_1\,z_1-5\,S^4\,T\,XX_1\,Y_1\,z_1+$$
$$12\,S\,T^2\,XX_1\,Y_1\,z_1-9\,S^2\,T^2\,XX_1\,Y_1\,z_1+21\,S^3\,T^2\,XX_1\,Y_1\,z_1-9\,S^4\,T^2\,XX_1\,Y_1\,z_1+$$
$$12\,S^5\,T^2\,XX_1\,Y_1\,z_1-2\,T^3\,XX_1\,Y_1\,z_1-5\,S\,T^3\,XX_1\,Y_1\,z_1-9\,S^2\,T^3\,XX_1\,Y_1\,z_1-$$
$$7\,S^3\,T^3\,XX_1\,Y_1\,z_1-7\,S^4\,T^3\,XX_1\,Y_1\,z_1-9\,S^5\,T^3\,XX_1\,Y_1\,z_1-5\,S^6\,T^3\,XX_1\,Y_1\,z_1-$$
$$2\,S^7\,T^3\,XX_1\,Y_1\,z_1+7\,S\,T^4\,XX_1\,Y_1\,z_1+21\,S^3\,T^4\,XX_1\,Y_1\,z_1-7\,S^4\,T^4\,XX_1\,Y_1\,z_1+$$
$$21\,S^5\,T^4\,XX_1\,Y_1\,z_1+7\,S^7\,T^4\,XX_1\,Y_1\,z_1-9\,S^2\,T^5\,XX_1\,Y_1\,z_1-9\,S^4\,T^5\,XX_1\,Y_1\,z_1-$$
$$9\,S^5\,T^5\,XX_1\,Y_1\,z_1-9\,S^7\,T^5\,XX_1\,Y_1\,z_1+7\,S^3\,T^6\,XX_1\,Y_1\,z_1-5\,S^4\,T^6\,XX_1\,Y_1\,z_1+$$
$$12\,S^5\,T^6\,XX_1\,Y_1\,z_1-5\,S^6\,T^6\,XX_1\,Y_1\,z_1+7\,S^7\,T^6\,XX_1\,Y_1\,z_1-2\,S^4\,T^7\,XX_1\,Y_1\,z_1-$$
$$2\,S^7\,T^7\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-7\,S\,z_1\,Z_1+9\,S^2\,z_1\,Z_1-5\,S^3\,z_1\,Z_1+2\,S^4\,z_1\,Z_1-7\,T\,z_1\,Z_1+$$
$$10\,S\,T\,z_1\,Z_1-12\,S^2\,T\,z_1\,Z_1+5\,S^3\,T\,z_1\,Z_1-2\,S^4\,T\,z_1\,Z_1+9\,T^2\,z_1\,Z_1-12\,S\,T^2\,z_1\,Z_1+$$
$$18\,S^2\,T^2\,z_1\,Z_1-12\,S^3\,T^2\,z_1\,Z_1+9\,S^4\,T^2\,z_1\,Z_1-3\,S^5\,T^2\,z_1\,Z_1-5\,T^3\,z_1\,Z_1+$$
$$5\,S\,T^3\,z_1\,Z_1-12\,S^2\,T^3\,z_1\,Z_1+14\,S^3\,T^3\,z_1\,Z_1-14\,S^4\,T^3\,z_1\,Z_1+9\,S^5\,T^3\,z_1\,Z_1-$$
$$2\,S^6\,T^3\,z_1\,Z_1+2\,S^7\,T^3\,z_1\,Z_1+2\,T^4\,z_1\,Z_1-2\,S\,T^4\,z_1\,Z_1+9\,S^2\,T^4\,z_1\,Z_1-14\,S^3\,T^4\,z_1\,Z_1+$$
$$14\,S^4\,T^4\,z_1\,Z_1-12\,S^5\,T^4\,z_1\,Z_1+5\,S^6\,T^4\,z_1\,Z_1-5\,S^7\,T^4\,z_1\,Z_1-3\,S^2\,T^5\,z_1\,Z_1+$$
$$9\,S^3\,T^5\,z_1\,Z_1-12\,S^4\,T^5\,z_1\,Z_1+18\,S^5\,T^5\,z_1\,Z_1-12\,S^6\,T^5\,z_1\,Z_1+9\,S^7\,T^5\,z_1\,Z_1-$$
$$2\,S^3\,T^6\,z_1\,Z_1+5\,S^4\,T^6\,z_1\,Z_1-12\,S^5\,T^6\,z_1\,Z_1+10\,S^6\,T^6\,z_1\,Z_1-7\,S^7\,T^6\,z_1\,Z_1+$$
$$2\,S^3\,T^7\,z_1\,Z_1-5\,S^4\,T^7\,z_1\,Z_1+9\,S^5\,T^7\,z_1\,Z_1-7\,S^6\,T^7\,z_1\,Z_1+4\,S^7\,T^7\,z_1\,Z_1\big)\,\epsilon\big)\Big/$$
$$\left(\left(1-S+S^2\right)^3\left(1-T+T^2\right)^3\left(1-S\,T+S^2\,T^2\right)^3\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

```
(*EndProfile[]; *)
```

The double multiplication tensor

For the sake of completeness, we give the explicit formula for ${}^{\mathbf{t}}dm_{ij}^{k}$. We denote by $\mathbf{a} = exp[-a^*]$, and similarly for b. As we did before, $\mathbb{A} = exp[-A]$, and the same convention holds for B.

