Expansions of quantum group invariants

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4. Towards constructing $U_q(sl_n^\epsilon)$

Introduction

In this chapter we repeat the construction of $U_q(sl_3^\epsilon)$ of the first chapter for sl_n, for general n. In the first section we quantize the upper Borel subalgebra of sl_n, and we construct a basis by using the Weyl-group action. The Weyl group is constructed for $U_q(sl_2^\epsilon)$ in section 4.2, after which we continue with constructing the Weyl group for $U_q(sl_n^\epsilon)$. We assume that ϵ is invertible in this chapter. We calculate the algebra relations up to any order of ϵ by taking the power series expansion of an expression.
In this chapter we assume that ϵ is invertible, as in the non-invertible, $\epsilon^k = 0$ case the construction of the quantum Weyl group breaks down. It is not possible to construct the usual highest weight representations when $\epsilon^k = 0$. Taking ϵ invertible provides an isomorphism between sl_n^ϵ and sl_n. The usual automorphisms T_i that originate from the Weyl group are not algebra automorphisms when ϵ is not invertible. When working over $\mathbb{R}(\epsilon)$, the maps T_i turn out to be algebra automorphisms.
In the last section we prove that one can define algebra maps $\tilde{T}_i$ from the automorphisms T_i for non-invertible ϵ. However, the $\tilde{T}_i$ can only be applied to simple generators times a factor of ϵ. This is familiar from chapter one, where we saw a similar term ϵZ in the commutator.
A different set of symmetries has been found by Bar-Natan and Van der Veen when $\epsilon^k = 0$, or more generally for $\mathbb{R}[[\epsilon]]$. The set of symmetries for non-invertible ϵ is isomorphic to the dihedral group D_n for sl_n^ϵ. See [37] for details.
It remains to be seen if this means that the invariants arising are stronger, as they might have less symmetry, or if this means there are more hidden symmetries that arise in the invariants. This symmetries only differs for $n \geq 4$, as $D_3 = S_3$. If $\epsilon \in \mathbb{R}[[\epsilon]]$, it has been noted that a quotient of an affine quantum group is obtained, see [37] and [5]. The Dynkin diagrams of affine Lie algebras have a circular form, so there are different symmetries than in the sl_n case. When $\epsilon^k = 0$ in an affine Lie algebra in some sense, these symmetries survive. See [37] and [5] for details.

The contents of this chapter is as follows. In the first section we provide the general sl_n^ϵ Lie algebra relations and its quantization $U_q(sl_n^\epsilon)$ for invertible ϵ. The construction of the $U_q(sl_n^\epsilon)$ is briefly covered. In the second section we cover the finite dimensional representation theory for $U_q(sl_2^\epsilon)$ for invertible ϵ, and an algebra automorphism is constructed. In the third section we proceed with the

$U_q(sl_n^\epsilon)$ case in the same way, following [29]. In the last section we scetch the connection between the Hopf algebras covered in this and the first chapter.

4.1. Quantizing a Lie subalgebra of sl_n

Let sl_n^ϵ be a Lie bialgebra over $\mathbb{R}(\epsilon)$ for an indeterminate ϵ with generators $H_i^\pm, X_i^\pm$, $i = 1 \cdots n$, Cartan-matrix a_{ij} and the relations (we introduce ϵ in the b^- side multiplication, as opposed to chapter 1)

$$[H_i^-, X_j^\pm] = \pm\epsilon a_{ij} X_j^\pm, [H_i^\pm, H_j^\mp] = 0, [H_i^+, X_j^\pm] = \pm a_{ij} X_j^\pm, \tag{4.1}$$

$$[X_i^+, X_i^-] = -\frac{1}{2}\delta_{i,j}(H_i^+ + \epsilon^{-1} H_i^-), (ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, (i \neq j), \tag{4.2}$$

$$\delta(X_i^+) = \epsilon X_i^+ \otimes H_i^+ - \epsilon H_i^+ \otimes X_i^+, \tag{4.3}$$

$$\delta(X_i^-) = X_i^- \otimes H_i^- - H_i^- \otimes X_i^-, \tag{4.4}$$

$$\delta(H_i^\pm) = 0. \tag{4.5}$$

In our convention, $a_{ii} = 2$, $a_{ij} = -1$ if $i = j \pm 1$ and else zero. We consider the double of the Lie algebra of upper triangular matrices $b^+ \subset gl_n$, so we will assume that the Cartan matrix has rank n. As noted in chapter 1, the Cartan matrix is well defined, even though the above algebra is not semisimple.
We observe furthermore that $H_i^\pm$ generate the Cartan subalgebra $\mathfrak{h}$, the biggest commutative subalgebra of $\mathfrak{g}$ (which is the case for semisimple Lie algebra's). Define $ad_X(Y) = [X, Y]$ as the adjoint action of $\mathfrak{g}$ on itself. Putting $\epsilon = 1$ and dividing out to $H_i^+ - H_i^- = 0$ yields the usual sl_n Lie bialgebra. In this chapter we consider the generalization of the classical double quasitriangular Lie bialgebra calculated in chapter 1, although with ϵ present in the b^- lower Borel subalgebra commutation relations.
The simple roots $\alpha_i : \mathfrak{h}^+ \to \mathbb{R}(\epsilon)$ are defined as the linear maps $\alpha_i(H_j^+) = a_{ij}$, and similarly for $\mathfrak{h}^- \subset \mathfrak{h}$, with an additional factor of ϵ. Since ϵ is invertible, the root space for $h^\pm$ are isomorphic and we may talk about the rootspace of sl_n^ϵ. As we are concerned with the quantization of b^+ in this section, we will use roots on $\mathfrak{h}^+$.
The fundamental reflections $s_i : \mathfrak{h} \to \mathfrak{h}$ are defined by $s_i(h) = h - \alpha_i(h) H_i^+$ for $h \in \mathfrak{h}$. The Weyl group of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h})$ generated by $s_1, \cdots, s_{n-1}$. The Lie algebra sl_n^ϵ is finite dimensional, so there is a unique element w of maximal length N. Write $w = s_{i_1} \cdots s_{i_N}$. Define the positive roots as the set $\Delta^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1}(\cdots s_{i_{N-1}}(\alpha_{i_N})\}$. Note that each element occurs exactly once.
Although sl_n^ϵ is not semisimple, the Killing form is nondegenerate on the subalgebras $\mathfrak{h}^\pm \subset \mathfrak{h}$ generated by $H_i^\pm$. The reason is that the upper an lower Borel subalgebras $b^\pm$ are embedded in sl_n. So the Cartan matrix and the root system corresponding to the $b^\pm$ algebra is well defined.
For a Lie algebra $\mathfrak{g}$ over a field, one can extend the fundamental reflections to act on $\mathfrak{g}$ instead of $\mathfrak{h} \subset \mathfrak{g}$, see [6], in which case we call the automorphism corre-

sponding to s_i T_i. In the case of sl_n^ϵ, we have the following definition of T_i. This definition is equivalent to the automorphisms T_i of sl_n, as one can check in for example [6], or [14].

Proposition 4.1.1. *Let sl_n^ϵ be the Lie algebra structure as specified above. The T_i defined in the following way*

$$T_i(X_i^\pm) = -X_i^\mp,\ T_i(H_j^+) = \epsilon^{-1}H_j^- - \epsilon^{-1}a_{ji}H_i^-, T_i(H_j^-) = \epsilon H_j^+ - \epsilon a_{ji}H_i^+, \quad (4.6)$$

$$T_i(X_j^+) = (-a_{ij})!^{-1}(ad_{X_i^+})^{-a_{ij}}(X_j^+), i \neq j \quad (4.7)$$

$$T_i(X_j^-) = (-1)^{a_{ij}}(-a_{ij})!^{-1}(ad_{X_i^-})^{-a_{ij}}(X_j^-), i \neq j. \quad (4.8)$$

are Lie algebra automorphisms of sl_n^ϵ.

Proof. The only relations that change in the presence of ϵ are the commutator $[H_i^-, X_j^\pm] = -\epsilon a_{ij}X_j^\pm$ and $[X_i^+, X_i^-] = -\frac{1}{2}(H_i^+ + \epsilon^{-1}H_i^-)$. Applying T_i and T_j on both sides of the first identity, we observe that T respects the relation. Here we make use of the Jacobi-identity to calculate commutators of commutators.
For the second identity, we observe that the right-hand side is invariant (modulo a global minus sign) under T_j if $i = j$ and if $j \neq i$ we gain a term $-a_{ij}(H_j^+ + \epsilon^{-1}H_j^-) = H_j^+ + \epsilon^{-1}H_j^-$, as $a_{ij} = -1$ if $i \neq j$. On the left hand side we obtain the term $[[X_j^+, X_i^+], [X_i^-, X_j^-]]$, which we can evaluate with applying the Jacobi identity twice. Note that $-a_{ij} = 1$, so $T_j(X_i^-) = [X_i^-, X_j^-]$. We see that $[[X_j^+, X_i^+], [X_i^-, X_j^-]] = -[[X_j^-, [X_i^-, X_i^+]], X_j^+] - [[[X_j^+, X_j^-], X_i^-], X_i^+]$. We only need to prove that $[X_j^+, [X_j^-, [X_i^+, X_i^-]]]$ yields a term $H_j^+ + \epsilon^{-1}H_j^-$, and similarly for the term $i \leftrightarrow j$. Using the commutator $[X^+, X^-]$ we obtain $-[[X_j^-, [X_i^-, X_i^+]], X_j^+] - [[[X_j^+, X_j^-], X_i^-], X_i^+] = \frac{1}{2}(-[[X_j^-, H_i^+ + \epsilon^{-1}H_i^-], X_j^+] + [[H_j^+ + \epsilon^{-1}H_j^-, X_i^-], X_i^+])$. With $[H_i^+, X_j^\pm] = \pm a_{ij}X_j^\pm$ and the relations for H_i^-, this yields the required result. This proves the theorem. □

These automorphisms obey the braid group relations $T_iT_jT_i = T_jT_iT_j$ for all $i \neq j$.

Proposition 4.1.2. *Let the T_i be as defined above, and let a_{ij} be the Cartan matrix corresponding to sl_n. Then $T_iT_jT_i = T_jT_iT_j$ for all $i \neq j$.*

Proof. As a_{ij} only takes nonzero values if i and j differ at most 1, we only need to check two non-trivial identities. The case for $H_i^\pm$ can be reduced to the sl_n case by counting the factors of ϵ on both sides and realizing T_i is linear in ϵ. We note that independently of the index i, T_i switches the sign of $H^\pm$. This is in fact the only thing that is different from sl_n for the Cartan subalgebra, together with ϵ we need to keep track of.
The case for $X_i^\pm$ is the same for both $b^\pm$, and can be reduced to the usual case by realizing that H^+ acts in the same way as $\epsilon^{-1}H^-$. We only need to count the factors of ϵ that are introduced when checking the Weyl condition on $X_j^\pm$. This is left as an exercise, as it follows by a straightforward calculation. □

We continue with constructing the quantization of the above sl_n^ϵ Lie bialgebra. We first quantize the upper triangular matrices Lie subalgebra b^+ of sl_n^ϵ. Consider the subalgebra b^+ generated by the simple root vectors X_i^+ and H_i^+ for all $i = 1 \cdots n-1$. remember that the cobracket on b^+ is multiplied with ϵ:

$$\delta(X_i^+) = \epsilon X_i^+ \wedge H_i^+. \tag{4.9}$$

The cobracket on the other positive root vectors is implicitly defined. To quantize a Lie bialgebra, only the cobracket on the simple generators are needed. We follow the usual construction of $U_q(sl_3)$ here to obtain a quantization of b^+. See chapter 6 and 8 of [6].

We are looking for a Hopf algebra with classical limit 4.9, so it is easiest to start with quantizing the cobracket. Firstly, let us take the trivial Hopf algebra structure on the universal enveloping algebra $U(b^+)$, as introduced earlier. $\delta(H_i^+) = 0$ yields

$$\Delta_h(H_i^+) = H_i^+ \otimes 1 + 1 \otimes H_i^+.$$

Continueing with X_i^+, we introduce a grading deg on b^+. Here $deg(H_i^+) = 0$ and $X_i^+ = 1$. In order to obtain a graded algebra, we need the (co)multiplication to preserve the grading, or at least not lowering the degree in the case of general positive roots. To this end, let us follow [6] and guess

$$\Delta_h(X_i^+) = X_i^+ \otimes e^{h\mu H_i^+} + e^{h\nu H_i^+} \otimes X_i^+, \tag{4.10}$$

where $\mu, \nu \in \mathbb{R}(\epsilon)[[h]]$, so that $e^{h\mu H_i^+}$ and $e^{h\nu H_i^+}$ are grouplike, meaning $\Delta_h(e^{h\mu H_i^+}) = e^{h\mu H_i^+} \otimes e^{h\mu H_i^+}$. Multiplying X_i^+ with $e^{-h\nu H_i^+}$ thus yields

$$\Delta_h(X_i^+) = X_i^+ \otimes e^{h\mu H_i^+} + 1 \otimes X_i^+,$$

so we can take $\nu = 0$ without loss of generality. We can use that the classical limit of $\Delta_h(X_i^+)$ equals 4.9, so we can take $\mu = \epsilon$ to obtain

$$\Delta_h(X_i^+) = X_i^+ \otimes e^{h\epsilon H_i^+} + 1 \otimes X_i^+. \tag{4.11}$$

Δ_h extends to an algebra homomorphism on the subalgebra generated by H_i^+ and X_i^+, since the H_i^+ has trivial comultiplication. Consequently, the multiplication (bracket) can be left unchanged. Hence we can directly write down the antipode for H_i^+ and X_i^+ from the calculated comultiplication.

$$S_h(H_i^+) = -H_i^+, S(X_i^+) = -X_i^+ e^{-h\epsilon H_i^+}. \tag{4.12}$$

We extend Δ_h to an algebra homomorphism on $U_h(b^+)$. Consider the classical

Serre-relations for $i \neq j$, that hold for the Lie algebra:

$$(ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (X_i^\pm)^k (X_j^\pm)(X_i^\pm)^{1-a_{ij}-k} = 0. \tag{4.13}$$

For sl_n, $1 - a_{ij} = 2$ for all the nontrivial relations (the case $i = j$ yields a vanishing commutator for X_i^+ and X_j^+). Note that in the case of sl_3, with the definition $[x, y] = z$ and $X_1^- = x$, $X_2^- = y$, gives $[z, x] = 0$.

