Expansions of quantum group invariants

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2. Perturbed Gaussians and their applications

Introduction

In this chapter we describe an approach to do calculations in the quantum group described in the previous chapter. The idea is to describe Hopf algebra maps such as (co)multiplication as a single object, instead of a number of seperate relations. This method has an advantage when computing the knot invariant, it provides a formalism through which we can do computations in Wolfram Mathematica. Also, since the information of the Hopf algebra is contained in only a few objects that can be checked with the computer, there are less errors to be made. The goal is to prove that this construction is isomorphic to the construction given in the previous chapter. A key ingredient is the map $\mathbb{O} : O \to H$ introduced in the previous chapter for some ring O of commutative power series and a Hopf algebra H. This operator gives a vector space isomorphism between the commutative ring O and the Hopf algebra H. The essential theorem is the PBW theorem that is proven in the previous chapter.

2.1. The tensor formalism

Let A and A' be Hopf algebras over $R_\epsilon[[h]]$, where $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. We write $Hom(A, A') := Hom_{R_\epsilon[[h]]}(A, A')$ for the algebra homomorphisms between A and A'. Let $B \subset A$ be a finite subset of A that generates A. In other words, we assume that A has a basis of ordered monomials in the elements of B. In the previous chapter we saw that $B = \{X, Y, Z, A, B, b, a, z, y, x\}$ generates $U_q(sl_3^\epsilon)$ topologically.
Let V be the vector space over $R_\epsilon[[h]]$ generated the (finite) basis B. Let V^* be the linear dual of V. As B is finite, this is well defined. We will refer to the basis of V^* dual to z_i as z_i^*. $z_i^* \in V^*$ are not to be confused with the generators of the dual Hopf algebra A^*. To make sure this confusion does not take place, when we refer to the generators of the dual Hopf algebra A^*, we will write them with capital letters $Z_i \in A^*$. This will become important mainly in the second section, where we will work with the Hopf algebra constructed in the previous chapter.
Consider the tensor algebra $T(V)$ associated with V. $T(V)$ is equipped with the trivial product, which concatenates tensor products of elements of V. To make this product commutative, let us divide out to the relation $u \otimes v - v \otimes u$, $u, v \in$

$T(V)$. This new space is denoted as $S(V)$, the symmetric tensor algebra of V. In the same way one can construct $S(V^*)$. Note that $S(V)$ is isomorphic to the ring of polynomials in the elements of $S(V) \cong R_\epsilon[[h]][B]$. If $B = \{z_1, \cdots, z_n\}$, then the ring of polynomials in z_i is also written as $R_\epsilon[[h]][z_1, \cdots, z_n] = P[z_1, \cdots, z_n] =: P[B]$. Here $P = R_\epsilon[[h]]$.
$S(V) = P[B]$ can be equipped with the J-adic topology, where J is the ideal $(z_1, \cdots, z_n)$ generated by the elements $z_i \in B$. Then the J-adic topology has a basis consisting of the sets $x + J^n$, $x \in P[B], n \in \mathbb{N}$. The sets J^n are the polynomials of which the monomials have minimal degree n. In this topology, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a cauchy sequence if for all n, there exists $N > 0$ such that for all $i, j > N$, $x_i - x_j \in J^n$. This is equivalent to saying that x_i and x_j differ by a polynomial which has monimials of minimal degree n, in the ring $R_\epsilon[[h]][z_1, \cdots, z_n]$. To take the completion in the J-adic topology amounts to adding the limits of Cauchy sequences to the space. These limits can have nonzero powers in an infinite number of terms. So the completion of $S(V) = P[B]$ is isomorphic to the ring of formal power series in the generators B. We use the notation $\hat{S}(B^*) = P[[B^*]]$ for the completion $\hat{S}(V^*)$ of $S(V^*)$.
We will mainly use the ring of power series in $z_i \in B$, and leave the symmetric tensor algebra on the background. Instead of $R_\epsilon[[h]][z_1, \cdots, z_n]$ and $R_\epsilon[[h]][[z_1, \cdots, z_n]]$ we may also write $P[B]$ and $P[[B]]$ respectively. R_ϵ in $P = R_\epsilon[[h]]$ has the discrete topology, so exponentials in h exist only as formal power series in the ring $R_\epsilon[[h]]$, since $R_\epsilon[[h]]$ has the h-adic topology.
On $P[B]$, one can define the map $\mathbb{O}(\cdot|p) : P[B] \to A$ for an ordering p on A. We state the definition of $\mathbb{O}$ for a general Hopf algebra A with PBW basis consisting of monomials in $z \in B$. Whenever it is obvious what ordering p we use we will leave it out of the notation, writing simply $\mathbb{O}(\cdots)$. When we write a general Hopf algebra A, we have $U_q(sl_3^\epsilon) = D(U_h(b^-))$ in mind, which was constructed in the previous chapter, with $B = \{X, Y, Z, A, B, b, a, z, y, x\}$. By the PBW theorem, we have an isomorphism of $D(U_q(b^-)) = U_q(sl_3^\epsilon) \cong P[B]$, as was mentioned in the previous chapter.

Definition 2.1.1. *Let B be a finite set with ordering p. Let A be a Hopf algebra with a basis consisting of ordered monomials in elements of B. Define the map $\mathbb{O}(\cdot|p) : P[B] \to A$ as the map that sends an unordered expression $T \in P[B]$ to an expression $T \in A$ in which all monomials are ordered via ordering p on B.*

We have the following proposition.

Proposition 2.1.1. *$\mathbb{O} : P[B] \to D(U_h(b^-))$ is an isomorphism of $R_\epsilon[[h]]$ modules, with $B = \{X, Y, Z, A, B, b, a, z, y, x\}$.*

Proof. The proposition is obvious from the PBW theorem in chapter 1. □

This proposition holds for $\mathbb{O}(\cdot|p) : P[B] \to U_q(sl_3^\epsilon)$, and for any Hopf algebra A with 'nice' properties in general. If we wish to extend $\mathbb{O}$ to multiple copies of $P[B]$, as we would for example do for the multiplication $m : P[B_1, B_2] \to$

$P[B']$, we have to specify on which indices $\mathbb{O}$ takes its input. Especially when there are more copies than the algebra map takes input on, for example. This is dependent on which entries we specify the algebra maps to take their input. As a consequence, when writing down a map for which this could be ambiguous, we have to specify the entries of its input.
Let $J \subset \mathbb{N}$ be a finite subset of $\mathbb{N}$. We define the map $\mathbb{O}_J$ as the map $\mathbb{O}$ extended to $A^{\otimes J}$ and $P[B_j]_{j\in J}$. We can define an inverse of $\mathbb{O}$ by forgetting the ordering of a normal ordered expression $Q \in A$. For multiple indices, this is only well-defined if we specify on which entries (index $j \in J$) the expression Q is put by $\mathbb{O}^{-1}$, so we write $\mathbb{O}_I^{-1}$, for some finite set indices I.
We want to use $\mathbb{O}$ to be able to calculate $F : P[B] \to P[B']$ for any Hopf algebra map $F : A \to A'$ or on $A^{\otimes J}$ in general. This happens by taking the pullback of F under $\mathbb{O}$. The notation for $\mathbb{O}_J^{-1}$ is a specification of the input of F when F acts on $A^{\otimes J}$. In particular when composing multiple maps it is important to keep track of the indices. We denote $A_i := 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1 \subset A^I$, with a copy of A only on entry $i \in I$.

Definition 2.1.2. *Let B_i for each $i \in J$, where J is a finite index set, be a finite set of generators of the Hopf algebra $A_i \subset A^{\otimes J}$. Denote an element in $P[B_i]$ as $\boldsymbol{z}_i^{\boldsymbol{n}_i} = (z_1)_i^{n_1}(z_2)_i^{n_2}\cdots(z_m)_i^{n_m}$, where $(z_j)_i \in B_i$. For a subset $J' \subset J$, define $\mathbb{O}_{J'} : P[B_i]_{i\in J} \to A^{\otimes J'} \otimes P[B_i]_{i\in J-J'}$ as $\mathbb{O}$ on each index $j \in J'$ and as the identity on $j \in J - J'$. Conversely, define $\mathbb{O}_{J'}^{-1}$ as the map that is equal to $\mathbb{O}^{-1}$ on $A^{\otimes J}$ for each index $j \in J'$ and acts as the identity on the other indices.*

Concretely, for any map F we will have to specify its input data in terms of the indices or tensor factor on which F takes its input. So instead of writing F, we will write F_J^I, where I and J are a finite set of entries corresponding to the domain and codomain of $F : A^{\otimes I} \to A^{\otimes J}$.

Definition 2.1.3. *Let $F : A^{\otimes I} \to A^{\otimes J}$ be a map between tensor products of the algebra A. Define the map $F_I^J : P[B_i]_{i\in I} \to P[B_j]_{j\in J}$ as $\mathbb{O}_J^{-1} \circ F \circ \mathbb{O}_I : P[B_i]_{i\in I} \to P[B_j]_{j\in J}$.*

There are no requirements for a general map F on multiple copies of A. We distinguish cases. First we cover the one-dimensional case, where there is only one copy on both sides: $F : A \to A'$. This may be an homomorphism or an antihomomorphism like the antipode. Later we will cover the cases where F has $A^{\otimes J}$ as a domain and A as codomain and vice versa.
Let $a \in A$, then we define $F : P[B] \to P[B']$ using the PBW ordering on A. So $F : P[B] \to P[B']$ is denoted in the same way as $F : A \to A'$, and is defined as $\mathbb{O}^{-1} \circ F \circ \mathbb{O}$. Since the input of F is assumed to be normal ordered, $F : P[B] \to P[B']$ corresponding to an algebra map F is automatically a ring-homomorphism on $P[B]$. So $F \in Hom(P[B], P[B'])$. Hence we can define the following.

Definition 2.1.4. *Let $F \in Hom(P[B], P[B'])$ and let $B = \{z_i\}_{i\in J'}$, and $B^* = \{z_i^*\}_{i\in J'}$, where $z_i^*(z_j) = \delta_{ij}$. Define a map ϕ between $Hom(P[B], P[B'])$ and $P[B'][[B^*]]$ which*

maps $F \in Hom(P[B], P[B'])$ to ${}^{\mathbf{t}}F := F(\exp[\sum_{i\in J'} z_i^* z_i]) =$, $z_i^* \in B^*, z_i \in B$. We say that ${}^{\mathbf{t}}F$ is the tensor corresponding to F.

In this definition we denote elements of $P[B'][[B^*]]$ as ${}^{\mathbf{t}}F$ with a bold **t**. This is not to be confused with the transposed matrix, which we will denote with the usual t, and we write it as A^t for a matrix A. The map that one should have in mind is the antipode.

Proposition 2.1.2. *ϕ is well-defined.*

Proof. Note that $\exp[\sum z_i^* z_i] \in S(B) \otimes \hat{S}(B^*) = P[B][[B^*]]$. A homomorphism $F : P[B] \to P[B']$ yields a well defined map $F : P[[B^*]][B] \to P[[B^*]][B']$, which is the identity on $P[[B^*]]$. Since F is an homomorphism on $P[B]$, F is specified by its image on B. For any generator $z \in B$, $F(z)$ is an element of $P[B']$. F is linear in the generators z_i, and by definition acts trivially on B^*, so ${}^{\mathbf{t}}F = \exp[\sum z_i^* F(z_i)] \in P[B'][[B^*]]$. In particular, the tensor $\exp[\sum z_i^* z_i] \in P[B'][[B^*]]$ corresponds to the identity in $Hom(P[B], P[B'])$.

□

A more general version that is defined in the same way is where $F : A \to A'^{\otimes I}$ is an algebra homomorphsim. Consider $\mathbb{O}_I^{-1} \circ F \circ \mathbb{O} : P[B] \to P[B'_i]_{i\in I} = P[B'_I]$. We introduce the notation $B_I = \bigcup_{i\in I} B_i$, where $B_i = \{z_j\}_{j\in J_i}$ for all $i \in I$, for some finite index set I. Then we can extend ϕ by the same definition.

Definition 2.1.5. *Let $F \in Hom(P[B], P[B'_I])$ and let $B = \{z_i\}_{i\in J'}$, and $B^* = \{z_i^*\}_{i\in J'}$, where $z_i^*(z_j) = \delta_{ij}$. Define $\phi : Hom(P[B], P[B'_I]) \to P[B'_I][[B^*]]$ by sending $F \in Hom(P[B], P[B'_I])$ to ${}^{\mathbf{t}}F^I := F(\exp[\sum_{i\in J'} z_i^* z_i])$.*

Where F is an algebra map and is extended B^*-linearly to $P[B_I][[B^*]]$, so ${}^{\mathbf{t}}F^I = \exp[\sum_{i\in J'} z_i^* F(z_i)] \in P[B'_I][[B^*]]$. The map that one should have in mind is the comultiplication in a Hopf algebra.
For a map $F : A^{\otimes I} \to A$ the calculation of ${}^{\mathbf{t}}F$ is slightly more difficult, as there are more generators to check. We explicitly use $\mathbb{O}$ in the definition of ${}^{\mathbf{t}}F$ in this case.

Definition 2.1.6. *Consider the isomorphism $\mathbb{O} : P[B_I] \to A^{\otimes I}$ for finite I. Let $B_i = \{z_j\}_{j\in J'_i}$, J'_i finite for all $i \in I$, and let $B_i^* = \{z_j^*\}_{j\in J'_i}$, where $z_l^*(z_k) = \delta_{lk}$. For $F : A^{\otimes I} \to A'$, define $F_I := \mathbb{O}^{-1} \circ F \circ \mathbb{O}_I : P[B_I] \to P[B']$. Define $\phi : Hom(P[B_I], P[B']) \to P[B'][[B_I^*]]$ which maps $F \in Hom(P[B], P[B'_I])$ to ${}^{\mathbf{t}}F_I := F(\exp[\sum_{j\in\cup_{i\in I} J'_i} z_j^* z_j]) =$*

$$\mathbb{O}^{-1} \circ F \circ \mathbb{O}_I(\exp[\sum_{j\in\cup_{i\in I} J'_i} z_j^* z_j]).$$

Here $P[[B]]\hat{\otimes}P[[B']] \cong P[[B, B']]$, and in this sense we interpret $P[[B_I]]$ for finite I. We note that $\mathbb{O}_I : \bigotimes_{i\in I} P[B_i] = P[B_I] \to A^{\otimes I}$ is an isomorphism. The tensor

products are over $P = R_\epsilon[[h]]$. Usually the calculation of ${}^{\mathbf{t}}F_I$ something complicated. The typical example is when F is the multiplication map. This will be the most important instant of this construction. To illustrate the definition we will cover two examples. Through these examples, one can develop some intuition for what we are calculating later in the case of $U_q(sl_3^\epsilon)$.

