Expansions of quantum group invariants

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3. A polynomial time sl_3-knot invariant

Introduction

In this chapter we explain how to turn the quasi-triangular Hopf algebra $U_q(sl_3^\epsilon)$ constructed in the first chapter into a knot invariant. We conclude with the proof that the constructed knot invariant can be calculated in polynomial time using the tensor formalism. The factorization of the knot invariant in Alexander polynomials in zeroth order of epsilon is proven.
For the general form of the invariant in the sl_2 case we refer to [35]. For $U_q(sl_3^\epsilon)$, it remains an important open question if we can find a general expression for the knot invariant. This means that one extracts the invariant part of the output of the calculation of the knot invariant. This greatly reduces the length of the polynomial for a knot K. This in turn will enable us to beter recognize the structure of the knot invariant. Another important factor is the potential reduction of the calculation time.

3.1. Knots diagrams and the Reidemeister theorem

We define the knot diagrams and we consider the usual embedddings of a knot K in $\mathbb{R}^3$. We restrict ourselves to the class of framed knots. For an unframed knot, one can always choose it to have writhe 0 and rotation number 0. If we use this normalization it is possible to choose a canonical snarl diagram, as defined in [35]. In this section we use the concept of framed knots, which enables us to define the rotation number. A knot diagram in our convention is an rv-tangle diagram in the language of [36]. We state the definition of (framed) knots.

Definition 3.1.1. *A knot K is an equivalence class of a continuous (C^∞) embedding of $i : S^1 \to \mathbb{R}^3$. The equivalence class on the space of continuous embeddings is defined by isotopies in $\mathbb{R}^3$.*

This definition uses C^∞-embeddings to exclude unrealistic posibilities such as wild knots from the space of knots. We will not go into details, see for example [4]. A knot is usually defined to be a piecewise linear embedding of the circle. This provides a way to formalize operations on knot diagrams on an elementary level. We skip this step and refer to [4] for an elementary treatment of the subject. Let $I = [0, 1]$ the unit interval. Instead of the embedding of S^1, one can embed $S^1 \times I$ into $\mathbb{R}^3$. Note that it is not essential to take the interval I as $[0, 1]$. Instead, one could also take this interval to be infinitesimally small. We wil consider I to be infinitesimally small, say $I = [0, \delta]$, for $\delta > 0$.

Definition 3.1.2. *A framed knot K is an equivalence class of a continuous embedding of $i : S^1 \times I \to \mathbb{R}^3$. The equivalence class on the space of continuous embeddings is defined by isotopies in $\mathbb{R}^3$.*

Let us define the framing of a framed knot.

Definition 3.1.3. *(Framing) Let K be a framed knot with boundary components $K^{\pm}$. The framing of K is defined as the linking number of the curves $K^{\pm}$.*

If one considers embeddings of S^1 (or $S^1 \times I$) with an orientation along S^1, one obtains an oriented knot. In this chapter we use framed oriented knots.

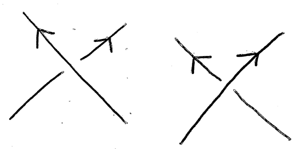
Definition 3.1.4. *Let $J = \{1, \cdots, n\}$ be a finite discrete index set, and fix 2n distinct points $x_i, y_i \in \mathbb{R}^3$. A link is the equivalence class under isotopies of a continuous embedding of the disjoint union $\phi : \amalg_{i \in J} I_i \to \mathbb{R}^3$ such that $x_i = \phi(0) \in \phi([0,1]_i)$ and $y_i = \phi(1) \in \phi([0,1]_i)$.*

Two knots or links K and K' are called isotopic if there exists a smooth family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0,1]$ such that h_0 is the identity on K and $h_1(K) = K'$.
The space of tangles will be most important in our discussion in this chapter, and is closely related to the concept of long knots. The following definition is taken from [25].

Definition 3.1.5. *(Tangle) An (m,n)-tangle is a compact 1-manifold properly embedded in $\mathbb{R} \times \mathbb{R} \times I$ such that the boundary of the embedded 1-manifold is a set of m disinct points in $\{0\} \times \mathbb{R} \times \{0\}$ and a set of n disinct points in $\{0\} \times \mathbb{R} \times \{1\}$. To (m,n)-tangles are said to be isotopic if there is an isotopy between the tangles that fixes the boundary points. A framed tangle is a tangle with a framing on each component, idem for an oriented tangle. A long knot is a $(1,1)$ tangle, where we exclude closed components.*

A knot can be obtained from a long knot by closing its endpoints. Conversely, we can obtain a long knot from a knot by cutting S^1 to obtain the interval I. This is independend from the cutting point of the knot. A knot invariant is an invariant if and only if it is an invariant of long knots, and the two invariants coincide when the knot is cut, or the long knot is closed respectively. See [18], for the context of classical knots, as we consider here.
More generally, it is not true that the closure of an (m,m) tangle without loops is in one to one correspondence with the links of m components. There are $n!$ options to close an (n,n)-tangle. This is the reason that it is usually easier to construct a knot invariant than a link invariant. In this chapter we construct an invariant of framed oriented long knots. See also paragraph X.5 of [17], or chapter 3 of [25] for more information.

Definition 3.1.6. *A pre-knot diagram is defined to be a finite oriented four-valent graph where each vertex is denoted as an over-crossing or an under crossing respectively. A*



(a) under-crossing (b) over-crossing

Figure 3.1.: The possible crossings in a knot, up to rotation.

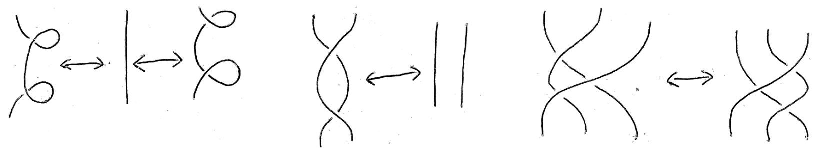


Figure 3.2.: The Reidemeister moves for a framed knot. From left to right the Reidemeister I, II and III move.

labeling of a vertex as an over- and undercrossing is a labeling of the vertex with a ± 1 *and the labeling of two opposite opposite edges ending at the vertex as the underpass. The remaining two edges are labeled as the overpass. We refer to two edges labeled as an over- or underpass as on the same strand, or as a strand.*

See figure 3.1 for the notation of an over and under-crossing.
The space of pre-knot diagrams will be subject to an equivalence relation. Two pre-knot diagrams are equivalent if they can be obtained from each other by applying a finite number of Reidemeister moves. The Reidemeister moves are depicted in figure 3.2. We are using framed knots in our theory, so the Reidemeister I move is different from the usual Reidemeister I move, in order to keep the rotation-number of the knot-diagram constant. We refer to the usual (unframed) RI move as the Reidemeister I′ move. This move is depicted in figure 3.3.