$$
\begin{aligned}
{}^{\mathbf{t}}dm_{ij}^{k} = \mathbb{E}\,\big[& a_k a^*{}_i + a_k a^*{}_j + A_k A^*{}_i + A_k A^*{}_j + b_k b^*{}_i + b_k b^*{}_j + B_k B^*{}_i + B_k B^*{}_j, \\
& \frac{x_k \mathbf{a}_j^2 x^*{}_i}{\mathbf{b}_j} + x_k x^*{}_j + X_k X^*{}_i + \frac{X_k \mathbf{a}_i^2 X^*{}_j}{\mathbf{b}_i} + \frac{\mathbb{A}_k^{-2\hbar}\left(\mathbb{A}_k^{2\hbar} - \mathbb{B}_k^{\hbar}\right) x^*{}_i X^*{}_j}{\hbar} + \frac{y_k \mathbf{b}_j^2 y^*{}_i}{\mathbf{a}_j} + \\
& y_k y^*{}_j + \frac{z_k \mathbf{a}_j^2 x^*{}_i y^*{}_j}{\mathbf{b}_j} + Y_k Y^*{}_i + \frac{Y_k \mathbf{b}_i^2 Y^*{}_j}{\mathbf{a}_i} + \frac{\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{\hbar} + \mathbb{B}_k^{2\hbar}\right) y^*{}_i Y^*{}_j}{\hbar} + z_k \mathbf{a}_j \mathbf{b}_j z^*{}_i + \\
& z_k z^*{}_j + Z_k Z^*{}_i + Z_k \mathbf{a}_i \mathbf{b}_i Z^*{}_j - \frac{Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i Z^*{}_j}{\mathbf{a}_i} + \frac{X_k \mathbf{a}_i^2 y^*{}_i Z^*{}_j}{\mathbf{b}_i} + \\
& \frac{\mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar}\left(\mathbb{A}_k^{\hbar} - \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Z^*{}_j}{\hbar} + \frac{\mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar}\left(-1 + \mathbb{A}_k^{\hbar} \mathbb{B}_k^{\hbar}\right) z^*{}_i Z^*{}_j}{\hbar}, \\
& 1 + \\
& \Bigg(-\frac{x_k \mathbf{a}_j^2 A^*{}_j x^*{}_i}{\mathbf{b}_j} - \frac{X_k \mathbf{a}_i^2 A^*{}_i X^*{}_j}{\mathbf{b}_i} - \frac{2\hbar x_k X_k \mathbf{a}_i^2 \mathbf{a}_j^2 x^*{}_i X^*{}_j}{\mathbf{b}_i \mathbf{b}_j} + a_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j + \\
& \frac{x_k \mathbb{A}_k^{-2\hbar}\left(-\mathbf{a}_j^2 \mathbb{A}_k^{2\hbar} + 3\mathbf{a}_j^2 \mathbb{B}_k^{\hbar}\right) (x^*)_i^2\, X^*{}_j}{\mathbf{b}_j} + \frac{X_k \mathbb{A}_k^{-2\hbar}\left(-\mathbf{a}_i^2 \mathbb{A}_k^{2\hbar} + 3\mathbf{a}_i^2 \mathbb{B}_k^{\hbar}\right) x^*{}_i\, (X^*)_j^2}{\mathbf{b}_i} + \\
& \frac{\mathbb{A}_k^{-4\hbar}\left(-\mathbb{A}_k^{4\hbar} + 4\mathbb{A}_k^{2\hbar} \mathbb{B}_k^{\hbar} - 3\mathbb{B}_k^{2\hbar}\right) (x^*)_i^2\, (X^*)_j^2}{2\hbar} - \frac{y_k \mathbf{b}_j^2 B^*{}_j y^*{}_i}{\mathbf{a}_j} + \frac{\hbar X_k y_k \mathbf{a}_i^2 \mathbf{b}_j^2 X^*{}_j y^*{}_i}{\mathbf{a}_j \mathbf{b}_i} - \\
& \frac{2 y_k \mathbf{b}_j^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j y^*{}_i}{\mathbf{a}_j} - \frac{\hbar x_k y_k \mathbf{a}_j^2 x^*{}_i y^*{}_j}{\mathbf{b}_j} - \frac{z_k \mathbf{a}_j^2 A^*{}_j x^*{}_i y^*{}_j}{\mathbf{b}_j} - \\
& \frac{2\hbar X_k z_k \mathbf{a}_i^2 \mathbf{a}_j^2 x^*{}_i X^*{}_j y^*{}_j}{\mathbf{b}_i \mathbf{b}_j} + \frac{z_k \mathbb{A}_k^{-2\hbar}\left(-\mathbf{a}_j^2 \mathbb{A}_k^{2\hbar} + 3\mathbf{a}_j^2 \mathbb{B}_k^{\hbar}\right) (x^*)_i^2\, X^*{}_j y^*{}_j}{\mathbf{b}_j} - \\
& \hbar y_k z_k \mathbf{a}_j \mathbf{b}_j x^*{}_i y^*{}_i y^*{}_j - \frac{\hbar y_k z_k \mathbf{a}_j^2 x^*{}_i\, (y^*)_j^2}{\mathbf{b}_j} + \frac{\hbar X_k Y_k \mathbf{a}_i^2 X^*{}_j Y^*{}_i}{\mathbf{b}_i} - \\
& \frac{Y_k \mathbf{b}_i^2 B^*{}_i Y^*{}_j}{\mathbf{a}_i} + \frac{\hbar x_k Y_k \mathbf{a}_j^2 \mathbf{b}_i^2 x^*{}_i Y^*{}_j}{\mathbf{a}_i \mathbf{b}_j} + \frac{2 Z_k \mathbf{b}_i^2 X^*{}_i Y^*{}_j}{\mathbf{a}_i} - \frac{2\hbar y_k Y_k \mathbf{b}_i^2 \mathbf{b}_j^2 y^*{}_i Y^*{}_j}{\mathbf{a}_i \mathbf{a}_j} + \\
& b_k \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} y^*{}_i Y^*{}_j + \frac{x_k \mathbb{B}_k^{-2\hbar}\left(-\mathbf{a}_j^2 \mathbb{A}_k^{\hbar} + \mathbf{a}_j^2 \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Y^*{}_j}{\mathbf{b}_j} + \\
& \frac{X_k \mathbb{B}_k^{-2\hbar}\left(-\mathbf{a}_i^2 \mathbb{A}_k^{\hbar} - \mathbf{a}_i^2 \mathbb{B}_k^{2\hbar}\right) X^*{}_j y^*{}_i Y^*{}_j}{\mathbf{b}_i} + \frac{y_k \mathbb{B}_k^{-2\hbar}\left(3\mathbf{b}_j^2 \mathbb{A}_k^{\hbar} - \mathbf{b}_j^2 \mathbb{B}_k^{2\hbar}\right) (y^*)_i^2\, Y^*{}_j}{\mathbf{a}_j} +
\end{aligned}
$$

$$\frac{\hbar Y_k z_k \mathbf{a}_j^2 \mathbf{b}_i^2 x^*{}_i y^*{}_j Y^*{}_j}{\mathbf{a}_i \mathbf{b}_j} + \frac{z_k \mathbb{B}_k^{-2\hbar} \left(-\mathbf{a}_j^2 \mathbb{A}_k^{\hbar} + \mathbf{a}_j^2 \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i y^*{}_j Y^*{}_j}{\mathbf{b}_j} +$$

$$\frac{Y_k \mathbb{B}_k^{-2\hbar} \left(3\mathbf{b}_i^2 \mathbb{A}_k^{\hbar} - \mathbf{b}_i^2 \mathbb{B}_k^{2\hbar}\right) y^*{}_i \left(Y^*\right)_j^2}{\mathbf{a}_i} + \frac{\mathbb{B}_k^{-4\hbar} \left(-3\mathbb{A}_k^{2\hbar} + 4\mathbb{A}_k^{\hbar} \mathbb{B}_k^{2\hbar} - \mathbb{B}_k^{4\hbar}\right) \left(y^*\right)_i^2 \left(Y^*\right)_j^2}{2\hbar} -$$

$$z_k \mathbf{a}_j \mathbf{b}_j A^*{}_j z^*{}_i - z_k \mathbf{a}_j \mathbf{b}_j B^*{}_j z^*{}_i - \frac{\hbar X_k z_k \mathbf{a}_i^2 \mathbf{a}_j \mathbf{b}_j X^*{}_j z^*{}_i}{\mathbf{b}_i} + \frac{2 y_k \mathbf{b}_j^2 X^*{}_j z^*{}_i}{\mathbf{a}_j} +$$

$$2 z_k \mathbf{a}_j \mathbf{b}_j \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j z^*{}_i - \frac{\hbar Y_k z_k \mathbf{a}_j \mathbf{b}_i^2 \mathbf{b}_j Y^*{}_j z^*{}_i}{\mathbf{a}_i} - \frac{2 x_k \mathbf{a}_j^2 \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} Y^*{}_j z^*{}_i}{\mathbf{b}_j} +$$

$$\frac{\mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-2\hbar} \left(-2\mathbb{A}_k^{2\hbar} + 2\mathbb{B}_k^{\hbar}\right) X^*{}_j Y^*{}_j z^*{}_i}{\hbar} - \frac{2 z_k \mathbf{a}_j^2 \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} y^*{}_j Y^*{}_j z^*{}_i}{\mathbf{b}_j} +$$

$$\frac{\hbar x_k z_k \mathbf{a}_j^2 x^*{}_i z^*{}_j}{\mathbf{b}_j} - \frac{\hbar y_k z_k \mathbf{b}_j^2 y^*{}_i z^*{}_j}{\mathbf{a}_j} + \frac{\hbar z_k^2 \mathbf{a}_j^2 x^*{}_i y^*{}_j z^*{}_j}{\mathbf{b}_j} - \frac{\hbar X_k Z_k \mathbf{a}_i^2 X^*{}_j Z^*{}_i}{\mathbf{b}_i} +$$

$$\frac{\hbar Y_k Z_k \mathbf{b}_i^2 Y^*{}_j Z^*{}_i}{\mathbf{a}_i} - Z_k \mathbf{a}_i \mathbf{b}_i A^*{}_i Z^*{}_j - Z_k \mathbf{a}_i \mathbf{b}_i B^*{}_i Z^*{}_j - \frac{\hbar x_k Z_k \mathbf{a}_i \mathbf{a}_j^2 \mathbf{b}_i x^*{}_i Z^*{}_j}{\mathbf{b}_j} -$$

