Expansions of quantum group invariants

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1. An expansion of the $U_q(sl_3)$ quantum group

Introduction

Hopf algebras are used to produce knot invariants such as the Jones polynomial and the Alexander polynomial. In this chapter we will construct a quasitriangular Hopf algebra that is in some sense a deformed version of the quantum group $U_q(sl_3)$. The knot invariant associated with this Hopf algebra is stronger than the Alexander polynomial, and is computable in polynomial time. These facts will be proven in a later chapter.
In order to arrive at the correct quasitriangular Hopf algebra, we first cover its classical limit, the deformed sl_3 Lie bialgebra. We proceed with quantizing these Lie bialgebras. Finally we cover the quantum or Drinfel'd double construction to obtain the deformed version of $U_q(sl_3)$.
We aim to quantize the algebras over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. Usually the quantization and Drinfel'd double construction is done over a field. Most theorems also hold for coordinate rings. The subject of this chapter, besides obtaining the $U_q(sl_3^\epsilon)$ algebra, is the question 'What is ϵ?'. There is no definite answer to this question. One may choose between the possible perspectives we provide in this thesis. Some perspectives do provide more information than others, however.
Most of the constructions presented here are covered in sources like [23] and [6]. It is advisable to consult these sources on the subject. The deformed Hopf algebra presented here is based on the research by Van der Veen and Bar-Natan in [35]. While [35] covers the $U_q(sl_2^\epsilon)$-quantum group invariant, we cover the $U_q(sl_3^\epsilon)$ quantum invariant. The construction is the same in essence. However, we hope to gain insight in the problems arising in the quantization of sl_n.

1.1. Lie bialgebras

In this section we treat quasi-triangular Lie bialgebras. From a general Lie bialgebra it is possible to construct a quasitriangular Lie bialgebra through the classical double construction. Using this construction, we turn the deformed lower Borel sub Lie bialgebra of sl_3 into a quasi-triangular Lie bialgebra. It is possible to obtain the same Lie algebra relations through Wigner group contraction on the upper Borel Lie subalgebra of gl_n. See appendix A.4 for more information.
By a vector space over a ring we mean a free module over a ring. In the case of

the Lie algebras considered here, the modules are finitely generated. When we say a Lie algebra or Lie bialgebra, we will mean a Lie (bi)algebra over a ring R. R will be specified when necessary. A ring is always commutative with unit in this thesis.
We will often work over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. A specific problem that arises is the non-degeneracy of a bilinear pairing $\langle , \rangle : M^* \otimes M \to R_\epsilon$, where M is a free R_ϵ module. In this and the next chapter, whenever we say that a pairing is nondegenerate, we will mean that it is nondegenerate over $\mathbb{R} = R_\epsilon/\epsilon R_\epsilon$. In other words, the map $\langle , \rangle : M^*/\epsilon M^* \otimes M/\epsilon M \to \mathbb{R}$ is nondegenerate.
Since $\epsilon^2 = 0$, we can only pair in M^* with expressions in M as an element in $M^*/\epsilon M$ and $M/\epsilon M$. In practice, this is what we will use the pairings in this chapter for. As noted in the prelimenaries, we can extend an $\mathbb{R}$-basis of a module $M/\epsilon M$ to an R_ϵ-basis of M, where M is an R_ϵ-module. The same is true for the dual basis, since we consider finite dimensional modules in this section.

Definition 1.1.1. *(Lie bialgebra) A Lie bialgebra $(\mathfrak{g}, [,], \delta)$ is a vector space $\mathfrak{g}$ over a ring R together with a bilinear map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (the bracket) and a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ (the cobracket) satisfying the following axioms:*

1. $[X, X] = 0 \; \forall X \in \mathfrak{g}$,
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$,
3. $\tau \circ \delta(X) = -\delta(X) \; \forall X \in \mathfrak{g}$, *where* $\tau(A \otimes B) = (B \otimes A)$,
4. $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ *is a bracket on the dual Lie algebra* $\mathfrak{g}^*$,
5. $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$ *for all* $X, Y \in \mathfrak{g}$.

$X.\delta(Y) = (ad_X \otimes 1 + 1 \otimes ad_X)(\delta(Y))$, and $ad_X(Y) = [X, Y]$ is the (left-)action of the Lie algebra on itself, for all $X, Y \in \mathfrak{g}$. We introduce the Sweedler notation $\delta(a) = \sum a_1 \otimes a_2 = a_1 \otimes a_2$, where we leave out the summation symbol, but only indicate the entry in the tensor product. Let us define the Lie bialgebra cohomology.