Definition 3.1.7. *(knot diagrams) A knot diagram is defined as the equivalence class of a labeled pre-knot diagram under the Reidemeister moves. With labeled we mean each edge is* $\mathbb{N}$*-labeled. The number* $\mathbb{N}$ *of an edge E is referred to as the rotation number of E.*

There is a similar definition of tangle diagrams, but we consider tangles without framing, so there is no rotation number indicated on the strands. We fix the ending and starting points in $\mathbb{R}^2$ as the projections of $x_i, y_i \in \mathbb{R}^3$. Two distinct vertices in tangle diagrams are required to have a distinct height. The height of a vertex is defined through the second projection $\pi : \mathbb{R}^2 \to \mathbb{R}$. This will be implicit in our definition of tangle and knot diagrams. One can allways present a knot or tangle in this way, see for example [17].

Definition 3.1.8. *(tangle diagrams) A tangle diagram is defined as the equivalence class* *of a pre-knot diagram under the Reidemeister moves, where the Reidemeister I move is replaced by the Reidemeister I′ move.*

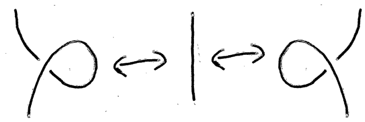


Figure 3.3.: The Reidemeister move I′

Given a framed knot K, we consider its projection on $\mathbb{R}^2$. We assume that the knot is embedded in such a way that no two crossings align with respect to the projection. The result can be presented as a knot diagram. For clarity, I is taken to be infinitesimally small. In this case, we denote the framing of the knot as an integer on the edges of the knot diagram of K, as the rotation number.
On the other hand, it is clear that a knot diagram F can be turned into a three dimensional framed knot $K(F)$. Given a knot diagram, we wish to prove two knot diagrams are equivalent if and only if the corresponding knots are equivalent. The Reidemeister theorem asserts this fact. A proof can be found in [25] for example. As a result of this theorem, we can work with knot diagrams instead of knots.
The same conventions hold in the case of a tangle, except that tangles do not have a framing, and hence there is no need to label the edges.

Theorem 3.1.1. *Two knot or tangle diagrams F and F′ are equivalent if and only if the knots (or tangles) corresponding to F and F′ are equivalent under isotopies.*

Let us be more clear about the about applying the Reidemeister moves to a knot (or tangle) diagram. Any knot diagram can be obtained from elementary tanngle diagrams. These elementary diagrams are shown in figure 3.4. The fifth and sixth diagrams are refered to as caps, and the last two diagrams in figure 3.4 are referred to as cups. Since we consider finite diagrams, we can put an or-

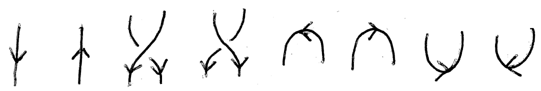


Figure 3.4.: The elementary tangles

dening on the nontrivial elementary tangle diagrams where a knot diagram K is constructed from. Concretely, we draw the diagrams in a way such that the crossings, cups and caps are ordered vertically. This way of drawing a knot diagram is referred to as a sliced knot diagram. See chapter 3 of [25]. On sliced diagrams, the Reidemeister moves take a slightly different form. These moves are referred to as the Turaev moves for oriented sliced diagrams, and are depicted in figure

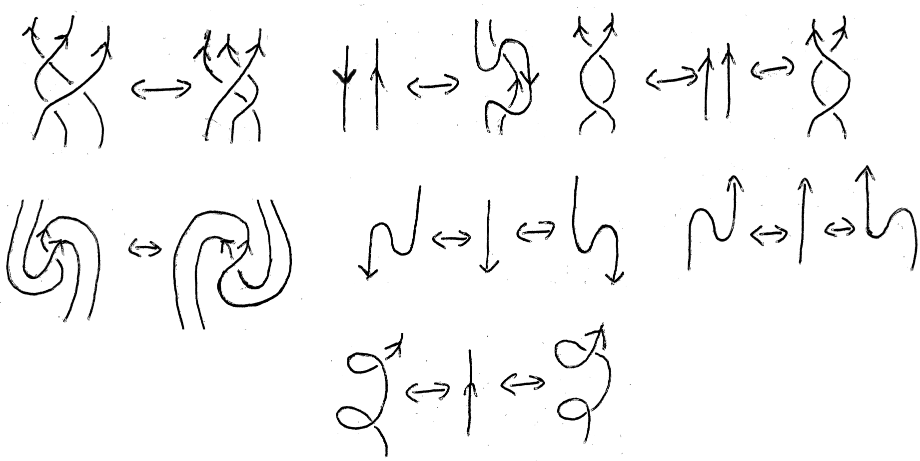


Figure 3.5.: The Turaev moves for a framed knot. The removal of trivial knots from the diagram is excluded from the pictures, but is formally a Turaev move. The last move is replaced by the Reidemeister I′ move in case of a non-framed knot. The Turaev moves are numbered in left to right, downward order as $T-1, \cdots T-7$.

3.5. We have the following theorem for sliced diagrams. For the proof we refer to [25].

Theorem 3.1.2. *The knots are isotopic if and only if their corresponding two oriented sliced knot diagrams are equivalent under the Turaev moves.*

We wish to label the crossings in a knot-diagram with $i \in \{\pm 1\}$. Let K be an oriented framed long knot, so the diagram of K is oriented as a graph, where the crossings are seen as vertices. To determine the sign of the crossing, we use the convention of the left hand rule.
We draw a crossing as having a ninety degree angles between both strands. Place the left hand thumb on the upper strand in the direction of the orientation, with the palm of the hand pointed towards the paper. Align the index finger along the lower strand of the crossing. If the index finger points in the direction of the oriented strand, the crossing has sign $+1$, if it points in the opposite direction, it has sign -1. Crossings with sign $+1$ are called over-crossings, -1 crossings are refered to as under crossings. We can define the following.