$$\frac{\hbar Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i Z^*{}_j}{\mathbf{a}_i} + \frac{\hbar a_k Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i Z^*{}_j}{\mathbf{a}_i} + \frac{Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} B^*{}_i x^*{}_i Z^*{}_j}{\mathbf{a}_i} +$$

$$\frac{2\hbar x_k Y_k \mathbf{a}_j^2 \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} \left(x^*\right)_i^2 Z^*{}_j}{\mathbf{a}_i \mathbf{b}_j} - \frac{2 Z_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_i Z^*{}_j}{\mathbf{a}_i} +$$

$$2\hbar X_k Y_k \mathbf{a}_i \mathbf{b}_i \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j Z^*{}_j + \frac{Y_k \mathbb{A}_k^{-4\hbar} \left(\mathbf{b}_i^2 \mathbb{A}_k^{2\hbar} \mathbb{B}_k^{\hbar} - \mathbf{b}_i^2 \mathbb{B}_k^{2\hbar}\right) \left(x^*\right)_i^2 X^*{}_j Z^*{}_j}{\mathbf{a}_i} -$$

$$\frac{\hbar y_k Z_k \mathbf{a}_i \mathbf{b}_i \mathbf{b}_j^2 y^*{}_i Z^*{}_j}{\mathbf{a}_j} - \frac{X_k \mathbf{a}_i^2 A^*{}_i y^*{}_i Z^*{}_j}{\mathbf{b}_i} - \frac{\hbar x_k X_k \mathbf{a}_i^2 \mathbf{a}_j^2 x^*{}_i y^*{}_i Z^*{}_j}{\mathbf{b}_i \mathbf{b}_j} -$$

$$b_k \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} x^*{}_i y^*{}_i Z^*{}_j + \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(\mathbb{A}_k^{\hbar} - \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Z^*{}_j +$$

$$a_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(-\mathbb{A}_k^{\hbar} + \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Z^*{}_j + \frac{x_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(-2\mathbf{a}_j^2 \mathbb{A}_k^{\hbar} + 2\mathbf{a}_j^2 \mathbb{B}_k^{2\hbar}\right) \left(x^*\right)_i^2 y^*{}_i Z^*{}_j}{\mathbf{b}_j} +$$

$$\frac{\hbar X_k^2 \mathbf{a}_i^4 X^*{}_j y^*{}_i Z^*{}_j}{\mathbf{b}_i^2} + \frac{X_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(-\mathbf{a}_i^2 \mathbb{A}_k^{\hbar} + 3\mathbf{a}_i^2 \mathbb{B}_k^{2\hbar}\right) x^*{}_i X^*{}_j y^*{}_i Z^*{}_j}{\mathbf{b}_i} +$$

$$\frac{\mathbb{A}_k^{-4\hbar}\mathbb{B}_k^{-\hbar}\left(-\mathbb{A}_k^{3\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}+\mathbb{A}_k^{2\hbar}\mathbb{B}_k^{2\hbar}-\mathbb{B}_k^{3\hbar}\right)(x^*)_i^2\, X^*{}_j y^*{}_i Z^*{}_j}{\hbar}+$$

$$\frac{y_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-\mathbf{b}_j^2\mathbb{A}_k^{\hbar}-\mathbf{b}_j^2\mathbb{B}_k^{2\hbar}\right)x^*{}_i\,(y^*)_i^2\, Z^*{}_j}{\mathbf{a}_j}-\frac{\hbar z_k Z_k\mathbf{a}_i\mathbf{a}_j^2\mathbf{b}_i x^*{}_i y^*{}_j Z^*{}_j}{\mathbf{b}_j}+$$

$$\frac{2\hbar Y_k z_k\mathbf{a}_j^2\mathbf{b}_i^2\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}\,(x^*)_i^2\, y^*{}_j Z^*{}_j}{\mathbf{a}_i\mathbf{b}_j}-\frac{\hbar X_k z_k\mathbf{a}_i^2\mathbf{a}_j^2 x^*{}_i y^*{}_i y^*{}_j Z^*{}_j}{\mathbf{b}_i\mathbf{b}_j}+$$

$$\frac{z_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-2\mathbf{a}_j^2\mathbb{A}_k^{\hbar}+2\mathbf{a}_j^2\mathbb{B}_k^{2\hbar}\right)(x^*)_i^2\, y^*{}_i y^*{}_j Z^*{}_j}{\mathbf{b}_j}+\frac{\hbar X_k Y_k\mathbf{a}_i^2 y^*{}_i Y^*{}_i Z^*{}_j}{\mathbf{b}_i}-$$

$$\frac{\hbar Y_k^2\mathbf{b}_i^4\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i Y^*{}_j Z^*{}_j}{\mathbf{a}_i^2}-\hbar X_k Y_k\mathbf{a}_i\mathbf{b}_i y^*{}_i Y^*{}_j Z^*{}_j+2Z_k\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{\hbar}\mathbb{B}_k^{-2\hbar}y^*{}_i Y^*{}_j Z^*{}_j-$$

$$\frac{4Y_k\mathbf{b}_i^2\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}x^*{}_i y^*{}_i Y^*{}_j Z^*{}_j}{\mathbf{a}_i}+\frac{X_k\mathbb{B}_k^{-2\hbar}\left(\mathbf{a}_i^2\mathbb{A}_k^{\hbar}-\mathbf{a}_i^2\mathbb{B}_k^{2\hbar}\right)(y^*)_i^2\, Y^*{}_j Z^*{}_j}{\mathbf{b}_i}+$$

$$\frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-3\hbar}\left(2\mathbb{A}_k^{\hbar}-2\mathbb{B}_k^{2\hbar}\right)x^*{}_i\,(y^*)_i^2\, Y^*{}_j Z^*{}_j}{\hbar}-2\hbar z_k Z_k\mathbf{a}_i\mathbf{a}_j\mathbf{b}_i\mathbf{b}_j z^*{}_i Z^*{}_j+$$

$$a_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}z^*{}_i Z^*{}_j+b_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}z^*{}_i Z^*{}_j+\frac{3\hbar Y_k z_k\mathbf{a}_j\mathbf{b}_i^2\mathbf{b}_j\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i z^*{}_i Z^*{}_j}{\mathbf{a}_i}+$$

$$\frac{x_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_j^2-\mathbf{a}_j^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i z^*{}_i Z^*{}_j}{\mathbf{b}_j}+\frac{X_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbf{a}_i^2+\mathbf{a}_i^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)X^*{}_j z^*{}_i Z^*{}_j}{\mathbf{b}_i}+$$

$$\frac{\mathbb{A}_k^{-3\hbar}\mathbb{B}_k^{-\hbar}\left(2\mathbb{A}_k^{2\hbar}-2\mathbb{B}_k^{\hbar}\right)x^*{}_i X^*{}_j z^*{}_i Z^*{}_j}{\hbar}-\frac{\hbar X_k z_k\mathbf{a}_i^2\mathbf{a}_j\mathbf{b}_j y^*{}_i z^*{}_i Z^*{}_j}{\mathbf{b}_i}+$$

$$\frac{y_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbf{b}_j^2+\mathbf{b}_j^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)y^*{}_i z^*{}_i Z^*{}_j}{\mathbf{a}_j}+$$

$$z_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-2\mathbf{a}_j\mathbf{b}_j\mathbb{A}_k^{\hbar}+2\mathbf{a}_j\mathbf{b}_j\mathbb{B}_k^{2\hbar}\right)x^*{}_i y^*{}_i z^*{}_i Z^*{}_j+$$

$$\frac{z_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_j^2-\mathbf{a}_j^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i y^*{}_j z^*{}_i Z^*{}_j}{\mathbf{b}_j}+\frac{Y_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{b}_i^2-\mathbf{b}_i^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)Y^*{}_j z^*{}_i Z^*{}_j}{\mathbf{a}_i}+$$