Definition 1.1.2. *(Chevalley-Eilenburg complex) Let M be a $\mathfrak{g}$-module, where $\mathfrak{g}$ is a Lie algebra over a ring R. Set*

$$C^n(\mathfrak{g}, M) := Hom_R(\bigwedge^n \mathfrak{g}, M), \; n > 0,$$

and $C_0(g, M) := M$, where $\bigwedge^n \mathfrak{g}$ is the n-th exterior power of $\mathfrak{g}$. This is the Chevalley-Eilenberg cochain complex.
The differential d on $c \in C^n(\mathfrak{g}, M)$ is defined as

$$dc(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} x_i.c(x_1, \cdots, \hat{x}_i, \cdots, x_n) + \sum_{1 \le i < j \le n+1} (-1)^{i+j} c([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}), \quad (1.1)$$

where $x_1, \cdots, x_{n+1} \in \mathfrak{g}$*, and* $x.d$ *is the module action of* $\mathfrak{g}$ *on* $d \in M$*.*

With this complex one can now define the cocycles and coboundaries.

Definition 1.1.3. *(Lie bi algebra cohomology) Define the space of cocycles*

$$Z^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) | dc = 0\},$$

and the space of coboundaries

$$B^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) | \exists c' \in C^{p-1}(\mathfrak{g}, M) \text{ s.t. } dc' = c\}.$$

Then define the Lie algebra cohomology as $H^p(\mathfrak{g}, M) := Z^p(\mathfrak{g}, M)/B^p(\mathfrak{g}, M)$*.*

The condition $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$ in the definition of a Lie bialgebra states that δ is a 1-cocycle in the Lie algebra complex $C^*(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$, with the adjoint action of $\mathfrak{g}$ on the tensor product module $\mathfrak{g} \otimes \mathfrak{g}$.
According to the definition, δ is a 1-cocycle, so we can look at the cases when δ is a coboundary: $\delta(X) = X.r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$ and for all $X \in \mathfrak{g}$. A Lie bialgebra where δ is a coboundary is called a coboundary Lie bialgebra. $\mathfrak{g}$ is coboundary if and only if r obeys (let $r = \sum r_{12} = \sum r^{[1]} \otimes r^{[2]}$):

1. $2r_+ = r_{12} + r_{21}$ is a invariant under the action of $\mathfrak{g}$.

2. $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.

Here $[r_{12}, s_{13}] = \sum [r^{[1]}, s^{[1]}] \otimes r^{[2]} \otimes s^{[2]}$. See proposition 8.1.3 in [23] for the proof that $(\mathfrak{g}, [,], r)$ defines a coboundary Lie bialgebra if and only if the above conditions hold. Conditon 2 is called the classical Yang-Baxter equation, and r is called the classical r-matrix. If the Lie bialgebra structure arises from a classical r-matrix, then we call the Lie bialgebra quasitriangular. The condition that $2r_+$ is ad-invariant is usually not included in the definition of an r-matrix, but for simplicity we will do so. Usually the following definition is taken for a triangular Lie algebra. See for example [23], chapter 8.

Definition 1.1.4. *Let* $\mathfrak{g}$ *be a Lie algebra. Define the classical r-matrix as an element* $r \in \mathfrak{g} \otimes \mathfrak{g}$ *which obeys (*$r = \sum r_{12} = \sum r^{[1]} \otimes r^{[2]}$*):*

1. $r_{12} + r_{21}$ *is a invariant under the action of* $\mathfrak{g}$*.*

2. $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$*.*

Note that $[r_{12}, s_{13}] = \sum [r^{[1]}, s^{[1]}] \otimes r^{[2]} \otimes s^{[3]}$ and similarly when other indices overlap. Let us now proceed with the construction of quasitriangular Lie bialgebras through the classical double construction. We remind the reader that a Lie algebra in this thesis is always finite dimensional. When this is the case, the dual space is well defined and is again a Lie bialgebra.

Definition 1.1.5. *The dual of a Lie bialgebra $\mathfrak{g}$ over a ring R is the dual vector space $\mathfrak{g}^*$ with bracket and cobracket and a R-linear pairing $\langle , \rangle : \mathfrak{g}^* \oplus \mathfrak{g} \to R$ satisfying the axioms*

$$\langle [a,b], c\rangle := \langle a \otimes b, \delta c\rangle \tag{1.2}$$
$$\langle \delta a, c \otimes d\rangle := \langle a, [c,d]\rangle, \tag{1.3}$$

where $a, b \in \mathfrak{g}^*$, and $c, d \in \mathfrak{g}$. We extend the bracket to the tensor-product $\mathfrak{g}^* \otimes \mathfrak{g}^* \oplus \mathfrak{g} \otimes \mathfrak{g}$ by $\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle$.

It is interesting to turn a dual pairing into an inner product $(,) : (\mathfrak{g} \oplus \mathfrak{g}^*) \oplus (\mathfrak{g} \oplus \mathfrak{g}^*) \to R$ by defining $(X, \eta) = \langle X, \eta\rangle$ and $(X, X) = (\eta, \eta) = 0$ for $X \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$. We record the following fact, see lemma 1.3.5 of [6].