In order for Δ_h to be an algebra homomorphism, 4.13 needs to be altered. Repeating the calculation we did in chapter 1, the correct form of the quantum Serre relations is obtained by replacing the binomial coefficients with quantum binomial coefficients, with $q = e^{\epsilon h} = 1 + \epsilon h$. For the calculation in the case of sl_n, see chapter 6 of [6]. In the presence of ϵ this calculation is the same.

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{e^{2\epsilon h}} (X_i^+)^k (X_j^+)(X_i^+)^{1-a_{ij}-k} = 0. \tag{4.14}$$

One of the ingredients for the proof that with these relations Δ_h does indeed become an algebra homomorphism is the commutation relation

$$e^{\epsilon h H_i^+} X_j^+ e^{-\epsilon h H_i^+} = e^{\epsilon h a_{ij}} X_j^+.$$

Together with the trivial counit, we have constructed the Hopf algebra structure on $U_q(b^+)$.

Theorem 4.1.1. *$U_h(b^+)$ is a quantization of the Lie bialgebra b^+, for invertible and non-invertible ϵ. Moreover, there exists an algebra isomorphism $U_q(b^+) \cong U(b^+)[[h]]$ in both cases.*

Proof. In order to prove that we have indeed found the quantization of b^+, observe that multiplication and comultiplication in $U_h(b^+)$ have b^+ as classical limit. It is also necessary to find a $\mathbb{R}(\epsilon)[[h]]$-module isomorphism between $U_h(b^+)$ and $U(b^+)[[h]]$. Since ϵ is invertible, this is equivalent to the sl_n case. This equivalence yields an $\mathbb{R}(\epsilon)[[h]]$-module isomorphism. Even stronger, we obtain an algebra isomorphism between $U_h(b^+)$ and $U(b^+)[[h]]$, as there exists an algebra isomorphism between $U_h(sl_n)$ and $U(sl_n)[[h]]$ by the rigidity theorem. See [6], chapter 6.1.
The case where ϵ is not invertible can be obtained from the first case by expanding the isomorphism in terms of ϵ. The fact that this can be done follows because the ϵ only occurs together with the h in the algebra relations on the b^+ side in the q-Serre relations. This implies that the isomorphism $U_h(b^+) \cong U(b^+)[[h]]$ is also defined over $\mathbb{R}[[\epsilon, h]]$. Moreover, ϵ is only present in q and q is invertible up to any order of ϵ^k. We note that after expansion of q there is no factor of ϵ^{-1} present in the relations of $U_h(b^+)$.

To prove that in finite order of ϵ we still have isomorphisms of $\mathbb{R}[\epsilon]/(\epsilon^k)[[h]]$-modules, note that injectivity follows from comparing the terms in each order of ϵ. The surjectivity follows in the same way. That we obtain an isomorphism of algebras follows from linearity of the isomorphism over ϵ. So we have a quantization of the Lie bialgebra b^+. □

We are now in a position to construct a PBW basis for $U_h(b^+)$, while also calculating the dual of $U_h(b^+)$, the partial R-matrices and the comultiplication on the PBW basis. These calculations are necessary in order to be able to write an algoritm that can calculate the algebra relations for general n. We could start with the dual basis to the simple roots X_i^+. Using these generators and the action of the Weyl group, we can calculate the necessary partial R-matrices. Using those (and their inverse), we can calculate the coproduct on basis elements associated with any positive root. Then we can find the (co)multiplication properties of the dual basis using the Hopf algebra pairing.

The action of the braid group can be defined straightforwardly on the algebra $U_h(n^+)$ spanned by X_i^+ in $U_h(b^+)$ through the Hopf algebra right-adjoint action Ad

$$Ad_x(y) = \sum x_{(1)} y S(x_{(2)})$$

of U_h and U_h^{cop}, the opposite coalgebra. Define $T_i(X_j^-) = Ad_{(-X_i^-)^{(-a_{ij})}}(X_j^-)$ and $T_i(X_j^+) = Ad^{cop}_{-(X_i^+)^{(-a_{ij})}}(X_j^+)$. As we will see, for invertible ϵ it is possible to define T_i on $U_q(sl_n^\epsilon)$. This will be covered in the next sections. For now we restrict ourselves to $U_q(n^+)$. Note that we define the T_i slightly different here than we will do in the next section. We leave out the central factor for cosmetic reasons. We note that this has no effect on the expression for $\Delta(X_\beta)$ we state here.
This action can be used to write down explicit generators of $U_q(b^+)$ for invertible ϵ. If $\beta = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta^+$, define $X_\beta^\pm = T_{i_1} \cdots T_{i_{k-1}}(X_{i_k}^\pm)$. Assuming that there are no redundant reflections in the notation for β, this is well defined, and yields generators $X_\beta^\pm$ for each positive root. The fact that this is well defined follows from the Weyl property for T_i. For the proof that T_i satisfy the Weyl property we refer to the next two sections.
Denote $X_\beta^\pm$ for the generators of $U_q(sl_n)$ corresponding to the root β, and denote $H_i^\pm$ for the generators of the quantized Cartan subalgebra of $U_q(sl_3^\epsilon)$, for $i = 1, \cdots n$. Let w be the longest root with decomposition $w = s_{i_1} \cdots s_{i_N}$. We denote the positive roots by $\beta_1, \cdots \beta_N$. Corresponding to this decomposition we have the non-simple generators $X_{\beta_i}^\pm$, where $i = 1, \cdots, N$
Monomials in X_β^+ and H_β^+ form a basis of $U_q(b^+)$. The folowing theorem is a generalization of the theorem we saw in chapter 1, however the proof is easier since we start with an algebra over $\mathbb{R}(\epsilon)$.

Theorem 4.1.2. *Let X_β^+ and H_β^+ be the generators corresponding to the positive roots*

β. Then the monomials $\prod_{i=1}^{N}(X^+_{\beta_i})^{m_i}\prod_{i=1}^{n}(H_i^+)^{p_i}$ form a basis of $U_q(b^+)$.

Proof. From the classical PBW theorem it follows that ordered monomials in $X^\pm_\beta$ constitute a linear basis of $U_q(sl_n^\epsilon)$. Now we use the fact that ϵ is invertible, so that we have an isomorphism between $U_q(sl_n^\epsilon)$ and $U(sl_n^\epsilon)[[h]]$, by the rigidity theorem. □

This finishes the construction of $U_q(b^+)$. To obtain $U_q(sl_n^\epsilon)$ we need to calculate the QUE-dual of $U_q(b^+)$, which we refer to as $U_q(b^-)$, consistent with chapter 1. Then one can form the quantum double of $U_q(b^\pm)$ to form $U_q(sl_n^\epsilon)$. We skip this construction and state the relations of $U_q(sl_n^\epsilon)$. The proof that these relations form a Hopf algebra can be found in many sources, since ϵ is invertible.
Notice that $U_h(b^+)$ is also well defined over $\mathbb{R}[[\epsilon,h]]$ by expanding q. Over this ring, it is possible to divide out to ϵ^k for some k. When we take the quantum double of $U_h(b^+)$, we can no longer work over $\mathbb{R}[[\epsilon]]$, due to the factor $\frac{1}{q-q^{-1}}$ present. In the last section of this chapter we cover this issue.

Theorem 4.1.3. *Let ϵ be invertible, and let $q=e^{\epsilon h}$. The following relations*

$$[X_i^-,H_j^-]=\epsilon a_{ij}X_i^-,\ [X_i^+,H_j^+]=-a_{ij}X_i^+,\ [X_i^-,H_j^+]=a_{ij}X_i^-,\ [X_i^+,H_j^-]=-\epsilon a_{ij}X_i^+$$

$$[X_i^-,X_j^+]=\frac{q^{H_i^+}-q^{-\epsilon^{-1}H_i^-}}{q-q^{-1}}\delta_{ij},\quad \sum_{k=0}^{k=1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q^2}(X_i^\pm)^kX_j^\pm(X_i^\pm)^{1-a_{ij}-k}=0$$

$$\Delta(X_i^-)=X_i^-\otimes e^{hH_i^-/2}+e^{-hH_i^-/2}\otimes X_i^-,\Delta(X_i^+)=X_i^+\otimes e^{\epsilon hH_i^+/2}+e^{-\epsilon hH_i^+/2}\otimes X_i^+$$

$$\Delta(H_i^\pm)=H_i^\pm\otimes 1+1\otimes H_i^\pm,S(X_i^+)=-e^{\epsilon h}X_i^+,S(X_i^-)=-e^{-h}X_i^-,S(H_i)=-H_i,$$

define an Hopf algebra $U_q(sl_n^\epsilon)$ over $\mathbb{R}(\epsilon)$, which is the quantization of the Lie bialgebra sl_n^ϵ. The monomials

$$\prod_{i=1}^{N}(X^+_{\beta_i})^{m_i}\prod_{i=1}^{n}(H_i^+)^{p_i}\prod_{i=1}^{n}(H_i^-)^{p_i'}\prod_{i=1}^{N}(X^-_{\beta_i})^{m_i'}$$

form a basis of $U_q(sl_n^\epsilon)$.

In general, the action of the Braid group is not compatible with the coproduct when extended to the full Hopf algebra. This means that one has to compute the action of the braid group, before one can compute the coproduct. There is another option. One can express the coproduct of the generators in terms of the R-matrices corresponding to the $U_q(sl_2)_i$ subalgebras of $U_q(sl_n)$. We refer these R-matrices as partial R-matrices.
Let $A=a_{ij}$ be the Cartan matrix. Define $\zeta_i=\sum(A^{-1})_{ij}H_j^-$. For simple roots α_i, $i=1,\cdots,n-1$, associated with (dual) generators X_i^+,X_i^- and H_i^+,ζ_i, one has

the following pairing ($q = e^{\epsilon h}$):

$$\langle (H_i^+)^o (X_i^+)^t, (\zeta_i)^{o'} (X_i^-)^{t'} \rangle = \delta_{o,o'} \delta_{t,t'} h^{-o-t} o! [t]_q!. \tag{4.15}$$

Where $[s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}$ and $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$ is the quantum factorial. We wish to calculate the pairing on general monomials in $X^\pm)_i$ and H_j^+, ζ_j. To this end, define $\mathcal{R}_{h,i}$ as

$$\tilde{\mathcal{R}}_{h,i} = \sum_{k=0,l=0}^{\infty} \frac{h^{k+l} (X_i^+)^l (H_i^+)^k \otimes (\zeta_i)^k (X_i^-)^l}{k! [l]_q!}. \tag{4.16}$$

By the quantum double construction, this is the R-matrix for $U_h(sl_2)$ for the i-th simple root. Using the braid group action, one can define the R-matrix for general positive root $\beta_r = T_{i_1}(\cdots T_{i_r-1}(\alpha_{i_r}))$ as follows. We note that the algebra automorphisms T_i can be defined on $U_q(sl_n^\epsilon)$. The definition can be found in section 4.3.

$$\tilde{\mathcal{R}}_{h,\beta_r} = (T_{i_1} \cdots T_{i_{r-1}} \otimes T_{i_1} \cdots T_{i_{r-1}})(\tilde{R}_{h,i_r}), \tag{4.17}$$

$$\tilde{\mathcal{R}}_{h,<\beta_r} = \tilde{\mathcal{R}}_{h,\beta_{r-1}} \cdots \tilde{\mathcal{R}}_{h,\beta_1}. \tag{4.18}$$

We have the following proposition, see section 4.3 for the proof.

Proposition 4.1.3. *(Comultiplication) For any $\beta \in \Delta^+$,*

$$\Delta_h(X_\beta^+) = \tilde{\mathcal{R}}_{h,<\beta}^{-1} (X_\beta^+ \otimes e^{\epsilon h H_\beta^-} + 1 \otimes X_\beta^+) \tilde{\mathcal{R}}_{h,<\beta}.$$

Note that if $\beta = \sum_i k_i \alpha_i$, then $H_\beta = \sum_i d_i k_i H_i$, where $d_i = 1$ are the Cartan integers, where we restrict ourselves to sl_n.

So it is possible to quantize the algebra on the simple generators and know the comultiplications on the non-simple generators. We turn the PBW basis consisting of monomials in generators into a dual basis to obtain the R-matrix. To this end, we define the following generators. Let $A = a_{ij}$ be the Cartan matrix. Remember $\zeta_i = \sum (A^{-1})_{ij} H_j^-$. The following pairing for monomials in the dual generators can be calculated by using the comultiplication.

Proposition 4.1.4.

$$\langle \prod_{i=1}^{N} (X_i^+)^{m_i} \prod_{i=1}^{n} (H_i^+)^{p_i}, \prod_{i=1}^{n} (\zeta_i)^{p_i'} \prod_{i=1}^{N} (X_i^-)^{m_i'} \rangle = \prod \delta_{m_i, m_i'} \prod \delta_{p_i, p_i'} \prod h^{m_i + p_i} \prod [m_i]_q!.$$

where $[n]_q = \frac{q^{-n} - q^n}{q^{-1} - q^1}$.

The proof makes use of proposition 8.3.7 in [6].

Proof. Let us scetch the proof of the general case. The proof is by induction, using proposition 8.3.7 in [6] that is proven in section 4.2.1. We apply Δ to the non-capital side, after which we only need to count the tensor-products that pair non-zero. Note that our algebra has the same pairing as the $U_q(b^\pm)$ dual pairing in [6], except for the factor of ϵ, and the correction with $q - q^{-1}$.
In particular, since ζ_i are dual to H_i^+, the basis of $U_q(b^-)$ corresponds to the $\{\zeta_i, \mu_{\beta_i}\}$ basis in [6]. The different conventions for the comultiplication of X_i^+ in [6] result in a factor of $q^{1/2t_r(t_r-1)}$ present in 8.3.7 in [6] that is absent here. Looking at our $\mathcal{R}_i$, in particular the prefactor R_{n_i} present in the sum, we get the required result.