$$\frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-3\hbar}\left(-2\mathbb{A}_k^{\hbar}+2\mathbb{B}_k^{2\hbar}\right)y^*{}_i Y^*{}_j z^*{}_i Z^*{}_j}{\hbar}+z_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_j\mathbf{b}_j-\mathbf{a}_j\mathbf{b}_j\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)(z^*)_i^2\, Z^*{}_j-$$

$$\frac{\hbar Y_k Z_k\mathbf{b}_i^2\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i Z^*{}_i Z^*{}_j}{\mathbf{a}_i}-\frac{\hbar X_k Z_k\mathbf{a}_i^2 y^*{}_i Z^*{}_i Z^*{}_j}{\mathbf{b}_i}+\hbar Y_k Z_k\mathbf{b}_i^3\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i\,(Z^*)_j^2-$$

$$\hbar X_k Z_k\mathbf{a}_i^3 y^*{}_i\,(Z^*)_j^2+2\hbar X_k Y_k\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i y^*{}_i\,(Z^*)_j^2+$$

$$Z_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-3\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{\hbar}+\mathbf{a}_i\mathbf{b}_i\mathbb{B}_k^{2\hbar}\right)x^*{}_i y^*{}_i\,(Z^*)_j^2+$$

$$\frac{Y_k \mathbb{A}_k^{-4\hbar}\left(3\mathbf{b}_i^2\mathbb{A}_k^{\hbar} - \mathbf{b}_i^2\mathbb{B}_k^{2\hbar}\right)(x^*)_i^2\, y^*{}_i\,(Z^*)_j^2}{\mathbf{a}_i} +$$
$$\frac{X_k \mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-2\mathbf{a}_i^2\mathbb{A}_k^{\hbar} + 2\mathbf{a}_i^2\mathbb{B}_k^{2\hbar}\right) x^*{}_i\,(y^*)_i^2\,(Z^*)_j^2}{\mathbf{b}_i} +$$
$$\frac{\mathbb{A}_k^{-4\hbar}\mathbb{B}_k^{-2\hbar}\left(-3\mathbb{A}_k^{2\hbar} + 4\mathbb{A}_k^{\hbar}\mathbb{B}_k^{2\hbar} - \mathbb{B}_k^{4\hbar}\right)(x^*)_i^2\,(y^*)_i^2\,(Z^*)_j^2}{2\hbar} +$$
$$Z_k \mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_i\mathbf{b}_i - \mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right) z^*{}_i\,(Z^*)_j^2 +$$
$$\frac{Y_k \mathbb{A}_k^{-3\hbar}\left(-4\mathbf{b}_i^2 + 2\mathbf{b}_i^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right) x^*{}_i z^*{}_i\,(Z^*)_j^2}{\mathbf{a}_i} + \frac{2X_k\mathbf{a}_i^2\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar} y^*{}_i z^*{}_i\,(Z^*)_j^2}{\mathbf{b}_i} +$$
$$\frac{1}{\hbar}\mathbb{A}_k^{-3\hbar}\mathbb{B}_k^{-2\hbar}\left(3\mathbb{A}_k^{\hbar} - \mathbb{A}_k^{2\hbar}\mathbb{B}_k^{\hbar} - 3\mathbb{B}_k^{2\hbar} + \mathbb{A}_k^{\hbar}\mathbb{B}_k^{3\hbar}\right) x^*{}_i y^*{}_i z^*{}_i\,(Z^*)_j^2 +$$
$$\left.\left.\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-3 + 4\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar} - \mathbb{A}_k^{2\hbar}\mathbb{B}_k^{2\hbar}\right)(z^*)_i^2\,(Z^*)_j^2}{2\hbar}\right)\epsilon + O[\epsilon]^2\right]$$

A.2. Poisson-Lie groups

In this section we will describe the connection between Poisson-Lie groups and Lie-bi algebras. A large part of this appendix is taken from the masterthesis “The two dimensional Ising Model” by the author. In this appendix we will introduce the notion of a Lie group, followed by the definition of a Poisson Lie group. We follow the construction of Lee [22] and [6]. A general knowledge about smooth manifolds is required.

Definition A.2.1. *A Lie group is a smooth manifold G without boundary that is a group with a smooth multiplication map $m : G \times G \to G$ and a smooth inversion map $i : G \to G$. Let $g, h \in G$, then $i(g) = g^{-1}$ is called the inverse of g and m(g,h)=gh. Denote with $L_g(h) = gh$ left translation and with $R_g(h) = hg$ right translation.*

Definition A.2.2. *Let G and H be Lie groups, then a Lie group homomorphism F from G to H is a map $F : G \to H$ that is a group homomorphism. It is called a Lie group isomorphism if it is a diffeomorphism.*

Definition A.2.3. *Let M be a smooth manifold, and let TM be the tangent bundle of M. A vectorfield X on M is a section of the map $\pi : TM \to M$. THat is, X is a map $X : M \to TM$, such that $\pi \circ X = Id_M$.*

One can add vector fields pointwise. If (U, x^i) is a chart of M, and $p \in M$, then $p \to \frac{\partial}{\partial x^i}|_p$ is a vector field on U, which we will call the i-th coördinate vector field, and it will be denoted by $\partial/\partial x^i$. A vector field X can be written out on chart as a linear combination of coördinate vector fields, and this will be denoted with $X = X^i \frac{\partial}{\partial x^i}$, where the summation symbol over i is omitted.

Definition A.2.4. *Let X and Y be smooth vector fields on a smooth manifold M. Let* $f : M \to \mathbb{R}$ *be a smooth function. Then the Lie bracket of X and Y is given by* $[Y, X]f = XYf - YXf$.

Given a smooth function $f : M \to \mathbb{R}$, it is possible to apply X and Y to f to obtain new smooth vector fields fX and fY respectively. On the other hand, by differentiation, a vector field can act on a function. To show that the Lie bracket is well defined, one has to show that $[X, Y]$ is again a vectorfield. This is equivalent to showing that it obeys the product rule, which will be omitted here.
From now on we will mean with M a smooth manifold with Lie bracket $[\cdot, \cdot]$, and with X, Y, Z smooth vectorfields on M. The space of smooth vector fields on M is denoted by $\mathcal{X}(M)$ and the space of smooth functions on M is denoted by $C^\infty(M)$.

Proposition A.2.1. *The Lie bracket satisfies the following identities:*

(a) (linearity) Let $a, b \in \mathbb{R}$. *Then*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]. \tag{A.1}$$

(b) (anti-symmetry)

$$[X, Y] = -[Y, X] \tag{A.2}$$

(c) (Jacobi identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{A.3}$$

(d) Let $f, g \in C^\infty(M)$, *then*

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X. \tag{A.4}$$

Definition A.2.5. *Let V be a finite dimensional vector space, and denote with GL(V) the group of invertible linear transformations on V, which is isomorphic to a Lie Group* GL_n *for some n. If G is a Lie group, then a finite dimensional representation of G is a Lie group homomorphism from G to GL(V) seen as Lie group for some V. if a representation* $\rho : G \to GL(V)$ *is injective, then the representation is said to be faithful.*

Definition A.2.6. *Let G be a Lie group. The Lie algebra of G is the set of all smooth left-invariant vector fields, and it is denoted by Lie(G).*

The Lie algebra of G is well defined because the Lie bracket of two left invariant vector fields (invariant under L_g for all g) is again left invariant. It turns out that $Lie(G)$ is finite dimensional and that the dimension of $Lie(G)$ is equal to $dim(G)$. [22] The representation of a Lie group yields a representation of the corresponding Lie algebra by taking the tangent map. We proceed with the definition of a Poisson manifold.