Example 2.1.1. *Let A be the coalgebra $\mathbb{R}[[z]]$ generated by the element z, and the map $\Delta : A \to A \otimes A$, with $\Delta(z) = z \otimes 1 + 1 \otimes z$, and the usual counit sending z to 0 and 1 to $1 \in \mathbb{R}$. It is clear that this obeys the coalgebra axioms. We define the pullback of Δ under $\mathbb{O}$ as $\Delta_1^{2,3} = \mathbb{O}_{2,3}^{-1} \circ \Delta \circ \mathbb{O}_1$, where $\mathbb{O}_{2,3}^{-1}(a \otimes b) = a_2 b_3$, and 1 is not written. So $\Delta_1^{2,3}(z_1) = z_2 + z_3$.*
Following the above definitions we obtain ${}^{\mathbf{t}}\Delta_1^{2,3} = e^{z_1^* \Delta_1^{2,3}(z_1)} = e^{z_1^*(z_2+z_3)} \in \mathbb{R}[[z_1^*]][z_2, z_3]$. Conversely, we can obtain $\Delta_1^{2,3}(z_1)$ from ${}^{\mathbf{t}}\Delta_1^{2,3}$ by substituting $z_1 \mapsto \partial_{z_1^*}$ in the expression $z_1 {}^{\mathbf{t}}\Delta_1^{2,3} = z_1 e^{z_1^*(z_2+z_3)}$ and putting z_1^* to zero after differentiating. We denote this process as $\langle z_1 {}^{\mathbf{t}}\Delta_1^{2,3} \rangle_{z_1}$, or $\langle z_1 {}^{\mathbf{t}}\Delta_1^{2,3} \rangle_1$ for short. The subscript indicates that the variables z_i with index 1 are substituted for a derivative $\partial_{z_1^*}$, and that z_i^* with subscript 1 are put to zero. We obtain

$$\langle z_1 {}^{\mathbf{t}}\Delta_1^{2,3} \rangle_1 = (\partial_{z_1^*} e^{z_1^*(z_2+z_3)})|_{z_1^*=0} = z_2 + z_3.$$

This is the so called Feynman-trick, and can be generalized $\Delta_1^{2,3}(z_1^n)$, or even to exponentials of z. Since z is grouplike, this is an easy exercise. When z is not grouplike, the process will become more complicated.

Let us consider a more difficult example, the $U_q(b^+) \subset U_q(sl_2^\epsilon)$. Instead of checking the relations in Mathematica and using other tricks, as we will do in the next section, one can see the correctness of the tensors and other identities used in the implementation by directly performing the multiplication of $\mathbb{O}_{i,j}({}^{\mathbf{t}}\mathrm{Id}_i {}^{\mathbf{t}}\mathrm{Id}_j)$, for example. The following example is taken from [36].
Observe that it is (in theory) a straightforward but lengthy exercise to generalize this example to $U_q(sl_3^\epsilon)$. In practice one runs into the problem of calculating the commutator between ordered exponentials. This would correspond to the naive way of multiplying R-matrices. For $\epsilon = 0$ this can be done explicitly, but for $\epsilon \neq 0$ trails so far have been unsuccesful. One has to use some kind of Feynman-trick, using differential operators to obtain these commutators. See [36] for more information. In fact, this is where the zipping theorem originates.

Example 2.1.2. *Consider the universal enveloping algebra A of the Lie algebra generated by x and a and the relation $[x, a] = -x$. This becomes an algebra when we take the universal enveloping algebra, and we choose to ignore the underlying tensor-algebra structure by just writing $u \otimes v = uv \in \mathbb{R}[x, a]$. Consider multiplication m in A. We wish to consider the description of m as a tensor ${}^{\mathbf{t}}m$. Let L be finite. Remember that in $\mathbb{R}[x, a]^{\otimes \{i,j\} \cup L}$ we label the variables with i and j to indicate in which tensor factor they are. We write $x_i = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1$, with the x on the i-th tensor factor. We have to add indices to ${}^{\mathbf{t}}m$ to indicate on which tensor factor m is acting. We denote ${}^{\mathbf{t}}m_{i,j}^k$ for the*

multiplication tensor of the multiplication map $m : \mathbb{R}[x,a]^{\otimes\{i,j\}\cup L} \to \mathbb{R}[x,a]^{\otimes\{k\}\cup L}$. *In order to find* ${}^{\mathbf{t}}m$, *observe that A has a basis consisting of ordered monomials in x and a, by the PBW theorem, so that we can define* $\mathbb{O}^{-1}$. *Now we can define* ${}^{\mathbf{t}}m$ *as the pullback under* $\mathbb{O}$, *as we did before. We use order ax in this example.*

$$\begin{aligned}
{}^{\mathbf{t}}m^k_{i,j} &= \mathbb{O}_k^{-1}(m(\mathbb{O}_{i,j}(exp(x_i x_i^* + a_i a_i^* + x_j x_j^* + a_j a_j^*)|p))) \\
&= \mathbb{O}_k^{-1}(\sum \frac{(x_i^*)^q (a_j^*)^r}{q!r!}(e^{a_i^* a} x^q a^r e^{x_j^* x})) \\
&= \mathbb{O}_k^{-1}(\sum \frac{(x_i^*)^q (a_j^*)^r}{q!r!}(e^{a_i^* a}(a-q)^r x^q e^{x_j^* x})) \\
&= \sum \frac{(x_i^*)^q}{q!} e^{a_i^* a_k} e^{a_j^*(a_k - q)} (x_k)^q e^{x_j^* x_k} \\
&= exp((a_i^* + a_j^*)a_k + (e^{-a_j^*} x_i^* + x_j^*)x_k).
\end{aligned}$$

We can now do some basic calculations with ${}^{\mathbf{t}}m$, *like checking associativity, or calculating commutation relations. For checking associativity in tensor-language, one needs to check that* ${}^{\mathbf{t}}m^k_{1,2} // {}^{\mathbf{t}}m^l_{k,3} = {}^{\mathbf{t}}m^k_{2,3} // {}^{\mathbf{t}}m^l_{1,k}$, *for example. The concatination operation // is defined as* ${}^{\mathbf{t}}F // {}^{\mathbf{t}}G := {}^{\mathbf{t}}(G \circ F)$. *This is the main subject of the next section. For calculating [x,a] using* ${}^{\mathbf{t}}m$, *we proceed as follows. We start with the expression* $x \otimes a \in A \otimes A$, *which is in the "wrong" order. We now map this to* $\mathbb{O}^{-1}_{1,2}(x \otimes a) = a_2 x_1 \in \mathbb{R}[x_1, a_1, x_2, a_2]$, *which is a choice of convention for* $\mathbb{O}^{-1}_{1,2}$.

$$\begin{aligned}
{}^{\mathbf{t}}m^k_{1,2}(a_2 x_1) &= \langle a_2 x_1 exp((a_1^* + a_2^*)a_k + (e^{-a_2^*} x_1^* + x_2^*)x_k)\rangle_{1,2} \\
&= \langle a_2 exp(((a_1^* + a_2^*)a_k)e^{-a_2^*} x_k\rangle_2 = -x_k + a_k x_k = (a_k - 1)x_k.
\end{aligned}$$

2.2. The ZIP-function

We proceed with translating the Hopf algebra structure to the tensor formalism, including the (co)multiplication maps and the antipode. Composition of tensors is the most important tool that is developed in this section. Furthermore, we will prove that a certain space of perturbed Gaussian exponentials is closed under composition. Care will have to be taken when attempting to interpret general exponentials as maps on Hopf algebras. Some conditions will have to be put in place.
We want to be able to calculate the composition tensor ${}^{\mathbf{t}}(G \circ F) = {}^{\mathbf{t}}F \,//\, {}^{\mathbf{t}}G = \exp[\sum z_i^* G(F(z_i))]$ from the two seperate tensors ${}^{\mathbf{t}}F, {}^{\mathbf{t}}G$. In this section we will define the concatination // of two tensors using the ZIP function. To do calculations we wish to specify how to compute ${}^{\mathbf{t}}F^I_{I'} // {}^{\mathbf{t}}G^J_I$ for maps F and G on $U_q(sl_3^\epsilon)^{\otimes I}$.

In this section we consider an Hopf algebra A over the ring R_ϵ, except when noted otherwise, with topological generators $B = \{z_1, \cdots z_n\}$. The ordered mono-

mials in the elements of B form an $R_\epsilon[[h]]$-basis of A by definition. Furthermore we consider the vector space V linearly generated by the elements of B, which is a finite dimensional vector space. We introduced the notation $P[B] = R_\epsilon[[h]][z_1, \cdots, z_n]$ in the previous section. By the ordering operator we obtained an isomorphism $\mathbb{O} : P[B] \to A$, which uses the PBW basis of A of ordered monomials in the elements of B. The algebra $A = U_q(sl_3^\epsilon)$ obeys the conditions. In particular it has a PBW basis, and a set of generators $B = \{X, Y, Z, A, B, b, a, z, y, x\}$, with an ordering.
This was generalized in the previous section to the case where multiple copies of A are present by introducing indices through the map $\mathbb{O}_I : P[B_I] \to A^{\otimes I}$. This map provides an isomorphism of R_ϵ-modules. By specifying on which tensor factors a map acts it is possible to define the same maps on the space $P[B_I]$. Given a map F on $P[B_I]$, we defined its corresponding tensor ${}^tF_I$. In this section we continue with the concatination of such tensors.

Definition 2.2.1. *Let V and V^* be the vectorspaces generated by the finite sets B and its dual B^* respectively. Denote with $\langle , \rangle$ the natural pairing on V and V^* which evaluates the functionals of V^* on elements of V. Consider the space $S(V^*) \otimes S(V)$. Using the leibniz rule one can use this pairing to define a map $\langle \cdot \rangle : S(V^*) \otimes S(V) \to R_\epsilon[[h]]$ where we define $\langle \phi_1 \otimes \cdots \otimes \phi_m \otimes f_1 \otimes \cdots \otimes f_n \rangle = \langle \phi_1, \langle \phi_2, \cdots \langle \phi_m, f_1 \otimes \cdots \otimes f_n \rangle \cdots \rangle$, and extend the map linearly. Here $\phi_i \in V^*$ and $f_i \in V$.*

Note that $V^{**} = V$. Using the isomorphism between $S(V)$ and $R_\epsilon[[h]][B]$ we see that this pairing acts the same way as substituting the formal derivative ∂_{v^*} for $v \in V$ since the pairing by definition obeys the Leibniz rule. To obtain a pairing, we put v^* to zero after differentiating. We see that the pairing is well defined whenever the expression converges in $R_\epsilon[[h]]$. Since $S(V^*) \otimes S(V) \cong R_\epsilon[[h]][B, B^*]$, and this ring consists of finite polynomials, we see that this is always the case. We present the following alternative definition, which we will use throughout this chapter. It is a more specific version of $\langle \rangle$ in which we can pair specific generators of B. We refer to this function as the ZIP-function.

Definition 2.2.2. *Consider the finite sets B and its dual B^* respectively, and let $\phi_i \in B^*$ and $f_i \in B$. Define the ZIP-function as*

$$\langle \cdot \rangle_{\phi_i} : P[B^*, B] \to R_\epsilon[[h]] : \phi_1 \cdots \phi_i \cdots \phi_m f_1 \cdots f_n \mapsto \phi_1 \cdots \partial_{f_i} \cdots \phi_n f_1 \cdots f_n|_{\phi_i = 0},$$

sending $\phi_i \mapsto \partial_{f_i}$, and acting as the identity on ϕ_j, for $j \neq i$ and for all $f_i \in B$, and putting $f_i \mapsto 0$ after differentiating.
Let $zs \subset B$, let $\zeta s = \{x^* | x \in zs\}$, and let $Q \in P[B^*, B]$. To implicate dependence on $z \in B, \zeta \in B^*$ we write $Q(z^*, z)$. The ZIP-function $\langle \rangle_{\zeta s} : P[B^*, B] \to P[B^*, B]$ is written as $\langle \rangle_{\zeta_i} : Q(\zeta_i, z_i) \mapsto Q(\partial_{z_i}, z_i)|_{z_i = 0}$. If $\zeta s = \{\zeta_i\}_{i=1,\cdots,m}$, then we then define $\langle \rangle_{\zeta s} = \langle \cdots \langle \cdot \rangle_{\zeta_1} \cdots \rangle_{\zeta_m}$.

This function is called the ZIP-function, and we may refer to applying the ZIP function as zipping, or more general with the verb to zip. In this case, it is usually

clear from the context on which variables the ZIP-function is applied. Observe that the pairing $\langle , \rangle$ is symmetric in the definition of V and V^*, so we can choose to substitute $b \in B$ or $b^* \in B^*$ with a partial derivative. It is not clear that the definition is independent of the order in which the zipping occurs. This will be proven later in this section.
Consider the ring $P[B][[B^*]]$, which is isomorphic to the subspace $S(V) \otimes \hat{S}(V^*)$. An element $w \in P[B][[B^*]]$ can be written as the limit of a sequence $\{w_n\}$ of elements $w_n \in P[B][B^*]$. In this way, the definition of $\langle , \rangle$ can be extended to $P[B][[B^*]]$. Note that the extension of the ZIP-function to $P[B][[B^*]]$ is not always well-defined.
For example we have for a polynomial $P(x^*, x) = \sum a_{nm} x^{*n} x^m$ the formula

$$\langle P(x^*, x) \rangle_x = \sum n! a_{nn},$$

by simple calculation. This shows that $\langle e^{zz^*} \rangle_z$ diverges as R_ϵ is equipped with the discrete topology in $R_\epsilon[[h]]$. Observe furthermore that we cannot complete both $P[B]$ and $P[B^*]$, since for example $e^{\partial_x \partial_y} e^{xy}$ does not converge. Since R_ϵ has the discrete topology, $\langle e^{\partial_x} e^x \rangle_x$ does not converge either as a series in R_ϵ.
However, $R_\epsilon[[h]]$ is completed in the h-adic topology. So $\langle e^{\partial_x} e^{hx} \rangle_x = e^h$ does converge in the h-adic topology. This trick allows us to extend $\langle \rangle$ to exponentials of the form $e^{z_i + z_i^* + z_j z_i^*} \in P[B][[B^*]]$, while assuring convergence. While $\langle e^{z_i + z_i^* + z_i z_i^*} \rangle$ does not converge in $P[B][[B^*]]$, $\langle e^{hz_i + hz_i^* + hz_i z_i^*} \rangle$ does converge in $P[B][[B^*]]$. The convergence follows in essence by the observation that

$$\langle e^{hzz^*} \rangle = e^{h \partial_{z^* z^*}} = \sum_n \frac{h^n}{n!} \partial_x^n x^n = \sum_n \frac{h^n n!}{n!} = \frac{1}{1-h}.$$

More generally, we introduce formal parameters $\eta_i, y_i, \Theta_{ij} \in P[[B']]$, where $i, j \in I$, the index set labeling elements in B. B' is some finite set, and $P[[B']]$ equipped with the the J'-adic topology, where $J' = (B')$ is an ideal generated by the elements $J' = (z'_1, \cdots, z'_m)$ of B'. For convergence we require that $\eta_i, y_i, \Theta_{ij} \in (z'_1, \cdots, z'_m, h) \subset P[[B']]$. In practice, η_i,y_i and Θ_{ij} will be other variables, part of a bigger space on which we can zip later. We now prove convergence of the ZIP-function for a specific domain.