□

Now this construction is finished, we can write down the universal R-matrix corresponding to $U_q(sl_n^\epsilon)$. The hardest part is calculating the PBW basis and the corresponding dual, and the multiplication relations between the generators. The quantum Serre relations together with the braid group action provide the multiplication relations between the PBW generators.

4.2. Representation theory of $U_q(sl_2^\epsilon)$

Let us proceed with calculating the comultiplication of the quantized Lie bialgebra $U_q(sl_2^\epsilon)$. Before we are able to properly calculate the comultiplication (in such a way that is generalizable, anyway), we need to look at the finite dimensional representations of $U_q(sl_2^\epsilon)$. In this section, we may write $U_q(sl_n^\epsilon)$, $U_h(sl_n^\epsilon)$ or $H_{n,\epsilon}$ for the quantization of sl_n^ϵ.
First we note that if $\epsilon^2 = 0$, it is impossible to define the q-Weyl group in the way it is usually done, since the finite dimensional highest weight representations cannot be constructed. A solution to this problem is to work over the field $\mathbb{R}(\epsilon)$, and prove afterwards that all components of the desired identity lie in $\mathbb{R}[[\epsilon]]$, so that we can divide out to (ϵ^2). For the remainder of this section we will work over the field $\mathbb{R}(\epsilon)[[h]]$.
If $\epsilon = 1$, and one divides out to $H^+ - H^-$ one gets $U_q(sl_2)$. This algebra is obtained by taking the quantum double of the upper triangular matrix subalgebra, with the Hopf structure calculated earlier. For a description of the representations of the regular $U_h(sl_2)$ see for example [6]. We will take the op quantum double construction in this section, instead of the cop construction.
We consider the algebra $U_q(sl_2^\epsilon)$, also denoted as $H_{2,\epsilon}$ for short, generated by X^+, X^-, H^+ and H^- and the following relations. Note that we introduce ϵ in the b^+ multiplication relations. Moreover, our conventions match the conventions used in [29]. In particular note the factor of $\frac{1}{q-q^{-1}}$ in the commutator between

$X^\pm$.

$$[X^-, H^-] = 2X^-,\ [X^+, H^+] = -2\epsilon X^+,\ [X^-, H^+] = 2\epsilon X^-,\ [X^+, H^-] = -2X^+ \tag{4.19}$$

$$[X^-, X^+] = \frac{q^{H^-} - q^{-\epsilon^{-1}H^+}}{q - q^{-1}}$$

Note that we scaled the generators X^- by a factor of $\frac{1}{(q-q^{-1})}$ with respect to the algebra in the first chapter. Substituting $H^+ = 2A - B$ and $H^- = a$ yields the familiar algebra structure, where B is left out when considering only the $U_q(sl_2^\epsilon)$ subalgebra. We take $q = e^{-h\epsilon}$, which will be useful when constructing the universal R-matrix. This is a different from the previous section. Note that ϵ is invertible.
Multiplying X^- with $q - q^{-1}$ yields an algebra over $\mathbb{R}[[\epsilon]]$ (formally we also have the parameter h, so it is an algebra over $\mathbb{R}[[\epsilon, h]]$). The final results of the construction are valid for non-invertible ϵ over the ring $\mathbb{R}[[\epsilon]]$, as discussed in the last section of this chapter.
We can take $\tilde{H}^+ = \epsilon^{-1}H^+$ instead of H^+. In this case, we have an algebra homomorphism with the algebra in [29], by sending our $\tilde{H}^+$ to Reshetikhin's H, (as well as sending our H^- to H) and substituting $q^{\frac{1}{2}}$ for q. $U_q(sl_2^\epsilon)$ agrees with example 3.2.1 in [23] in the same way. The comultiplication, antipode and R-matrix are given by the following formulas.

$$\Delta(X^-) = X^- \otimes e^{\epsilon h H^-/2} + e^{-\epsilon h H^-/2} \otimes X^-, \Delta(X^+) = X^+ \otimes e^{hH^+/2} + e^{-hH^+/2} \otimes X^+ \tag{4.20}$$

$$\Delta(H^\pm) = H^\pm \otimes 1 + 1 \otimes H^\pm, S(X^+) = -e^h X^+, S(X^-) = -e^{-\epsilon h} X^-, S(H) = -H.$$

We also introduce the q-commutator as

$$[A, B]_q = qAB - q^{-1}AB. \tag{4.21}$$

In constructing the representations of $U_q(sl_2^\epsilon)$, we will follow [29] (see Reshetikhin's website for this paper). Denote the representation map by $\pi : H_{2,\epsilon} \to End(V)$, where V is the 2-dimensional vector space generated by $\{e_{\frac{1}{2}}, e_{-\frac{1}{2}}\}$. We obtain the following actions, denoted in matrix notation, where we use the order of the basis as indicated.

$$\pi(H^+) = \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix} \tag{4.22}$$
$$\pi(X^+) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\pi(H^-) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\pi(X^-) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

To prove this is a representation, one needs to prove that the maps given above are indeed algebra homomorphisms. This is a straightforward exercise. Remember that we take $q = e^{-\epsilon h}$. In the case of $j = \frac{1}{2}$ we obtain the simplest case of [29], if we identify $\tilde{H}^+$ and H^-.
Following [29], we can denote the representation in the following more general way. For this section, we use the convention that $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. Furthermore, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, as usual.
For a finite dimensional module V^j of dimension $2j+1$ we get, where j is a positive integer or half integer and V^j is generated by the basis vectors e^j_m, $-j \leq m \leq j$,

$$\pi^j : U_h(sl_2^\epsilon) \to End(V^j) \tag{4.23}$$
$$\pi^j(X^+)(e^j_m) = ([j-m]_q[j+m+1]_q)^{1/2} e^j_{m+1} \tag{4.24}$$
$$\pi^j(X^-)(e^j_m) = ([j+m]_q[j-m+1]_q)^{1/2} e^j_{m-1} \tag{4.25}$$
$$\pi^j(H^+)(e^j_m) = 2m\epsilon e^j_m \tag{4.26}$$
$$\pi^j(H^-)(e^j_m) = 2m e^j_m. \tag{4.27}$$

Again, checking that this yields a representation is straightforwardly writing out the relations 4.19. So 4.19 becomes a quasitriangular Hopf algebra with the following R-matrix. See [30], or [6], if you use the variables $E = e^{hH^+/2}X$ and $F = e^{-\epsilon h H^-/2}Y$, and identify $\epsilon^{-1}H^+$ with H^-. Note that the factor 1/2 in the exponential comes from the pairing between H^+ and H^-. Ultimately this is due to a different definition of the comultiplication, which is used in calculating the pairing between monomials. We know that the R-matrix is defined as the universal R-matrix of the Drinfel'd double. Hence it must be of the form $\sum e_a \otimes f^a$, where e and f are dual bases. We obtain after a correction with a factor $\frac{1}{2}$

$$\mathcal{R} = \exp(hH^+ \otimes H^-/2) \sum R_n(h)(e^{hH^+/2}X^+)^n \otimes (e^{-\epsilon hH^-/2}X^-)^n \qquad (4.28)$$
$$R_n(h) = \frac{q^{1/2n(n-1)}(1-q^{-2})^n}{[n]_q!}.$$

An application of the Hopf algebra automorphism $S \otimes S$ to the R-matrix shows that we obtain (note the plus instead of the minus sign in $\tilde{R}_n$):

$$\mathcal{R} = \sum \tilde{R}_n(h)(e^{-hH^+/2}X^+)^n \otimes (e^{\epsilon hH^-/2}X^-)^n \exp(hH^+ \otimes H^-/2). \qquad (4.29)$$
$$\tilde{R}_n(h) = \frac{q^{1/2n(n+1)}(1-q^{-2})^n}{[n]_q!}. \qquad (4.30)$$

An essential element we can construct with the R-matrix is the ribbon element, or more importantly, the inverse of the ribbon element. See [23] for the precise definition. In particular we can write $v = pu$ for the ribbon element, where $u = \sum \mathcal{R}^{(2)}S(\mathcal{R}^{(1)})$, and $p^2 = u^{-1}v$, $u^{-1} = \sum \mathcal{R}^{(2)}S^2(\mathcal{R}^{(1)})$, $v = S(u)$. Using these identities, we get

$$p^{-1} = e^{\frac{1}{2}(H^+ + \epsilon H^-)}.$$

The square root w of the inverse ribbon element is not a part of the algebra $H_{2,\epsilon}$ since it cannot be expressed in terms of $X^\pm$. Writing these out in matrix notation makes clear that $\pi(X^\pm)$ don't generate the entire $End(V^j)$, except in the case of $j = \frac{1}{2}$ and $j = 1$, the standard representation of $H_{2,\epsilon}$. We will write out the action of w in the representations V^j later in this section, but it will be the case that w sends a basis vector e^j_m to the vector e^j_{-m}. In matrix notation this is the element with only non-zero entries on the 'mirrored' diagonal. Since this is true for any m, $\pi(w)$ cannot be written as a linear combination of $(H^\pm)^b$ and $(X^\pm)^a$, a and b positive integers, as any combination of these will yield non-zero off-diagonal entries.
This means that we have to add w to the Hopf algebra. To prove that this makes sense, we can use proposition 6.3.12 and example 6.3.13 of [23]. It turns out that we obtain another Hopf algebra, called the quantum Weyl group ([6]), and denoted by $\overline{H}_{2,\epsilon}$, in some sense the completion of $H_{2,\epsilon}$[30]. Proposition 6.3.12 can be used for $U_q(sl_n^\epsilon)$ as well, but one obtains a Weyl element associated with the longest root in sl_n, so this lemma will be of less use there.

Proposition 4.2.1. *Define the algebra automorphism T by*

$$T(H^+) = -\epsilon H^-, T(\epsilon H^-) = -H^+,\ T(X^\pm) = -q^{\pm 1}X^\mp,$$

and let v^{-1} be the inverse of the Ribbon element. Then these data together with the quasitriangular structure $\mathcal{R}$ on $H_{2,\epsilon}$ define the Hopf algebra $\overline{H_{2,\epsilon}}$, which is generated by

$H_{2,\epsilon}$ and w, obeying the relations

$$wgw^{-1} = T(g),\ w^2 = \nu, \Delta(w) = \mathcal{R}^{-1}w \otimes w, \varepsilon(w) = 1, S(w) = we^{\frac{h}{2}(\epsilon H^- + H^+)}.$$

Here $g \in H_{2,\epsilon}$, and ε is the counit.

Before proving the proposition, let us note that if we take $\epsilon = 1$, and we identify H^+ with H^-, taking the sl_2 limit, the antipode agrees with [30], remembering that we have introduced a factor of 2 in our conventions. The conventions here agree directly with [6], however they use an asymmetric comultiplication. The expressions for T can be checked in representations by explicitly taking the square root in representations of ν^{-1}.

Proof. We check the conditions for proposition 6.3.12 in [23].
We have $T^2 = id$ and $T \otimes T(\mathcal{R}) = \mathcal{R}_{21}$, which is a consequence of the expression for $\mathcal{R}$ given before, and can be checked by explicit calculation. This means that ν^{-1} has to obey

$$\nu^{-1} \text{ is central, } \Delta(\nu^{-1}) = ((\nu^{-1} \otimes \nu^{-1})\mathcal{R}_{21}\mathcal{R}),\ T(\nu^{-1}) = \nu^{-1}.$$

The first two conditions are satisfied by definition of the ribbon element. See prop. 2.1.8 in [23]. As noted before, $\nu^{-1} = p^{-2}u^{-1}$. Also it is useful to observe $T \circ S = S^{-1} \circ T$. Taking the automorphism T of this expression, we get $T(p^{-2})T(u^{-1}) = p^2 T(\mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)})) = p^2 \mathcal{R}^{(1)} S^{-2}(\mathcal{R}^2) = p^2 v^{-1} = \nu^{-1}$.
We have to prove that T defines an algebra map and an anti-coalgebra map, and we have to prove that $\mathcal{R}$ is a 2-cocycle. The last condition follows by definition of a quasitriangular structure, as usual. The fact that T is an algebra automorphism is checked by checking the algebra relations, as is the case with a anti-coalgebra map. This is a straightforward exercise and is left to the reader. This shows that our map T and the ribbon element obey the relations of proposition 6.3.12 in [23]. As a result we obtain a Hopf algebra $\overline{H}_{2,\epsilon}$ which is generated by $H_{2,\epsilon}$ and w^{-1}, which obeys the following relations

$$wgw^{-1} = T(g),\ w^2 = \nu, \Delta(w) = \mathcal{R}^{-1}w \otimes w, \varepsilon(w) = 1, S(w) = wuS(w^{-2})$$

We are left with the calculation of the antipode of the Weyl element.
For the calculation of the element $p^2 = u^{-1}v$ we used the Mathematica implementation, for which we refer to the appendix. In the program we used the cop-convention of the double construction instead of the op. Note that the antipode S on $U_q(b^+)$ provides an isomorhpism between the two double constructions. Under this isomorphism, the R-matrix $\mathcal{R}_{12}$ is taken to $\mathcal{R}_{12}^{-1}$, see exercise 7.1.2 in [23]. A simple calculation using proposition 2.1.8 in [23] shows that p is invariant under this isomorphism.
Using similar arguments together with the general expression $S(w) = wuS(w^{-2})$ from proposition 6.3.12 in [23], we can show that in general the antipode of the

Weyl group element is given by $S(w) = wp^{-1}$. Observe that the w we use here is the inverse of the w defined in [23], by definition. This ends the proof of the proposition. □

The T defined here for sl_2 is not a braid group generator. However, this is not important in the sl_2 case since there is only one simple root. It is possible to make T a braid group generator in the sl_n case by performing a simple transformation on w. For this construction we will have to do more work.
Let us now calculate the action of the inverse ribbon element on a basis-vector e_m^j of a module V^j. We leave the representation-map π^j out of the notation.