Definition A.2.7. *(Poisson Structure) Let M be a smooth manifold of finite dimension m, and denote with C(M) the algebra of smooth real valued functions on M. A Poisson structure on M is an* $\mathbb{R}$ *bilinear map* $\{,\} : C(M) \times C(M) \to C(M)$ *(the Poisson bracket) satisfying for all* $f_1, f_2, f_3 \in C(M)$*:*

1. $\{f_1, f_2\} = -\{f_2, f_1\}$
2. $\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$
3. $\{f_1 f_2, f_3\} = \{f_1, f_3\} f_2 + f_1 \{f_2, f_3\}$

One needs to consider maps between Poisson structures as well.

Definition A.2.8. *(Poisson Maps) A smooth map* $F : M \to N$ *between Poisson manifolds is a Poisson map if it preserves the Poisson brackets of M and N:* $\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N$.
(Product Poisson structure) The Product Poisson structure is given by

$$\{f_1(x,y), f_2\}_{M\times N}(x,y) = \{f_1(.,y), f_2(.,y)\}_M(x) + \{f_1(x,.), f_2(x,.)\}_N(y),$$

where $f_1, f_2 \in C(M \times N)$.

Finally we are able to define Poisson-Lie groups.

Definition A.2.9. *A Poisson-Lie group G is a Lie group which also has a Poisson structure that is compatible with the Lie structure, i.e. the multiplication map* $\mu : G \times G \to G$ *is a Poisson map. A homomorphism of Poisson Lie groups is a homomorphism of Lie groups that is also a Poisson map.*

Now let us go into the relation between Poisson-Lie groups and Lie bialgebras.

Theorem A.2.1. *Define on a Poisson-Lie group G* $Ad(x)(y) = xyx^{-1}$ *for all* $x, y \in G$. *Then the tangent space at the unit element e of G is a Liealgebra* $\mathfrak{g}$ *with Lie bracket* $[X,Y] = T_e Ad(X)(Y)$. *Define the cobracket* δ *by the relation*

$$\langle X, d\{f_1, f_2\}_e \rangle = \langle \delta(X), (df_1)_1 \otimes (df_2)_e \rangle.$$

Then $(T_e G, [,], \delta)$ *is a Lie bialgebra.*

The proof consists of checking the definitions. (See [6], page 25.) Note that if a Lie algebra corresponding to a Lie group G (not necessarily a Poisson-Lie group) is quasitriangular, i.e. if δ is a coboundary, then one can use the classical r-matrix to define the Poisson bracket on G. See proposition 2.2.2 on page 61 of [6]. On the other hand one can define from a classical r-matrix $r \in \mathfrak{g} \times \mathfrak{g}$ a corresponding R-matrix $\mathcal{R} : G \times G \to G \times G$ which is a solution of the quantum Yang Baxter equation: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$. See page 67 of [6] for more details. Confusingly, $\mathcal{R}$ is called a classical R-matrix in [6].
The dual of the universal enveloping algebra of a semisimple Lie algebra corresponds to the function algebra on its corresponding Poisson-Lie group. See

chapter 7 of [6]. This is not the case for $U_q(sl_3^\epsilon)$, since this algebra is not semisimple. Suppose this were the case, then the space of functions on the quantum group $U_q(sl_3^\epsilon)$ would be spanned by the representation-matrices of finite dimensional representations, and each function would be fully determined by its action on finite dimensional representations. We know that this is not the case by looking at central elements in $U_q(sl_3^\epsilon)$, so the dual of $U_q(sl_3^\epsilon)$ cannot correspond to the function algebra of a Poisson-Lie group.
It would be interesting to consider the corresponding construction of $\mathcal{F}(G)$ with a non-invertible term epsilon, and quantize it. This might give insight in $U_q(sl_3^\epsilon)$. When we consider ϵ in the ring $\mathbb{R}[[\epsilon]]$ it turns out to be equivalent to the quantization of a quotient of an affine Lie algebra where the central extension is quotiented out, see [37] and [5]. This suggests that a geometric interpretation of the dual of $U_q(sl_3^\epsilon)$ over the ring $\mathbb{R}[[\epsilon]]$ is possible.

A.3. Lie algebras and root systems

In this section we will give the definitions of a root system corresponding to a Lie algebra. This appendix is taken from the master thesis "The two dimensional Ising Model" by the author. It is not our aim to introduce the reader to Lie theory, so we will only state a few definitions and results. For a good introduction in Lie algebras and finite dimensional representation of Lie algebras, see for example [14].

Definition A.3.1. *(Lie algebra) Let L be a vector space over a commutative ring R, with a bracket operation $[\cdot,\cdot] : L \times L \to L$ with the following properties:*

(L1) The bracket operation is bilinear.

(L2) [xx]=0 for all $x \in L$.

(L3) The Jacobi identity is satisfied: [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.

Then L is called a Lie algebra.

From now on, when we write L, we will always mean a Lie algebra L.

Definition A.3.2. *A derivation of L is a linear map $\delta : L \to L$ satisfying the product rule: $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in L$. The collection of all derivation on L is denoted by Der(L).*

Since $Der(L) \subset End(L)$, we can define a representation on L by sending an element $x \in L$ to its derivation $ad(x) = [x, \cdot]$. This representation (a representation of a Lie algebra L is a linear map tp $\mathfrak{gl}(L)$ respecting the bracket operation) is called the adjoint representation, and plays an important role. Using this representation, we can define a symmetric, bilinear form on L.

Definition A.3.3. *(Killing Form) For $x, y \in L$, define the Killing form $\kappa(x,y) = Tr(ad(x)ad(y))$, where Tr denotes the trace.*

A special class of Lie algebras are the so called semisimple Lie algebras. This class has certain nice properties, which we will need.

Definition A.3.4. *Let $L^{(i)}$ be the sequence obtained by $L^{(0)} = L$ and $L^{(i+1)} = [L^{(i)}, L^{(i)}]$. We call L sovable if $L^{(n)} = 0$ for some n.*

The unique maximal solvable ideal of L is called the radical of L and is denoted by $Rad(L)$. Its existence follows from the property that if I and J are solvable ideals, then so is $I + J$.

Definition A.3.5. *(semisimple Lie algebra) Let L be a Lie algebra such that rad(L)=0. Then L is called semisimple.*

For semisimple Lie algebras, the Killing form is nondegenerate (i.e. the adjoint representation is faithful, i.e. 1 to 1). This is also true for a general faithful representation ϕ of L. Define a symmetric, bilinear form $\beta(x, y) = Tr(\phi(x)\phi(y))$. If ϕ is faithful and L is semisimple, then β is nondegenerate and associative. For a proof of this, see [14].
It can be checked, by using the Jacobi identity, that the Killing form is invariant under the adjoint action of L on itself, defined by $ad : L \times L \to L : (x, y) \mapsto [x, y]$. So the Killing form satisfies: $\kappa(ad_x(y), ad_x(z)) = \kappa(y, z)$, for all x, y, z in L. It is interesting to look at a general adjoint action invariant, bilinear form β. One can define the Casimir element associated to this form the following way.

Definition A.3.6. *(Casimir element) Let L be semisimple, with basis $(x_1, x_2, \cdots, x_n)$. Let β be an adjoint invariant bilinear form on L, and let $(y_1, \cdots, y_n)$ be the dual basis with respect to this two form: $\delta_{ij} = \beta(x_i, y_j)$. Then define the Casimir element associated with β as follows:*

$$c_\beta = \sum_{i=1}^{n} y_i \otimes x_i \in \mathfrak{U}(L), \tag{A.5}$$

where $\mathfrak{U}(L)$ is the universal enveloping algebra of L.

The construction of the Casimir element can be generalized, at least in theory, for any semisimple Lie algebra to higher degree Casimir elements. This might be trivial in some cases, whereas in other cases it might not be.