Lemma 1.1.1. *Let $\mathfrak{g}, \mathfrak{g}^*$ be Lie algebras with inner product $(,)$ on the space $\mathfrak{g} \oplus \mathfrak{g}^*$. Then $\mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra structure with $\mathfrak{g}$ and $\mathfrak{g}^*$ as Lie subalgebras, where the inner product $(,)$ is invariant under the adjoint action of $\mathfrak{g}$ and $\mathfrak{g}^*$ if and only if $\mathfrak{g}$ is a Lie bialgebra. Moreover, the Lie algebra structure is unique.*

There is a natural candidate for this Lie bialgebra structure when $\mathfrak{g}$ is a Lie bialgebra: the classical double. Before defining the quantum double, we cover some examples. The following examples are the classical versions of the $U_q(sl_3^\epsilon)$ lower and upper Borel subalgebras, as will become clear in the next section.

Example 1.1.1. *Consider the Lie bialgebra $(b^-, [,], \delta)$ over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$ generated by the elements $\{b, a, z, y, x\}$ as a R_ϵ-module and the relations*

$$[a, x] = -2x,\ [a, y] = y,\ [a, z] = -z \tag{1.4}$$
$$[b, x] = x,\ [b, y] = -2y,\ [b, z] = -z \tag{1.5}$$
$$[x, y] = z \tag{1.6}$$
$$\delta(x) = \epsilon(x \otimes a - a \otimes x) \tag{1.7}$$
$$\delta(y) = \epsilon(y \otimes b - b \otimes y) \tag{1.8}$$
$$\delta(z) = \epsilon(z \otimes (a+b) - (a+b) \otimes z + 2x \otimes y - 2y \otimes x), \tag{1.9}$$

and all other identies on generators zero. Concretely, we define the Lie algebra b^- as the R_ϵ-module generated by b, a, z, y, x, divided out to the ideal generated by the algebra relations stated above. The algebra-relations are equal to the relations in the lower triangular subalgebra of the Lie algebra $sl_3(\mathbb{R})$. The cobracket is the usual cobracket multiplied by ϵ. Hence it satisfies the Lie bialgebra axioms.

The b^- Lie bialgebra is a central object in this chapter. We might denote the generators $\{b, a, z, y, x\}$ of b^- as the more general

$$\{H_1^-, H_2^-, X_1^-, X_2^-, X_3^-\},$$

with $H_3^- = H_1^- + H_2^-$. Although b^- is not semisimple, as can be seen from calculating the Killing form, where the diagonal block-matrix corresponding to

(z, y, x) (using the order (b, a, z, y, x)) vanishes, the Killing form κ restricted to the maximal toral subalgebra H of b^- is nondegenerate over $\mathbb{R}$. See appendix A.3 for the definition of the Killing form.
The Killing form is not nondegenerate over R_ϵ since the Killing form is bilinear in ϵ. Formally, as remarked in the preliminaries, we can extend an $\mathbb{R}$-basis to an R_ϵ-basis. So we can consider $\{b, a, z, y, x\}$ as elements in $b^-/\epsilon b^-$. In this sense is the Killing form nondegenerate. Note that the construction of the Killing form is independent of ϵ.
In fact $\kappa(a, a) = \kappa(b, b) = 6$ and $\kappa(a, b) = \kappa(b, a) = -3$. This is due to the fact that b^- is derived from a semisimple Lie algebra. The following lemma holds. Again, the remark is that the $\mathbb{R}$-basis can be extended to an R_ϵ-basis.

Lemma 1.1.2. *Let $\phi \in H^*$ and κ be the Killing form on H, the maximal toral subalgebra of b^-. Then there exists a unique $t_\phi \in H$ such that $\phi(h) = \kappa(t_\phi, h)$.*

This lemma allows us to identify H with H^*, and also enables us to define a nondegenerate form on H^* by transferring the Killing form. Since the set of roots $\Phi \in H^*$ consists of 3 roots $\alpha, \beta, \alpha + \beta$ corresponding to the root-spaces generated by x, y and z, respectively (the root-spaces corresponding to these roots are nonzero), the set of roots Φ of b^- is the same as the set of sl_3. Moreover, the Cartan matrix is the same as the Cartan matrix for sl_3, which reads

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{1.10}$$

Another example of a Lie bialgebra is the lower Borel subalgebra of sl_2.

Example 1.1.2. *Let $\mathfrak{g}$ be the algebra over $\mathbb{R}$ generated by A and X and the relations*

$$[X, A] = X, [X, X] = 0, [A, A] = 0, \tag{1.11}$$

$$\delta(X) = X \otimes A - A \otimes X, \tag{1.12}$$

$$\delta(A) = 0. \tag{1.13}$$

Let us check the axioms explicitly for this example. Clearly the first two Lie bialgebra axioms are satisfied, as there are only two generators. By definition, the third axioms is satisfied. If we use the formula for δ, the last axiom is equivalent to

$$\delta([A, X]) = [A, X] \otimes A - A \otimes [A, X] = X \otimes A - A \otimes X.$$

Let us now calculate the dual of $\mathfrak{g}$, the generators of which are denoted by a and x and are dual to A and X respectively. Using the properties of the dual pairing ($\mathfrak{g}$ being finite dimensional) we get

$$\langle [a, x], X \rangle = \langle a \otimes x, \delta(X) \rangle = \langle a \otimes x, X \otimes A - A \otimes X \rangle \tag{1.14}$$

$$= \langle a \otimes x, -A \otimes X \rangle = -1. \tag{1.15}$$

Hence the algebra is generated by $[a, x] = -x$, and the other relations zero. In the same way we get $\delta(x) = x \otimes a - a \otimes x$. Checking the Jacobi identity, we see that δ^* is indeed a bracket on $\mathfrak{g}^*$, and hence $\mathfrak{g}$ is a Lie-bi algebra. This last exercise is left to the reader.