$$\begin{aligned}\nu^{-1}(e_m^j) &= u^{-1}p^{-1}(e_m^j) = \sum \mathcal{R}^{(2)}S^2(\mathcal{R}^{(1)})e^{-\frac{1}{2}(H^+ + \epsilon H^-)}(e_m^j) \\ &= exp(hH^-H^+/2)\cdot \\ &\sum R_n(h)(e^{-\epsilon hH^-/2}X^-)^n((-1)^2e^{2h\epsilon}e^{hH^+/2}X^+)^n e^{-\frac{1}{2}(H^+ + \epsilon H^-)}(e_m^j)\end{aligned} \tag{4.31}$$

Observe that since X^+ acts as a raising operator, only the terms with $0 \leq n \leq j-m$ act nonzero on e_m^j. Hence we obtain for such an n-term in the R-matrix, which we will sum over afterwards,

$$\begin{aligned}&= exp(hH^-H^+/2)R_n(h)(e^{-\epsilon hH^-/2}X^-)^n((-1)^2e^2he^{hH^+/2}X^+)^n e^{\epsilon(2m)}(e_m^j) \quad (4.32)\\ &= exp(hH^-H^+/2)R_n(h)(e^{-\epsilon hH^-/2}X^-)^n(-1)^{2n}e^{2\epsilon hn}(e^{h\epsilon(m+1+\cdots+m+n)})e^{\epsilon(2m)}(e_{m+n}^j) \\ &= exp(hH^-H^+/2)R_n(h)(-1)^{2n}e^{2\epsilon hn} \\ &e^{h\epsilon\frac{1}{2}((m+n)(m+n+1)-m(m+1))}(e^{\epsilon hH^-/2}X^-)^n e^{-\epsilon(2m)}e_{m+n}^j \\ &= exp(hH^-H^+/2)R_n(h)(-1)^{2n}e^{2\epsilon hn}\cdot \\ &e^{h\epsilon\frac{1}{2}((m+n)(m+n+1)-m(m+1))}e^{h\epsilon\frac{1}{2}(m(m-1)-(m+n)(m+n-1))}e^{\epsilon(2m)}e_m^j\end{aligned}$$

Remember that the module V^j is generated by the highest weight vector e_j^j. Since the inverse of of the ribbon element is central (see [23]), it is enough to check equality on the highest weight vector. Then only the $n = 0$ term contributes, and we get the following identity

$$\nu^{-1}e_m^j = e^{h\epsilon(2j(j+1))}e_m^j. \tag{4.33}$$

By Schur's lemma the action of ν^{-1} is proportional to the identity. Note in particular the contribution from p^{-1}. The following expression is a square root of ν^{-1}, and as it turns out the only one that meets the requirements of proposition 4.2.1. It can be proven by direct calculation that it satisfies $wgw^{-1} = T(g)$ and $w^2 = \nu$. We get the required result:

$$w^{-1}e_m^j = (-1)^{-j+m}e^{h\epsilon(j(j+1)-m)}e_{-m}^j.$$

We have the following lemma.

Lemma 4.2.1. *Let w^{-1} be as in proposition 4.2.1. Then the action of w^{-1} in the highest weight module V^j is given by*

$$w^{-1}e^j_m = (-1)^{-j+m}e^{h\epsilon(j(j+1)-m)}e^j_{-m}. \tag{4.34}$$

Proof. By definition, w is the non-central square root of the inverse ribbon element ν^{-1}. To calculate the action of w in the representation we have to know which square-root we have to use, given that we are working with matrices, so there are multiple options, a priori. We claim that the square root in the representation is uniquely determined by two equations:

$$wgw^{-1} = T(g) \text{ and } w^2 = \nu.$$

This is proved by looking at the action of ν in the representation, and is translated into the following lemma, which will not be proven here, but can be proved by looking at the Jordan decomposition of $w(j)$, or by counting the degrees of freedom, alternatively.

Lemma 4.2.2. *Suppose w(j) is a 2j+1 by 2j+1 invertible matrix, and let $xp(j) = \pi(X^+)$ and $xm(j) = \pi(X^-)$, the action of $X^\pm$ in the representation V^j. Let λ be any invertible element of the underlying ring. Then w(j) is uniquely defined by the following two equations*

$$\begin{aligned} w(j)^2 &= \lambda Id(j) \\ w(j)xp(j)w(j)^{-1} &= xm(j). \end{aligned} \tag{4.35}$$

Here Id(j) is the 2j+1 times 2j+1 identity matrix.

Proof. □

It is clear that the scuare root w given above satisfies

$$wgw^{-1} = T(g) \text{ and } w^2 = \nu.$$

This finishes the proof. □

We know the explicit action of w in any finite dimensional $H_{2,\epsilon}$ module, and we can compare it with other definitions. One can check that the given square root of the inverse ribbon element yields the correct T, as stated in proposition 4.2.1. Usually, w^{-1} is defined by its action on all finite representations V^j of $H_{2,\epsilon}$, for example in [29] and others. This is possible if $H_{2,\epsilon}$ is semisimple as an algebra. An algebra is said to be semisimple if the set of elements that act as zero in every irreducible representation contains only zero. We know that $H_{2,\epsilon}$ is not semisimple however, since $\epsilon^{-1}H^+ - H^-$ acts as zero in every representation. This element exactly generates the ideal we need to divide out to, in order to get the $U_q(sl_2)$, which is a semisimple algebra. The non-semisimplicity implies that w is well defined up to terms $H^-, \epsilon^{-1}H^+$, which have the same action in any representation,

if we would define w by its action in the representation. Of course, this is a consequence of the fact that $\epsilon^{-1}H^+ - H^-$ is central in $H_{2,\epsilon}$. This element will turn out to be the term we gain in our final expression, with respect to the sl_n case.
The w defined here agrees, after the 'semisimplification' of $H_{2,\epsilon}$, with the quantum Weyl element in [30], and is the inverse of the w defined in example 6.3.13 in [23], as mentioned before. This can be seen from the action of w in the representation. Note that $U_q(sl_2)$ in [23] agrees with the conventions of [30].

4.3. Constructing the q-Weyl group of $U_q(sl_n^\epsilon)$

We proceed with constructing quantum Weyl group of $U_q(sl_n^\epsilon)$, with Cartan matrix a_{ij} of sl_n. We follow [30]. Note that the relations here agree with [21] and [30], when we divide out to $\epsilon^{-1}H_i^+ - H_i^-$. Before reading this section it is advised to study [30] in full detail, since we copy a large part of his calculations.
This algebra is non semi-simple. The Weyl group elements used in the main part of this section come from the $H_{2,\epsilon}$ case, where we defined it to be the square root of ν^{-1}. Later in this section we introduce $\overline{w}_i$, where w_i and $\overline{w}_i$ are related by a simple transformation with $q^{-\epsilon^{-1}H^+H^-}$, like the $U_q(sl_n)$ case [30]. It turns out to be the case that the algebra automorphisms obtained this way will yield a braid group representation.
$U_q(sl_n^\epsilon)$ is generated by $\{X_i^\pm, H_i^\pm\}$ and the relations

$$[X_i^-, H_j^-] = a_{ij}X_i^-,\ [X_i^+, H_j^+] = -a_{ij}\epsilon X_i^+,\ [X_i^-, H_j^+] = a_{ij}\epsilon X_i^-,\ [X_i^+, H_j^-] = -a_{ij}X_i^+ \tag{4.36}$$

$$[X_i^-, X_j^+] = \frac{q^{H_i^-} - q^{-\epsilon^{-1}H_i^+}}{q - q^{-1}}\delta_{ij},\quad \sum_{k=0}^{k=1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^2} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-a_{ij}-k} = 0$$

$$\Delta(X_i^-) = X_i^- \otimes e^{\epsilon h H_i^-/2} + e^{-\epsilon h H_i^-/2} \otimes X_i^-, \Delta(X_i^+) = X_i^+ \otimes e^{hH_i^+/2} + e^{-hH_i^+/2} \otimes X_i^+$$

$$\Delta(H_i^\pm) = H_i^\pm \otimes 1 + 1 \otimes H_i^\pm, S(X_i^+) = -e^h X_i^+, S(X_i^-) = -e^{-\epsilon h}X_i^-, S(H_i) = -H_i.$$

Where $q = e^{-\epsilon h}$. Using proposition 4.2.1, we will write down the action of the Weyl group in $U_q(sl_n^\epsilon)$ for each of the $H_{2,\epsilon}$ subalgebras. The quantum Weyl elements are defined via the $H_{2,\epsilon}$ submodules of the representations of $U_q(sl_n^\epsilon)$, for each simple root α_i, $i = 1, \cdots, rank(sl_n) = n-1$ of sl_n.
$U_q(sl_n^\epsilon)$ is a quasitriangular Hopf algebra, as has been noted in the first section of this chapter. In this section we denote the R-matrix as $\mathcal{R}$, and its inverse as $\mathcal{R}^{-1}$. The notation $\overline{\mathcal{R}}$ is reserved for other purposes. The same notation will be used for the partial R-matrices of $U_q(sl_2^\epsilon)$.

In the notation of [30], let V^λ be a representation of $U_q(sl_n^\epsilon)$. We know that V^λ are highest weight representations [6]. $U_q(sl_n^\epsilon)$ is generated by $n-1$ copies of the subalgebras $H_{2,\epsilon,i}$ corresponding to the simple roots. In each of these copies we can find a corresponding Weyl element w_i with proposition 4.2.1. We know that V^λ factorizes into irreducable $H_{2,\epsilon}$-submodules V^j. Checking this fact comes down to checking relations on simple generators. This is equivalent to lemma 2 in [30], which proves that if V is an $H_{2,\epsilon,i}$ module $X_j^\pm(V)$ is still an $H_{2,\epsilon,i}$ submodule. In general we have, for V^j irreducible $H_{2,\epsilon,i}$-submodules for the usual half integer values for j,

$$V^\lambda = \bigoplus_j (Hom(V^j, V^\lambda) \otimes V^j) = \bigoplus_j (W_j^\lambda \otimes V^j).$$

We define $W_j^\lambda = Hom_k(V^j, V^\lambda)$, with $k = \mathbb{R}(\epsilon)$. The isomorphism

$$f : \bigoplus_j (Hom(V^j, V^\lambda) \otimes V^j) \to V^\lambda$$

is given by $(\cdots, 0, \phi_j \otimes e_m^j, 0, \cdots) \mapsto \phi_j(e_m^j)$, evaluation. Since the $H_{2,\epsilon,i}$ modules V^j are irreducible submodules of V^λ, the homomorphisms $\phi_j \in Hom(V^j, V^\lambda)$ are, by Schurs lemma, the identity on V^j, scaling the constant to 1 without loss of generality.
Since $U_q(sl_n^\epsilon)$ is generated by the $H_{2,\epsilon,i}$ subalgebras corresponding to simple roots, we can assume that $Im(\phi_i)$ is a copy of V^i in V^λ as $H_{2,\epsilon,i}$-submodule, where $i = 1, \cdots n-1$. The action of $H_i^\pm$ on an element $\phi \otimes e_m^j$ is then given by $H_i^\pm \circ \phi \otimes e_m^j = H_i^\pm(\phi(e_m^j)) = -a_{ij}(\epsilon)^{(1\pm 1)/2} m(\phi_j(e_m^j))$, since $[H_i^\pm, X_j^\pm] = \mp(\epsilon)^{(1\pm 1)/2} a_{ij} X_j^\pm$, so the submodules $V^j \subset V^\lambda$ are invariant under the action of $H_i^\pm$. In general, the submodules V^j are not invariant under the action of $X_j^\pm$. This action is more complicated, and although it yields another $H_{2,\epsilon}$-submodule, it may not be the same submodule.
Define the elements w_i acting on $U_q(sl_n^\epsilon)$ via representations by

$$w_i = \bigoplus_j (I_{w_j^\lambda} \otimes (w_i)_j),$$

where $I_{w_j^\lambda}$ is the identity on W_j^λ. The $(w_i)_j$ are then defined via proposition 4.2.1, where $(w_i)_j$ acts on the $H_{2,\epsilon,i}$ submodule V^j by the action calculated in lemma 4.2.1. Note that in the semisimple case, like in [30], this definition would uniquely define w_i. In the non-semisimple case we need to require that the Weyl property holds for conjugation with w_i. Then the w_i become well-defined on the Cartan subalgebra.
Let us calculate $w_i H_j^+ w_i^{-1}$ from the definition of w_i, by comparing the action on irreducable modules. By the previous discussion we can check this for the factorization of V^λ into $H_{2,\epsilon,i}$-submodules, the i corresponding to the simple roots. We denote the vectors as e_m^n, omitting the ϕ. In this case, let $e_m^j \in V^n$ be any vector in any irreducible $H_{2,\epsilon,j}$-submodule $V^n \subset V^\lambda$ of highest weight n. Let $i \neq j$ (in the

case $i = j$ we get the action from proposition 4.2.1), then H_j acts only nonzero if $i = j \pm 1$.

$$w_i H_j^+ w_i^{-1}(e_m^n) = w_i H_j^+(e_{-m}^n) = a_{ij}\epsilon m w_i(e_{-m}^n) = a_{ij}\epsilon m e_m^n = -\epsilon m e_m^n.$$

Note that $a_{ij} = -1$. On the other hand we have

$$(\epsilon)^{-(-1\mp1)/2}(H_j^\pm - a_{ij}H_i^\pm)(e_m^n) = (-a_{ji} + a_{ij}a_{ii})\epsilon m e_m^n = -m\epsilon e_m^n.$$

From this we can conclude that $w_i H_j^+ w_i^{-1}$ acts as $(\epsilon)^{-(-1\mp1)/2}(H_j^\pm - a_{ij}H_i^\pm)$ in $H_{2,\epsilon,i}$-modules. From proposition 4.2.1 we obtain the $T_j(H_j)$ relations. Since the algebra is not semisimple, we can always add a term $H_i := \epsilon^{-1}H_i^+ - H_i^-$ (and idem for H_j) and get the same action in representations.
However, if we take the $T_j(H_j)$ relations together with the requirement that $T_iT_jT_i = T_jT_iT_j$ we get the following relations. The proof is by straightforwardly checking the Weyl-relation. We leave this to the reader. The requirement that the Weyl-property holds could be seen as a definition of the action of w_i on H_j, since it uniquely determines this action.