Definition 3.1.9. *(Writhe) We define the writhe of an oriented knot diagram of a knot K as the difference between the number of over-crossings and the number of under-crossings. We denote the writhe of K by writhe(K).*

Lemma 3.1.1. *writhe(K) is well-defined for a framed knot.*

To prove this, we look at the admissable moves on knot diagrams, the Reidemeister moves. We observe that the number of positive and negative crossings is conserved under these moves for a framed knot.

Definition 3.1.10. *(mutants) Consider a knot diagram K, and consider a disc D in K such that there are exactly four edges crossing the boundary of the disc. Let T be the tangle in the disc D. Consider the operations $\mathcal{S}$ on the disc D where the four crossings (equally spaced on the circle, w.l.o.g.) are mapped to each other. By applying these operations to T we obtain a different tangle T′. A mutant of the knot K is a knot K' where the tangle T is replaced by a tangle T′ obtained from T by any of the operations $\mathcal{S}$ on D. We call K′ a mutation of K.*

Of course it does not matter which definition one takes in rotating the crossing. We are now in a position to define a knot invariant for a knot K.

Definition 3.1.11. *(knot invariant) Let S be a set. Let $Z(K) \in S$ be an expression in S corresponding to any knot K. Then Z(K) is called a knot invariant if for any two isotopic knots K and K′, Z(K) = Z(K′).*

Note that for a framed knot K, $writhe(K)$ is a knot invariant where $S = \mathbb{Z}$. Let us now define the invariant corresponding to $U_q(sl_3)$. Defining a knot invariant is equivalent to defining how to compute it. For the following considerations, we follow chapter 4 of [25]. For a more concise treatment of knot invariants coming from ribbon Hopf algebras we refer to this source, although there exist many other books that treat the subject as well.
A quasitriangular ribbon Hopf algebra A is equipped with an R-matrix $\mathcal{R}$, its inverse $\mathcal{R}^{-1}$, multiplication m, comultiplication Δ, unit $\mathbf{1}$ and counit ε. Note that ε is different from the parameter ϵ introduced in chapter 1. As introduced in chapter 1, we have $u = \sum S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}$, and $v = S(u)$. Since A is a ribbon Hopf algebra we have the square root ν of the element uv which is called the ribbon element.
We introduce the graphical calculus for a ribbon Hopf algebra A. Consider a framed oriented tangle diagram T. The graphical calculus is a way to denote operations in A. For a rigorous introduction of the graphical calculus, see [25]. The idea is to label the strands of T expressions in $A^{\otimes S}$, where each strand stands for a tensor factor. The concatination of strands as taking the product of the boxed quantities in the order of the orientation of the strand.
The labels of the strands are written in coupons. An (n, m)-coupon (or box) is a rectangle with n inputs and m outputs, and corresponds to a map $A^{\otimes\{1,\cdots,n\}} \to A^{\otimes\{1,\cdots,m\}}$. Multiplication with an element $w \in A^{\otimes S}$ is considered as a map $A^{\otimes S} \to A^{\otimes S}$, and is denoted as a coupon with $|S|$ in- and outputs. A strand with no coupon (or box) is the unit of A.

Definition 3.1.12. *An expression in the graphical calculus of A is a collection of (n, n)-coupons, where n may be any positive integer, where the in- and outputs of the coupon are connected by elementary oriented framed tangle diagrams. A graphical expression with n strands is referred to as an n-ribbon graph.*

Note that a ribbon graph without coupons is an oriented framed tangle diagram. An example is depicted in figure 3.6. The graphical calculus is a way to denote

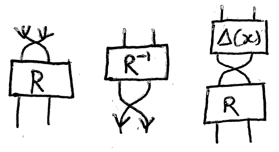


Figure 3.6.: Graphical calculus for A. From left to right: $\mathcal{R}$, $\mathcal{R}^{-1}$ and $\Delta^{op}(x)\mathcal{R}$.

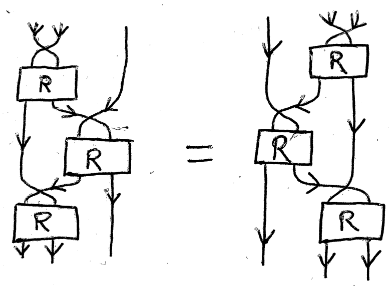


Figure 3.7.: Graphical Yang-Baxter equation.

multiplication on $A^{\otimes S}$. We denote the R-matrix of A as a crossing with sign ± 1 for $\mathcal{R}^{\pm 1}$ respectively. By the Reidemeister theorem, this is in fact well defined on the equivalence classes of tangle diagrams, since $\mathcal{R}$ obeys the Yang-Baxter equation. For clarity, we may add a coupon containing $\mathcal{R}^{\pm 1}$, to indicate multiplication with the R-matrix.

The graphical language is useful to prove properties of A. One can for example prove nicely that the Drinfel'd double is a quasi-triangular Hopf algebra. See chapter 4.1 in [25]. We state the Yang-Baxter equation in this graphical language. For more elaborate examples we refer to [25]. We may define operations on a ribbon graph. We only state the most important operations. Obviously one may multiply two n-ribbon graphs by putting the two diagrams together. To avoid confusion, we label each of the $2n$ strands with a different integer, to indicate which entries are multiplied. The multiplication of two n-ribbon graphs is referred to as the concatination of the strands.

The comultiplication in the Hopf algebra A is a map $\Delta : A \to A \otimes A$. Its operation on an unlabeled strand, or a strand labeled with a grouplike element such as uv^{-1}, is doubling the strand. By the quasitriangularity axioms, when a crossing occurs, doubling a strand results in two of the same crossings. This follows from $\Delta \otimes id(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$.

Since the antipode $id \otimes S$ inverts the R-matrix, we may define the action of S on a strand as inverting the orientation of a strand. This is also well defined when the strands are labeled with grouplike elements in A. Later in this chapter we will use these operations to prove some properties of the knot invariant we are about to define. We also note that it is customary in Hopf theory to denote calculations in a graphical way, using these oberations. See for example [10].