Let us construct the dual $(b^-)^*$ of b^-. $(b^-)^*$ is needed for the classical double construction.

Example 1.1.3. *The dual of b^-, which we will call b^+ suggestively, can be defined by using a pairing $\langle , \rangle : (b^-)^* \oplus b^- \to k$. If we extend the dual basis over $\mathbb{R}$ obtained through this pairing R_ϵ-linearly, we obtain the dual $(b^-)^*$ that is generated by the dual basis $\{X, Y, Z, A, B\} \subset b^+$. Since b^- is finite dimensional, so is its dual.*

$$\langle X, x\rangle = 1,\ \langle Y, y\rangle = 1,\ \langle Z, z\rangle = 1,\ \langle A, a\rangle = 1,\ \langle B, b\rangle = 1, \tag{1.16}$$

and relations between other generators zero. The generators of b^+ satisfy the following relations

$$[X, Y] = 2\epsilon Z, \tag{1.17}$$
$$[X, A] = \epsilon X,\ [X, B] = 0, \tag{1.18}$$
$$[Y, A] = 0,\ [Y, B] = \epsilon Y, \tag{1.19}$$
$$[Z, A] = \epsilon Z,\ [Z, B] = \epsilon Z, \tag{1.20}$$
$$\delta(A) = \delta(B) = 0 \tag{1.21}$$
$$\delta(X) = X \otimes (2A - B) - (2A - B) \otimes X \tag{1.22}$$
$$\delta(Y) = Y \otimes (2B - A) - (2B - A) \otimes Y \tag{1.23}$$
$$\delta(Z) = Z \otimes (A + B) - (A + B) \otimes Z + X \otimes Y - Y \otimes X. \tag{1.24}$$

The relations between generators that are not mentioned here are zero. It can be checked that these relations indeed satisfy the pairing axioms in a similar fashion as the previous example (in fact, there is only one relation that is different from the previous example, namely $[X, Y] = 2\epsilon Z$). It follows that b^+ is a Lie bialgebra. Note that b^+ is constructed in such a way that the generators are dual with respect to the pairing $\langle , \rangle$. It is interesting to see that this algebra is solvable.

Let us now define the classical double construction. For a proof that this structure indeed defines a quasitriangular Lie bialgebra in the sense of definition 1.1.4 and definition 1.1.1 see 8.2.1 in [23].

Definition 1.1.6. *(classical double) Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra over a ring R with Lie-dual $\mathfrak{g}^*$ (i.e. there exists a dual pairing obeying 1.16). The classical dual $D(\mathfrak{g})$ is the vector space $\mathfrak{g}^* \oplus \mathfrak{g}$ together with Lie bracket, cobracket and classical* *r-matrix*

$$[a \oplus b, c \oplus d]_D = ([c,a] + \sum c_1 \langle c_2, b\rangle - a_1\langle a_2, d\rangle) \oplus ([b,d] + \sum b_1\langle c, b_2\rangle - d_1\langle a, d_2\rangle) \tag{1.25}$$

$$\delta_D(a \oplus b) = \sum (a_1 \oplus 0) \otimes (a_2 \oplus 0) + \sum (0 \oplus b_1) \otimes (0 \oplus b_2), \tag{1.26}$$

$$r_D = \sum_a (f^a \oplus 0) \otimes (0 \oplus e_a). \tag{1.27}$$

Here, $a, c \in \mathfrak{g}^*$ and $b, d \in \mathfrak{g}$. The elements $f^a \in \mathfrak{g}^*$ form a basis dual to the basis $e^a \in \mathfrak{g}$. We use the Sweedler notation for $\delta(a) = \sum a_1 \otimes a_2$, where we often forget the summation symbol.

Note that $\mathfrak{g}$ and $(\mathfrak{g}^*)^{op}$ are included in $D(\mathfrak{g})$ as sub Lie bialgebras. See Lemma 1.4.2 of [6] and lemma 1.1.1 for the connection between the classical double and b^- and $(b^-)^*$.

Example 1.1.4. *Let us construct the classical double of the upper Borel of sl_2 from the previous example. From the definition we get*

$$[x \oplus 1, 1 \oplus X] = a \oplus A \tag{1.28}$$

$$[x \oplus 1, 1 \oplus A] = -x \tag{1.29}$$

$$[a \oplus 1, 1 \oplus X] = -X, \tag{1.30}$$

$$r = x \oplus X + a \oplus A. \tag{1.31}$$

The r-matrix follows from the definition of the dual generators. The algebra relations follow by direct calculation.