Proposition 2.2.1. *Consider the ring $P[[B', B^*]][B]$, where $B = \{z_1, \cdots, z_n\}$ and B^* its dual, and B' is some finite set. $P[[B']]$ is equipped with the (B')-adic topology, where (B') is the ideal generated by the elements of B'.Let (h, B') be the ideal generated by B' and h. Consider the subspace of exponentials*

$$\begin{aligned} W_B = &\{Q(B, B^*) \exp[c + \sum \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] | Q \in P[B', B, B^*], \\ &c, \eta_1, \cdots, \eta_n, y_1, \cdots, y_n, \Theta_{11}, \cdots, \Theta_{nn} \in (h, B') \subset P[[B']], \ det(1 - \Theta) \neq 0\}. \end{aligned}$$

Summation over indices appearing twice is assumed. Extend $\langle \rangle$ linearly in $P[[B']]$ to the ring $P[[B', B^*]][B]$. Let $w \in W_B$, and let $\zeta s = \{z_1^*, \cdots, z_m^*\} \subset B^*$. Then $\langle w \rangle_{\zeta s}$ is

convergent in $P[[B', B^*]][B]$.

Proof. Let $w = Q(B, B^*)\exp[c + \sum \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \in W_B$. Without loss of generality we can take $\zeta s = B$, and $Q = 1$, since we can obtain Q by differentiating the exponential a finite number of times to y_i and η_i. We will prove the proposition by induction to the number of elements of B. We have to prove that after one zip, the result is a power series in W_B. Then the result will follow by induction. Let us specify an order when zipping to B for now, as we do not know yet if zipping is independent of the order of zipping.
The case where $B = \{z\}$ will be proven in a seperate lemma. Let us state the result here. The idea is to reduce the general case to the case where $B = \{z\}$. Let us zip to $z_n^* \in \zeta s = \{z_1^*, \cdots, z_n^*\} = B^*$, so we substitute $z_n^* \mapsto \partial_{z_n}$. The one dimensional result is given by

$$\langle \exp[c + \eta z + y z^* + \Theta z^* z] \rangle_{\{z\}} = \det((1-\Theta)^{-1}) \langle \exp[c + \eta(1-\Theta)^{-1}(z+y)] \rangle_{\{z\}}.$$

The last expression is clearly convergent in $P[[B', B^*]][B]$. We now proceed with the general case, where sums over i and j are from 1 to n, unless stated otherwise. Let us take $c = 0$ without loss of generality.

$$\begin{aligned}
&\langle \exp[\sum_{i,j} \eta_i z_i + y_j z_j^* + \Theta_{ji} z_j^* z_i] \rangle_{z_n} = \\
&\langle exp[\sum_{i<n} \eta_i z_i + \sum_{j<n} y_j z_j^* + \sum_{i,j<n} \Theta_{ij} z_j^* z_i + (\eta_n + \sum_{j<n} \Theta_{nj} z_j^*) z_n \\
&+ (y_n + \sum_{i<n} \Theta_{in} z_i) z_n^* + \Theta_{nn} z_n^* z_n] \rangle_{z_n}.
\end{aligned}$$

Since we only zip to z_n^*, the variables z_i^*, $i < n$ are unaffected by the zipping. Hence we can take $c = \sum_{i<n} \eta_i z_i + \sum_{j<n} y_j z_j^* + \sum_{i,j<n} \Theta_{ji} z_j^* z_i$, $\eta = \eta_n + \sum_{j<n} \Theta_{nj} z_j^*$, $y = y_n + \sum_{i<n} \Theta_{in} z_i$, and $\Theta = \Theta_{nn}$ and apply the one dimensional formula. This yields

$$\langle \exp[\sum_{i,j} \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_n} = \frac{1}{1-\Theta_{nn}} \langle exp[c + \frac{1}{1-\Theta_{nn}} \eta(z_n + y)] \rangle_{z_n}.$$

To check that the result is an element of W_B, we note that Θ_{ij} is of a positive degree in $P[[B']]$, and likewise for y_n and η_n. Furthermore,

$$\frac{1}{1-\Theta_{nn}} \eta y = \frac{1}{1-\Theta_{nn}} (\sum_{i,j<n} \Theta_{jn}\Theta_{ni} z_j^* z_i + \eta_n y_n + \eta_n (\sum_{i<n} \Theta_{in} z_i) + y_n (\sum_{j<n} \Theta_{nj} z_j^*)).$$

As a coefficient of $z_j^* z_i$ we now get $(\Theta_{nj}\Theta_{in} + \Theta_{ij})$. Since $\Theta_{ij} \in (B', h) \neq P[[B']]$, and the ideal does not include 1 since none of the generators is invertible in $P[[B']]$, we know that the matrix $\mathbf{1} - \Theta_{ij} - \Theta_{jn}\Theta_{ni}$ is invertible. Note that $\mathbf{1} - \Theta$ is invertible for any matrix Θ with $\Theta_{ij} \in (B', h) \subset P[[B']]$. We conclude that the

result is indeed an element of W_B. This concludes the proof of the proposition. □

From now on, we restrict ourselfves to the space of formal power series specified in the proposition. We state a formula for the case where $B = \{z\}$. The derivatives in this section should be interpreted as formal derivatives. This interpretation makes sense as long as the variables are part of a formal power series ring. In W_B, the parameters and the elements of B only occur in the exponentials in pairs or triples. This means that when taking a derivative of the exponential to either an element of B or a parameter, the result will be an element of the formal series ring. We use this property in the proofs in this section.

Lemma 2.2.1. *Consider the ring $P[[B', B^*]][B]$, where $B = \{z\}$ and B^* its dual, and B' is some finite set. $P[[B']]$ is equipped with the (B')-adic topology, where (B') is the ideal generated by the elements of B'. Let (h, B') be the ideal generated by B' and h. Consider the subspace of exponentials*

$$\begin{aligned} W_z =&\{Q(z,z^*)\exp[c+\sum \eta z + yz^* + \Theta z^* z] | Q \in P[B', z, z^*], \\ &c, \eta, y, \Theta \in (h, B') \subset P[[B']], (1-\Theta) \neq 0\}. \end{aligned}$$

Let $w \in W_z$. Then

$$\begin{aligned} &\langle Q(z,z^*)\exp[c+\eta z + yz^* + \Theta z^* z]\rangle_{\{z^*\}} = \\ &\qquad (1-\Theta)^{-1}\langle Q((1-\Theta)^{-1}(z+y), z^*)\exp[c+\eta(1-\Theta)^{-1}(z+y)]\rangle_{\{z^*\}}. \end{aligned}$$

Proof. Let $w = \exp[\sum \eta z + yz^* + \Theta z^* z] \in W_z$. The proof of the lemma is by explicitly expanding $\langle w \rangle_{z^*}$. By differentiating to η, and $z \mapsto \partial_\eta$, we obtain the case where Q is not equal to 1. We can put $c = 0$ as zipping acts trivially on e^c. We now proceed with the actual proof.

$$\begin{aligned} \langle \exp[\eta z + yz^* + \Theta z^* z]\rangle_{\{z^*\}} &= \sum_{k,l,m} \frac{y^m \eta^l \Theta^k}{l!m!k!} \partial_z^{m+k} z^{l+k}|_{z=0} \\ &= \sum_{l,m} \frac{y^m \eta^l}{l!m!} \partial_z^m z^l \sum_k \binom{k+l}{k} \Theta^k|_{z=0} \\ &= \sum_{l,m} \frac{y^m \eta^l}{l!m!} \partial_z^m z^l (\frac{1}{1-\Theta})^{l+1}|_{z=0} \\ &= \frac{1}{1-\Theta} \sum_{p,m} \frac{y^m}{(p+m)!m!} \partial_z^m (\frac{\eta z}{1-\Theta})^{p+m}|_{z=0} \\ &= \frac{1}{1-\Theta} \sum_m \frac{y^m}{m!} (\frac{\eta}{1-\Theta})^m \sum_p \frac{(\frac{\eta z}{1-\Theta})^p}{p!}|_{z=0} \\ &= \frac{1}{1-\Theta} \langle \exp[(1-\Theta)^{-1}\eta(z+y)]\rangle_{z^*}. \end{aligned}$$

This finishes the proof. □

Consider an Hopf algebra A over $R_\epsilon[[h]]$ and a finite set I. As noted in the introduction, $A \cong P[B]$, which is completed in the h-adi topology. One can generalize the zip function to a finite tensor product $P[[B', B^*]][B]^{\otimes I} = P[[B'_I, B^*_I]][B_I]$, for a finite index set I and the completed tensor product. Since I is finite, convergence is clear by the previous proposition. In fact it is essentially identical to the case where we extend B. Only the map $\mathbb{O}$ is more complicated, as one has to keep track of the multiple tensor factors by introducing additional indices. This has been covered in the previous section.
For simplicity we will not write the tensor products explicitly, as $P[[B]] \otimes P[[B']] \cong P[[B, B']]$ when B and B' are finite and the tensor product is completed. We label elementary elements in $B^{\otimes I}$ with an index $i \in I$ to indicate the tensor factor on which they live. This is not to be confused with the labeling used earlier for the elements of B. Implicit is the isomorphism between $S(V)^{\otimes I}$ and $R_\epsilon[[h]][(z_j)_i | i \in I, (z_j)_i \in B_i]$. In practice we will denote the topological generators of A without an index, but as $B = \{X, Y, Z, A, B, b, a, z, y, x\}$ in the case of $U_q(sl_3^\epsilon)$.
We proceed with the general case of the previous lemma, which is what is used to calculate ${}^{\mathbf{t}}(G \circ F)$. We aim to calculate the ZIP-function explicitly, so we will restrict the definition of $\langle\rangle$ to the space where we can explicitly calculate the concatination ${}^{\mathbf{t}}F // {}^{\mathbf{t}}G = {}^{\mathbf{t}}(G \circ F)$ of two tensors, as we did in the case where $I = \{1\}$. The method we use to calculate explicit zippings on W_B is called the zipping formula. The theorem is true for any finite set B, given the conditions on W_B, independently of the underlying Hopf algebra.

Theorem 2.2.1. *(The zipping-formula) Let $B = \{z_1, \cdots, z_n\}$, and its dual B^*. Let*

$$Q(z_1^*, \cdots, z_n^*, z_1, \cdots, z_n) \exp[c + \sum_{i,j} (\eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i)] \in W_B.$$

Then

$$\begin{aligned} &\langle Q(z_1^*, \cdots, z_n^*, z_1, \cdots, z_n) \exp[c + \sum_{i,j} \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{B^*} \\ &= det(\tilde{\Theta}) \langle Q(z_1^*, \cdots, z_n^*, \sum_k (\tilde{\Theta}_{1k}(z_k + y_k)), \cdots, \sum_k (\tilde{\Theta}_{nk}(z_k + y_k))) \\ &\quad \exp[c + \sum_{i,k} \eta_i \tilde{\Theta}_{ik}(z_k + y_k)] \rangle_{B^*}. \end{aligned} \tag{2.1}$$

Denote by $\tilde{\Theta}$ the inverse of the matrix $1 - \Theta$. Sums are from 1 to n unless stated otherwise.

Proof. Let $w = Q \exp[\cdots] \in W_B$. Since Q is only a polynomial of finite degree, we can reduce the theorem to the case where $Q = 1$. Like before, we also can take $c = 0$ without loss of generality.