Definition A.3.7. *(generalized Casimir element) Let L be semisimple, and let $(x_{\alpha_1}), \cdots, (x_{\alpha_n})$ be bases of L. Define the multilinear form $\beta(x_1, \cdots, x_n) = Tr(ad(x_1) \cdots ad(x_n))$. Then define the generalized casimir element c_β by*

$$c_\beta = \sum_{\alpha_1, \cdots \alpha_n} \frac{x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}}{\beta(x_{\alpha_1}, \cdots, x_{\alpha_n})}. \tag{A.6}$$

The degrees for which these generalized Casimir elements exist minus one are called the exponents of the Lie algebra. The next concept we want to define is the Coxeter number. In order to define this concept, we need to introduce roots and the Weyl group.

Definition A.3.8. *Let L be semisimple, and let κ be the killing form on L. Let H be the maximal subalgebra of L consisting of elements x for which ad(x) is diagonalizable (such an element x is called semisimple, and and algebra consisting of such elements is called Toral). Let $\alpha, \beta \in H^*$, such that $L_\alpha = \{x \in L | [hx] = \alpha(h)x \text{ for all } h \in H\} \neq 0$ (such α are called roots, the set of roots is denoted by Φ). Denote by $P_\alpha = \{\beta \in H^* | (\beta, \alpha) = 0\}$ the reflecting hyperplane of α (here $(\cdot, \cdot)$ denotes the Killing form transferred from H to H^*, which we may do since the killing form is nondegenerate on H, see [14]), and define $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.*

As it turns out, the set Φ of all roots of L obeys the axioms of a root system.

Definition A.3.9. *(Root system) A subset Φ of an euclidean space E is called a root system in E if the following axioms are satisfied:*

R1 *Φ is finite, spans E and does not contain 0.*

R2 *If $\alpha \in \Phi$, then the only multiples of α contained in Φ are $\pm\alpha$.*

R3 *If $\alpha \in \Phi$, then σ_α leaves Φ invariant.*

R4 *If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \langle \alpha, \beta \rangle \in \mathbb{Z}$.*

Here, σ_α is defined similarly as the case in which $E = H^*$, since any Euclidian space is equipped with a nondegenerate, positive definite symmetric, bilinear form. Let us now define the notion of a coroot $\alpha^\vee$ for a root α as follows

$$\alpha^\vee = \frac{2\alpha}{|\alpha|^2}. \tag{A.7}$$

We need the definition of simple roots.

Definition A.3.10. *Let Δ be a subset of a root system Φ of a Euclidian space E such that*

B1 *Δ is a basis of E,*

B2 *Each root can be written as a linear combination of elements of Δ, such that the coefficients are all nonnegative or all nonpositive.*

Then Δ is called a base, and its elements are called simple roots.

Fix a base $\{\alpha_1, \cdots, \alpha_r\}$ for the roots of L, and let θ be the highest root of L, in the sense that the sum of the coefficients a_i, when θ is written out as a linear combination of simple roots is maximized. The coeffients a_i are called marks. The coefficients $a_i^\vee$, when θ is decomposed in terms of $\alpha_i^\vee$ are called comarks.
With a base fixed for L, we can define the Cartan matrix as $A_i j = \kappa(\alpha_i, \alpha_j^\vee)$, where i and j run between 1 and r. Now let us define the Weyl group.

Definition A.3.11. *(Weyl group) Let Φ be a root system, and let $\mathcal{W}$ be the group generated the reflections σ_α, for $\alpha \in \Phi$. We call $\mathcal{W}$ the Weyl group of Φ.*

From the definition of a root system, it is clear that $\mathcal{W}$ permutes the roots, and hence can be seen as a subgroup of the symmetric group on Φ. To define the Coxeter element and the Coxeter number, we need a few more definitions.

Definition A.3.12. *(Base) A subset $\Delta \subset \Phi$ is called a base if Δ is a basis of Φ and if each root β can be written as $\beta = \sum k_\alpha \alpha$ with the integral coefficients k_α all nonnegative or nonpositive. The roots in Δ are called simple roots. The reflections corresponding to these roots are called simple reflections.*

Now we can define the Coxeter element.

Definition A.3.13. *(Coxeter element) Let Φ be a root system of a semisimple Lie algebra L with a fixed base $\Delta = (\alpha_1, \cdots, \alpha_n)$. Then $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ is called a Coxeter element of L. The order of w is called the Coxeter number.*

Note that one can define several Coxeter elements in given group, so it is important to prove that these elements have the same order. This will not be done here, but the proof that all Coxeter elements are conjugate to each other can for example be found in for example [14].

A.4. Wigner group contraction

In this appendix we describe the process of Wigner group contraction. In 1953 Wigner et al. came up with this method to transform Lie groups and their corresponding Lie algebras into different Lie groups. This is accomplished by a continuous transformation with a function $t(\epsilon)$ on the generators of which the limit $\epsilon \to 0$ is taken. Wigner proved that this limit exists under certain conditions. We follow [12]. We will use Wigner group contraction for the construction of the Lie algebra $sl_2^{\epsilon=0}$. This gives some inspiration for the origin of the parameter ϵ.
Let $\epsilon \in [0,1]$, and let $\mathfrak{g}, \mathfrak{f}$ be Lie algebras. Let $t_\epsilon : \mathfrak{g} \to \mathfrak{f}$ be a one to one Lie algebra map for all $\epsilon \neq 0$ such that $t_1 = id$ and $\det(t_0) = 0$. Let $a, b, c \in \mathfrak{g}$. Then we have

$$t_\epsilon^{-1}[t_\epsilon(a), t_\epsilon(b)] = c.$$

We may now take the limit $\epsilon \to 0$. If this limit exists, this results in a Lie algebra $\mathfrak{g}'$ for any $\epsilon \in [0,1]$. For $\epsilon = 0$ the result is nonisomorphic to $\mathfrak{g}$, with bracket $[a,b] = \lim_{\epsilon \to 0} et_\epsilon^{-1}[t_\epsilon(a), t_\epsilon(b)]$. In this case we call $\mathfrak{g}'$ the contraction of $\mathfrak{g}$, and we say that $\mathfrak{g}$ is contracted with respect to t_ϵ. Suppose we have a basis a_i of $\mathfrak{g}$. When the contraction of $\mathfrak{g}$ exists, define the basis a_i' of $\mathfrak{g}'$ as $a_i' = t_\epsilon(a_i)$.
The following theorem is taken from [12], we will not prove it here.

Theorem A.4.1. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ be a Lie algebra and t_ϵ a transformation as specified above such that*

$$t_0(\mathfrak{h}) = \mathfrak{h},$$
$$t_0(\mathfrak{h}') = 0.$$

Then $\mathfrak{g}$ can be contracted with respect to $\mathfrak{h}$ if and only if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, in this case $\mathfrak{h}$ is a subalgebra of the contraction $\mathfrak{g}'$ of $\mathfrak{g}$, and $\mathfrak{h}'$ is a commutative subalgebra of $\mathfrak{g}'$. In particular $\mathfrak{g}'$ is not semisimple.

We will now treat the example relevant for us, the case where $\mathfrak{g} = \mathfrak{gl}_2$.

Example A.4.1. *Define $\mathfrak{gl}_2$ as the Lie algebra with generators $\{X, A, a, x\}$ and the relations*

$$[A, X] = X, [x, a] = x,$$
$$[a, X] = X, [x, A] = x,$$
$$[X, x] = A + a, [a, A] = 0.$$

Define the Lie algebra map t_ϵ as $t_\epsilon(a) = a, t_\epsilon(x) = x$ on the subalgebra $\mathfrak{h}$, and as $t_\epsilon(X) = \epsilon X, t_\epsilon(A) = \epsilon A$. We define the elements $A' = \epsilon A, X' = \epsilon X, x' = x, a' = a$. Then we find the following relations for $\{X', A', a', x'\}$:

$$[x', a'] = [x, a] = x \tag{A.8}$$
$$[x', A'] = [x, \epsilon A] = \epsilon x' \tag{A.9}$$
$$[X', A'] = [\epsilon X, \epsilon A] = -\epsilon X' \tag{A.10}$$
$$[X', a'] = [\epsilon X, a] = X' \tag{A.11}$$
$$[X', x'] = [\epsilon X, x] = \epsilon(A + a) = A' + \epsilon a'. \tag{A.12}$$

In these relations we already recognize a subalgebra of the Lie algebra constructed in section 1.1 of chapter 1, in the case where $\epsilon^2 \neq 0$. This is also the sl_2^ϵ algebra as constructed in [35]. Since the elements $\{X, A\}$ generate a subalgebra of $\mathfrak{gl}_2$, by theorem A.4.1 we can take the limit of $\epsilon \to 0$. The result is the Lie algebra $sl_2^{\epsilon=0}$.