The classical double $D(\mathfrak{g})$ is a quasitriangular Lie bialgebra built on the vector space $(g)^* \oplus g$ with bracket, cobracket and r-matrix. Note that g^* has the negated (opposite) bracket in $D(\mathfrak{g})$. This specific Lie bialgebra structure is related to the invariance of the inner product on $\mathfrak{g}^* \oplus \mathfrak{g}$ under the adjoint action. See [23] chapter 8.2 for example. Let us briefly describe what we mean.
Let $2r_+$ be the symmetric part of $r \in D(\mathfrak{g}) \otimes D(\mathfrak{g})$ as defined before. Since $\mathfrak{g}$ is finite dimensional, $2r_+$ can be interpreted as a map $2r_+ : D(\mathfrak{g})^* \to D(\mathfrak{g})$. This map is given by $2r_+(\xi \oplus \phi) = \langle \xi \oplus \phi, r^{[1]}\rangle r^{[2]} + \langle \xi \oplus \phi, r^{[2]}\rangle r^{[1]}$. Note that since e_a and f^a are dual basis with respect to $\langle , \rangle$ we get $2r_+(\xi \oplus \phi) = \phi \oplus \xi$, so $2r_+$ is an linear isomorphism (over $\mathbb{R}$) with an inverse $2r_+^{-1} : D(\mathfrak{g}) \to D(\mathfrak{g})^*$ which is invariant under the adjoint action because r is an r-matrix, so $2r_+ = r_{12} + r_{21}$ is invariant under the adjoint action of $\mathfrak{g}$. This implies that $2r_+$ gives rise to a Lie algebra isomorphism between $D(\mathfrak{g})$ and $D(\mathfrak{g})^*$.
Because we are not working over a field but over the ring R_ϵ, the definition of $2r_+^{-1}$ does not follow automaticaly from $2r_+$. However, as noted in the prelimenaries, we can extend an $\mathbb{R}$-basis to an R_ϵ-basis. In this way we can define $2r_+^{-1}$ to be the inverse of $2r_+$ as a map of $\mathbb{R}$-modules and extend it to an R_ϵ map. This map is injective. We could also construct $2r_+^{-1}$ by introducing ϵ as an invertible

parameter. Since this changes nothing in the algebra-relations on generators, we can take $\epsilon^2 = 0$ after defining $2r^+$.
This means that we can write $2r_+^{-1}(\phi \oplus \xi) = K(\phi \oplus \xi, \cdot)$ for an adjoint-invariant element $K \in D(\mathfrak{g})^* \otimes D(\mathfrak{g})^*$. Since K is a bilinear map this will define a bilinear form. So in short, if r is quasitriangular, we can define a bilinear symmetric form on $D(\mathfrak{g}) \otimes D(\mathfrak{g})^*$ that is invariant under the adjoint action.
The question arises when is r quasitriangular. In particular, when is $2r_+$ ad-invariant. It turns out that this is the case if we use the opposite multiplication on $\mathfrak{g}^*$. Let $\phi \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. We identify $\phi \oplus 0 = \phi$ and $0 \oplus \xi = \xi$ in the notation. For the moment we denote $\delta(x) = x_{[1]} \otimes x_{[2]}$.

$$\begin{aligned} ad_\phi(r) &= [\phi, f^a]_{D(\mathfrak{g})} \otimes e_a + f^a \otimes [\phi, e_a]_{D(\mathfrak{g})} \\ &= [f^a, \phi] \otimes e_a - f^a \otimes \phi_{[1]} \langle \phi_{[2]}, e_a \rangle - f^a \otimes e_{a[1]} \langle \phi, e_{a[2]} \rangle \\ &= f^a \otimes e_{a[1]} \langle \phi, e_{a[2]} \rangle - f^a \otimes \phi_{[1]} \langle \phi_{[2]}, e_a \rangle - f^a \otimes e_{a[1]} \langle \phi, e_{a[2]} \rangle \\ &= -f^a \otimes \phi_{[1]} \langle \phi_{[2]}, e_a \rangle \\ &= -\phi_{[2]} \otimes \phi_{[1]} = \delta(\phi). \end{aligned}$$

We used the properties of a Lie bialgebra pairing 1.1.5. The same result holds for $\delta(\xi)$, only the minus signs change in + signs, since the product on $\mathfrak{g}$ is the multiplication on the double. However, this is exactly the right order to obtain $ad_\xi(r) = \delta(\xi)$. Since δ is anti-symmetric we see that $ad_\phi(2r_+) = ad_\xi(2r_+) = 0$. It follows that the opposite multiplication is essential to obtain a coboundary Lie bialgebra.
We denote this sub-Lie algebra with $\mathfrak{g}^{*op}$. One can take the dual of $D(\mathfrak{g})$, the double construction this results in is called the co-double. The resulting Lie algebras are equivalent in the sense that there exists an explicit Lie algebra isomorphism relating the two.