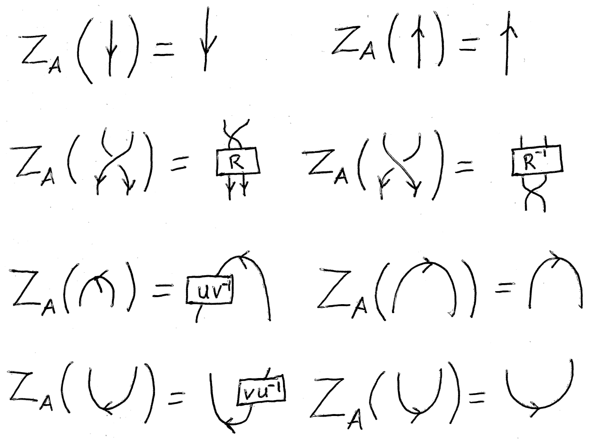


Figure 3.8.: Graphical depiction of Z_A for a ribbon Hopf algebra A.

Let us consider a ribbon Hopf algebra A and a framed sliced oriented tangle T. We will define the tangle invariant $Z_A(K)$ for any ribbon Hopf algebra A. This yields a knot invariant. This is independent from the cutting of the knot, as the knot consists of one strand. In general, this invariant is ill defined on links, as it depends on the cutting point on the embedding of the copies of S^1. We will not attempt to construct a link invariant, but this an interesting topic for future research.
Let S be the set labeling the strands of T. We define

$$Z_A : \{tangle\ diagrams\}/\sim \rightarrow A^{\otimes S}$$

as the map that takes tangle diagrams and assigns $\mathcal{R}$ to a positive crossing and $\mathcal{R}^{-1}$ to a negative crossing. To a left oriented cap we assign the element $C = uv^{-1}$, and to the left oriented cup we assign multiplication with $\overline{C} = vu^{-1}$, the product of the ribbon element v with the inverse of u. The right oriented cup and cap are left as they are, as are the single strands. Multiplication now takes place according to the graphical calculus by concatinating (or glueing) the elementary diagrams to each other in the order as they appeared in T. In algebraic terms, when two strands are concatinated, multiplication takes place on the same copy of A labeled according to the label of the strand in K. For clarity we may assign seperate labels to each side of two concatinated strands. Z_A is depicted in figure 3.8.

When considering a map $F : \{tangle\ diagrams\}/\sim \rightarrow \{tangle\ diagrams\}/\sim$ of tangle diagrams in $|S|$ strands, we introduce $Z_A(F) : A^{\otimes S} \rightarrow A^{\otimes S}$ as the corresponding map on Hopf algebras. $Z_A(F)(Z_A(T)) := Z_A(F(T))$. By the Reidemeister theorem (or Turaev's theorem, depending on the diagrams under consideration) this is well defined. We note furthermore that we may identify $Z_A(T)$ with ${}^{\mathbf{t}}Z_A(T)$ for any tangle T using the map $\mathbb{O}$, since $Z_A(T) \in A^{\otimes S}$. In what fol-

lows, we leave the $\mathbb{O}$ out of the notation.
Usually, for a closed knot diagram we have to take the quotient of $A^{\otimes S}$ with the space of commutators in A. So for a framed oriented sliced knot diagram K, $Z_A(K) \in A/J$. Here $J = \{xy - yx | x, y \in A\}$ is the vector space of all commutators in A. We quotient out to J since there is a choice how to map each elementary tangle to an element in A. In other words, if $Z_A(K)$ were not commutative, this construction would be ill-defined. For a proof of this fact we refer to [25], paragraph 4.2. However, we do not close the knot so we are safe.

Theorem 3.1.3. *Let K be a framed oriented sliced knot, and let A be a ribbon Hopf algebra. Then $Z_A(K) \in A$ is an isotopy invariant of the knot K. We refer to this invariant as the universal A invariant of K, and write it as $Z_A(K)$.*

We are now in a position to define the knot invariant for the case $A = U_q(sl_3^\epsilon)$.

Definition 3.1.13. *Let K be a sliced framed oriented knot diagram. We define $Z_3^\epsilon(K) = Z_{U_q(sl_3^\epsilon)}(K)$.*

We now state the main theorem of this section.

Theorem 3.1.4. *For any framed oriented long knot K, $Z_3^\epsilon(K)$ is invariant under the isotopies of K.*

Proof. To prove this theorem, it is enough to prove invariance of $Z_3^\epsilon(K)$ under the Turaev moves, by the above discussion. These moves are checked explicitly in Mathematica, and we refer to the implementation in appendix A.1.
We state the appropriate equations here for clarity. The equations are matched to the diagrams in figure 3.5 by reading from left to right. We use the notation introduced in earlier chapters, where $\mathcal{R}_{ij}$ is the R-matrix $\mathcal{R} = \sum R^{(1)} \otimes R^{(2)}$ acting on the i-th and j-th tensor factor by $1 \otimes \cdots \otimes R^{(1)} \otimes 1 \otimes \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$. Similarly we write C_i for a copy of C on the i-th tensor factor ('strand'). We define $\mathcal{K} = R^{(2)} C R^{(1)}$ and $\overline{\mathcal{K}} = \overline{R}^{(2)} \overline{C} \overline{R}^{(1)}$.

$$\begin{aligned}
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} &= \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \\
\overline{\mathcal{R}}_1^{(1)}\overline{C}_1\mathcal{R}_{12}\overline{\mathcal{R}}_2^{(2)} &= \overline{C}_2, \\
\mathcal{R}_{12}\overline{\mathcal{R}}_{12} &= 1 = \overline{\mathcal{R}}_{12}\mathcal{R}_{12}, \\
\overline{C}_1\overline{C}_2\mathcal{R}_{12}C_1C_2 &= \mathcal{R}_{12}, \\
\overline{C}C &= 1, \\
C\overline{C} &= 1, \\
\overline{\mathcal{K}}\mathcal{K} &= 1 = \mathcal{K}\overline{\mathcal{K}}.
\end{aligned}$$

The second equality can also be written as $\overline{R}_1^{(1)}\overline{C}_1\mathcal{R}_{12}C_1\overline{R}_2^{(2)} = 1 \otimes 1$, which is equal to the corresponding Turaev move by the using graphical calculus. The fourth identity can be rewritten in the same way.