It is possible to do the same thing for the sl_n case, covered in chapter 4. In this case, one could start with the algebra of section 4.4.2 in [10] to obtain the quasi-triangular Lie bialgebra covered in chapter 1, which one would need to quantize in the manner of chapter 4. This is a straightforward exercise for the reader.

A.5. Rings

In this appendix we follow [19]. By a ring R we always mean a commutative ring with identity 1 and of characteristic zero. The characteristic of a ring is the smallest number such that $1^n = 1 + 1 + \cdots + 1 = 0$.
An element $r \in R$ is called a zero devisor of R if there exists a nonzero element $x \in R$ such that $rx = 0$. An element of R is called regular if it is not a zero devisor. We define an integral domain as a ring without zero devisors.
An ideal $I \subset R$ of R is a set I containing 0 such that I is closed under addition, and such that if $i \in I$ and $r \in R$, $ir \in I$. $\mathfrak{m}$ is the maximal ideal $\mathfrak{m}$ of a ring R if $\mathfrak{m} \neq R$ and if for any ideal $I \subset R$ such that $\mathfrak{m} \subset I$, either $I = \mathfrak{m}$ or $I = R$.

Definition A.5.1. *An ideal $I \subset R$ is called a prime ideal if for any $a, b \in R$ such that $ab \in I$, $a \in I$ or $b \in I$, and if $I \neq R$. Define the spectrum $Spec(R)$ of R as the set of prime ideals of R.*

Denote by $R[x_1, \cdots, x_n]$ the ring of polynomials in n indeterminates with coefficients in R.

Definition A.5.2. *Let k be a field, let $S \subset k[x_1, \cdots, x_n]$. Define the affine variety of S as $\mathcal{V}_{k^n}(S) := \{(\xi_1, \cdots, \xi_n) \in k^n | f(\xi_1, \cdots, \xi_n) = 0 \, \forall f \in S\}$. For $X \in k^n$, define the ideal of X as $\mathcal{I}(X) = \mathcal{I}_{k[x_1, \cdots x_n]}(X) := \{f \in k[x_1, \cdots x_n] | f(\xi_1, \cdots, \xi_n) = 0$ for all $(\xi_1, \cdots, \xi_n) \in X\}$.*

We can now define the coordinate ring of a set $X \subset k^n$.

Definition A.5.3. *Let $X \subset k^n$ be an affine variety. Define the coordinate ring of X as $k[X] := K[x_1, \cdots, x_n]/\mathcal{I}(X)$.*

We define a module over a ring R as one defines a vector space over a field k.

Definition A.5.4. *Let R be a ring. A (left-)module M over R is an abelian group $(M, +)$ together with an operation $\cdot : R \times M \to M$ such that for $r, s \in R$ and $x, y \in M$,*

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1_R \cdot x = x.$

A module over R is called free if it has an R-basis. An R-basis of a module M is a generating set of M that is linearly independent over R. Denote for a subset $S \subset M$ of an R-module M, (S) for the submodule of M generated by S. By definition this is equal to the set of all linear combinations of S. If $S = \{m_1, cdotsm_n\}$, we may write $(S) = (m_1, \cdots m_n)$. In the same way we may define an ideal $(m_1, \cdots m_n) \subset R$ generated by the set $\{m_1, \cdots m_n\} \subset R$.
Define the formal power series ring in the variable x over a ring R as $R[[x]] := \{\sum_{i=0}^{\infty} a_i x^i | a_i \in R\}$, and similarly for any finite number of indeterminates x_i, $i \in I$.

Definition A.5.5. *A ring R is called local if it has precisely one maximal ideal. R is called Noetherian if for every strictly ascending chain of subideals $I_i \subset M$ such that $I_i \subset I_{i+1}$ there exists an integer n such that $I_i = I_n$ for all $i \geq n$.*

If R is a local Noetherian ring with maximal ideal $\mathfrak{m}$, we can define the residual class field $K := R/\mathfrak{m}$. Furthermore if $\mathcal{M}$ is a set of sets, we define a chain in $\mathcal{M}$ as a subset $\mathcal{C} \subset \mathcal{M}$ that is totally ordered by inclusion. The length of a chain $\mathcal{C}$ is defined as $length(\mathcal{C}) := |\mathcal{C}| - 1 \in \mathbb{N}_0 \cup \{-1, \infty\}$. We then define

$$length(\mathcal{M}) := \sup\{length(\mathcal{C}) | \mathcal{C} \text{ is a chain in } \mathcal{M}\}.$$

Define the dimension of R as $dim(R) = length(Spec(R))$.
It turns out that $dim_K(\mathfrak{m}/\mathfrak{m}^2) \geq dim(R)$.

Definition A.5.6. *A local ring R is called regular if $dim_K(\mathfrak{m}/\mathfrak{m}^2) = dim(R)$.*

For a ring R, we define an R-algebra A to be a ring A with a homomorphism $\alpha : R \to A$. So an R-algebra is a commutative, associative algebra with unit. For a field k, an affine k-algebra is a finitely generated k-algebra. A k-algebra is finitely generated if it is isomorphic to the ring $k[x_1, \cdots, x_n]/I$, where $I \subset k[x_1, \cdots, x_n]$ is an ideal. It is clear that $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$ is an affine $\mathbb{R}$-algebra.

Definition A.5.7. *Let A be an algebra over a field k. Define the transcendence degree of A as $sup\{|T||T \subset A$ is finite and algebraically independent$\}$.*

For a k-algebra A, we define a set $a = \{a_1, \cdots, a_n\}$ to be algebraically independent if for all $f \in k[x_1, \cdots, x_n], f(a) \neq 0$. An example is the ring R_ϵ. We see that R_ϵ has transcendence degree 0 over $\mathbb{R}$, as $\epsilon^2 = 0$. Moreover, R_ϵ is local, with unique maximal ideal (ϵ) (observe that any regular element is invertible).
For affine k-algebras, $dim(A) = trdeg(A)$. The proof can be found in e.g. [19], chapter 5. So $dim(R_\epsilon) = 0$. However, in R_ϵ, (ϵ) is the maximal ideal. Since $(\epsilon)^2 = 0$, $dim_{\mathbb{R}}((\epsilon)/(\epsilon)^2) = dim_{\mathbb{R}}((\epsilon)) = 1$. So we see that R_ϵ is not regular.

Definition A.5.8. *Let M be an R-module and let $m \in M$. m is called a torsion element of M if there exists a regular $r \in R$ such that $rm = 0$. M is called torsion-free if 0 is the only torsion element of M.*

In the ring R_ϵ, the set of regular elements is given by $\{r = a + \epsilon b \in R_\epsilon | a \neq 0\}$.
Let R be any ring, and let M be a free R-module. It is clear from the definition of linear independence that M is torsion-free. Let M be a free R-module. Define the dual M^* of M as $M^* = Hom_R(M, R)$. Observe that M^* has a natural R-module structure. Let $\phi \in M^*$ and $r \in R, m \in M$, then $r\phi(m) = \phi(rm)$. Let r be a regular element of R, $r\phi = 0$ implies that $\phi(rm) = 0$ for all $m \in M$. However, since r is regular and M is torsion free, $rm \neq 0$ if $m \neq 0$. It is easy to show (by induction, for example) that if $R = K[X]/(X^n)$ for a field K and an integer $n > 0$, and if r is regular, $\{rm | m \in M\} = M$. This implies that $\phi = 0$. So M^* is torsion free.
We continue with the discription of freeness and flatness of a module M over the ring R_ϵ.