Definition 1.1.7. *Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be finite dimensional dual Lie algebras over a ring R. Let $a, d \in \mathfrak{g}^*$ and $b, c \in \mathfrak{g}$. Define the Lie algebra pairing $\langle , \rangle_D : D(\mathfrak{g}) \times D(\mathfrak{g})^* \to R$: $\langle a \oplus b, d \oplus c \rangle_D = \langle a, c \rangle + \langle d, b \rangle$.*

With this pairing it is possible to calculate the dual of a classical double $D(\mathfrak{g})$. To this end we state the following lemma, which simplifies this task.

Lemma 1.1.3. *Let $a, d \in \mathfrak{g}^*$ and $b, c \in \mathfrak{g}$. Then*

$$\langle [a, b]_D, c \rangle_D = \langle a, [b, c] \rangle, \langle a, [d, c]_{D*} \rangle_D = \langle [d, a], c \rangle.$$

The bracket [,] denotes the bracket in $\mathfrak{g}$ and $\mathfrak{g}^*$.

Proof. The proof is by direct verification, and can be found in [23] for example. We use the fact that $\langle , \rangle$ is a Lie bialgebra pairing, and the definition of $[,]_D$. □

We proceed to use the classical double construction on b^- and its dual. We are using the co-double construction, which is the dual of the classical double. Let us first give the classical double $D(b^-)$, by combining examples 1.1.1 and 1.1.3, where we put the opposite bracket on the $b^+ = (b^-)^*$ side. Only the trivial relations and the relations of b^- and b^{-*} are left out. Remember that $(b^-)^*$ has the opposite bracket in $D(b^-)$. The bracket on b^- is the usual bracket.

$$[X,b] = X,\ [X,a] = -2X,\ [X,z] = 2\epsilon y,\ [X,x] = 2A - B + \epsilon a, \tag{1.32}$$
$$[Y,b] = -2Y,\ [Y,a] = Y,\ [Y,z] = -2\epsilon x,\ [Y,y] = 2B - A + \epsilon b, \tag{1.33}$$
$$[Z,b] = -Z,\ [Z,a] = -Z,\ [Z,z] = A + B + \epsilon a + \epsilon b,\ [Z,y] = -X,\ [Z,x] = Y \tag{1.34}$$
$$[A,z] = -\epsilon z,\ [A,x] = -\epsilon x,\ [B,z] = -\epsilon z,\ [B,y] = -\epsilon y, \tag{1.35}$$
$$r_D = A \otimes a + B \otimes b + X \otimes x + Y \otimes y + Z \otimes z, \tag{1.36}$$

Lemma 1.1.4. *$D(b^-)$ is a quasitriangular Lie bialgebra, and is the classical double of the Lie bialgebras b^- and $(b^-)^*$.*

Proof. $\mathfrak{g} = D(b^-)$ is a quasitriangular Lie bialgebra by construction. The bracket-relations follow by direct calculation from the definition, and can be checked manually. Observe that the relations follow from group contraction on the standard gl_n structure. See for example 4.4.1 in [10]. See appendix A.4 for an example of Wigner group contraction.
The axioms for the cobracket are satisfied by examples 1.1.1 and 1.1.3, together with the definition of the cobracket on the classical double. The r-matrix follows from the definition of the algebra relations of the classical double. The fact that $D(b^-)$ is the double of b^- and $(b^-)^*$ can be seen from the uniqueness of the classical double (see lemma 1.1.1) and the fact that $D(b^-)$ contains b^- and $(b^-)^*$ as Lie subalgebras. This finishes the proof. □

The relations of the algebra $D(b^-)^*$ are given by

$$[X,b] = X,\ [X,a] = -2X,\ [X,z] = -2\epsilon y,\ [X,x] = -2A + B - \epsilon a, \tag{1.37}$$
$$[Y,b] = -2Y,\ [Y,a] = Y,\ [Y,z] = 2\epsilon x,\ [Y,y] = -2B + A - \epsilon b, \tag{1.38}$$
$$[Z,b] = -Z,\ [Z,a] = -Z,\ [Z,z] = -A - B - \epsilon a - \epsilon b,\ [Z,y] = -X,\ [Z,x] = Y \tag{1.39}$$
$$[A,z] = \epsilon z,\ [A,x] = \epsilon x,\ [B,z] = \epsilon z,\ [B,y] = \epsilon y, \tag{1.40}$$
$$r_D = A \otimes a + B \otimes b + X \otimes x + Y \otimes y + Z \otimes z, \tag{1.41}$$

and the bracket as defined above on the Lie subalgebras $b^{\pm}$. In particular, b^+ does not have the opposite bracket in $D(\mathfrak{g})^*$. The cobracket δ is negated on $(b^-)^* \subset D(\mathfrak{g})^*$ (as Lie algebras), and stays the same on b^-. The cobracket in general is very complicated, and can be calculated by using the dual pairing. We will not describe the cobracket of $D(\mathfrak{g})^*$ explicitly.