Lemma 4.3.1.

$$T_i(H_j^+) = \epsilon H_j^- - \epsilon a_{ij}H_i^-, T_i(\epsilon H_j^-) = H_j^+ - a_{ij}H_i^+,\ T_i(X_i^\pm) = -q^{\pm1}X_i^\mp. \quad (4.37)$$

For sl_n, $a_{ij} = 2$ if $i = j$, $a_{ij} = -1$ if $i = j \pm 1$ and zero else. By proposition 4.2.1 we now have

$$T_i(e) = w_i e w_i^{-1}, \Delta(w_i) = \mathcal{R}(i)^{-1} w_i \otimes w_i, \quad (4.38)$$

where $\mathcal{R}(i)$ is the partial R-matrix on the i-th $H_{2,\epsilon}$-subalgebra defined by 4.28. Remember that the adjoint action of $U_q(sl_n^\epsilon)$ on itself is given by

$$ad_e(f) = \sum e_{(1)} f S(e_{(2)}).$$

We denote the adjoint action for short as $\circ$, in multiplicative notation. Using this definition, we calculate $w_i X_j^\pm w_i^{-1}$.

Let us define two sets of generators that make the comultiplication anti-symmetric, and correspond to the two ways to write the R-matrix in the $H_{2,\epsilon}$ case. Note that our definitions agree with the definitions of [30].

$$E_i = q^{\epsilon^{-1}H_i^+/2}X_i^+, F_i = q^{-H_i^-/2}X_i^-, K_i^+ = q^{\epsilon^{-1}H_i^+/2}, K_i^- = q^{H_i^-/2},$$
$$\overline{E}_i = q^{-\epsilon^{-1}H_i^+/2}X_i^+, \overline{F}_i = q^{H_i^-/2}X_i^-.$$

Via the adjoint action we have an action of the q-Weyl element on these generators. We now have the following lemmas, the proof of which are equivalent to the proofs given in [30], except for a factor of two in the definitions of $H^\pm$ resp. H in [30], after reducing the Cartan subalgebra. The reason we cannot directly follow Reshetikhin and Kirrilov's proof is that we have to introduce the +/- back into the equations, so we have to check all the relations manually. It is insightful to study this proof, and the proof of proposition 2.2.1 in chapter 4 of [21], which are roughly the same. The proof of lemma 4.3.2 is by explicit calculation of the adjoint action of the Weyl group element.

Lemma 4.3.2.

$$\begin{aligned} w_i \circ \overline{E}_j &= w_i \overline{E}_j (K_i^+)^{a_{ij}} w_i^{-1} \\ w_i \circ F_j &= S(w_i^{-1})(K_i^-)^{-a_{ij}} F_j S(w_i). \end{aligned} \tag{4.39}$$

Proof. We will only scetch the proof here, since this proof is exactly the same as the proof of lemma 1 in [30], where we are keeping track of $H^\pm$ and the different factors 2. Note that we start with the expression 4.29 and let $\mathcal{R}^{-1} = S \otimes id(\mathcal{R})$. Then we use $\mathcal{R} = S \otimes S(\mathcal{R})$ to rewrite the adjoint action of w_i on $\overline{E}_j$ and F_j. After using the commutation relations, the final result is obtained by using the fact that $u^{-1} = \sum \mathcal{R}_2 S^2(\mathcal{R}_1)$, and $uhu^{-1} = S^2(h)$ for any $h \in H-2,\epsilon$, so that u commutes with $K_i^\pm$. Note that $w^{-1} = (up)^{-1/2} = (up)^{-1}w = u^{-1}S(w)$. This ends the proof. □

Lemma 4.3.3. *The sets $V_{ij} = \{\overline{E}_j, \cdots, \overline{E}_i^{-a_{ij}} \circ \overline{E}_j\}$ and $\overline{V}_{ij} = \{F_j, \cdots, F_i^{-a_{ij}} \circ F_j\}$ are irreducible $H_{2,\epsilon,i}$-modules of weight $-a_{ij}$.*

Proof. Lemma 4.3.3 can be concluded directly from the algebra relations. Since we used the same conventions as [30], the relations are exactly the same. An explicit isomorphism between both sets and $V^{a_{ij}}$ is given by the maps

$$\begin{aligned} \phi(F_i^n \circ F_j) &= c_{ij}^- \sqrt{\frac{[n]_q!}{[-a_{ij}-n]_q!}} e^{\frac{-a_{ij}}{2}}_{\frac{-a_{ij}}{2}-n} \\ \psi(\overline{E}_i^n \circ \overline{E}_j) &= c_{ij}^+ \sqrt{\frac{[-a_{ij}-n]_q!}{[n]_q!}} e^{\frac{-a_{ij}}{2}}_{\frac{a_{ij}}{2}+n}. \end{aligned} \tag{4.40}$$

Note that because $\epsilon^{-1}H^+ - H^-$ acts as zero in V^j, we can make multiple choices for V_{ij} and $\overline{V}_{ij}$ that are isomorphic to V^j. We will parametrize these choices by the two parameters $c_{ij}^\pm$ in the future, where $c_{ij}^\pm$ stands for a central factor. □

Lemma 4.3.4.

$$\begin{aligned} \overline{E}_i^n \circ \overline{E}_j &= (K_i^+)^{-n}(K_j^+)^{-1}[X_i^+, \cdots [X_i^+, X_j^+]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{(a_{ij}+2n)/2-1}} \\ S(F_i^n \circ F_j) &= -q^{(-n-1)}[X_i^-, \cdots, [X_i^-, X_j^-]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{(a_{ij}+2n)/2-1}}(K_i^-)^n(K_j^-) \end{aligned} \tag{4.41}$$

Proof. The relations follow by induction to n from the algebra relations. We write down the case for $n = 1$ here by using the definition of the adjoint action.

$$\begin{aligned}\overline{E}_i \circ \overline{E}_j &= \overline{E}_i\overline{E}_j - e^{-H_i^+}\overline{E}_j e^{H_i^+}\overline{E}_i \\ &= \overline{E}_i\overline{E}_j - q\overline{E}_j\overline{E}_i = q^{1/2}(K_i^+)^{-1}(K_j^+)^{-1}q^{-1/2}(q^{-1/2}X_i^+X_j^+ - q^{1/2}X_j^+X_i^+) \\ &= (K_i^+)^{-1}(K_j^+)^{-1}[X_i^+, X_j^+]_{q^{-1/2}}.\end{aligned} \tag{4.42}$$

The case for higher n follows in the same fashion. The second formula follows in the same way, except we take the antipode on both sides afterwards. This proves the lemma. □

We can use the lemmas to calculate the algebra automorphisms associated with the quantum Weyl group explicitly. To this end let us introduce

$$\overline{w}_i = w_i q^{-\epsilon^{-1}H_i^+H_i^-/4}.$$

Lemma 4.3.5.

$$\Delta(\overline{w}_i) = \overline{\mathcal{R}}(i)^{-1}\overline{w}_i \otimes \overline{w}_i, \tag{4.43}$$

$$\overline{\mathcal{R}}(i) = q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum R_n(h)(e^{-hH_i^+/2}X_i^+)^n \otimes (e^{\epsilon hH_i^-/2}X_i^-)^n, \tag{4.44}$$

$$R_n(h) = \frac{q^{\frac{1}{2}n(n+1)}(1-q^{-2})^n}{[n]_q!}. \tag{4.45}$$

Proof. This can be calculated by straightforward computation from proposition 4.2.1, where it is important to remember that $q^{-\epsilon^{-1}H^+H^-/4}$ is not group-like, but that a correction appears when taking the coproduct. □

Note from the previous section that the partial R-matrix

$$\overline{\mathcal{R}}(i) = q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum \frac{(1-q^{-2})^n}{[n]_q!}E_i^n \otimes F_i^n.$$

Since the E, F have an antisymmetric coproduct, it is this form in which we will later recognize the algebra we are using, only with X, Y and Z instead of E_i and lowercase letters instead of F_i. Denote algebra automorphisms of $U_q(sl_n^\epsilon)$ as

$$T_i(h) = \overline{w}_i^{-1}h\overline{w}_i, \forall h \in U_q(sl_n^\epsilon). \tag{4.46}$$

We have the following formula, which is a direct consequence of the above lemma. (Denote $\mathcal{R}(i) = \mathcal{R}_i$).

$$\Delta(T_i(X_j^\pm)) = \overline{\mathcal{R}}_i T_i \otimes T_i(\Delta(X_j^\pm))\overline{\mathcal{R}}_i^{-1}.$$

The following theorem holds true and is proven by combining the above lemmas, together with the action of w_i in the irreducible modules. Note that the T_i are denoted in the opposite way as T in proposition 4.2.1. Observe that in the $U_q(sl_n)$ case, T_i are the quantization of the classical Weyl group action [6] in the sense that the action of T_i corresponds to the action of the simple reflections s_i corresponding to the simple roots α_i. Looking at the first order in h we obtain the same fact. One can compare the T_i with the action of the simple reflections given in the first section and convince oneself of this fact.

Theorem 4.3.1. *The T_i as defined above are well-defined up to central factors c_{ij} depending on $K^{\pm}_{i,j}$. Moreover, the T_i are $U_q(sl_n^\epsilon)$ algebra automorphisms, given by*

$$\begin{aligned}
&T_i(K_j^+) = K_j^-(K_i^-)^{-a_{ij}}, T_i(K_j^-) = K_j^+(K_i^+)^{-a_{ij}}, \\
&T_i(X_i^+) = -X_i^-(K_i^-)^{-1}(K_i^+)^{-1}, T_i(X_i^-) = -(K_i^-)(K_i^+)X_i^+, \\
&T_i(X_j^+) = c_{ij}(-1)^{a_{ij}}[-a_{ij}]_q!((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}}, \\
&\qquad [X_i^+, \cdots, [X_i^+, X_j^+]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{-a_{ij}/2-1}}((K_i^-)(K_i^+)^{-1})^{a_{ij}/2}, i \neq j \\
&T_i(X_j^-) = c_{ij}^{-1}\frac{1}{[-a_{ij}]_q!}((K_j^+)^{-1}(K_j^-))^{-1}((K_i^-)(K_i^+)^{-1})^{a_{ij}}, \\
&\qquad [X_i^-, \cdots, [X_i^-, X_j^-]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{-a_{ij}/2-1}}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}, i \neq j
\end{aligned} \tag{4.47}$$

and when $a_{ij} = 0$, $T_i(X_j^{\pm}) = X_j^{\pm}$.

Proof. The main objective is to prove that conjugation with the Weyl element respects the algebra structure. We will first prove that the T_i are of the form given above. This follows directly from lemmas 4.3.3 and 4.3.2 together with the action of w_i. We can then rewrite $\overline{E}_i^{-a_{ij}} \circ \overline{E}_j$ with lemma 4.3.4, taking $n = -a_{ij}$, since w_i takes the lowest weight vector $\overline{E}_j$ in the module homomorphic to $V^{a_{ij}}$ (and the same for F_j) to the highest weight vector, which is $\overline{E}_i^{-a_{ij}} \circ \overline{E}_j$, by the action of w_i in irreducible modules. This gives the desired relations for T_i applied on $X^{\pm}$ up to a central factor depending on i and j, due to the non-semisimplicity. Making this choice is equivalent to choosing a root of ν^{-1}. This shows that the T_i are not well defined, when defined from the quantum Weyl group, up to a central factor. Once we choose a root of ν^{-1}, we make a choice for c_{ij}. Concretely, this choice corresponds to choosing isomorphisms ϕ and ψ in lemma 4.3.3.
Let us get some specific values for c_{ij}. Firstly, the relations $T_i(K_j^{\pm}) = (K_j^{\mp})(K_i^{\mp})^{-a_{ij}}$ and $T_i(X_i^{\pm})$ follow from proposition 4.2.1 and lemma 4.3.1. Consider $T_i([X_j^{\pm}, X_j^{\mp}])$. Since we want T_i to be an algebra homomorphism, we get the requirement $c_{ij}^+ c_{ij}^- = 1$ for all i and j, since c_{ij} is central (it consists of powers of $\epsilon^{-1}H_i^+ - H_i^-$). In particular, c_{ij} must be invertible.

Hence we have the following relations.

$$\begin{aligned}
&T_i(K_j^+) = K_j^-(K_i^-)^{-a_{ij}}, T_i(K_j^-) = K_j^+(K_i^+)^{-a_{ij}}, \qquad (4.48)\\
&T_i(X_i^+) = -X_i^-(K_i^-)^{-1}(K_i^+)^{-1}, T_i(X_i^-) = -(K_i^-)(K_i^+)X_i^+\\
&T_i(X_j^+) = c_{ij}(-1)^{a_{ij}}[-a_{ij}]_q!((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}}\\
&\qquad [X_i^+,\cdots,[X_i^+,X_j^+]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}((K_i^-)(K_i^+)^{-1})^{a_{ij}/2}, i \neq j\\
&T_i(X_j^-) = \frac{1}{[-a_{ij}]_q!}c_{ij}^{-1}((K_j^+)^{-1}(K_j^-))^{-1}((K_i^-)(K_i^+)^{-1})^{a_{ij}}\\
&\qquad [X_i^-,\cdots,[X_i^-,X_j^-]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}, i \neq j.
\end{aligned}$$

Where we will write simply c_{ij} for c_{ij}^+. Comparing T_i with [30] we can conclude that the T_i must be algebra-automorphisms, since $(K_i^-)(K_i^+)^{-1}$ is a central element for all $i = 1, \cdots n-1$. The proof that T_i are automorphisms coincides then with the proof in [30]. This concludes the proof. □

From the lemmas in this section together with proposition 4.2.1 we obtain the following fact. It can be proven by direct verification, when we remember that $a_{ij} = -1$ for $i \neq j$, since we are working with the sl_n Cartan matrix. To prove it for a general Cartan matrix, it is enough to consider rank 1 and 2 cases [6], [30].