Before attempting the general case, let us prove the simpler case

$$\langle Q(z_j^*, z_i) \exp[\eta_i z_i + y_j z_j^*] \rangle_{z_j} = \langle Q(z_j^*, (z_i + y_i)) \exp[\eta_i (z_i + y_i)] \rangle_{z_j}.$$

We will prove this formula by replacing y_i by $\hbar y_i$. Introducing $\hbar$ yields

$$\langle Q(z_j^*, z_i) \exp[\eta_i z_i + \hbar y_j z_j^*] \rangle_{z_j} = \langle Q(z_j^*, (z_i + \hbar y_i)) \exp[\eta_i (z_i + \hbar y_i)] \rangle_{z_j}. \quad (2.2)$$

Evaluating at $\hbar = 0$ gives an identity that is obviously true. We will now prove that the identity holds when both sides are differentiated with respect to $\hbar$. Integrating to $\hbar$ will then prove the identity.
First, for any power series $Q(z)$, the operator $e^{\hbar y \partial_z}$ has the effect of shifting the argument of Q by $\hbar y$:

$$e^{\hbar y \partial_z} Q(z) = Q(z + \hbar y). \quad (2.3)$$

One proves the statement for any polynomial of finite degree by a straightforward induction argument on the degree. Then the statement is also clear for power series.
Now we can prove 2.2. Evaluating the left and right hand side of 2.2 at $\hbar = 0$, the equality holds clearly, as observed before. Let us differentiate the left hand side with respect to $\hbar$:

$$\begin{aligned} \partial_\hbar \langle Q(z_j^*, z_i) \exp[\eta_i z_i + \hbar y_j z_j^*] \rangle_{z_j} &= \langle y_j z_j^* Q(z_j^*, z_i) \exp[\eta_i z_i + \hbar y_j z_j^*] \rangle_{z_j} && (2.4) \\ &= \langle y_j z_j^* Q(z_j^*, z_i + \hbar y_i) \exp[\eta_i (z_i + \hbar y_i)] \rangle_{z_j} && (2.5) \\ &= \partial_\hbar \langle Q(z_j^*, z_i + \hbar y_i) \exp[\eta_i (z_i + \hbar y_i)] \rangle_{z_j}. && (2.6) \end{aligned}$$

Where we use the definition of the ZIP-function in the second and third equality, and 2.3 in the second equality. If we integrate 2.4 and 2.6 with respect to $\hbar$ we get 2.2, since we already know that 2.2 holds at $\hbar = 0$.
Let us proceed with proving 2.1. We will prove it by differentiating to Θ_{ij}, and show that both sides obey the same differential equiations

$$\partial_{\Theta_{ij}} \Psi = \partial_{\eta_i} \partial_{y_j} \Psi, \quad (2.7)$$

and that both sides agree on $\Theta_{ij} = 0$. If we can prove this, then both sides are actually the same, since this set of differential equations fully determine a solution. To see this, note that Ψ is a power series in $\Theta_{i,j}$ and η_i and y_j, so that both the right and the left hand side is a power series. The set of equation becomes a set of ordinary differential equations in the variables $\Theta_{i,j}$.
One can also try to prove a more general shift lemma, however to prove this lemma is much more cumbersome. It is insightful to prove the general shift lemma for the case where Θ is a 1×1 or a 2×2 matrix and P is a degree 1 polynomial. This gives a nice insight in where the determinant originates. The 1-dimensional case has been done in the previous lemma. The higher dimensional

calculations are left as an exercise for the curious reader.
Firstly, for $\Theta_{ij} = 0$, 2.1 reduces to 2.2, and hence holds true. Let Θ_{ij} be nonzero such that the matrix $1 - \Theta$ is invertible. We obtain on the left hand side

$$\partial_{\Theta_{ij}} \langle Q(z_j^*, z_i) \exp[c + \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_j} \tag{2.8}$$
$$= \langle z_i z_j^* Q(z_j^*, z_i) \exp[c + \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_j}$$
$$= \partial_{\eta_i} \partial_{y_j} \langle Q(z_j^*, z_i) \exp[c + \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_j}. \tag{2.9}$$

Let $A_{ij} = (1 - \Theta)_{ij}$ be an invertible matrix. Since $A_{ij}^{-1} = \frac{A_{ji}^{adj}}{det(A)}$, $\partial_{\Theta_{ij}}(det(A)) I_{nn} = \partial_{\Theta_{ij}}(A_{nk} A_{nk}^{adj}) = \partial_{\Theta_{ij}}(A_{nk}) A_{nk}^{adj} = -I_{in} I_{jk} A_{nk}^{adj} = -I_{in} A_{nj}^{adj} = -A_{ij}^{adj}$, where A^{adj} is the adjugate matrix, or the matrix with the determinants of the minors of A (A_{ij}^{adj} is the determinant of the minor of A that arises by deleting the i-th row and the j-th column and multiplying with a sign). We use the einstein summation convention if the same index appears twice in different matrices. However, I_{nn} means the n, n-th entry of the identity matrix. Hence, if A_{nk} is dependent on Θ_{ij}, then A_{nk}^{adj} is independent of Θ_{ij}, and so by the product rule, $\partial_{\Theta_{ij}}(det(A)) = -A_{ij}^{adj}$.

Using this identity, on the right hand side of 2.1 we get using the product rule,

$$\partial_{\Theta_{ij}}(\det((1-\Theta))^{-1}\cdot \tag{2.10}$$
$$\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))\exp[c+\eta_{i'}((1-\Theta)^{-1})_{ik}(z_k+y_k)]\rangle_{z_j})$$
$$=\partial_{\Theta_{ij}}(\det((1-\Theta))^{-1})\cdot$$
$$\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))\exp[c+\eta_i((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j})$$
$$+\frac{\partial_{\Theta_{ij}}((1-\Theta)^{-1}_{i'k})}{\det((1-\Theta))}\langle(z_k+y_k)\partial_{x_{i'}}(Q(z_j^*,x_{i'}))|_{x_{i'}=((1-\Theta)^{-1})_{i'k}(z_k+y_k)}$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}+$$
$$\frac{\partial_{\Theta_{ij}}((1-\Theta)^{-1}_{i'k})}{\det((1-\Theta))}\langle\eta_{i'}(z_k+y_k)Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}$$
$$=\frac{(1-\Theta)^{-1}_{ij}}{\det(1-\Theta)}\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j} \tag{2.11}$$
$$+\frac{(1-\Theta)^{-1}_{ik}(1-\Theta)^{-1}_{i'j}}{\det((1-\Theta))}\langle(z_k+y_k)\partial_{x_{i'}}(Q(z_j^*,x_{i'}))|_{x_{i'}=((1-\Theta)^{-1})_{i'k}(z_k+y_k)}$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}+$$
$$\frac{(1-\Theta)^{-1}_{ik}(1-\Theta)^{-1}_{i'j}}{\det((1-\Theta))}\langle\eta_{i'}(z_k+y_k)Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}$$
$$=\frac{1}{\det((1-\Theta))}\partial_{\eta_i}\partial_{y_j}\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k)) \tag{2.12}$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}.$$

To calculate the derivative of $(1-\Theta)^{-1}$ we used the identity $A^{-1}_{ij}=\frac{A^{adj}_{ji}}{det(A)}$. The third equality follows from the product rule. This finishes the proof of 2.1. □

The theorem implies that the ZIP-function as defined above is actually independent of the order in which we zip, so we can define the ZIP function for any set $\zeta s \subset B^*$ as substituting and differentiating to the corresponding elements of B.

Corollary 2.2.1. *Let B be a finite set of generators of a Hopf algebra A as above. Let W_B be the corresponding space of Gaussian exponentials in $P[[B',B^*]][B]$ for some finite set B'. Let $\zeta s \subset B^*$, and let $w \in W_B$. The ZIP-function $\langle w\rangle_{\zeta s}$ is independend of the order on ζs.*

We consistently chose to zip on elements of B^* in this section, and completed B^*. This was only a convention, as noted before. One might equally well choose to

substitute partial derivatives for elements of B, and complete the ring of power series in elements of B. When zipping to one set of variables this does not make a difference. However, when one chooses to zip to elements of B' afterwards for example, convergence can be an issue. One has to make sure that the exponentials that are being zipped on are convergent in the appropriate formal power series ring. Especially in maps on more than one copy of the Hopf algebra, this choice might make a difference.
In the next chapter we will have to specify which variables are completed as formal power series. This will be indicated by the phrase 'zipping to variables $\cdots$', where the polynomials in the variables $\cdots$ are then completed as power series. It is then a matter of checking that the ZIP-function is defined in each case. This mostly boils down to verification that the matrix Θ is invertible. Whenever this will come up, it will be clearly stated.
Let B and B' be finite as before, and let $F \in hom(P[B], P[B'])$. Let ${}^{\mathfrak{t}}F = \phi(F)$ as defined in the previous section. Define the map $\psi : im(\phi) \subset P[[B^*]][B'] \to Hom(P[B], P[B'])$ by sending ${}^{\mathfrak{t}}F \in im(\phi)$ to the map $\psi(F) : z \mapsto \langle zF \rangle_{B^*}$ where $z \in P[B]$. Since ${}^{\mathfrak{t}}F \in im(\phi) \subset P[[B^*]][B']$ is of the form $e^{\sum_i z_i^* F(z_i)}$, we can apply the zip-formula to see that $\langle z^{\mathfrak{t}}F \rangle_B$ converges when $z \in B$. Note here that $F(z_i) \in P[B']$, and that we only zip on the elements of B. When $z \in P[B]$ convergence is only clear when $z = Q \exp[c + \sum_i y_i z_i] \in P[B]$, where Q is a finite polynomial. In this case $z^{\mathfrak{t}}F \in W_B$, and we can use the above theorem.
More generally, when $F : A \to A^{\otimes I}$ is a map of algebras, ϕ is defined in the same way. Convergence follows in the same way as the elementary case where $F : A \to A$. The most complicated case is when $F : A^{\otimes I} \to A$. There is no way to define a universal inverse of ϕ, since it depends on the conventions chosen in $\mathbb{O}_I$ which indices have to be zipped on. The process works in the same way, but one has to consider the specific case at hand. This is what one needs to keep track of when working with tensors in practice. For the following propositions we restrict ourselves to the elementary case where $F : A \to A$ for simplicity. In this case we label the generators of the image of F differently, and denote the basis of A with B' instead of B. This is to make sure that we do not zip on the wrong set of generators.

Proposition 2.2.2. *The composition $\psi \circ \phi : Hom(P[B], P[B']) \to Hom(P[B], P[B'])$ is the identity.*

Proof. Both ψ and ϕ are continuous functions in the h-adic topology. The first because the pairing is continuous, and the second map is applying the homomorphism, which is clearly continuous. From the zipping formula and the definition of the zip formula, we can write down the following formula for monomials of generators. Since the composition $\psi \circ \phi$ is continuous, and the ZIP function is linear, we can then take the limit, so the formula will apply to elements in the completion. One can apply the ZIP function on each generator z_i sperately, and since F is a homomorphism, without loss of generality we can prove the formula for monomials of the form z_i^n, where $z_i \in B$. Note further that $F(z_i) \in P[B']$ is

independent of $z_i \in B$.

$$\begin{aligned}\psi(\phi(F))(z_i^n) &= \langle z_i^n \exp[\sum_j z_j^* F(z_j)]\rangle_{B^*} \\ &= (\frac{F(z_i)^n}{n!}(\partial_{z_i})^n z_i^n)|_{z_i=0} \\ &= F(z_i)^n = F(z_i^n).\end{aligned}$$

□

The following result follows by similar reasoning.

Proposition 2.2.3. *(composition law) Let $F : P[B] \to P[B']$ and $G : P[B'] \to P[B'']$ be homomorphisms, and let the sets B,B' and B'' be finite. Then whenever $\langle \phi(F)\phi(G)\rangle_{B'^*}$ is convergent,*

$$\langle \phi(F)\phi(G)\rangle_{B'^*} = \phi(G \circ F). \tag{2.13}$$

Proof. Because the ZIP function is a linear function on monomials of generators, we can take the limit of a sequence of monomials to obtain the result in the completion in the h-adic topology. Let J be the set of indices of the elements in B', and I the index set for B. Since F and G are functions on power series, we can take without loss of generality $F(z_k) = \prod_{j\in J}(z'_j)^{n_j}$, as we can write a general homomorphism as a sum of these elementary functions. An essential observation here is that we can indeed take the limit when considering infinite power series such as the exponential.
Let us take $F(z_i) = f_i(z'_i)^{m_i}$, where f_i is a constant, for all $i \in I$. The general case follows in the same way, one only has to pick out the right term in the exponential. We can write $\phi(F) = {}^{\mathbf{t}}F = \exp[\sum_{i\in I} z_i^* f_i (z'_i)_i^m]$, and similarly ${}^{\mathbf{t}}G = \exp[\sum_{i\in I}(z'_i)^* G(z'_i)]$. Since we can zip on one element of B' at the time, it is enough to check

$$\langle ((z_i^*)^n F(z_i)^n)^{\mathbf{t}} G\rangle_{(z'_i)^*} = (z_i^*)^n G \circ F(z_i^n).$$

This follows by a simple calculation.

$$\begin{aligned}\langle (z_i^*)^n F(z_i)^n \phi(G)\rangle_{(z'_i)^*} &= \langle (z_i^*)^n F(z_i)^n \exp[\sum_j z_j^* G(z_j)]\rangle_{(z'_i)^*} \\ &= \frac{G((z'_i)^{m_i n})(\partial_{z'_i})^{nm_i}}{(m_i n)!}(z_i^*)^n f_i^n (z'_i)^{nm_i}|_{z'_i=0} \\ &= (z_i^*)^n G(z_i)^{nm_i} f_i^n \\ &= (z_i^*)^n G(F(z_i^n)).\end{aligned}$$

This finishes the proof. □

We should note that when composing maps, one has to be careful with conver-

gence of the zip-function. The composition of tensors does not always converge. In particular, the spaces of Gaussian exponentials do not form a category. Whenever care should be taken, we will clearly mention this. In most cases that we encounter it is clear that the composition of tensors converges. We will however have to prove that ${}^tF \in W_B$ explicitly in these cases. When ${}^tF \in W_B$, it is clear that composing tensors converges. This tactic will be used in the next section. We cover a simple example of this method here, to introduce the reader to the general setting in the next chapter. This example continues the examples of the previous section.
We write ${}^tF_I^J //{}^tG_J^K$ for the composition of tensors ${}^tF$ and ${}^tG$. When writing $//$, it is implied that zipping takes place on the input J of G and the output J of F. Convergence will have to be proven in this case, and the zipping takes place on the corresponding elements of B_I. The variables on which is being zipped are left implicit, and will be clear from the tensors that are composed.