Theorem 1.1.1. *The algebra $D(b^-)^*$ constructed above is a quasitriangular Lie bialgebra, and is the dual of $D(b^-)$. We refer to this Lie bialgebra as sl_3^ϵ.*

Proof. The bracket of $b^{\pm} \subset D(\mathfrak{g})^*$ is calculated from the cobracket of $D(\mathfrak{g})$. By construction of $b^+ = (b^-)^*$, it follows that the bracket on $b^{\pm}$ is the usual (non-opposite) bracket. The other relations follow by using the ad-invariance of $\langle , \rangle : D(\mathfrak{g})^* \times D(\mathfrak{g}) \to R_\epsilon$. This enables us to directly calculate the bracket of $D(\mathfrak{g})^*$. The fact that $D(b^-)^*$ is dual to $D(b^-)$ follows from this calculation. We will do one example with the generators X and z. In the classical double $[X, z] = 2\epsilon y$. y pairs dually with Y, hence it follows by ad-invariance of the inner product $\langle , \rangle$ that

$$\begin{aligned} \langle X \oplus 1, [z \oplus 1, 1 \oplus Y]_{D^*} \rangle_D &= \langle [z, X], Y \rangle \\ &= \langle -2\epsilon y, Y \rangle \\ &= -2\epsilon. \end{aligned}$$

So we can conclude that $[Y, z]$ at least contains a term $2\epsilon x$, since x is the only generator that pairs nonzero with X. Since there are no generators in $D(b^-)$ that commute with z to yield y except X, we can conclude that $[Y, z] = 2\epsilon x$. The other relations follow in a similar fashion. On generators, the cobracket of $D(\mathfrak{g})^*$ is negated on $(b^-)^* \subset D(\mathfrak{g})^*$. From the pairing we can define the cobracket on mixed terms. This will not be done explicitly, but it is also not necessary. We conclude that $D(\mathfrak{g})^*$ is a Lie bialgebra.
To obtain a quasitriangular Lie bialgebra we need to check that $\delta(u) = u \cdot r$ for all $u \in D(\mathfrak{g})^*$, and moreover that $2r_+ = r_{12} + r_{21}$ is a invariant under the action of $\mathfrak{g}$, and $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$. Define $\mathfrak{a} = \mathfrak{g}^{*cop}$, where $\mathfrak{g}^{*cop}$ refers to $\mathfrak{g}^*$ with the negated cobracket, then $\mathfrak{a}^{*op} = \mathfrak{g}$. This follows from the Lie bialgebra pairing axioms. One can do the usual double construction on the Lie bialgebra $\mathfrak{a}$ to obtain a quasitriangular Lie bialgebra structure on $\mathfrak{g} \oplus \mathfrak{g}^{*cop}$ with classical r-matrix r_D. By lemma 1.1.1 this Lie algebra structure is unique and coincides with the Lie algebra $D(\mathfrak{g})^*$. The adjoint action on r_D coincides in both algebras. Moreover the cobracket of $\mathfrak{a}^{*op}$ agrees with the cobracket on $\mathfrak{g}$ by definition (and similarly for the dual), and hence the cobracket of $D(\mathfrak{g})^*$ is identical to the cobracket on $D(\mathfrak{a})$. So we see that $D(\mathfrak{g})^*$ is indeed quasitriangular. □

For completeness we also mention the definition of the universal enveloping algebra of a Lie bialgebra. The universal enveloping algebra $U(\mathfrak{g})$ is the noncommutative algebra generated by 1 and the elements of $\mathfrak{g}$. Formally, we have the following definition. One can also define $U(\mathfrak{g})$ by a universal property, see for example page 90 of [14]. The concept of a Hopf algebra will be defined in the next section. In this definition we only define the relevant maps without proving that they in fact define a Hopf algebra.

Definition 1.1.8. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a ring R. Let $S^n(\mathfrak{g}) = \bigotimes_{i=1}^{n}$ the n-th tensor space, and define the tensor algebra $T(\mathfrak{g}) = \lim_{n\to\infty} \bigoplus_{i=0}^{n} S^i(\mathfrak{g})$ (with tensor products over R). Define the universal enveloping algebra $U(\mathfrak{g})$ as $T(\mathfrak{g})$ modulo the relations $[a, b] = a \otimes b - b \otimes a$ for all $a, b \in \mathfrak{g}$. Let $a \in U(\mathfrak{g})$. The coproduct*

$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, counit $\varepsilon : U(\mathfrak{g}) \to R$ and antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ are given by

$$\Delta a = a \otimes 1 + 1 \otimes a, \varepsilon a = 0, Sa = -a,$$

where Δ, ε are extended as algebra maps, and S as an antialgebra map.

Note that this bialgebra is cocommutative, so we can take the R-matrix to be trivial to make $U(\mathfrak{g})$ a quasitriangular Hopf-algebra, as we will see in the next section. On a seperate note, observe that $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \otimes U(\mathfrak{h})$. This gives a nice insight in the relation between the classical double and the Drinfel'd double, which uses the tensor product instead of the sum. We have extended the bracket of a Lie bialgebra $\mathfrak{g}$ to $U(\mathfrak{g})$, and we have equipped $U(\mathfrak{g})$ with a Hopf algebra structure, but we have not yet extended δ to $U(\mathfrak{g})$.