Theorem 4.3.2. *The c_{ij} in theorem 4.3.1 are group-like elements in H_i and H_j, with $H_i = \epsilon^{-1}H_i^+ - H_i^-$, $i = 1, \cdots n-1$.*

Proof. We know that c_{ij} are invertible from the proof of the last theorem. Note that for any invertible element c, we have $\Delta(c^{-1}) = \Delta(c)^{-1}$. Also, c_{ij} are central elements, and live in the Cartan subalgebra of $U_q(sl_n^\epsilon)$. As a consequence, c_{ij} must be a power-series in the elements $\epsilon^{-1}H_i^+ - H_i^- = H_i$, since these are the only central elements that are contained in the Cartan subalgebra. This follows from the relations $[(H_i^+)^k, X_j^\pm] = (H_i^+ \pm a_{ij})^k X_j^\pm - (H_i^\pm)^k X_j^\pm \neq X_j^\pm$.
We see that $\Delta(c_{ij})$ must also be central, since $\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i$. As a consequence of this, $\Delta^{op}(c_{ij}) = \mathcal{R}^{-1}\Delta(c_{ij})\mathcal{R} = \Delta(c_{ij})$. So if we can prove that $\Delta(c_{ij}) = (c_{ij} \otimes c_{ij})\Psi$ for some invertible 2-cocycle Ψ, then we know that Ψ is also a central element, with $\Psi_{21} = \Psi$ that lives in the tensor product of the Cartan-subalgebra with itself, since the tensor algebra of the Cartan part of $U_q(sl_n^\epsilon)$ is closed under taking the coproduct. That Ψ is a 2-cocycle means that $1 \otimes \Psi(id \otimes \Delta)(\Psi) = \Psi \otimes 1(\Delta \otimes id)(\Psi)$.
Since the c_{ij} are invertible, let us write $\Delta(c_{ij}) = (c_{ij} \otimes c_{ij})\Psi$. Since c_{ij} are central, so is Ψ. Let us write $\Psi = c \otimes c\Delta(c^{-1})$, where $c = c_{ij}^{-1}$ depends on H_i for some i without loss of generality. The fact that Ψ is central and symmetric and only dependent on elements of the Cartan subalgebra follows from the properties of

c_{ij}. We prove now that Ψ is a 2-cocycle.

$$\begin{aligned}1 \otimes \Psi(id \otimes \Delta)(\Psi) &= 1 \otimes c \otimes c(1 \otimes \Delta(c^{-1}))c \otimes \Delta(c) id \otimes \Delta(\Delta(c^{-1})) \\ &= 1 \otimes c \otimes c(c \otimes \Delta(cc^{-1})) id \otimes \Delta(\Delta(c^{-1})) \\ &= c \otimes c \otimes c(\Delta \otimes id)(\Delta(c^{-1})) \\ &= \Psi \otimes 1(\Delta \otimes id)(\Psi).\end{aligned}$$

The second-last equality follows from coassociativity and homomorphism property of Δ. The last equality follows from the definition of Ψ and the fact that Δ is a homomorphism.
Since Ψ is central it is a power series $\sum_{l_i,k_i \in \mathbb{N}} \prod_{i=1\cdots n-1} f_{\mathbf{k},\mathbf{l}} H_i^{k_i} \otimes H_i^{l_i}$, for central elements H_i and vectors $\mathbf{k} = (k_1, \cdots, k_{n-1})$ and $\mathbf{l} = (l_1, \cdots, l_{n-1})$, where $n-1$ indicates the rank of a_{ij}. However, the case where Ψ is dependend on multiple H_i can be reduced to the case of one variable.
The reason is that Ψ factorizes into terms corresponding to the simple roots. This fact follows from requiring that Ψ is a 2-cocycle and the fact that H_i^n are linearly independend for different i and n. This yields $f_{\mathbf{k},\mathbf{l}} = \prod_i f_{(0,\cdots,k_i,0,\cdots),(0,\cdots,0,l_i,0,\cdots,0)}$, by comparing different terms and using a straightforward induction argument. Conceptually, the general argument is equivalent to the case for two variables H_1 and H_2. This is left to the reader.
Let us note that we may take $f_{0,0} = 1$ without loss of generality, as the invertibility of c_{ij} implies that Ψ is invertible. This is the case if and only if $f_0 \neq 0$. So it is enough to look at the case where $\Psi = \sum_n f_{n_i} H_i^{n_i} \otimes H_i^{n_i}$.
Furthermore, we know that Ψ is symmetric, in the sense that when the tensor factors of Ψ are interchanged, we obtain Ψ. This forces

$$\Psi = \sum_{l_i,k_i \in \mathbb{N}} \prod_{i=1\cdots n-1} f_{\mathbf{k}+\mathbf{l},\mathbf{l}} (H_i^{k_i} \otimes 1 + 1 \otimes H_i^{k_i})(H_i^{l_i} \otimes H_i^{l_i}).$$

We claim that $\mathbf{k} = 0$ in the above expression. To see this it is enough to consider the case where Ψ only depends on one central variable, say H_i, by the previous discussion. By plugging in $\Psi = \sum_{l,k \in \mathbb{N}} f_{k+l,l}(H_i^k \otimes 1 + 1 \otimes H_i^k)(H_i^l \otimes H_i^l)$ into the cocycle condition (putting $H_i = H$ for simplicity) we get

$$\begin{aligned}&\sum_{i,j,k,l \geq 0} f_{i+k,k} f_{j+l,l} (H^{i+k} \otimes H^k \otimes H^l + H^k \otimes H^{i+k} \otimes H^l) \\ &(\Delta(H)^{j+l} \otimes 1 + \Delta(H)^l \otimes H^j) \\ &= \sum_{i,j,k,l \geq 0} f_{i+k,k} f_{j+l,l} (H^l \otimes H^{i+k} \otimes H^k + H^l \otimes H^k \otimes H^{i+k}) \\ &(1 \otimes \Delta(H)^{j+l} + H^j \otimes \Delta(H)^l).\end{aligned}$$

Let us look at the prefactor of the term $H \otimes 1 \otimes 1$. Since we assumed $f_{0,0} = 1$, we get $2f_{1,0} = 3f_{1,0}$, so $f_{1,0} = 0$. We use here that $\Delta(H) = H \otimes 1 + 1 \otimes$

H. In general, we get as a prefactor of $H^m \otimes 1 \otimes 1$ the equation $2f_{n,0} = 3f_{n,0} + f_{n-1,0}f_{1,0} + f_{n-2,0}f_{2,0} + \cdots + f_{1,0}f_{n-1,0}$. Assuming $f_{i,0} = 0$ for $i < n$ as induction hypothesis, we get $f_{n,0} = 0$. This proves the claim. So Ψ can only be a power series in $H_i \otimes H_i$. Hence we can write $\Psi = \sum_{n_i,n_j} f_{n_i} H_i^{n_i} \otimes H_i^{n_i}$.
Without loss of generality we assume that Ψ depends only on $H_i = H$. We wish to prove that $\Psi = 1 \otimes 1$ if Ψ is an invertible, central, symmetric 2-cocycle that is an element of the Cartan subalgebra. Then we are finished, since the only elements that have a coproduct of the form $\Delta(c) = c \otimes c$ are elements $c = e^{eH}$. See for example lemma 6.4.1 in [6].
Let us now prove that $\Psi = 1 \otimes 1$ if Ψ is an invertible, central, symmetric 2-cocycle. We will do this by explicitly checking the 2-cocycle condition for $\Psi = \sum_n f_n H^n \otimes H^n$. From inserting the power series for Ψ into the 2-cocycle condition we get, with a straightforward substitution of summetion variables,

$$\sum_{n,k,p=0}^{p=n} f_n f_k \binom{n}{p} H^{k+p} \otimes H^{k+n-p} \otimes H^n$$
$$= \sum_{n,k,p=0}^{p=n} f_{k+p} f_{n-p} \binom{k+p}{p} H^{k+p} \otimes H^{k+n-p} \otimes H^n.$$

We take $f_0 = 1$ without loss of generality. By linear independence of the generators we can compare term by term. We claim that this relation is satisfied only if $\Psi = \sum_n f_1 \frac{H^n \otimes H^n}{n!}$. We have to prove that the equation above holds only if $f_{n+1} = \frac{f_1^{n+1}}{n+1!}$. We observe that a given combination of exponents of the Hs appears only once in the equation above. The base case follows from the $k = 1$ and $n = 1$, $p = 1$-term: $f_1^2 = 2f_2 f_0$. Suppose that for some $l > 0$ the formula holds. Then we look at the terms with $n = l$ such that $H^{k+p} \otimes H^{k+n-p} \otimes H^n = H^{l+1} \otimes H \otimes H^l$. This implies that $k = 1$ and $p = l$. Writing down the coefficient of this term we immediately get $f_{l+1} = \frac{f_l f_1}{l+1}$, the desired result.
However, implementing $f_k = f_i^k / k!$ into the equation we see that (again comparing terms) for any pair non-negative integers n, k and $0 \leq p \leq n$, we have

$$\frac{f_1^{n+k}}{n!k!} \binom{n}{p} = \frac{f_1^{k+n}}{(k+p)!(n-p)!} \binom{k+p}{p}.$$

We quickly realize that $f_1 \neq 0$ implies $(n-p)!p! = n!$, which is something that people in highschool might wish is true, but fortunately for us, it is not. So $f_1 = 0$. But this means that c_{ij} is group-like. □

Theorem 4.3.3. ***(Weyl property)*** *Let the T_i be as in theorem 4.3.1, and let a_{ij} be the sl_n Cartan matrix. Denote $c_{ij} = c_i d_j$. Then $d_j = c_j^2$ for all $j = 1, \cdots n-1$ if and ony if for all $i, j = 1, \cdots n-1$,*

$$T_i T_j T_i = T_j T_i T_j. \tag{4.49}$$

We call this property the Weyl-property. The Weyl-property implies that we can use the T_i for defining Hopf algebra generators. In the case of sl_3 there are only two simple roots, so for a decomposition of the longest root vector into simple roots $w_0 = s_{i_1}s_{i_2}s_{i_1}$, where $i_1, i_2 = 1, 2, i_1 \neq i_2$, there is a unique way to write down the corresponding algebra generator: $X^{\pm}_{w_0} = T_{i_1}(T_{i_2}(X^{\pm}_{i_1}))$. Hence we do not need the Weyl property for sl_3^ϵ.
In general this is not the case. In that case, for each reduced decomposition of the longest classical Weyl group element, there exist multiple ways to write down the corresponding root-vector in the Hopf algebra. We need the Weyl-property in order for the construction of higher order generators to be well-defined. See for example proposition 8.1.3 in [6] and proposition 5, chapter 4, paragraph 1.5 in [3]. In general, the quantum Weyl group construction depends on a choice of the longest root decomposition. If the Weyl-property is not satisfied, then it is impossible to know which order of T_i belongs to a given decomposition, so the construction of higher order basis vectors is ill-defined. Let us prove the theorem.

Proof. First note that $T_k(T_j(X_i))$ is nonzero only if $i = j$ or $i = k$, so we only need to check one relation on the generators X_i^+ (the Weyl-relation follows for X_i^- from the X_i^+-relations, since the prefactors are inverted in this case). Secondly, if the automorphisms have no central prefactor, then the proof that they obey the Weyl property is similar to the proof in [30], by explicit verification. We leave this to the reader, since the computations are exactly the same. It is only needed to follow the additional central factors present in T_i as compared to the T_i with no central pre-factor. We include the central factor $((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}$ in c_{ij} in this proof for simplicity.
First note that c_{ij} are group-like by the previous theorem. Denote this central factor by $c_{ij} = c_i d_j$, where $c_i = e^{(\epsilon^{-1}H_i^+ - H_i^-)l}$, for some $l \in k$, since c_{ij} is group-like. Idem for d_i. We now prove

$$T_i(T_j(T_i(X_i^+))) = T_j(T_i(T_j(X_i^+))).$$

On the left hand side, no central factors are introduced: $T_i(X_i^+) = X_i^-(K_i^-)^2$. Since $i \neq j$, $a_{ij} = -1$. So $T_j(X_i^-) = c_{ji}^{-1}[X_j^- X_i^-]_{q^{-1/2}}$. Applying T_i to this expression yields a central factor of $c_{ij}^{-1}T_i(c_{ji}^{-1}) = c_i^{-1}d_j^{-1}T_i(c_{ji}^{-1}) = c_i^{-1}d_j^{-1}d_i^{-1}c_ic_j = d_j^{-1}d_i^{-1}c_j$.
On the right-hand side we get $T_j(X_i^+) = c_{ji}[X_j^+, X_i^+]_{q^{-1/2}}$. Applying T_i yields

$$T_i(c_{ji})c_{ij}[[X_i^+, X_j^+]_{q^{-1/2}}, X_i^-]_{q^{-1/2}},$$

where $T_i(c_{ji}) = T_i(c_jd_i) = d_ic_i^{-1}c_j^{-1}$. So as a central factor we have $d_ic_i^{-1}c_j^{-1}c_id_j = d_id_jc_j^{-1}$. Note that applying T_j to the commutator gives no additional central factor, since both X_i^+ and X_i^- are present. We have as a central factor $T_j(d_id_jc_j^{-1}) = d_i^{-1}d_j^{-1}d_jc_j^{-1} = d_i^{-1}c_j^{-1} = c_{ji}^{-1}$. We see that the left and right handside are equal if and only if $d_j = c_j^2$.