In appendix A.1 we number the Turaev moves as $T-1,\cdots,T-7$, as indicated in the picture. The equations checked there are the exact same equations written down in this proof. Since we already know that the Hopf algebra structure of $U_q(sl_3^\epsilon)$ is implemented in Mathematica in the program *sl3invariant.nb* as shown in appendix A.1, we can directly conclude that $Z_3^\epsilon(K)$ is indeed invariant under the Turaev moves. This ends the proof. □

3.2. Computing the Alexander polynomial

In this section, the Seifert surface $S(K)$ of a knot diagram K is constructed, and we compute its Alexander polynomial. The Alexander polynomial is computed from the band representation of the Seifert surface. The idea is that we compute the linking matrix of the generators of the fundamental group of $S(K)$. For details we refer to [25] and [4]. We will follow page 17-22 of [25] and [4], page 107.
Let K be a framed oriented knot. To construct its Seifert surface, consider the planar representation of K. In the knot diagram, we replace a crossing with two untangled strands in the same direction. The result is a disjoint union of S^1. We consider the discs in $\mathbb{R}^3$ bound by these discs.
When a disc within a disc occurs, we elevate one of the two discs in the direction perpendicular to the disc. Finally, connect the discs according to the crossings present in the diagram of K. For a positive crossing we attach a band with a positive half twist, and for a negative crossing in the diagram of K we attach the discs with a negative half twist. The surface we obtain is the Seifert surface $S(K)$ of K.
Note that any Seifert surface can be expressed in a band form. This is called the

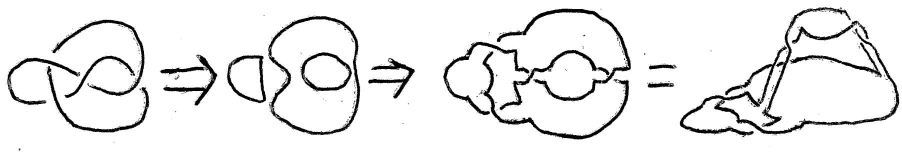


Figure 3.9.: The Seifert surface for the Trefoil knot.

band representation of the Seifert surface, see figure 3.10. This fact is proven by considering one of the discs as the base of the band representation, and contracting other discs to bands. Twists occuring will be denoted as curls in the band representation. For a proof of this elementary fact, see [4], page 105.
We observe that the boundary of the Seifert surfact of a knot K is equal to K. The boundary of the bands of the Seifert surface consists of two strands that are linked together with linking number +1 or -1, depending on the convention chosen. Moreover, the two strands always have opposite orientation, if we compare them to the orientation of the band. This motivates the definition of the map B in the next section.

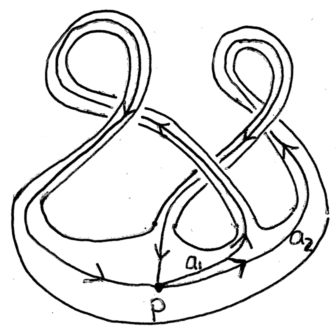


Figure 3.10.: Band representation for the Trefoil knot.

To calculate the Alexander polynomial Δ_K of K, consider the fundamental group $\pi^1(S)$ of $S = S(K)$. We wish to compute the linking number of the generators of $\pi^1(S)$. For simplicity we chose a basebpoint p of $\pi^1(S)$. Since $S(K)$ is connected the final result is independent of the coice of p. Let p be in the 'base' of $S(K)$, i.e. the rectengular part on which the bands are attached. Consider an orientation of each generator $a_i \in \pi^1(S)$. Here, i runs through the number of bands attached to the base. For the Trefoil for example, $i = 1, 2$.
Considering this orientation, we define the numbers l_{ij} and r_{ij}. Consider two bands B_i and B_j with orientation $a_i, a_j \in \pi^1(S)$, respectively. When B_i overcrosses B_j from left to right, define $l_{ij} = 1$, else $l_{ij} = 0$. Similarly, when B_i overcrosses B_j from right to left, define $r_{ij} = 1$. Then define the Seifert matrix V of S by $V_{ij} = l_{ij} - r_{ij}$. Finally, we define the Alexander polynomial $\Delta_K(t)$ in the indeterminate t as $det(t^{-1/2}V^T - t^{1/2}V)$. This normalization forces $\Delta_K(t)$ to be symmetric under $t \mapsto t^{-1}$. See [4] for details.
We calculate the Alexander polynomial for the Trefoil knot with the loops in figure 3.10. We see that a_1 overcrosses a_2 from right to left, a_1 overcrosses a_1 (itself) from right to left and similarly, a_2 overcorsses itself from right to left. We have $V_{11} = V_{22} = V_{12} = -1$ and $V_{21} = 0$. Computing Alexander's polynomial we get

$$\begin{aligned} det((t^{-1/2}V^T - t^{1/2}V) &= det(\begin{pmatrix} -t^{-1/2} & 0 \\ -t^{-1/2} & -t^{-1/2} \end{pmatrix} + \begin{pmatrix} t^{1/2} & t^{1/2} \\ 0 & t^{1/2} \end{pmatrix}) \\ &= t^{-1} - 1 + t. \end{aligned}$$

3.3. Multiplying R-matrices

When $\epsilon = 0$, we know that the invariant connected to $U_q(sl_2)$ is the Alexander polynomial [35]. The $U_q(sl^{\epsilon}_{2,i})$ algebra relations are identical to the algebra in [35], so we can connect the $U_q(sl^{\epsilon}_3)$ invariant to the Alexander polynomial as well. Note that for $\epsilon = 0$, and a knot K, $Z^0_3(K)$ is a polynomial in the variables

S, T, ST [35].
Let K be a knot, and let G be the tangle associated to the band representation of the Seifert surface. The tangle G is obtained from the Seifert surface, disconnecting the bands from the central disc, and labeling them with indices $1 \cdots 2g$, where g is the genus of K. Let us define the operation $B = \prod_{n=0}^{g} B^{k_n}_{i_n,j_n}$ as the operation that turns G into the knot K. We can describe this operation as doubling the strands $1, \cdots, 2g$, reversing the orientation on one of the strands and then connecting the strands as they fit on the Seifert-surface to match the orientation of K. We have the following lemma.