Example 2.2.1. *Consider the set $B = \{x, y, z\}$ and its dual $B^* = \{x^*, y^*, z^*\}$, and let $\mathbb{R}[[h]][[B_1^*, B_2^*]][B_1, B_2, B_3]$ be the polynomial ring in variables*

$$\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$$

over the formal power series in an indeterminate h and the variables of B_1^*, B_2^*. In this ring consider the element ${}^tZ = \mathbb{E}[x_1^* y_2^* z_3]$, where we introduce the notation $e^x = \mathbb{E}[x]$. In general ${}^tZ$ occurs in a sl_3-like multiplication tensor such as the following (simplified) tensor

$${}^tm_{1,2}^3 = \mathbb{E}[x_3({x^*}_1 + {x^*}_2) + y_3({y^*}_1 + {y^*}_2) + z_3({x^*}_1 {y^*}_2 + {z^*}_1 + {z^*}_2)].$$

Let $\zeta s = \{x_1^*, y_2^*\}$, naively. We wish to perform the following zip

$$\langle (x_1 y_2)\mathbb{E}[z_3 {x^*}_1 {y^*}_2]\rangle_{\zeta s}.$$

However ${}^tZ$ is not of the form $e^{\Theta z z^*}$, so we cannot apply the zipping theorem to this expression at once. We are forced to zip to x^* and y^* separately. First we calculate

$$\begin{aligned}\langle (x_1 y_2)\mathbb{E}[z_3 {x^*}_1 {y^*}_2]\rangle_{x_1^*} &= (x_1 y_2)\mathbb{E}[z_3 \partial_{x_1} {y^*}_2]|_{x_1=0} \\ &= (x_1 + y_2^* z_3) y_2|_{x_1=0} \\ &= y_2^* z_3 y_2.\end{aligned}$$

Zipping to y_2^* now yields z_3. In this calculation we used the zipping theorem on the second line. For bigger expressions this takes a lot of computations. With a smart choice of the zipping variables we might be able to zip a cubic term like ${}^tZ$ at once, using the zipping theorem.
Observe that ${}^tZ \in \mathbb{R}[[h]][[y_1, y_2, y_3, x_1^*, x_2^*, z_1^*, z_2^*, z_3]][y_1^*, y_2^*, x_1, x_2, x_3, z_1, z_2]$. In this ring, let us take $\zeta s = \{y_2, x_1^*\} \subset \{y_1, y_2, y_3, x_1^*, x_2^*, z_1^*, z_2^*\}$. Now ${}^tZ$ is of the form $e^{\sum \Theta_{ij} z_j^* z_i}$. Concretely, we can take $\Theta_{12} = z_3$, and all other entries of Θ zero. Note that

$x_1^* \in \zeta s$ plays the role of the 'starred' variable and y_2^* plays the role of the 'z'.
Now $1-\Theta = \begin{pmatrix} 1 & -z_3 \\ 0 & 1 \end{pmatrix}$, with inverse $(1-\Theta)^{-1} = \begin{pmatrix} 1 & z_3 \\ 0 & 1 \end{pmatrix}$ and determinant 1. We can apply the zipping theorem to the formula, where we have to substitute $\sum_k (1-\Theta)^{-1}_{1k}(x_k) = (x_1 + y_2^* z_3)$ for x_1 in the polynomial in front of the exponential. This yields

$$\begin{aligned} \langle (x_1 y_2) \mathbb{E}[z_3 {x^*}_1 {y^*}_2] \rangle_{\zeta s} &= 1 \cdot \langle ((x_1 + z_3 y_2^*) y_2) \rangle_{\zeta s} \\ &= z_3 \end{aligned}$$

We are now ready to calculate more complicated examples, such as the multiplication of e^x and e^y by using ${}^t m$ and zipping $e^{x_1+y_2} {}^t m$ to $\zeta s = \{y_2, x_1^*\}$. This is now a simple application of the zipping theorem. This is left to the curious reader.

There is not much difference between the two ways of zipping in this example. However, when the multiplication becomes more difficult, when we wish to implement the R-matrix for example, it does matter much what choice of variables we make. There will be only a few correct choices in starred and unstarred variables to make that will lead to convergent expressions. Moreover, as we illustrated in the first half of the example, sometimes one can only zip in steps.
In the next section we will encounter a combination of these problems. As the number of variables is increased, we will make a specific choice to which variables we zip, and we will have to zip to those variables in three steps. This way of zipping is referred to as the three-way or three-stage zip. It is this method that will enable us to implement the $U_q(sl_3^\epsilon)$ Hopf algebra in Mathematica. One must not forget the essential role ϵ plays here. ϵ cuts of higher order terms that would occur in the exponential otherwise. This allows us to use the zipping theorem in calculating these zips.

2.3. The tensor formalism for $U_q(sl_3^\epsilon)$

In this section we give the explicit tensors for the $U_q(sl_3^\epsilon)$ quantum group, and prove that the formulas we use in the implementation in Mathematica are correct. We have proven that the $U_q(b^-)$ is a Hopf algebra in the previous chapter. In the previous section we have proven the essential zip-theorem. In the implementation of this theorem in Mathematica, we check the axioms for the tensors associated with the $U_q(b^-)$ algebra. We prove in this section that the given tensors indeed correspond to the $U_q(b^-)$ algebra.
The tensors and algebra relations are calculated by implementing the Drinfel'd-double construction in the tensor formalism. We define the tensors up to order ϵ. We can obtain the order ϵ^k term from the usual $U_q(b^-) \subset U_q(sl_n)$ QUE algebra. See chapter 1 and chapter 4.
We denote $T \in W_B$, for $B = \{X, Y, Z, A, B, b, a, z, y, x\}$ as $\mathbb{E}[L, Q, P]$, where P is the term proportional to ϵ, which is finite, since we work over $R_\epsilon[[h]]$. L is the ϵ independent part that only depends on A, B, b and a. Q is the ϵ independent part

of T in the exponential that is not in L.
Instead of h, we may use $\hbar$. To simplify the notation, we introduce $\mathbb{A} = exp[-hA]$, and similarly for B. For the explicit tensor ${}^{\mathbf{t}}dm_{ij}^{k}$ for the multiplication in the quantum double we refer to the appendix.
We should warn the reader that the starred variables in this section correspond to the dual variables of elements of $U_q(sl_3^\epsilon)$, as to indicate on which entry a tensor ${}^{\mathbf{t}}F$ takes its input and output. This should not be confused with a choice for the set of variables on which we zip.
In this section we use the three stage ZIP function. We first zip to the variables $\{A^*, B^*, b, a\}$, then to $\{X, Y^*, y, x^*\}$ and at last to $\{Z^*, z\}$. In this section we prove that this choice, together with the stated expressions leads to a convergent implementation of $U_q(sl_3^\epsilon)$ in the tensor-formalism.

Theorem 2.3.1. *The quantum double multiplication dm of the Hopf algebra $D(U_q(b^-))$, where $b^- \subset sl_3^\epsilon$ is the lower borel sub Lie bialgebra, can be constructed as a tensor ${}^{\mathbf{t}}dm$ by composing ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$, ${}^{\mathbf{t}}\overline{\mathcal{R}}$ and ${}^{\mathbf{t}}P$ as*

$$\begin{aligned}
{}^{\mathbf{t}}dm_{i,j}^{k} = (\mathbb{E}_{\{i,j\}}^{\{i,j\}}[&a_j a^*{}_j + A_i A^*{}_i + b_j b^*{}_j + B_i B^*{}_i, \\
&X_i X^*{}_i + Y_i Y^*{}_i + Z_i Z^*{}_i + x_j x^*{}_j + y_j y^*{}_j + z_j z^*{}_j, 1] \\
&\left({}^{\mathbf{t}}a\Delta_i^{\{1,2\}} //{}^{\mathbf{t}}a\Delta_2^{\{2,3\}} //{}^{\mathbf{t}}\overline{aS}_3\right)\left({}^{\mathbf{t}}b\Delta_j^{\{-1,-2\}} //{}^{\mathbf{t}}b\Delta_{-2}^{\{-2,-3\}}\right)) \\
&// \left({}^{\mathbf{t}}P_{-1,3}{}^{\mathbf{t}}P_{-3,1}{}^{\mathbf{t}}am_{2,j}^{k}{}^{\mathbf{t}}bm_{i,-2}^{k}\right).
\end{aligned}$$

Moreover, the composition of tensors converges. In particular, the tensor ${}^{\mathbf{t}}am$ corresponds to multiplication on $U_q(b^-)$, ${}^{\mathbf{t}}bm$ corresponds to multiplication on $U_q(b^-)^*$, ${}^{\mathbf{t}}\mathcal{R}$ corresponds to the R-matrix of $U_q(sl_3^\epsilon)$, ${}^{\mathbf{t}}\overline{\mathcal{R}}$ to its inverse and ${}^{\mathbf{t}}P$ to the pairing between $U_q(b^-)$ and $U_q(b^-)^*$. ${}^{\mathbf{t}}\overline{aS}$ is the inverse of the antipode on $U_q(b^-)$ and ${}^{\mathbf{t}}a\Delta$ and ${}^{\mathbf{t}}b\Delta$ are the comultiplication on $U_q(b^-)$ and $U_q(b^+)$ respectively.

${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t P$ have the following explicit form

$$
\begin{aligned}
{}^t am_{ij}^k &= \mathbb{E}\left[a_k\left(a^*{}_i + a^*{}_j\right) + b_k\left(b^*{}_i + b^*{}_j\right),\right.\\
&\quad x_k\left(e^{2a^*{}_j - b^*{}_j}x^*{}_i + x^*{}_j\right) + y_k\left(e^{-a^*{}_j + 2b^*{}_j}y^*{}_i + y^*{}_j\right) +\\
&\quad z_k\left(e^{2a^*{}_j - b^*{}_j}x^*{}_i y^*{}_j + e^{a^*{}_j + b^*{}_j}z^*{}_i + z^*{}_j\right),\\
&\quad 1 - e^{-a^*{}_j - b^*{}_j}\hbar\left(e^{3a^*{}_j}x_k y_k x^*{}_i y^*{}_j + e^{2a^*{}_j + 2b^*{}_j}y_k z_k x^*{}_i y^*{}_i y^*{}_j + e^{3a^*{}_j}y_k z_k x^*{}_i\left(y^*\right)_j^2 -\right.\\
&\quad \left.\left.e^{3a^*{}_j}x_k z_k x^*{}_i z^*{}_j + e^{3b^*{}_j}y_k z_k y^*{}_i z^*{}_j - e^{3a^*{}_j}z_k^2 x^*{}_i y^*{}_j z^*{}_j\right)\epsilon + O[\epsilon]^2\right]\\
{}^t bm_{ij}^k &= \mathbb{E}\left[A_k A^*{}_i + A_k A^*{}_j + B_k B^*{}_i + B_k B^*{}_j,\right.\\
&\quad X_k X^*{}_i + X_k X^*{}_j + Y_k Y^*{}_i + Y_k Y^*{}_j + Z_k Z^*{}_i + Z_k Z^*{}_j,\\
&\quad 1 + \left(-X_k A^*{}_i X^*{}_j + \hbar X_k Y_k X^*{}_j Y^*{}_i - Y_k B^*{}_i Y^*{}_j + 2Z_k X^*{}_i Y^*{}_j - \hbar X_k Z_k X^*{}_j Z^*{}_i +\right.\\
&\quad \left.\left.\hbar Y_k Z_k Y^*{}_j Z^*{}_i - Z_k A^*{}_i Z^*{}_j - Z_k B^*{}_i Z^*{}_j\right)\epsilon + O[\epsilon]^2\right]\\
{}^t\mathcal{R}_{ij} &= \mathbb{E}\left[\hbar a_j A_i + \hbar b_j B_i, \hbar x_j X_i + \hbar y_j Y_i + \hbar z_j Z_i,\right.\\
&\quad \left.1 + \frac{1}{2}\left(\hbar^3 x_j^2 X_i^2 + \hbar^3 y_j^2 Y_i^2 + \hbar^3 z_j^2 Z_i^2\right)\epsilon + O[\epsilon]^2\right]\\
{}^t\overline{\mathcal{R}}_{ij} &= \mathbb{E}\left[-\hbar a_j A_i - \hbar b_j B_i, -\hbar x_j X_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{-\hbar} + \hbar^2 X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} -\right.\\
&\quad \hbar y_j Y_i \mathbb{A}_i^{-\hbar}\mathbb{B}_i^{2\hbar},\\
&\quad 1 +\\
&\quad \left(-2\hbar^4 x_j X_i^2 Y_i z_j \mathbb{A}_i^{3\hbar} + \frac{3}{2}\hbar^3 x_j^2 X_i^2 \mathbb{A}_i^{4\hbar}\mathbb{B}_i^{-2\hbar} - \hbar^2 a_j x_j X_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{-\hbar} -\right.\\
&\quad 2\hbar^3 x_j X_i y_j Y_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar^3 X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + \hbar^3 a_j X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} +\\
&\quad \hbar^3 b_j X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + 2\hbar^2 x_j y_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + 2\hbar^2 z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar^2 a_j z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} -\\
&\quad \hbar^2 b_j z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar^2 b_j y_j Y_i \mathbb{A}_i^{-\hbar}\mathbb{B}_i^{2\hbar} + 2\hbar^5 X_i^2 Y_i^2 z_j^2 \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} -\\
&\quad 2\hbar^4 X_i Y_i z_j^2 Z_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} + \frac{3}{2}\hbar^3 z_j^2 Z_i^2 \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} - \hbar^4 X_i y_j Y_i^2 z_j \mathbb{B}_i^{3\hbar} +\\
&\quad \left.\left.\frac{3}{2}\hbar^3 y_j^2 Y_i^2 \mathbb{A}_i^{-2\hbar}\mathbb{B}_i^{4\hbar}\right)\epsilon + O[\epsilon]^2\right]\\
{}^t P^{ij} &= \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{a^*{}_j A^*{}_i}{\hbar} + \frac{b^*{}_j B^*{}_i}{\hbar}, \frac{x^*{}_j X^*{}_i}{\hbar} + \frac{y^*{}_j Y^*{}_i}{\hbar} + \frac{z^*{}_j Z^*{}_i}{\hbar},\right.\\
&\quad \left.1 + \left(-\frac{\left(x^*\right)_j^2\left(X^*\right)_i^2}{2\hbar} - \frac{\left(y^*\right)_j^2\left(Y^*\right)_i^2}{2\hbar} - \frac{\left(z^*\right)_j^2\left(Z^*\right)_i^2}{2\hbar}\right)\epsilon + O[\epsilon]^2\right].
\end{aligned}
$$

The proof of this theorem consists of multiple steps, and will be the subject of this section. We prove that the comutiplication and the antipode of the quantum double can be obtained from the multiplication on $U_q(b^-)$ and $U_q(b^-)^*$, the R-matrix $\mathcal{R}$, its inverse $\mathcal{R}^{-1}$ (also denoted as $\overline{\mathcal{R}}$ to avoid confusion in the indices)

and the pairing P in proposition 2.3.1.
We then provide an iterative formula (in orders of ϵ) to obtain the pairing and the inverse R-matrix for any order of ϵ, in propositions 2.3.2 and 2.3.3. Note that although we work modulo ϵ^2, we will provide proofs for general order of ϵ, whenever possible.
For the R-matrix we have an explicit formula due to the Faddeev-Quesne formula, which was proven in the first chapter. We then check that the stated tensors correspond to the pullback of the Hopf algebra maps, and that zipping on the tensors converges. In particular proving that zipping on the pairing converges is an important step in the proof of theorem 2.3.1.
Convergence of the above stated zip follows by explicitly checking that the tensors are of the correct format. When we have proven that the zips converge in the necessary compositions of tensors, we use the composition lemma in the previous section to compose tensors.
We start with providing formulas for the comultiplication and the antipode. In lemma, we will provide the corresponding tensor identities, and prove that the zipping of these identities indeed converges. This way we obtain ${}^{t}a\Delta$, ${}^{t}b\Delta$, ${}^{t}aS$ and ${}^{t}bS$. The inverse of the antipode on $U_q(b^-)$ is computed in the same way as the inverse R-matrix in a seperate proposition.