□

Note in particular that in 4.3.1 the central factor $((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}$ obeys $d_j = c_j^2$, since for $i \neq j$ $a_{ij} = -1$ or $a_{ij} = 0$, in which case Weyl property is trivially satisfied. So choosing $c_{ij} = 1$ is admissable. One might want to make a different choice for the constant c_{ij} where $\Delta(z)$ is of a different form, more like that of the semisimple case. To this end consider an element $\check{w}_i$ with the properties

$$\begin{aligned}\Delta(\check{w}_i) &= \check{\mathcal{R}}(i)^{-1}\check{w}_i \otimes \check{w}_i, \\ \check{\mathcal{R}}(i) &= \sum R_n(h)(e^{-hH_i^+/2}X_i^+)^n \otimes (e^{\epsilon hH_i^-/2}X_i^-)^n, \\ \check{T}_i(h) &= \check{w}_i^{-1}h\check{w}_i,\end{aligned} \tag{4.50}$$

such that there is no central term is present in the T_i as compared to the T_i in [29]. This is possible if one introduces an abstract Weyl element obeying the above properties, however it is not clear if such an element exists at all. We could define the explicit automorphisms $\check{T}_i$ by requiring that under the identification of $\epsilon^{-1}H^+$ and H^-, $T_i = \check{T}_i$. In particular this means that

$$\Delta(\check{T}_i(X_j^\pm)) = \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_j^+))\check{\mathcal{R}}_i^{-1},$$

for some 2-cocycle $\check{\mathcal{R}}_i$. Note that $\check{\mathcal{R}}_i$ is an element of $\overline{U_q(sl_n^\epsilon) \otimes U_q(sl_n^\epsilon)}$, so we can always write down this formula in the algebra $\overline{U_q(sl_n^\epsilon)}$. Note that $\check{T}_i$ by construction correspond to a choice for c_{ij}, since we want $\check{T}_i$ to obey the Weyl-property. We now prove the following fact.

Proposition 4.3.1. *The automorphisms T_i as defined in theorem4.3.1 obey the Weyl property and are of the form $T_i(h) = \overline{w}_i^{-1}h\overline{w}_i$, where $\overline{w}_i$ are defined as in lemma* **??***, if and only if $c_{ij} = 1$.*

Proof. Under the identification of $\epsilon^{-1}H^+$ and H^-, in $U_q(sl_n)$, we have $T_i = \check{T}_i$, so $\Psi = \mathcal{R}_i^{-1}\check{\mathcal{R}}_i$ is equal to $1 \otimes 1$ under this identification, by semisimplicity of $U_q(sl_n)$. We claim that Ψ is a central element.
We now use the fact that $T_i(X_j^+) = c_{ij}\check{T}_i(X_j^+)$ for some group-like element c_{ij} if $i \neq j$. From the identities $\Delta(\check{T}_i(X_j^\pm)) = \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_j^+))\check{\mathcal{R}}_i^{-1}$ and idem for T_i it follows that (noting that T_i and $\check{T}_i$ agree on $H_{2,\epsilon,i}$ and on the Cartan subalgebra of

$U_q(sl_n^\epsilon) = H_{n,\epsilon}$ by definition, as this action was calculated from the $H_{2,\epsilon}$ action)

$$\begin{aligned}
&(c_{ij} \otimes c_{ij})(T_i(X_j^+) \otimes T_i(e^{H_j^+}) + T_i(e^{-H_j^+}) \otimes T_i(X_j^+)) \\
&= R_i^{-1}\check{\mathcal{R}}_i(c_{ij}T_i(X_j^+) \otimes T_i(e^{H_j^+}) + T_i(e^{-H_j^+}) \otimes c_{ij}T_i(X_j^+))\check{\mathcal{R}}_i^{-1}\mathcal{R}_i \\
&= c_{ij}T_i(X_j^+) \otimes T_i(e^{H_j^+}) + T_i(e^{-H_j^+}) \otimes c_{ij}T_i(X_j^+).
\end{aligned}$$

But this implies $T_i(X_j^+) \otimes c_{ij}^{-1} = T_i(X_j^+) \otimes 1$, by invertability of c_{ij}, and linear independence of the terms involved. So $c_{ij} = 1$. It should be remarked that
We now prove the claim. Therefore look at a similar expression, $\Delta(T_i(X_i^+))$. We know that $\check{T}_i(X_i^\pm) = T_i(X_i^\pm)$. Furthermore $\Psi / \sim = 1 \otimes 1$, and $\mathcal{R}_i$ and $\check{\mathcal{R}}_i$ are elements of $H_{\epsilon,2,i}$. Let $H_i = \epsilon^{-1}H_i^+ - H_i^-$. Then Ψ must be a power series in $H_i \otimes H_i$, in $\epsilon^{-1}(H_i^+ \otimes H_i^- - H_i^- \otimes H_i^+)$ and $H_i^+ \otimes H_i^+ - H_i^- \otimes H_i^-$. We have

$$e^{H_i^\pm \otimes H_i^\pm} e^{H_i^+} \otimes X_i^+ = e^{H_i^+} \otimes X_i^+ e^{H_i^\pm \otimes 1} e^{H_i^\pm \otimes H_i^\pm}.$$

This observation is generalizable to general power series in $H^\pm \otimes H^\pm$, and holds also for the opposite case. Moreover, we observe that

$$\begin{aligned}
\Delta(\check{T}_i(X_i^+)) &= \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_i^+))\check{\mathcal{R}}_i^{-1} \\
&= \check{\mathcal{R}}_i T_i \otimes T_i(\Delta(X_i^+))\check{\mathcal{R}}_i^{-1} \\
&= R_i T_i \otimes T_i(\Delta(X_i^+))\mathcal{R}_i^{-1}.
\end{aligned}$$

Note that $H^\pm$ only introduce an aditional term when commutated with X_i^+, so the terms of $\Delta(X_i^+) = X_i^+ \otimes e^{H_i^+/2} + e^{H_i^+/2} \otimes X_i^+$ do not get mixed by commutating with Ψ, so by linear independence of the two terms in the coproduct we can compare them term by term. Since the $H^\pm$ commute with each other and Ψ is a power series in tensor products of $H_i^\pm$, we get the following identities

$$\begin{aligned}
\Psi X_i^- \otimes 1 \Psi^{-1} &= X_i^- \otimes 1, \\
\Psi 1 \otimes X_i^- \Psi^{-1} &= 1 \otimes X_i^-.
\end{aligned}$$

This means that Ψ commutes with $X_i^- \otimes 1$ and $1 \otimes X_i^+$, and with the above observation this means that Ψ must be a power series in $H_i \otimes H_i$. However, this means that Ψ is central in $U_q(sl_n^\epsilon) \otimes U_q(sl_n^\epsilon)$. This ends the proof. □

So there is no choice in c_{ij} if we want to have a quantum Weyl group. Such an algebra would have the above mentioned property by construction, since $\Delta(w_i)$ is definend explicitly in the $H_{2,\epsilon}$ case. Since we have an Hopf algebra we automatically obtain for any generalization of $\overline{H_{2,\epsilon}}$,

$$\Delta(\check{T}_i(X_j^\pm)) = \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_j^+))\check{\mathcal{R}}_i^{-1}.$$

We now have

$$\begin{aligned}
\Delta(T_i(X_j^+)) &= \Delta(\overline{w}^{-1}X_j^+\overline{w}_i)\\
&= \overline{\mathcal{R}}_i T_i \otimes T_i(\Delta(X_j^+))\overline{\mathcal{R}}_i^{-1},\\
\overline{\mathcal{R}}_i &= q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum R_n(h)(e^{-hH_i^+/2}X_i^+)^n \otimes (e^{\epsilon h H_i^-/2}X_i^-)^n,\\
R_n &= \frac{q^{1/2n(n-1)}(1-q^{-2})^n}{[n]_q!}
\end{aligned}$$

This is the result with which we can calculate the comultiplication for $U_q(sl_n^\epsilon)$ for non-simple generators $T_i(X_j^\pm)$, corresponding to the roots $\alpha_i + \alpha_j$. Of course this can be generalized to higher order generators for $U_q(sl_n^\epsilon)$. This is something straightforward and will not be done here. An example of the more general construction can be found in [6].
We rewrite the automorphisms T_i to apply them to the generators E_i and F_i with non-symmetric comultiplication. This yields the algebra we use in chapter 1, when $S \otimes id$ is applied to the quantum double. We rewrite the expressions in theorem 4.3.1, with $c_{ij} = 1$.

$$\begin{aligned}
\Delta(T_i(E_j)) &= \mathcal{R}_i T_i \otimes T_i(\Delta(E_j))\mathcal{R}_i^{-1},\\
\mathcal{R}_i &= q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum \frac{(1-q^{-2})^n}{[n]_q!}(E_i)^n \otimes (F_i)^n,\\
T_i(K_j^+) &= K_j^-(K_i^-)^{-a_{ij}}, T_i(K_j^-) = K_j^+(K_i^+)^{-a_{ij}},\\
T_i(E_i) &= -F_i(K_i^-)^{-1}(K_i^+)^{-1}, T_i(F_i) = -(K_i^-)(K_i^+)E_i,\\
T_i(E_j) &= (-1)^{a_{ij}}[-a_{ij}]_q!((K_i^-)^{-1}(K_i^+))^{-a_{ij}/2}\\
&\quad [E_i,\cdots,[E_i,E_j]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}, i \neq j,\\
T_i(F_j) &= \frac{1}{[-a_{ij}]_q!}((K_i^-)(K_i^+)^{-1})^{-a_{ij}/2}\\
&\quad [F_i,\cdots,[F_i,F_j]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}, i \neq j.
\end{aligned}$$

In particular, this formula is also valid for non-invertible ϵ, when we remember that $q^{-\epsilon h}$. The terms present are elements of the ring of power-series of ϵ. This allows for use in chapter 1.
To match the expressions with the algebra in chapter 1, note that we are using a different convention for $[n]_q$ in the last two sections of this chapter, then the factor $(1-q^{-2})^n$ is absorbed. The factor of $q^{1/2n(n-1)}$ is absorbed by the commutating of the group-like factor $e^{H^-/2} \otimes e^{H^-/2}$, with index i and j and the R-matrix. Furthermore, we change the definition of q to $e^{\epsilon h}$, and substitute $H_1^+ \mapsto 2A - B$ and $H_2^+ \mapsto 2B - A$. The change in q implies that we should keep track of all exponentials e^h and change the sign to e^{-h} and vice versa. Any factor q is substi-

tuted.
After performing these substitutions, there is one fundamental difference between the Hopf algebra used in this chapter and the one used in chapter 1. The Hopf algebra $U_q(sl_n^\epsilon)$ as defined in this chapter cannot be expanded modulo ϵ, as can be seen from the factor ϵ^{-1} present in the commutator between $X^\pm$. To solve this problem, we have to scale the X^- generator. This is the subject of the next section.
Once the conventions are correct, we are well on our way to implementing $U_q(sl_n^\epsilon)$ in the tensor formalism however. Especially when the program is more optimized, this is a very interesting topic of research. Theoretically, it is also interesting to be able to do calculations in $U_q(sl_n^\epsilon)$ for any n, and any order of ϵ, even when the program is not much faster than it is now.

4.4. Epilogue

Define gl_n^ϵ as the Lie bialgebra over $\mathbb{R}(\epsilon)$ with generators $X_i^\pm, H_i^\pm, i = 1, \cdots, n-1$ and the relations

$$[H_i^-, X_j^\pm] = \pm a_{ij} X_j^\pm, [H_i^\pm, H_j^\mp] = 0, [H_i^+, X_j^\pm] = \pm a_{ij} X_j^\pm, \tag{4.51}$$

$$[X_i^+, X_i^-] = -\frac{1}{2}\delta_{i,j}(H_i^+ + H_i^-), (ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, (i \neq j), \tag{4.52}$$

$$\delta(X_i^+) = X_i^+ \otimes H_i^+ - H_i^+ \otimes X_i^+, \tag{4.53}$$

$$\delta(X_i^-) = X_i^- \otimes H_i^- - H_i^- \otimes X_i^-, \tag{4.54}$$

$$\delta(H_i^\pm) = 0. \tag{4.55}$$

a_{ij} is the usual sl_n Cartan matrix.
In gl_n^ϵ, ϵ is an invertible indeterminate. The Lie bialgebra gl_n^ϵ is a quasitriangular Lie algebra that can be obtained through the classical double on the Lie bialgebras of upper and lower triangular matrices $b^\pm \subset gl_n^\epsilon$, generated by $\{H_i^{pm}, X_j^\pm\}$ respectively. This procedure is described in many standard sources, and follows the same procedure as described in chapter 1.
To obtain sl_n^ϵ from gl_n^ϵ, multiply H_i^- in the relations 4.51 and 4.52 with $\epsilon\epsilon^{-1}$. We define $\epsilon H_i^- =: \tilde{H}_i^-$, in the spirit of the Wigner contraction described in appendix A.4. When the redundant factors of ϵ^{-1} in $\epsilon^{-1}[\tilde{H}_i^-, X_j^\pm] = \pm a_{ij} X_j^\pm$ are transferred by multiplying both sides with ϵ, one obtains the familiar Lie algebra relations, and the slightly different cobracket

$$[\tilde{H}_i^-, X_j^\pm] = \pm\epsilon a_{ij} X_j^\pm, [H_i^+, X_j^\pm] = \pm a_{ij} X_j^\pm,$$

$$[X_i^+, X_i^-] = -\frac{1}{2}\delta_{i,j}(H_i^+ + \epsilon^{-1}\tilde{H}_i^-), (ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, (i \neq j),$$

$$\delta(X_i^+) = X_i^+ \otimes H_i^+ - H_i^+ \otimes X_i^+,$$

$$\delta(X_i^-) = \epsilon^{-1}(X_i^- \otimes \tilde{H}_i^- - \tilde{H}_i^- \otimes X_i^-).$$

We can multiply δ with any constant in $\mathbb{R}(\epsilon)$, this will yield a cobracket on the same Lie algebra. To this end, consider the b^- Lie bialgebra where we multiply δ with ϵ. We obtain

$$\delta(X_i^-) = X_i^- \otimes H_i^- - H_i^- \otimes X_i^-.$$

Let us introduce the dual Lie algebra b^+ of b^- with generators $\{X_i^+, \tilde{H}_i^+\}$ by $\langle X_i^+, X_j^- \rangle = \delta_{ij}$ and $\langle \tilde{H}_i^+, \tilde{H}_j^- \rangle = a_{ij}$. The algebra relations of b^+ are defined through the cobracket of b^-, and so we obtain the relations (note that ϵ is invertible, to obtain the Serre relation)

$$[\tilde{H}_i^+, X_j^+] = +a_{ij}X_j^+, (ad_{X_i^+})^{1-a_{ij}}(X_j^+) = 0, (i \neq j).$$

The cobracket is defined through the Lie algebra relations of b^- and takes the form

$$\delta(X_i^+) = \epsilon(X_i^+ \otimes \tilde{H}_i^+ - \tilde{H}_i^+ \otimes X_i^+).$$