Lemma 3.3.1. ${}^{t}Z_3^{\epsilon}(B)^k_{i,j} = {}^{t}\Delta_i^{l1,r1}\,{}^{t}\Delta_j^{l2,r2} // {}^{t}\bar{S}_{r1} // {}^{t}S_{r2} // {}^{t}m^k_{l1,r1,r2,l2}$.

Proof. The proof follows from considering the action of Δ, S and m on tangle diagrams. The action of multiplication on two given strands, as we already saw, is that of concatinating the two strands. The comultiplication doubles a given strand without changing its orientation. The antipode reverses the orientation of a given strand. From the definition of a quasitriangular Hopf algebra, we know that $\mathcal{R} \in U_q(sl_3^{\epsilon}) \otimes U_q(sl_3^{\epsilon})$ obeys the Reidemeister moves. On knot diagrams we define the action of multiplication with the R-matrix as a crossing.
Now we easily see that the action of B is equivalent to

$$^{\mathbf{t}}\Delta^{i}_{l1,r1}\,^{\mathbf{t}}\Delta_j^{l2,r2} // {}^{\mathbf{t}}\bar{S}_{r1} // {}^{\mathbf{t}}S_{r2} // {}^{\mathbf{t}}m^k_{l1,r1,r2,l2}$$

in $U_q(sl_3^{\epsilon})$. This finishes the proof. □

In what follows, we denote ${}^{\mathbf{t}}Z_3^{\epsilon}(B)^k_{i,j}$ as ${}^{\mathbf{t}}B^k_{i,j}$ for short, sometimes leaving out the indices. We write down the following explicit form of ${}^{\mathbf{t}}B^k_{ij}$. For convenience, a^* and b^* have been put to zero. We will see that they do not play a role. We used

Mathematica to calculate ${}^{\mathbf{t}}B^k_{ij}$, see the appendix for more information.

$$
\begin{aligned}
{}^{\mathbf{t}}B^k_{ij} = \mathbb{E}\Bigg[0,\ & \frac{\mathbb{A}_k^{-2\hbar}\left(-\mathbb{A}_k^{2\hbar}+\mathbb{B}_k^{\hbar}\right)x^*{}_jX^*{}_i}{\hbar}+\frac{\mathbb{A}_k^{-2\hbar}\left(\mathbb{A}_k^{2\hbar}-\mathbb{B}_k^{\hbar}\right)x^*{}_iX^*{}_j}{\hbar}-z_kx^*{}_jy^*{}_i+\\
& z_kx^*{}_iy^*{}_j+\frac{\mathbb{B}_k^{-2\hbar}\left(\mathbb{A}_k^{\hbar}-\mathbb{B}_k^{2\hbar}\right)y^*{}_jY^*{}_i}{\hbar}+\frac{\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{\hbar}+\mathbb{B}_k^{2\hbar}\right)y^*{}_iY^*{}_j}{\hbar}+\\
& Y_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_jZ^*{}_i+\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbb{A}_k^{\hbar}-\mathbb{B}_k^{2\hbar}\right)x^*{}_jy^*{}_iZ^*{}_i}{\hbar}-X_k\mathbb{A}_k^{\hbar}\mathbb{B}_k^{-2\hbar}y^*{}_jZ^*{}_i+\\
& \frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(\mathbb{A}_k^{3\hbar}-\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_iy^*{}_jZ^*{}_i}{\hbar}+\frac{\mathbb{A}_k^{-2\hbar}\left(\mathbb{A}_k^{2\hbar}-\mathbb{B}_k^{\hbar}\right)x^*{}_jy^*{}_jZ^*{}_i}{\hbar}+\\
& \frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(1-\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)z^*{}_jZ^*{}_i}{\hbar}-Y_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_iZ^*{}_j+X_k\mathbb{A}_k^{\hbar}\mathbb{B}_k^{-2\hbar}y^*{}_iZ^*{}_j+\\
& \frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{3\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_iy^*{}_iZ^*{}_j}{\hbar}+\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{3\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_jy^*{}_iZ^*{}_j}{\hbar}+\\
& \frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}+\mathbb{B}_k^{3\hbar}\right)x^*{}_iy^*{}_jZ^*{}_j}{\hbar}+\frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(-1+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)z^*{}_iZ^*{}_j}{\hbar},1\Bigg]
\end{aligned}
$$

Note that in particular we get

$$
Z_3^0(K) = Z_3^0(G)//{}^{\mathbf{t}}B.
$$

Now we can prove the following. The proof is based on the proof in [36]. We use the term mixing, which refers to the $U_q(sl^{\epsilon}_{2,i})$ subalgebras of $U_q(sl^{\epsilon}_3)$ that are invariant under certain algebra maps. This translates to the tensor-formalism by looking at specific terms in the tensors that are used. To see this, we invite the reader to inspect the zipping-formula closely. In particular, we observe that terms like e^{uu^*} leave the $U_q(sl^{\epsilon}_{2,i})$ subalgebras invariant.

Theorem 3.3.1. *Let K be an oriented framed long knot. The knot invariant $Z_3^0(K)$ is the product of inverse of the Alexander polynomial of K in the variables S, T and ST.*

Proof. Let K be a knot, and let G be the tangle associated to the band representation of $S(K)$. We have $Z_3^{\epsilon}(K) = Z_3^{\epsilon}(G)//{}^{\mathbf{t}}B$. We first prove that the invariant factorizes into sl_2 parts by showing that the only terms that contribute to $Z_3^0(G)//{}^{\mathbf{t}}B$ are the terms of the form $\mathbb{E}[0, uU, 1]$, where $u = x, y, z$, and U stands for the dual (capital) generator.

By symmetry of the x, y, z terms occurring in ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}dm$ and the absence of mixing terms like x_iY_j in the non-perturbative part of the exponentials we will obtain the factorization. Note that we connect all strands of G to one strand, since K is a knot. Observe furthermore that we only need to consider $Z_3^0(G)//{}^{\mathbf{t}}B^k_{i,j}$, the case where G consists of two strands. By induction to the number of strands the theorem will follow for general G, with the same argument.