Proposition 2.3.1. *(Comultiplication and antipode) Let $u \in U_q(b^-)$, $v \in U_q(b^+)$ and $w \in U_q(sl_3^\epsilon)$. It is understood that $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ can be written without summation sign. The following relations hold for $U_q(sl_n^\epsilon)$*

$$\begin{aligned}
S_a(u) &= \overline{\mathcal{R}}^{(2)} \langle \overline{\mathcal{R}}^{(1)}, u \rangle, \\
S_b(v) &= \mathcal{R}^{(1)} \langle v, aS(\mathcal{R}^{(2)}) \rangle \\
\Delta(u)_a &= \mathcal{R}^{(2)} \otimes \mathcal{R}^{(2)} \langle \mathcal{R}^{(1)} \mathcal{R}^{(1)}, u \rangle \\
\Delta(v)_b &= \mathcal{R}^{(1)} \otimes \mathcal{R}^{(1)} \langle v, \mathcal{R}^{(2)} \mathcal{R}^{(2)} \rangle
\end{aligned}$$

Proof. Let S be the antipode in the double $U_q(sl_3^\epsilon)$. For the first identity, we have $\overline{\mathcal{R}}_{1,2} = \overline{S} \otimes Id(\mathcal{R})$. This result follows from the double construction with the opposite comultiplication and the quasitriangularity axioms. Now we get $\overline{\mathcal{R}}^{(2)} \langle \overline{\mathcal{R}}^{(1)}, u \rangle = \mathcal{R}^{(2)} \langle bS(\mathcal{R}^{(1)}), u \rangle$. $S \otimes S(\mathcal{R}) = \mathcal{R}$, so

$$\mathcal{R}^{(2)} \langle bS(\mathcal{R}^{(1)}), u \rangle = aS(\mathcal{R}^{(2)}) \langle \overline{bS}(bS(\mathcal{R}^{(1)})), u \rangle = aS(\mathcal{R}^{(2)} \langle \mathcal{R}^{(1)}, u \rangle.$$

Since $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ are the sums of dual bases with respect to the pairing by construction of the R-matrix, we get the desired result. The results for bS follows similarly.
For the formulas for $a\Delta$ and $b\Delta$ we use the quasitriangular property of the comultiplication Δ of $U_q(sl_3^\epsilon)$ $\Delta(\mathcal{R}^{(1)}) \otimes \mathcal{R}(2) = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(1)} \otimes (\mathcal{R}^{(2)} \mathcal{R}(2))$. Applying $\langle \cdot, u \rangle$ on both sides to the third factor yields the desired result. Again we use that

the R-matrix is constructed by summing over dual bases of $U_q(b^\pm)$. □

The inverse R-matrix, the pairing and the inverse of the antipode aS can be recursively calculated from the zeroth order operation in ϵ, the R-matrix, the multiplication and the antipode on both sides of the double. In the case of the pairing there is also a recursive formula, which we will prove here.
For a generator $u \in U_q(b^\pm)$, define Δ_k as the expansion of Δ in ϵ up to and including the k-th order term. If $v = u_1 \cdots u_n$ is a product of generators, not necessarily in the PBW ordering, define $\Delta_k(v) = \Delta_k(u_1) \cdots \Delta_k(u_n)$. Define $\Delta_k^{(n)}$ similarly ($\Delta^{(n)}$ stands for Δ applied n times).
Remember that $\langle UV, u'v' \rangle = \langle \Delta(UV), u' \otimes v' \rangle$. Define the k-th order pairing as

$$\begin{aligned}\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_k &= \langle \Delta_k^{(n)}(U_1 \cdots U_n), u_1 \otimes \cdots \otimes u_n \rangle \\ &= \langle U_1 \otimes \cdots \otimes U_n, \Delta_k^{(n)}(u_1 \cdots u_n) \rangle.\end{aligned}$$

For $n \neq n'$, $\langle U_1 \cdots U_{n'}, u_1 \cdots u_n \rangle = 0$ by duality. Further observe that

$$\langle \Delta^{(n)}(U_1 \cdots U_n), \Delta^{(n)}(u_1 \cdots u_n) \rangle = \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle,$$

as there is only one term in $m^{(n)}_{1,\cdots n}(\Delta^{(n)}(U_1 \cdots U_n))$ that has the correct order.

Proposition 2.3.2. *Let $U_i \in U_q(b^+)$ and $u_i \in U_q(b^-)$ for $i = 1, \cdots n$ and where we take ϵ invertible for the moment (See chapter 1 or 4). Then*

$$\begin{aligned}&\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle \ mod\ \epsilon^{k+1} \\ =&\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_0 + \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_{k-1} \\ &- \langle U_1 \cdots U_n, \mathcal{R}^{(2)} \rangle_0 \langle \mathcal{R}^{(1)}, u_1 \cdots u_n \rangle_{k-1} \ mod\ \epsilon^{k+1}.\end{aligned}$$

Proof. If we prove that $\langle U_1 \cdots U_n, \mathcal{R}^{(2)} \rangle_0 \langle \mathcal{R}^{(1)}, u_1 \cdots u_n \rangle_{k-1}$ equals

$$\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_0 + \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_k - \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_{k-1},$$

then we are finished. To prove this, observe that since $\mathcal{R}$ is the summation of dual basis, i.e. it is the canonical R-matrix, we have $\langle \cdot, \cdot \rangle = \langle \cdot, \mathcal{R}^{(2)} \rangle \langle \mathcal{R}^{(1)}, \cdot \rangle = \langle \Delta^{(n)}(\cdot), \mathcal{R}^{(2)}_{\otimes n} \rangle \langle \mathcal{R}^{(1)}_{\otimes n}, \Delta^{(n)}(\cdot) \rangle$, for any order of k. Here $V_{\otimes n}$ stands for the expression V split into n tensor factors. By coassociativity this is well-defined inside the pairing.
To prove the formula, we look at the expression

$$\begin{aligned}&\langle U_1 \cdots U_n, \mathcal{R}^{(2)} \rangle_0 \langle \mathcal{R}^{(1)}, u_1 \cdots u_n \rangle_{k-1} \\ &= \langle \Delta_0^{(n)}(U_1 \cdots U_n), \mathcal{R}^{(2)}_{\otimes n} \rangle \langle \mathcal{R}^{(1)}_{\otimes n}, \Delta_{k-1}^{(n)}(u_1 \cdots u_n) \rangle = \langle \Delta_0^{(n)}(U_1 \cdots U_n), \Delta_{k-1}^{(n)}(u_1 \cdots u_n) \rangle.\end{aligned}$$

We notice that for $n = 1$ the relation holds trivially, since $\Delta_0(U_i) = \Delta(U_i)$, because in $U_q(b^+)$, there is no factor of ϵ present in Δ. For $n > 1$, we look at the

ordering of $\Delta_0^{(n)}(U_1 \cdots U_n) = \Delta^{(n)}(U_1 \cdots U_n)$ and we observe that the prefactor of this expression mod ϵ^k equals the inverse-prefactor of $\Delta_{k-1}^{(n)}(u_1 \cdots u_n) = \Delta_{k-1}^{(n)}(u_1 \cdots u_n)$ mod ϵ^k, since

$$\langle \Delta^{(n)}(U_1 \cdots U_n), \Delta_{k-1}^{(n)}(u_1 \cdots u_n) \rangle = \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle \bmod \epsilon^k.$$

So we are finished if we can prove that the prefactor of the ϵ^k term equals minus the k-th term of the prefactor of $\Delta_k^{(n)}(u_1 \cdots u_n)$. This follows by induction. Assume this is true for $n-1$ generators. Adding one generator on both sides, which has to be dual on both sides, only is non-trivial in the case of $U_n = X,Y$ or Z. Since there is only one correct order to put them, calculating the prefactor of $\Delta(n)(U_1 \cdots U_n)$ (and similarly that of the u_i side) now follows by commuting exponentials q^H past a number of U_i in other tensorfactors. Now observe for any exponent p, where $q = e^{-\epsilon h}$, we have (denoting q_k^p for the expansion of q^p in ϵ up to and including the k-th order term) $q^{-p} q_{k-1}^p = -q_k^p + q_{k-1}^p$. To see this, observe that $q^{-1} q_{k-1} = -q_k + q_{k-1}$. This formula follows from the expression $\sum_{l=1}^{k} \frac{(-1)^{k-l}}{(k-l)!l!} = -\frac{(-1)^k}{k!}$, which follows from Newtons binomial expansion of $(1-1)^k = 0$. Furthermore observe that the commutation relations in $U_q(b^{\pm})$ between Cartan generators and X, Y, Z (respectively x, y, z) have a minus sign compared to each other. By induction the result follows. □

It is possible to prove this proposition more directly using the generating function of the pairing, which we will write as P_{ij}. When writing multiplication, we mean applying the zip function on the appropriate indices. One should take the following proof only as a suggestion, since we are not allowed to zip the R-matrix and the pairing on both indices.
The canonical R-matrix sums over the q-PBW basis of $U_q(b^-)$ and the dual basis as $\mathcal{R} = \sum f^a \otimes e_a$. In chapter 1 the R-matrix was constructed in such a way that the monomials are normalized to pair to one: $\langle f^a, e_a \rangle = 1$. Pairing each term in $\mathcal{R}$ then yields $\sum_{\mathbb{N}} 1$. This is remeniscent of certain quantum field theoretical techniques. The difference between the k-th order and the $k-1$-th order of $(P_{ij})(\mathcal{R}_{ij})$ then is not equal to 0 as the following proof suggests, but it is presumably equal to the zipping of the zeroth order in P_{ij}, to match the expression in the proposition.
The zipping in the zeroth order of P_{ij} is left unevaluated in the Mathematica implementation, since this would diverge for the reasons mentioned above. We could try to do the computation of $(P_{ij})(\mathcal{R}_{ij})$ here explicitly, but this would amount to the proof given above. For this reason the following proof is only a suggestion, meant to give an intuition for the result.

Proof. Suppose $(P_{ij})_{k-1} = a^{-1}$ is the generating function of the pairing up to the $k-1$ th order in h. Let $(\mathcal{R}_{ij}^{-1})_{k-1} = a$ be the R-matrix up to the $k-1$-th order in h.

Define the k-th order term of the R-matrix respectively the pairing as $x = (\mathcal{R}_{ij})_k$ and $y = (P_{ij})_k$. Then mod h^{k+1}, we have $1 = (a + x)(a^{-1} + y) = 1 + ay + xa^{-1}$, since we work mod h^{k+1}. So $y = -a^{-1}xa^{-1}$. Since x is divisable by h^k, $y = -a_0^{-1}xa_0^{-1}$, where a_0 stands for the zeroth order in h of a. □

For the following proposition, which proves an iterative formula for the inverse R-matrix, we use the technique of tensor generating functions legally, contrary to the previous 'proof'. The proof is taken from [36].

Proposition 2.3.3. *Define $Y_n = \mathcal{R}^{-1}$ mod h^{n+1}. Then $Y_{n+1} = 1 - \mathcal{R}_{ij}Y_n + Y_n$.*

Proof. The proof is by induction. Suppose $\mathcal{R}_{ij}^{-1} = Y_n$ mod h^{n+1}, and that $Y_{n+1} = Y_n + Zh^{n+1}$ mod h^{n+2}. Let $\mathcal{R}_{ij}Y_n = 1 + Eh^{n+1}$ mod h^{n+2}. We know that $\mathcal{R}_{ij} = 1$ mod h, so the zeroth order in h does not contribute to the product $\mathcal{R}_{ij}\mathcal{R}_{ij}^{-1} = 1 = \mathcal{R}_{ij}(Y_n + Zh^{n+1})$ mod $h^{n+2} = (E + Z)h^{n+1} + 1$ mod h^{n+2}. So we conclude that $Z = -E$, and $Y_{n+1} = Y_n - Eh^{n+1}$. This finishes the proof. Since ϵ occurs only together with h in the expression for the R-matrix, it is clear that the proposition also holds when working mod ϵ. □

We can now proceed with the implementation of the zipping formula, and proving that it converges for the specific implementation of the quantum double we use. This implementation uses the previous two propositions. See the appendix for the implementation.
In the Mathematica file we use a specific implementation of the zipping theorem, namely we zip in three stages. First we zip the Cartan Lie subalgebra, then we zip the simple generators x, y, Y, X and at last we zip the non-simple z and Z. We prove that this way of zipping called the three-stage zip converges in each step.

Theorem 2.3.2. *(Miracle-theorem) The three-stage zip converges for all finite combinations of the tensors ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t dm$.*

Proof. The perturbation is finite, so the zipping of these terms converges by the zipping theorem. To see that the zipping of exponentials without the perturbation converges we inspect the exponentials of ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t dm$ term by term. The only terms for which it is not clear that they yield a result in W_B for $B = \{X, Y^*, y, x^*\}$ or $B = \{Z^*, z\}$ are the cubic terms.
The only non-perturbative cubic term arises in the ${}^t am$ tensor and the ${}^t\overline{R}_{i,j}$ tensor. Both are essentially dual to each other. Note at first that in every non-perturbed exponential of $U_q(sl_3^\epsilon)$ the x and the y are only present with an X and an Y in front, respectively. Zipping the $x_i^* y_j^* z_k$ term to the variables x and y first yields terms like $CX_{i'}Y_{j'}$, for some i' and j', with a term C. Zipping $e^{x_i y_j z_k}$ is well-defined in the three-stage zip because we choose $B = \{X, Y^*, y, x^*\}$, and so $e^{x_i y_i z_k}$ is of the form $e^{bc^* z_k}$, $b, c \in B$.
Let us be more concrete, and look at the structure of the matrix Θ. We observe that with the choice $B = \{X, Y^*, y, x^*\}$, there are never any diagonal terms in Θ

when zipping any of the tensors ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t dm$. It is important to note here that in one application of the zipping theorem, we only zip any tensor on either its input or on its output. Moreover, remember that the lowercase generators on the input-entries are only present in the exponential with an uppercase generator in front. An exception is z, but this results in another cubic term, as mentioned before.
Observe that $1 - \Theta$ is always upper-diagonal (or lower-diagonal, depending on the conventions and assuming normal ordering). This follows since the lowercase multiplication tensor has a cubic term in the exponential, but the uppercase multiplication tensor has not. Applying the zipping theorem for an uppertriangular matrix $1 - \Theta$ with only 1's on the diagonal yields a prefactor of $det(1 - \Theta) = 1$, which is non-singular. As mentioned before, the other cubic term which is present in the inverse R-matrix has the same property that Θ is upper-diagonal.
When combining both terms, it is important to mention that the upper- and lowercase starred variables are chosen such that for example X is a 'starred' variable, while x is a non-starred variable. With this choice, $1 - \Theta$ remains of uppertriangular shape.
So far, we have not mentioned the cubic terms present in ${}^t dm$, but it can be seen that these terms are in fact equivalent to the above case by zipping once. In this case, we are again saved by the xX terms in the R-matrix, and the fact that these are the only occurances of the non-starred lowercase variables. This finishes the proof.