Definition A.5.9. *Let R be a ring, and let $M_1, M2, M$ be R-modules. Let $f_i : M_1 \to M_2$ be an injective map. Define the map $\phi_f : M_1 \otimes M \to M_2 \otimes M : x \otimes m \mapsto f(x) \otimes m$. We call M flat if for any injective map f, ϕ_f is injective.*

A consequence of this definition is that if $M_1 \to M_2 \to M_3$ is an exact sequence, then $M_1 \otimes M \to M_2 \otimes M \to M_3 \otimes M$ is also an exact sequence. We will now prove that over R_ϵ, the notions of flatness and freeness coincide.

Proposition A.5.1. *Let M be an R_ϵ-module. Then M is flat if and only if M is free.*

Before proving the proposition, observe that for modules over any ring it is true that free modules are also flat. The converse is not always the case. When M is finitely generated, the conditions of flatness and freeness are identical. We will not prove these facts here. See for example [8], chapter 6. We will prove these facts for the ring R_ϵ here.

Proof. We first prove freeness $\implies$ flatness for an R_ϵ-module M. This is a well known fact, but it is proven here nonetheless. Suppose that M has an R_ϵ-basis $\{m_i\}_{i\in I}$. Let K and L be two R_ϵ-modules, let $\phi : K \to L$ be an injective map, and let $k, k' \in K, m \in M$, and $m \neq 0, k \neq 0$. Let $l \in L$. Tensor products are over R_ϵ.
We wish to prove that $\phi' : K \otimes M \to L \otimes M, k \otimes m \mapsto \phi(k) \otimes m$ is an injective map. Assume $\phi(k) \otimes m = 0$. We want to prove that $\phi(k) = 0$. Let us write $m = \sum_i c_i m_i$, to obtain $\phi(k) \otimes m = 0$ if and only if $\sum_i c_i \phi(k) \otimes m_i = 0$. Since the m_i form a basis of M, we can define $\pi^{-1} : L \otimes M \to L \times M$ on elements of the form $l \otimes m_i$ by sending $l \otimes m_i \mapsto (l, m_i) \in L \times M$. Now we define $\psi : L \times M \to K \otimes M$: $\psi(l, m_i) = k' \otimes m_i$, where k' is chosen such that $\phi(k') = \phi(k)$. Since ϕ is injective, $k = k'$, so this map is well-defined. We extend the map $\psi \circ \pi^{-1} : L \otimes M \to K \otimes M$, which is only defined on the set $\{l \otimes m_i | l \in \phi(K), i \in I\} \subset L \otimes M$ as a linear map. By construction $\psi \circ \pi^{-1}(\phi(k) \otimes m) = k \otimes m$, and $\phi' \circ \psi \circ \pi^{-1}(l \otimes m) = l \otimes m$, where $l \otimes m \in \phi'(K \otimes M)$. This implies that ϕ' is injective.
For the other implication, we assume that M is flat. We use the fact that any module over a field (i.e. a vector space) is free. This can be proven by using the maximal principle on a chain of linealry independent sets to construct a maximal linearly independent subset. Taking the union of all the sets in this chain provides a maximal element in this chain. Its elements are linearly independent, and it must span the vector space by maximality. We refer to other sources for the extended proof.
Proceding with a flat R_ϵ-module M, we observe that $M/\epsilon M$ is an $\mathbb{R}$-module. Concretely, if $(\epsilon) \subset R_\epsilon$ is the ideal generated by ϵ, we observe that $\mathbb{R} = \frac{R_\epsilon}{(\epsilon)}$. Taking the tensor product with M yields $M/\epsilon M$ as an $\mathbb{R}$-module.
Consider the short exact sequence

$$0 \to \epsilon \cdot \mathbb{R} \overset{f}{\hookrightarrow} R_\epsilon \overset{g}{\twoheadrightarrow} \mathbb{R} = \frac{R_\epsilon}{(\epsilon)} \to 0.$$

g is given by $a + \epsilon b \mapsto a$, and f is the inclusion. All spaces are considered as R_ϵ-modules. Since M is flat we can take the tensor product with $\otimes_{R_\epsilon} M$ to obtain

$$0 \to \epsilon \cdot M \hookrightarrow M \overset{\pi}{\twoheadrightarrow} \frac{M}{\epsilon M} \to 0.$$

We can form another exact sequence $R_\epsilon \overset{\cdot\epsilon}{\to} R_\epsilon \to R_\epsilon/(\epsilon)$. This implies $M \overset{\cdot\epsilon}{\to} M \to M/\epsilon M$ is exact, since M is flat. So $\epsilon M = \ker(M \overset{\epsilon\cdot}{\to} M)$, so we obtain an injective map $M/\epsilon M \overset{h}{\hookrightarrow} \epsilon M \subset M$, that is also surjective. So $M/\epsilon M \cong \epsilon M$.

Suppose that $\{\bar{m}_i\}_{i\in I}$ is an $\mathbb{R}$-basis of $M/\epsilon M$. Choose a set $\{m_i\}_{i\in I} \subset M$ such that $\pi(m_i) = \bar{m}_i$ for all $i \in I$. We claim that $\{m_i\}_{i\in I}$ is an R_ϵ-basis of M. To see that $\{m_i\}_{i\in I}$ spans M, we consider an element $m \in M$, then $\pi(m) = \sum c_i \bar{m}_i$, where $c_i \in \mathbb{R}$. Then we know that $m = \sum c_i m_i + \epsilon n$, for some $\epsilon n \in \ker(M \xrightarrow{\cdot\epsilon} M) = \epsilon M$. Because there is an isomorphism $M/\epsilon M \cong \epsilon M$, we can express n as a linear combination $n = \sum \epsilon \bar{c}_i m_i$, for $\bar{c}_i \in \mathbb{R}$. This proves that $\{m_i\}_{i\in I}$ spans M.
To prove linear independence of $\{m_i\}_{i\in I}$, we proceed in a similar fashion. Suppose $\sum\limits_{i\in I'} c_i m_i =$ for $c_i \in R_\epsilon$, where i runs over a finite set I'. We wish to prove that $c_i = 0$ for all i. We know that $g(c_i) = 0$ for all i, as $\pi(m_i)$ is an $\mathbb{R}$-basis of $M/\epsilon M$. We interpret $\pi(c_i m_i) = \pi(c_i \otimes m_i) = g(c_i) \otimes m_i$. This implies that $c_i \in \epsilon M$, so $c_i = \epsilon d_i$ for some $d_i \in \mathbb{R}$. Denote $\bar{m}_i = \pi(m_i)$. Since $\epsilon M \cong M/\epsilon M$ through multiplication with ϵ, we know that $\epsilon \bar{m}_i$ form an $\mathbb{R}$-basis of ϵM as an $\mathbb{R}$-module. Hence $d_i = 0$ for all i, and we have proven linear independence. This finishes the proof. □

As a concrete application we wish to extend an $\mathbb{R}$-basis of $M/\epsilon M$ to an R_ϵ-basis of M. We will use this construction in the thesis, for example in chapter 1.

Corollary A.5.1. *Let M be a free (and flat) R_ϵ-module. Let $\{\tilde{m}_i\}_{i\in I}$ be an $\mathbb{R}$-basis of $M/\epsilon M$. Let $\{m_i\}_{i\in I}$ be such that under the projection $\pi : M \to M/\epsilon M$, $\pi(m_i) = \tilde{m}_i$, for all $i \in I$. Then $\{m_i\}_{i\in I}$ is an R_ϵ-basis of M.*

This finishes the discussion of the ring R_ϵ.

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