The second part is proving that each of the sl_2 terms zips to the Alexander poly-

nomial. This is done in a similar fashion, and is done explicitly in [36]. We will not do this here explicitly, but refer to [36] for the argument.
We start with rewriting the multiplication tensor ${}^{\mathbf{t}}dm$ and R-matrices with (note $\epsilon = 0$) $s = 2A - B$ and $t = 2B - A$, where we observe that in ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1} = {}^{\mathbf{t}}\bar{\mathcal{R}}$ the s and t only occur with an a or a b in front. Note that since the antipode S is the convolution inverse of Δ, and $\bar{S}(p) = S(p) = -p$ for p an element of the Cartan subalgebra, ${}^{\mathbf{t}}B^k_{i,j}$ has no terms that consist only of elements of the Cartan-subalgebra. This follows from lemma 3.3.1, since S and $\bar{S}$ are applied on the same index of Δ. After multiplication the indices are changed to k, and the a and b drop out. Observe that $Z(G)$ only consists of products of R-matrices and trivial curls (which are central elements).
In ${}^{\mathbf{t}}\mathcal{R}$, we see that s and t only occur in combination with a and b respectively. Hence we can set s_n and t_n for $n = i, j$ to zero in ${}^{\mathbf{t}}B^k_{i,j}$. That is, we put s and t with the'incoming' indices i and j in ${}^{\mathbf{t}}B^k_{i,j}$ to zero. This is equivalent to setting S and T to 1, since s and t will never occur from zipping a and b. This is because the a and b dependence in the Cartan part of the exponential ${}^{\mathbf{t}}Z_3^{\epsilon=0}(B)^k_{i,j}$ drops out. Since we take $\epsilon = 0$, we can put a^* and b^* to zero in $Z_3^0(G)$ (which follows from the format of ${}^{\mathbf{t}}dm$).
Now we look at the non-square terms in the R-matrices and the multiplication tensor ${}^{\mathbf{t}}dm$. A calculation in Mathematica shows that before the zipping the only cubic terms are (looking at ${}^{\mathbf{t}}dm^k_{i,j}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1}$):

$$X_i Y_i z_j, \tag{3.1}$$
$$x_i^* y_j^* z_k - Y_k x_i^* Z_j^* + X_k y_i^* Z_j^*. \tag{3.2}$$

3.2 arises in ${}^{\mathbf{t}}dm$, and 3.1 arrises in ${}^{\mathbf{t}}\mathcal{R}^{-1}$. If we look at the last two terms 3.2, we see that by symmetry of x and y occurring (and X and Y terms, as a result, since these only occur together) in ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1}$ the two will cancel out. Here we use that in the end we are left with one strand, i.e. one index. The first term of 3.2 and 3.1 are similar in the sense that after zipping, $x_i^* y_j^* z_k \mapsto X_{i'} Y_{j'} z_k$.
From ${}^{\mathbf{t}}dm$ we can see that the cubic terms are unchanged by zipping with ${}^{\mathbf{t}}B$, since X^* only occurs with an x in ${}^{\mathbf{t}}B$, and similarly for Y. The only way these terms could contribute is after zipping of ${}^{\mathbf{t}}Z_3^0(B)^k_{i,j}$ (without loss of generality we can take $i' = i$ and $j' = j$).
In ${}^{\mathbf{t}}B$, ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1}$, z^* does not occur in combination with x and y. We once again remind the reader that $\epsilon = 0$ for the duration of this proof, so we only consider the non-perturbative part of the exponentials. Similarly, x_i does not occur in combination with y_j. So there is no mixing of the variables. The reader is encuraged to check this for themselves using the Mathematica implementation in the appendix. This finishes the proof. □

The following theorem uses a rough upper bound for the number of variables zipped. The proof follows [35] by looking the sizes of the matrices in the zipping theorem. We are using the three-stage zip and the zippinng theorem on 8 generators in total, in the most optimal implementation of $U_q(sl_3^\epsilon)$. The computational complexity of calculating an m by m determinant is assumed to be m^3. Inverting the matrix is assumed to be of complexity m^3. Differentiating a monomial of degree m times is of complexity c^m for some constant c. So differentiating a polynomial of m' terms will take $O(m')$ computations. There exist faster algorithms, but we calculate an upper-bound. It is possible to generalize this argument to the case $\epsilon^k = 0$, see [36] for an argument in the sl_2 case.

Theorem 3.3.2. *The calculation of $Z_3^\epsilon(K)$ for a knot K with n crossings has a computational complexity of at most $O(n^{10})$.*

Proof. Let K be a knot with n crossings. We will assume that it has rotation-number 0 and writhe 0. This can be done by inserting curls after the last crossing, which does not infect the final result, apart from a normalization of the Alexander polynomial. Note that conjugating an element x in the Hopf algebra $U_q(sl_3^\epsilon)$ with C equals $S^2(x)$. So putting the curls that occur in K after the final crossing will only change the normalization with factors of q. Moreover, the curls cancel out. Note that S^2 multiplies the non-Cartan generators with a factor of q, so this operation does not contribute to the upper bound. So we can assume that the knot invariant is calculated by zipping over and under crossings and multiplication tensors.

We assume that the zipping is done at once for K, meaning there are n crossings where each crossing has 2 indices to be concatinated. This follows from the relation between edges and vertices in any 4-valent graph. Furthermore we observe that s and t are central, so we can take them to be coefficients, and leave them out of the zipping. As a consequence we drop the index from t_i, s_i and s_i^*, t_i^*.

The first step is zipping a, a^*, b^*, b. Consider a tensor of the form $E = e^{L+Q}P$, where L stands for the Cartan-part, Q stands for rest of terms in the ϵ-independent part. As we put s and t constant, the only contribution to L comes from ${}^{\mathbf{t}}dm$, and is of the form uu^*. This is a diagonal matrix, so differentiating and computing the inverse and determinant takes $O(n^2)$ operations. The contraction of the perturbation and differentiating takes $O(n)$ operations, as in each monomial of P, a, b have a degree of at most 2, and replacement of a with ∂_{a^*} also takes $O(n)$ computations.

The contraction of $\{X, Y, y^*, x^*\}$ takes the most computations, as it has the most generators that need to be contracted. Note that it is of the same complexity as zipping Z, z^*. Only the prefactor differs because of the number of variables and the number of monomials in P with the corresponding variables differ. Zipping E to the variables $\{X, Y, x^*, y^*\}$ computes the matrix q of Q by differentiating Q with ∂_{u,u^*} and taking its inverse. This takes $O(n^4)$ computations, as there are $\sim n$ generators in Q.