□

We have proven that zipping on the space spanned by the above tensors and all finite zips is well-defined. The subspace of W_B formed by these exponentials is closed under zipping.

Corollary 2.3.1. *If ${}^t T$ and ${}^t S$ are two tensors in the ring $R_\epsilon[z_1, \cdots, z_n][[h, z_1^*, \cdots, z*_n]]$ obtained from the tensors ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\mathcal{R}^{-1}$ and ${}^t dm$ by a finite number of zips, then $ZIP_{B'}({}^t T {}^t S)$ converges.*

Proof. We know by the previous proposition that zipping converges on all tensors arising from finite zips of these tensors, this implies the above result. □

In the implementation of the quantum double, we need the pairing for the calculation of the multiplication in the double and for checking the pairing-axioms. It is also used in the computation of the antipode and the comultiplication. After the calculation of the double, the pairing is not used anymore.
To prove that the zipping of the pairing tensor with the appropriate tensors converges, we explicitly check convergence for these relations. The double multiplication is obtained from the multiplication on the the lower- and uppercase Borel subalgebra, together with the pairing and the (inverse-)R-matrix. In this case we will prove convergence as well.

To circumvent the troubles with the pairing, it turrns out to be possible to define the pairing from the R-matrix. See [36] for details. In particular theorem 8 of [36]. In this method, one implicitly defines the multiplication on the Drinfel'd double by proving the existence of a dual element. Although this this would be a way to circumvent the zipping of the pairing, it does not show that we may explicitly compute the double multiplication in our implementation. As this is what we are aiming for, we will explicitly prove convergence.

Proposition 2.3.4. *(Pairing-zip) Let ${}^{\mathbf{t}}P_{i,j}$ be the tensor corresponding to the pairing on the entries i and j. The zip of ${}^{\mathbf{t}}P_{i,j}$ along one of the indices i or j but not both i and j with the tensors ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$, ${}^{\mathbf{t}}\mathcal{R}^{-1}$ is well defined.*

Proof. We are zipping the pairing along the i or the j, but never along both indices. Both multiplication tensors have two input strands and one output strand. The input-variables correspond to starred variables, the output variables are the non-starred variables in our convention. In the notation of the tensors, the starred variables are not necessarily the variables on which is zipped. The non starred variables correspond to elements of $U_q(sl_3^\epsilon)$. The R-matrix and its inverse are elements of $U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$, where the exponent of the exponential is proportional to h.
Any term in the perturbation is finite and the zip of this term with ${}^{\mathbf{t}}P$ converges. Looking at non-perturbative $\epsilon = 0$ part, we note that we need to make a consistent choice for which variables to zip to. Observe that the pairing tensor does not contain non-perturbative cubic terms, so if we can make a consistent choice such that we only have $z_i z_j^*$ terms, we only need to prove that the given exponential is a power series in $R_\epsilon[[\frac{1}{h}]][[B', B^*]][B]$.
We know by the previous proposition that the zips excluding the pairing converge, and that a consistent choice is made so that cubical terms in the exponentials can also be zipped. To this end we choose $\{X, Y^*, y, x^*\}$ in our implementation, as noted before. the tensors ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$ or ${}^{\mathbf{t}}\mathcal{R}^{-1}$.
The convergence of a zip including the pairing and the tensor ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$ or ${}^{\mathbf{t}}\mathcal{R}^{-1}$, is clear from the fact that we only zip on one of the two indices in ${}^{\mathbf{t}}P_{i,j}$. In the case of the multiplication, it is clear that the exponents are not proportional to h. In the case of the R-matrix we have to be more careful, as the exponents are proportional to h. However, when zipping the pairing and the R-matrix on one index, the result includes a term BB^* or bb^*, where $b, B \in B$, depending on which index is zipped. Independently of the choice of starred and non starred variables this is proportional to an element of the formal power series ring $R_\epsilon[[h]][[B^*]]$. So this choice is consistent with zipping the pairing on one index. This proves the proposition. □

Before providing the tensors for the comultiplication and the antipode, we have to know that we are working with the correct multiplication tensors.

Proposition 2.3.5. *${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ are the pullback under $\mathbb{O}$ of multiplication on $U_q(b^-)$ and $U_q(b^+)$ respectively.*

Proof. We verified both the associativity of the multiplication $am^k_{i,j}$ and $bm^k_{i,j}$ in $U_q(b^-)$ and $U_q(b^+)$ respectively. We verify the associativity of ${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ tensors separately in Mathematica (see appendix A.1). The proposition can be seen to be true for generators by explicitly using the zipping theorem in Mathematica. Note that $B = \{b, a, z, y, x\}$ for $U_q(b^-)$ and $B = \{X, Y, Z, A, B\}$ for $U_q(b^+)$. We get

$$\mathbb{O}_k(am^k_{i,j}) = {}^{\mathbf{t}}\mathrm{Id}_i\,{}^{\mathbf{t}}\mathrm{Id}_j // \mathbb{O}_{i,j} // am^k_{i,j},$$

and the same for $bm^k_{i,j}$. But this implies the result.
Note that it is enough to check that ${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ agree on generators. Since sl_3 has only one non-simple generator, z in our case, this follows directly from the relation $[x, y] = z$ and its q-equivalent. This finishes the proof. □

For higher n it is necessary perform the check on monomials of higher order. The same proposition is true for the R-matrix and its inverse.

Proposition 2.3.6. *${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\overline{\mathcal{R}}$ are the pullback under $\mathbb{O}$ of the R-matrix and its inverse of the quasitriangular Hopf algebra $D(U_q(b^-))$ respectively.*

Proof. The proof of the statement for the R-matrix is easy, as $\mathcal{R}$ is an element of $U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$, and by the Fadeev-Quesne formula proven in the previous chapter we obtain the given formula.
The expression for ${}^{\mathbf{t}}\overline{\mathcal{R}}$ can be obtained by applying the iterative formula for $Y_n = \mathcal{R}^{-1} \bmod \epsilon^{n+1}$, $Y_{n+1} = 1 - \mathcal{R}_{ij}Y_n + Y_n$. The $\epsilon^2 = 0$ case we are working with then corresponds to $Y_1 = 1 - \mathcal{R}_{ij}Y_0 + Y_0$. We claim that $Y_0 \in U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$ is given by

$$\begin{aligned} Y_0 = \mathbb{O}_{i,j}(\mathbb{E}^{\{i,j\}}[&-\hbar a_j A_i - \hbar b_j B_i, -\hbar \mathrm{X}_i x_j \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{-\hbar} \\ &+ \hbar^2 \mathrm{X}_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar Y_i y_j \mathbb{A}_i^{-\hbar}\mathbb{B}_i^{2\hbar} - \hbar Z_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar}]). \end{aligned}$$

Observe that in $\mathcal{R}_{ij}Y_0$, entry-wise multiplication is implied. Multiplication here is interpreted in the Hopf-algebraic sense. The multiplication of the R-matrix and Y_0 takes place entirely in $U_q(b^+) \otimes U_q(b^-) \subset U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$, as we constructed the Drinfel'd double with the opposite comultiplication. We already proved that the zipping of the corresponding tensors ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$ and ${}^{\mathbf{t}}\mathcal{R}$ converges. We now observe that $Y_0 \in W_B$ as a tensor. To see this, first observe that $\mathbb{A} = e^{-hA}$ and $1 - hA$ act as the same element of $U_q(b^+)$. So we can replace the two in the expression for $\mathbb{O}(Y_0)$, and consequently in the corresponding tensor Y_0. The terms seem like higher order products of generators. However, when we first zip to $\zeta s = \{A, B^*, b, a^*\}$, secondly to $\zeta s = \{X^*, Y, y^*, x\}$ and at last to $\zeta s = \{Z^*, z\}$ we see that in each stage Y_0 is of the appropriate form.
We check by zipping the tensors that Y_0 is the zeroth order in ϵ of the inverse R-matrix. We use the fact that ${}^{\mathbf{t}}bm$ and ${}^{\mathbf{t}}am$ are the pullback under $\mathbb{O}$ of the multiplication in $U_q(b^\pm$ respectively. It then follows from the implementation in Mathematica that ${}^{\mathbf{t}}\overline{\mathcal{R}}$ is indeed the pullback of $\overline{\mathcal{R}}$. This ends the proof. □

We can now construct the comultiplication and antipode tensors from the multiplication and R-matrix, and the pairing.

Proposition 2.3.7. *${}^t a\Delta$, ${}^t b\Delta$, ${}^t aS$ and ${}^t\overline{aS}$ are the tensors corrsponding to respectively the comultiplication on $U_q(b^-)$, $U_q(b^+)$, the antipode on $U_q(b^-)$ and the inverse of the antipode on $U_q(b^-)$. They can be constructed as a composition of tensors as follows.*

$$
\begin{aligned}
{}^t a\Delta_i^{j,k} &= ({}^t R_{1,j}\, {}^t R_{2,k}) // {}^t b m_{1,2}^3 // {}^t P_{3,i} \\
{}^t b\Delta_i^{j,k} &= \left({}^t R_{j,1}\, {}^t R_{k,2}\right) // \left({}^t a m_{1,2\to 3}\right) // {}^t P_{i,3} \\
{}^t\overline{aS}_i &= {}^t aS|_{\epsilon=0} - {}^t\overline{aS}_i|_{\epsilon=0} // {}^t aS_i // {}^t\overline{aS}_i|_{\epsilon=0} \\
{}^t aS_i &= \langle {}^t\overline{R}_{i,j}\, {}^t P_{i,j}\rangle_j,
\end{aligned}
$$

and have the following explicit form. We introduce the notation $\boldsymbol{a} = e^{-a^*}$, and similarly

for b.

$$
\begin{aligned}
{}^{t}a\Delta_i^{j,k} = \mathbb{E}_{\{i\}}^{\{j,k\}} \Big[& a_j a^*{}_i + a_k a^*{}_i + b_j b^*{}_i + b_k b^*{}_i, x_j x^*{}_i + x_k x^*{}_i + y_j y^*{}_i + y_k y^*{}_i + z_j z^*{}_i \\
& + z_k z^*{}_i, 1 + \Big(-\hbar a_j x_k x^*{}_i - \hbar x_j x_k \left(x^*\right)_i^2 - \hbar b_j y_k y^*{}_i + \hbar x_k y_j x^*{}_i y^*{}_i - \hbar y_j y_k \left(y^*\right)_i^2 + \\
& 2\hbar x_j y_k z^*{}_i - \hbar a_j z_k z^*{}_i - \hbar b_j z_k z^*{}_i - \hbar x_k z_j x^*{}_i z^*{}_i + \hbar y_k z_j y^*{}_i z^*{}_i - \hbar z_j z_k \left(z^*\right)_i^2 \Big) \epsilon \Big] \\
{}^{t}b\Delta_i^{j,k} = \mathbb{E}_{\{i\}}^{\{j,k\}} \Big[& A_j A^*{}_i + A_k A^*{}_i + B_j B^*{}_i + B_k B^*{}_i, \\
& X_k X^*{}_i + X_j \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} X^*{}_i + Y_k Y^*{}_i + Y_j \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} Y^*{}_i + Z_k Z^*{}_i + Z_j \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} Z^*{}_i + \\
& \hbar X_j Y_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} Z^*{}_i, \\
& 1 + \Big(-\hbar X_j X_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} \left(X^*\right)_i^2 - \hbar X_j Y_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} X^*{}_i Y^*{}_i - \hbar Y_j Y_k \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} \left(Y^*\right)_i^2 + \\
& \hbar X_j Z_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} X^*{}_i Z^*{}_i + \hbar^2 X_j^2 Y_k \mathbb{A}_k^{-4\hbar} \mathbb{B}_k^{2\hbar} X^*{}_i Z^*{}_i - \hbar Y_j Z_k \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} Y^*{}_i Z^*{}_i - \\
& \hbar^2 X_j Y_j Y_k \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} Y^*{}_i Z^*{}_i - \hbar^2 X_j Y_k Z_j \mathbb{A}_k^{-3\hbar} \left(Z^*\right)_i^2 - \hbar Z_j Z_k \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} \left(Z^*\right)_i^2 + \\
& \frac{1}{2} \hbar^3 X_j^2 Y_k^2 \mathbb{A}_k^{-4\hbar} \mathbb{B}_k^{2\hbar} \left(Z^*\right)_i^2 \Big) \epsilon + O[\epsilon]^2 \Big] \\
{}^{t}\overline{aS}_i = \mathbb{E}_{\{i\}}^{\{i\}} \Big[& -a_i a^*{}_i - b_i b^*{}_i, -e^{-2a^*{}_i + b^*{}_i} x_i x^*{}_i - e^{a^*{}_i - 2b^*{}_i} y_i y^*{}_i + e^{-a^*{}_i - b^*{}_i} z_i x^*{}_i y^*{}_i - \\
& e^{-a^*{}_i - b^*{}_i} z_i z^*{}_i, \\
& 1 + \Bigg(-\frac{2\hbar x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} - \frac{\hbar a_i x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} + \frac{\hbar x_i^2 \boldsymbol{b}_i^2 \left(x^*\right)_i^2}{\boldsymbol{a}_i^4} - \frac{2\hbar y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} - \frac{\hbar b_i y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} - \\
& \frac{2\hbar x_i y_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{3\hbar z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{\hbar a_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{\hbar b_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \\
& \frac{\hbar x_i z_i \left(x^*\right)_i^2 y^*{}_i}{\boldsymbol{a}_i^3} + \frac{\hbar y_i^2 \boldsymbol{a}_i^2 \left(y^*\right)_i^2}{\boldsymbol{b}_i^4} + \frac{\hbar z_i^2 \left(x^*\right)_i^2 \left(y^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{2\hbar x_i y_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \\
& \frac{2\hbar z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar a_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar b_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{2\hbar z_i^2 x^*{}_i y^*{}_i z^*{}_i}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{\hbar z_i^2 \left(z^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} \Bigg) \epsilon + O[\epsilon]^2 \Big] \\
{}^{t}aS_i = \mathbb{E}_{\{i\}}^{\{i\}} \Big[& -a_i a^*{}_i - b_i b^*{}_i, -\frac{x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} - \frac{y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} + \frac{z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i}, \\
& 1 + \\
& \Bigg(-\frac{\hbar a_i x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} + \frac{\hbar x_i^2 \boldsymbol{b}_i^2 \left(x^*\right)_i^2}{\boldsymbol{a}_i^4} - \frac{\hbar b_i y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} - \frac{2\hbar x_i y_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \\
& \frac{\hbar a_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{\hbar b_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar x_i z_i \left(x^*\right)_i^2 y^*{}_i}{\boldsymbol{a}_i^3} + \frac{\hbar y_i^2 \boldsymbol{a}_i^2 \left(y^*\right)_i^2}{\boldsymbol{b}_i^4} + \\
& \frac{\hbar z_i^2 \left(x^*\right)_i^2 \left(y^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{2\hbar x_i y_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{2\hbar z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar a_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar b_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \\
& \frac{2\hbar z_i^2 x^*{}_i y^*{}_i z^*{}_i}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{\hbar z_i^2 \left(z^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} \Bigg) \epsilon + O[\epsilon]^2 \Big] .
\end{aligned}
$$