Taking the classical double of b^+ and b^- we obtain the Lie bialgebra sl_n^ϵ with relations 4.1. As noted in the first section of this chapter, we have a set of Lie algebra automorphisms T_i on sl_n^ϵ, which are defined with the adjoint action on sl_n^ϵ.
We can do a Wigner contraction on sl_n^ϵ by multiplying $X_i^- \in sl_n^\epsilon$ with $\epsilon\epsilon^{-1}$ and defining $\tilde{X}_i^- := \epsilon X_i^-$. This has no effect on the Lie algebra relations of $b^\pm$ and the cobracket of $b^\pm$, as can be seen by multiplying the relations with ϵ on both sides. It has an effect on the pairing between $\tilde{X}_i^+$ and $\tilde{X}_i^-$, which yields ϵ.
In sl_n^ϵ, this changes the bracket between $\tilde{X}_i^\pm$ to $[\tilde{X}_i^-, \tilde{X}_i^+] = \frac{1}{2}(\epsilon\tilde{H}_i^+ + \tilde{H}_i^-)$. This relation should remind the reader of the definition of sl_n^ϵ in chapter one. For now, let us denote this algebra as $\tilde{sl}_n^\epsilon$. In particular we observe that with these Lie bialgebra relations, it is possible to divide out to ϵ^k, as there are no explicit factors of ϵ^{-1} present in the algebra relations of $\tilde{sl}_n^\epsilon$.
However, another effect of this Wigner contraction is that the bracket $[,]$ no longer defines a set of automorphisms T_i of $\tilde{sl}_n^\epsilon$. When writing out the requirement that T_j is a Lie algebra map for the $[\tilde{X}_i^+, \tilde{X}_i^-]$ relation, one finds that T_j is an algebra map only when one multiplies with a factor $\epsilon^{a_{ij}} = \epsilon^{-1}$ when T_j is applied to $\tilde{X}_i^-$. So we have to define $\tilde{T}_j(\tilde{X}_i^-) = \epsilon^{-1}T_j(\tilde{X}_i^-)$ when $i = j \pm 1$ and $\tilde{T}_j(u) = T_j(u)$ else.
This means that the Lie algebra automorphisms $\tilde{T}_i$ of sl_n^ϵ, when defined on $\tilde{sl}_n^\epsilon$ gain a factor ϵ^{-1} with respect to the Lie algebra automorphisms on sl_n^ϵ that are used in this chapter. We note that in the definition of non-simple generators this yields also a power of ϵ^{-1}. In particular, for $\tilde{sl}_3^\epsilon$ this yields $\tilde{X}_{\alpha_1+\alpha_2}^- = \tilde{T}_1(\tilde{X}_2^-) = \epsilon^{-1}T_1(\tilde{X}_2^-)$. Multiplying both sides wth ϵ now gives $\epsilon X_{\alpha_1+\alpha_2}^- = [X_1^-, X_2^-]$, which is the relation familiar from chapter one.
A complication from this specific Wigner contraction on sl_n^ϵ is that the automorphisms $\tilde{T}_i$ do not obey the Weyl-property. Writing out $\tilde{T}_i\tilde{T}_j\tilde{T}_i(X_i^\pm) = \tilde{T}_j\tilde{T}_i\tilde{T}_j(X_i^\pm)$

gives a different factor of ϵ on both sides. Since the $\tilde{T}_i$ reduce to the usual automorphisms on sl_n when ϵ is put to one, one can only compensate by introducing a factor of ϵ^p in $\tilde{T}_i$. However, one quickly sees that it is not possible to make such a choice such that $\tilde{T}_i$ obey the Weyl property.
As has been noted before, in the case of sl_3 this is not a problem, since there is a unique way to decompose the reflection corresponding to the longest root. For higher $n > 3$, one has to make a choice for a decomposition of the longest Weyl group element and live with this. The result for different choices of decomposition yields different Lie algebras that are presumably isomorphic, although this is not directly clear. This is an interesting subject of future study.

When one quantizes sl_n^ϵ, this yields the Hopf algebra $U_q(sl_n^\epsilon)$ of theorem 4.1.3. The Hopf algebra $U_q(\tilde{sl}_3^\epsilon)$ can be obtained from $U_q(sl_3^\epsilon)$ by multiplying X_i^- with $q - q^{-1}\frac{1}{q-q^{-1}}$ and defining $\tilde{X}_i^- = (q - q^{-1})X_i^-$. This scaling only infuenzes the relations between $X_i^\pm$. The comultiplication, antipode and the other relations stay the same.
Let T_i be as in equation 4.48 with $c_{ij} = 1$. As proved in the third section of this chapter, T_i obey the Weyl property and are automorphisms of $U_q(sl_3^\epsilon)$ when $c_{ij} = 1$. Note that ϵ was introduced on the b^+ side in the previous sections. This does not change the properties of the T_i, since the relation between the two algebras is a scaling of $H_i^\pm$. We have encountered this fact in the classical case, and the quantum case follows in exactly the same way.
If we wish to define automorphisms $\tilde{T}_i$ on $U_q(\tilde{sl}_n^\epsilon)$, we have to correct in the same way as in the Lie algebra case, by introducing an additional factor of $\frac{1}{q-q^{-1}}$ when T_i is applied to $\tilde{X}_i^-$. We define $\tilde{T}_j(\tilde{X}_-^i) = \frac{1}{q-q^{-1}}T_j(\tilde{X}_i^-)$ for $i = j \pm 1$ and $\tilde{T}_j(u) = T_j(u)$ in any other case, for an elementary generator $u \in U_q(\tilde{sl}_n^\epsilon)$. In exactly the same way as the classical $\tilde{T}_i$ failed to have the Weyl-property, so do the quantum Weyl group automorphisms $\tilde{T}_i$. This can be seen by applying $\tilde{T}_j$ to $\tilde{X}_i^-$ and counting the terms $\frac{1}{q-q^{-1}}$ that are introduced.
Equivalent to the classical case, when defining non-simple generators in $\tilde{sl}_n^\epsilon$, one has to choose a decomposition of the longest Weyl group element. Moreover, on the $U_q(b^-) \subset U_q(\tilde{sl}_n^\epsilon)$ side, a number of factors $\frac{1}{q-q^{-1}}$ are introduced in the definition of nonsimple generators. This yields relations like $(q^{-1} - q)\tilde{X}_{\alpha_1+\alpha_2}^- = T_1(X_2^-)$ in the case of $U_q(\tilde{sl}_3^\epsilon)$. Here, α_i are the simple roots of $\tilde{sl}_3^\epsilon$. This should remind the reader of the relations in chapter 1, although the $b^\pm$ algebras switched place there, among some other details. For a generator corresponding to a root of length $k = 1, \cdots, n-1$, we obtain $k-1$ factors of $q - q^{-1}$. This yields $\tilde{T}_{\alpha_{i_1}}(\cdots(\tilde{T}_{\alpha_{i_{k-1}}}((q - q^{-1})^{k-1}\tilde{X}_{\alpha_{k-1}}^-) = (q - q^{-1})^{k-1}\tilde{X}_\beta^-$, where $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$.
We can use the results of the previous section to obtain an expression of the comultiplication of nonsimple elements such as $\tilde{X}_{\alpha_1+\alpha_2}^-$ in $U_q(\tilde{sl}_3^\epsilon)$. In the case of $\tilde{X}_{\alpha_1+\alpha_2}^-$ for example we get $(q^{-1} - q)\tilde{X}_{\alpha_1+\alpha_2}^- = \overline{\mathcal{R}}_1 T_1 \otimes T_1(\Delta((q - q^{-1})\tilde{X}_2^-))\overline{\mathcal{R}}_i^{-1}$ in

the notation of the previous section.
When we wish to consider $U_q(\tilde{sl}_n^\epsilon)$ over the ring $R_{\epsilon^k} = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$, this implies that we do not get a direct expression for $\Delta(\tilde{X}^-_{\alpha_1+\alpha_2})$ for example, but only for $\Delta((q-q^{-1})\tilde{X}^-_{\alpha_1+\alpha_2})$, since $q - q^{-1} = 2\epsilon h + \cdots$, and ϵ is not invertible. So to obtain the comultiplication of $\tilde{X}^-_{\alpha_1+\alpha_2}$ modulo ϵ^{k+1}, to take a specific example, we need to consider $U_q(\tilde{sl}_3^\epsilon)$ over the ring $R_{\epsilon^{k+1}}$, since over R_{ϵ^k} (which is working mod ϵ^{k+1}), terms proportional to ϵ^{k+1} vanish. So working over R_{ϵ^k} would only give us $\Delta(\tilde{X}^-_{\alpha_1+\alpha_2})$ up to and including order ϵ^{k-1}, since $(q-q^{-1})X^-_{\alpha_1+\alpha_2} = [X_1^-, X_2^-]$. One sees that one can work mod ϵ^{k+1} when k is even, as the ϵ^{2k} term vanish in the expansion of $q - q^{-1}$.
This observation is particularly useful when attempting to construct a general $U_q(sl_n^{\epsilon^k})$ knot invariant. Using formula 4.48 of the previous section, we can obtain expressions for the comultiplication of the non-simple generators. We have to work modulo ϵ^{k+n-1} to obtain the comultiplication of every non-simple generator, since for an element of maximal length $n-1$, there are $n-2$ factors of ϵ introduced, yielding a prefactor of ϵ^{k+n-2} for the comultiplication of the longest Weyl group element generator. When computing the knot invariant itself, so for the multiplication of R-matrices, one can then work modulo ϵ^{k+1} again.
A particular surprise when specializing to $\epsilon^k = 0$, is that the $\tilde{T}_i$ are not algebra automorphisms of $U_q(\tilde{sl}_n^\epsilon)$, due to the noninvertible factor of $q - q^{-1}$ present. In particular, we cannot apply $\tilde{T}_i$ to a non-simple generator such as $\tilde{X}_3^-$, but only to $(q-q^{-1})\tilde{X}_3^-$.
In general, the exact properties of T_i become more complicated as more factors of ϵ are introduced in the definition of the generators associated with positive non-simple roots. For $\tilde{sl}_3^\epsilon$ there is one non-invertible factor introduced when working over $\epsilon^k = 0$, but for $\tilde{sl}_n^\epsilon$ there are $n-2$ factors introduced in the definition of the element corresponding to the longest classical Weyl element. So if we wish to calculate the comultiplication of this generator in the first order of ϵ, say $\tilde{X}^-_\beta \in U_q(\tilde{sl}_n^\epsilon)$, we have to work modulo ϵ^n.

This explains why the usual symmetries $U_q(\tilde{sl}_n^\epsilon)$ for invertible ϵ are not symmetries of $U_q(\tilde{sl}_n^\epsilon)$ for $\epsilon^k = 0$. Some symmetries of $\tilde{sl}_n^\epsilon$ for non-invertible ϵ were found by Roland van der Veen and Dror Bar-Natan in [37], the classical case. We do not know if the symmetries in [37] provide a full discription of the symmetries of $\tilde{sl}_n^\epsilon$, or if there is a bigger set of hidden symmetries. This remains an interesting topic of research. It is also interesting to find the explicit quantum group analogue of the $\tilde{sl}_n^\epsilon$ symmetries.

Conclusion

In this chapter we constructed $U_q(b^-)$ and $U_q(sl_n\epsilon)$ from $b^+ \subset sl_n$ for invertible ϵ. We observed that in the algebra relations ϵ occured only in $q = e^{\epsilon h}$, and hence

one can take the expansion to the k-th order in ϵ for any k when the algebra generators are rescaled with a suitable factor of $(q - q^{-1})^m$, for some positive m.
The fact that $U_q(sl_n^\epsilon)$ is not semisimple does not change the symmetries of $U_q(sl_n^\epsilon)$ for invertible ϵ. They are equal to the symmetries of $U_q(sl_n)$ for invertible epsilon. In the last section we found that only when we specialize to $\epsilon^k = 0$ for $k > 0$ we lose most of the usual symmetries. It turns out that in this case, instead of S_n, we obtain D_n as the group of automorphisms of $U_q(sl_n^\epsilon)$. See [37].
However, important equalities to calculate the comultiplication remain true when ϵ is not invertible, and even when one specializes to $\epsilon^k = 0$. The main purpuse of the last sections of this chapter was to prove these formula for the coproduct in terms of partial R-matrices. We observed that after rescaling, this formula can be expanded in terms of ϵ, so that it is also valid for non invertible ϵ.
Using this formula, we constructed a dual PBW basis of $U_q(sl_n^\epsilon)$ in the first section of this chapter and we gave the pairing between monomials. This enables one to construct the universal R-matrix of $U_q(sl_n^\epsilon)$. We observed that when one wants to know the coproduct (and antipode) of $U_q(\tilde{sl}_n^\epsilon)$ modulo ϵ^{k+1}, one has to work modulo ϵ^{k+n-1}.
In the previous chapter we gave an upper bound for the computational complexity of the $U_q(sl_n^\epsilon)$ invariant. In short, this provided the insight that for small knots (i.e. less than say 20 crossings) the contribution of the number of crossings is smaller than the contribution of rank of sl_n. It remains to be seen if this problem can be overcome. On the other hand, it is interesting to gain insight in the symmetries of $U_q(sl_n^\epsilon)$ in the case where $\epsilon^k = 0$. This may reduce the number of computations one might have to do. This will also give insight in the quantum invariants that are obtained.
A concrete topic of future research is the implementation of $U_q(sl_n^\epsilon)$ using the comultiplication calculated in this chapter. As a first step, we wish to implement these formulas for the case $U_q(sl_3^\epsilon)$, to be able to match the conventions we used earlier with the ones used by [29] and [6] and others. Especially with the last section in mind, this should be possible in the current implementation of $U_q(sl_3^\epsilon)$, when one defines the (co)multiplication tensors for general order of ϵ. In the last section we concluded that the higher n, the higher the order of ϵ one needs to work over in order to obtain the full Hopf algebra structure. Once the Hopf algebra structure has been found, one can restrict oneself to any (lower) order of ϵ to calculate the invariant for actual knots.