To compute the contraction of P, we count the number of monomials in P with the variables X, Y, y and x. Observe that there are $32n$ variables (counting the indices and the fact that we are always contracting two different indices) and

the degree of each monomial is 8 at most, looking at the multiplication tensor in the quantum double. Now we get a total of at most $\binom{32n+8}{8} \leq (32n+8)^8$ monomials. Differentiating each monomial is of a constant complexity. However, for contracting P we first need to substitute the variables by applying the zipping theorem. As this is a vector of length $16n$ at worst, this takes $O(n^2)$ operations at worst, since the substitution could depend on $\sim n$ variables. So we find that zipping to $\{X, Y, x^*, y^*\}$ takes $O(n^{10})$ operations at most.
We observe that applying the zipping theorem multiple times is more efficient than performing the zip for all variables at once (in the absence of cubic terms of course). In particular since this reduces the size of the matrices we work with. In this case we have 3 implementations of the zipping theorem, the contribution of each we can add together. This finishes the proof. □

Although in theory our computation is very efficient in terms of the number of crossings, the biggest problem occurs when scaling the sl_3 invariant to an sl_N invariant. The number of variables m per strand (currently 8) will increase with $m = (N-1)(N+1)$, if we count the number of generators in sl_N, compensating for the central elements in the algebra, of which there are $N-1$. Since we have to compute the inverse of a matrix of size m times m by differentiating, this takes $\sim m^4$ operations. Adding in the substitution in the zipping theorem which takes m^2 operations we get a complexity of $O(m^{10}) = O(N^{20})$ for an sl_N invariant where all the variables are zipped in one implementation (in the assumed absence of troublesome terms), and $O(N^{22})$ if the zips are splitted into N different zips (one zip for each group of generators associated to a root-vector of a particular length $< N$). So in terms of the number of crossings, the sl_N invariant is less efficient as say the sl_2 invariant, although not much less. But the problem arises from the constant prefactor.
In the case of sl_3 the prefactor arising from the number of generators in the $\{X, Y, y^*, x^*\}$-zip can be estimated to be approximately $\sim 8(2 \cdot 16)^{10}$. The factor of 4 arises from the number of operations it takes to differentiate each term (with a degree of at most 2 in each generator). The factor $2 \cdot 16$ counts the number of generators. If we start with $\{X, Y, y, x\}$ and the dual variables, we should add in a factor of 4 since we are always contracting 2 indices, and each crossing has two strands. For a small number of crossings we see that the contribution from the number of generators per crossing is very big in comparison to the contribution of n^{10} from the number of crossings. In total, we see that for sl_m this prefactor will be roughly $\sim 4^{10}m^{10}$. In particular the factor of 4 contributes a lot when m is small.
In conclusion, although the sl_3 invariant computes as n^{10} where n is the number of crossings, in practice, for $n = 6$, the prefactor will contribute a term n^{13}, making the effective complexity roughly $O(n^{23})$ for small knots up to 6 crossings. For knots up to 10 crossings this reduces to $O(10^{20})$. For bigger knots, the contribution will be smaller still. Only for knots of ~ 40 crossings, $n^{10} \sim 8(32)^{10}$.
We see that for the knots we are interested in the contribution of the number of

generators and the size of ${}^{\mathbf{t}}dm$ is more important than the number of crossings in the knots. This makes the computations very slow. It is therefore necessary to look at a way to reduce the number of monomials in the perturbation P by for example introducing a second parameter $\gamma^k = 0$ on the b^- side, to make the algebra nilpotent. γ is equivalent to ϵ, it is only present on the "other side" of the algerba $U_q(sl_3^\epsilon)$.
The effect will be that we compute terms of the expansion of Z_ϵ^3 in terms of γ, which are finite type invariants, which are less powerfull than the Z_3^ϵ invariant. It may be expected however, that when computed to a sufficient order of γ, these invariants will give enough information to for example prove that Z_3^ϵ distinguishes mutants.

Conclusion

Starting with the definition of knots, knots diagrams and the Alexander polynomial we have proven that the invariant Z_3^0 factorizes into Alexander polynomials. We used the Seifert surface to prove this fact. An algebraic implementation of the Seifert surface was used to compute the knot invariant. This was used, together with the action of the (co)multiplication and antipode on knot diagrams.
The main result of this chapter is the theorem that we can compute Z_3^ϵ in polynomial time, and in fact in $O(n^{10})$ computations, where n is the number of crossings in a given knot. The proof of this theorem was by considering the explicit zipping of R-matrices corresponding to a knot diagram of a knot K. We concluded wiith the observation that although this might seem a small cost, in practice this cost is much larger. This is mainly because of the number of generators of sl_3. For a general sl_n invariant this complexity will increase with $O(n^{12})$, where n is the rank of the algebra $U_q(sl_n)$.
More research is needed to bring this cost down. In particular the zipping of R-matrices could possibly be improved upon, to bring down the $O(n^{10})$ even further. It seems unlikely that the cost of the number of generators can be reduced significantly, but this is the most important contribution to the complexity of computing Z_3^ϵ. In comparison, for a knot of 6 crossings, this factor contributs roughly as $O(n^{20})$. This is much larger than the $O(n^{10})$ contribution of the zipping of the R-matrices. One way to reduce this cost greatly is by cutting off the multiplication in the quantum double. This can be done by introducing a second parameter dual to γ for example. Although in practice this reduces the strength of the knot invariant greatly, this seems to be the best bet to compute Z_n^ϵ for bigger knots.
Another possibility would be to reduce the number of variables involved in the zipping of the R-matrices. One way to do this, is to isolate the actual sl_3 invariant from the knot-polynomial $Z_3^\epsilon(K)$ of a knot K. For example, in the sl_2 case ane can isolate this part from a long expression in a central element. It is concievable that we can drop some zips by smartly zipping the R-matrices of a knot diagram and

still obtain the 'new part' of the sl_3 invariant. To obtain an idea of the general form of $Z_3^\epsilon(K)$ we need more data of course, but we can also look at the invariance of parts of $Z_3^\epsilon(K)$ under the q-Weyl group.
Another way to improve the computation speed would be to implement the computation of the determinants involved in the zipping of knots in C++ for example, or another efficient computer language. Since Mathematica is Python-based it may not be as efficient in computing and handling large expressions. Implementing the computation in C++ would also enable one to use larger computers for the computation of the knot diagrams.