Proof. For the tensors corresponding to the comultiplication, the pairing is only zipped on one index. By the previous proposition this is well-defined. As for the antipode, the same holds true. From the explicit form of ${}^{\mathbf{t}}aS$ we see that ${}^{\mathbf{t}}\overline{aS_i}|_{\epsilon=0} // {}^{\mathbf{t}}aS_i // {}^{\mathbf{t}}\overline{aS_i}|_{\epsilon=0}$ is well-defined, and that the zipping converges.
The fact that the given tensors are the pullback of the comultiplication and the (inverse) antipode follows by explicitly checking the Hopf algebra axioms in Mathematica. We refer to the implementation $U_q(sl_3^\epsilon)$ in the appendix for this calculation. We already proved that the multiplication tensors ${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ correspond to multiplication in the Hopf algebras $U_q(b^\pm)$. Hence it follows that the tensors given in this proposition must correspond to the comultiplication and antipode in $U_q(b^\pm)$, as they agree on generators. □

We now proceed with the pairing, the most problematic part of this section.

Proposition 2.3.8. *${}^{\mathbf{t}}P$ is the pullback under $\mathbb{O}$ of the pairing between $U_q(b^\pm)$.*

Proof. When pairing on generators, which is a straightforward application of the zipping formula, we see that the pairing agrees on generators, and makes $Z_i \in U_q(b^+)$ and $z_i \in U_q(b^-)$ dual.
What is left now is to prove that ${}^{\mathbf{t}}P$ obeys the pairing axioms. This means proving that the zips necessary to check the pairing axioms in the tensor formalism converge. Then we are finished, for the explicit check we refer to Mathematica.
The proof that the checking of the axioms converges in the tensor formalism boils down to proving that for the expressions in W_B for the appropriate set B have an invertible $1 - \Theta$. This is an issue here as we are dealing with a formal power series in $R_\epsilon[[\frac{1}{h}]][B]$, the inverse of h.
The check that the zipping formula converges is equivalent to checking if the prefactor in the final result is still dependent on $\frac{1}{h}$. This isequivalent to checking that the exponents of the tensors that the pairing is zipped with are not proportional to h, but to a constant in $\mathbb{R} \subset R_\epsilon$. When the result of an (intermediate) zip is not dependent on h, the power series are still well defined in $W_B \subset R_\epsilon[[\frac{1}{h}]][[B^*]][B]$ since the exponent consists of pairs zz^*, where $z^* \in B^*$. However, the prefactor Θ might yield an uninvertible matrix $1 - \Theta$ in the following zips. This leads to divergences.
To show that the result of the zip in both inputs of the pairing tensor ${}^{\mathbf{t}}P$ has invertible $1 - \Theta$, we do not have to zip, we only need to write down the tensors, which is a product in the formal power series ring. By the previous propositions we do have the explicit form of ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}a\Delta$, ${}^{\mathbf{t}}b\Delta$, ${}^{\mathbf{t}}aS$ and ${}^{\mathbf{t}}bS$ and the inverse antipodes. Moreover, we know that they are the pullback of the (co)multiplication and antipodes on both halves of the Drinfel'd double $U_q(sl_3^\epsilon)$, remembering that $U_q(b^+)$ has the opposite comultiplication. So if we prove convergence of the zip we know that ${}^{\mathbf{t}}$ is the pullback of the pairing, as it agrees on generators and it obeys the pairing axioms. In this last observation we use $\mathbb{O}$.
The pairing axioms that need to be checked are stated in the first chapter, and are

as follows

$$\begin{aligned}\langle ab, c\rangle &= \langle a \otimes b, \Delta c\rangle\\ \langle \Delta a, c \otimes d\rangle &= \langle a, cd\rangle\\ \langle 1, c\rangle &= \epsilon(c)\\ \langle a, 1\rangle &= \epsilon(a)\\ \langle Sa, c\rangle &= \langle a, Sc\rangle.\end{aligned}$$

We write down the corresponding tensor identity with the appropriate indices of the first relation $\langle ab, c\rangle = \langle a \otimes b, \Delta c\rangle$. The relations are checked in the Mathematica implementation. The check that the zip converges is then trivial, and consists of checking the tensors stated above on their dependence of h. The other relations are similar to the first one, and are left to the reader. The concrete expressions can be found in the Mathematica implementation in appendix A.1.
The terms we need to check are the cubic terms in ${}^{\mathbf{t}}b\Delta$ and ${}^{\mathbf{t}}am$ of the form $\hbar X_j Y_k \cdots$. Terms like this are zipped twice (or three times in the case of the term $\hbar^2 X_i Y_i z_j$ in $\mathcal{R}^{-1}$), and hence get a factor of $\frac{1}{h}^2$ and $\frac{1}{h}^3$ respectively from the pairing. We never encounter a term like $X_i X_j$ in the exponent, so there are always multiple terms in the pairing involved in zipping cubic terms.
For example for the term $\hbar^2 X_i Y_i z_j$ combines with the terms $\frac{1}{h} X_i^* x_j^*$ and $\frac{1}{h} Y_i^* y_j^*$ and $\frac{1}{h} Z_i^* z_j^*$ in the pairing to $\frac{1}{h} x_j^* y_j^* Z_i^*$. So we obtain a prefactor of $\frac{1}{h}$ in both cases. In particular, $1 - \Theta$ is invertible for a suitable choice of variables in the three-stage zip.
To check the pairing axioms we apply ${}^{\mathbf{t}}P$ to the identity tensor

$$\mathbb{E}_{\{3\}}^{\{3\}}\left[a_3 a^*{}_3 + b_3 b^*{}_3, x_3 x^*{}_3 + y_3 y^*{}_3 + z_3 z^*{}_3, 1\right]$$

. We obtain that zipping with ${}^{\mathbf{t}}$ obeys the pairing axioms for a general expression in $U_q(sl_3^\epsilon)$ by differentiating both sides, as usual.
We now give the first pairing axiom in tensor form.

$$\begin{aligned}&\langle {}^{\mathbf{t}}\mathrm{bm}_{1,2}^4 \mathbb{E}_{\{3\}}^{\{3\}}\left[a_3 a^*{}_3 + b_3 b^*{}_3, x_3 x^*{}_3 + y_3 y^*{}_3 + z_3 z^*{}_3, 1\right] {}^{\mathbf{t}}P_{4,3}\rangle_{3,4}\\ &= \langle {}^{\mathbf{t}}\mathrm{a}\Delta_3^{4,5} \mathbb{E}_{\{1\}}^{\{1\}}\left[A_1 A^*{}_1 + B_1 B^*{}_1, \mathrm{X}_1 \mathrm{X}^*{}_1 + Y_1 Y^*{}_1 + Z_1 Z^*{}_1, 1\right]\\ &\mathbb{E}_{\{2\}}^{\{2\}}\left[A_2 A^*{}_2 + B_2 B^*{}_2, \mathrm{X}_2 \mathrm{X}^*{}_2 + Y_2 Y^*{}_2 + Z_2 Z^*{}_2, 1\right] {}^{\mathbf{t}}P_{1,4} {}^{\mathbf{t}}P_{2,5}\rangle_{\{1,2,4,5\}}.\end{aligned}$$

The other expressions are left to the reader to check. This finishes the proof. □

We are now able to prove the main theorem of this chapter, theorem 2.3.1. The proof combines the lemmas and propositions in this section.

Proof. (Theorem 2.3.1) The main problem is to check convergence of the zipping of the pairing ${}^{\mathbf{t}}P$ on both indices in the formula stated in the theorem. To prove

convergence we once again look at the tensors that are taken as input. Since ${}^{\mathsf{t}}P$ is the pullback of the pairing, when the zip converges we immediately know that the corresponding tensor ${}^{\mathsf{t}}dm$ is the multiplication tensor on the Drinfel'd double.
By Drinfel'ds theorem, we know that the multiplication on $D(U_q(b^-))$ converges in the h-adic completion. This implies that the $\frac{1}{h}$ cancels. For the explicit proof we refer to [23] for example, and we invite the reader to check this for themselves. The observation is that the antipode plays an essential role in canceling the factors of $\frac{1}{h}$ by introducing a minus sign.
We are looking for a similar fact here. A straightforward way out of this situation is to calculate the explicit form of the input of the pairing in the tensor formalism, and check the h-dependence, to make sure that the exponents in both inputs are not proportional to h. We do not need to calculate this explicitly. We apply a similar tactic as the previous proposition. Let us consider the term

$$\left({}^{\mathsf{t}}\mathrm{a}\Delta_i^{\{1,2\}} /\!/ {}^{\mathsf{t}}\mathrm{a}\Delta_2^{\{2,3\}} /\!/ {}^{\mathsf{t}}\overline{\mathrm{aS}}_3\right)\left({}^{\mathsf{t}}\mathrm{b}\Delta_j^{\{-1,-2\}} /\!/ {}^{\mathsf{t}}\mathrm{b}\Delta_{-2}^{\{-2,-3\}}\right)$$

in the construction of ${}^{\mathsf{t}}dm$. We observe that the exponentials ${}^{\mathsf{t}}a\Delta$, ${}^{\mathsf{t}}b\Delta$ and ${}^{\mathsf{t}}\overline{aS}$ do not have a factor of h in the exponential apart from the terms involving more than one 'unstarred' variable. These are zipped on more than once in the three-way zip with the pairing, resulting in the appropriate factors of $\frac{1}{h}$. The essential observation here is that we never encounter a term like X_iX_j in the exponent. By applying the zipping formula we see that the resulting terms $\left({}^{\mathsf{t}}\mathrm{a}\Delta_i^{\{1,2\}} /\!/ {}^{\mathsf{t}}\mathrm{a}\Delta_2^{\{2,3\}} /\!/ {}^{\mathsf{t}}\overline{\mathrm{aS}}_3\right)\left({}^{\mathsf{t}}\mathrm{b}\Delta_j^{\{-1,-2\}} /\!/ {}^{\mathsf{t}}\mathrm{b}\Delta_{-2}^{\{-2,-3\}}\right)$ also do not have a factor of h in the exponentials. We provided the explicit form of the tensors corresponding to the (co)multiplication, antipode, pairing and the (inverse) R-matrix by calculations in Mathematica in appendix A.1. We also proved that these tensors correspond to the pullback under $\mathbb{O}$. This finishes the proof of the main theorem. □

As a corollary of the main theorem we obtain the multiplication relations of $U_q(sl_3^\epsilon)$ stated at the end of the first chapter. These relations are obtained in the Mathematica implementation by explicitly zipping ${}^{\mathsf{t}}dm$ to generators. See the appendix for details.

Conclusion

We defined and proved the convergence of a formalism that implements Feynmans trick. In other words, following [36], we calculated the commutator between exponentials in an indirect way using the zipping theorem. This formalism gives a correspondence between exponentials in the ring of formal power series and the Hopf algebra maps. The zipping theorem enables us to implement the quasitriangular Hopf algebra $U_q(sl_3^\epsilon)$ in an efficient way. We proved that that the given tensors correspond under to the multiplication maps in $U_q(b^\pm)$,

the pairing P and the R-matrix and its inverse. We proved furthermore that the quantum double $U_q(sl_3^\epsilon)$ can be constructed using only these maps. We noted that one should be careful when working with the pairing in the tensor formalism, as this can lead to divergence. When one restricts to zipping P on one index with any other tensor however, we proved that we are safe. When the pairing is zipped to on both indices one has to check convergence explicitly.
A way to circumvent this problem is by defining the pairing P indirectly by solving a differential equation. See [36] for the argument. We noted however that we still need to prove convergence of the zip function on single index zipping of P. The observation that P zipped on both indices with $\mathcal{R}$ gives a divergence means also that it is not directly possible to generalize the zipping formula to the zipping of quadratic exponentials. It might be possible to find such a formula for a subspace of these exponentials, but this is subject to future research.
A particular case of cubic zipping that is used in this chapter is the so called three-stage zip. By splitting the zipping of the variables X and Y the zipping of Z into two different stages, we managed to prove convergence for the zipping of certain exponentials associated to the (co)multiplication and antipode in the Hopf algebra $U_q(sl_3^\epsilon)$. In particular the cubic terms xyZ and XYz required a special choice of the variables on which was to be zipped. The main reason for the convergence of the three-stage zip on these terms is the fact that X and Y do not occur together in one term in $\mathcal{R}$ and P. In other words, no mixing occurs. It is interesting to see if we can define a specific subclass of tensors for which the three stage zip converges, or in general understand the zipping of cubic terms better.
It is possible that the cubic terms can be put in the perturbation P of a tensor in the case of certain solvable or nilpotent Lie algebras. A particular example of this is of course the factor of ϵ, which is a way to construct a nilpotent subalgebra. There might by a way to be able to zip cubic terms with losing slightly less information by introducing another such factor. In particular this will be helpful for reducing the computation time, but also for theoretical purposes it is valuable to understand the cubic terms.