Definition 1.1.9. *(Co-Poisson Hopf algebras) A co-Poisson Hopf algebra over a ring R is a cocommutative Hopf algebra H with a skew symmetric R-module map $\delta : H \to H \otimes H$ (the Poisson cobracket) satisfying*

1. *$\sigma \circ \delta \otimes id \circ \delta = 0$, where σ means summing over cyclic permutations of the tensor product.*
2. *$(\Delta \otimes id)\delta = (id \otimes \delta)\Delta + \sigma_{23}(\delta \otimes id)\Delta$, where σ_{23} means switching the second and third factor.*
3. *For all $a, b \in H$, $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$.*

This definition is natural, as follows from the following proposition. This proposition can be found as proposition 6.2.3 in [6]. Although we state the proposition for a specific Lie algebra over the ring R_ϵ, it is expected to hold for a general ring of characteristic zero and a general Lie algebra, with a proof similar to the proof in [6]. Of course we have the same proposition for the Lie bialgebras constructed in example 1.1.2. The proof is identical, and we will not state this proposition here, as it is only a formality.

Proposition 1.1.1. *Let $\mathfrak{g} = b^{\pm}$ be the Lie bialgebra over the ring R_ϵ as defined above. Then the Lie cobracket extends uniquely to a Poisson cobracket δ on $U(\mathfrak{g})$, making $U(\mathfrak{g})$ a co-Poisson Hopf algebra.*
Conversely, if $U(\mathfrak{g})$ has a Poisson cobracket δ, then $\delta|_{\mathfrak{g}}$ is a Lie cobracket on $\mathfrak{g}$.

Proof. First consider $b^+/\epsilon b^+$ as a Lie bialgebra. According to proposition 6.2.3 in [6] the cobracket extends uniquely to a Poisson bracket on $U(b^+/\epsilon b^+)$. The cobracket on $b^+/\epsilon b^+$ is also a cobracket when extended to b^+. We obtain the proposition for the ring R_ϵ by considering the universal enveloping algebra of the Lie algebra $\mathfrak{g} = b^+$ over R_ϵ. One can check that this yields the correct cobracket on the Lie algebra generators, trivially. The co-Poisson bracket obeys the first axiom, as δ is antisymmetric. Since $\delta(1) = 0$, the second axiom follows straightforwardly. The last axiom follows from the fact that δ is 1-cocycle.

Secondly, consider the Lie bialgebra $b \subset sl_3$ of lower triangular matrices over $\mathbb{R}$. This extends uniquely to a co-Poisson Hopf algebra by proposition 6.2.3 in [6]. Consider the map $\phi : b^- \to b$ by taking the cobracket δ on b^- and forgetting about ϵ. This map extends to the co-Poisson Hopf algebra $U(b^-)$, since $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$, and Δ is trivial in $U(b^-)$. Suppose that δ_{b^-} cannot be extended uniquely to $U(b^-)$. The difference between the two extensions is proportional to ϵ. Taking its image under ϕ, we get a contradiction with the uniqueness of the cobracket on $U(b)$. This proves the proposition. □

In the following theorem and in later chapters we will use the language of roots and root systems. In particular it is important to know that a Lie bialgebra is characterised by its Cartan matrix (a_{ij}). The Cartan matrix has integer coefficients. For more information on this subject see for example [14]. The most important notions are collected in appendix A.3. Remember that to each simple root one can associate a simple generator, as we showed before. Non-simple roots are sums of simple roots, and to non simple roots one can associate non-simple generators. We will denote the two roots of the Lie algebra sl_3 with greek letters α and β. More generally, we may write β_i or α_i for the roots of the Lie algebra .
Root systems are interesting for the classification of semisimple Lie algebras. They also provide a way of ordering the generators of a Lie algebra. We present a general way in which these generators form a basis for the universal enveloping algebra of a Lie algebra .

Theorem 1.1.2. *Let $\mathfrak{g} = b^+ \oplus b^-$ be a Lie bialgebra over R_ϵ with simple generators $X_i^\pm, H_i^\pm, i = 1, \cdots, n$, where n is the rank of $\mathfrak{b}^\pm$. $\mathfrak{g}$ is defined as the classical double of the Lie bialgebras $b^\pm$. b^- is defined as the lower Borel subalgebra of sl_n, where the cobracket is multiplied by ϵ, and b^+ is defined as its dual. Then $U(\mathfrak{g})$ is spanned as a vector space by the monomials $(X_1^+)^{i_1} \cdots X(_n^+)^{i_n}(H_1^+)^{l_1} \cdots (H_n^+)^{l_n}(H_1^-)^{j_1} \cdots (H_n^-)^{j_n}(X_1^-)^{k_1} \cdots (X^+)_n^{k_n}$. This basis is called the PBW basis.*

Proof. We first observe that $U(\mathfrak{g}) = U(b^+) \otimes U(b^-)$, so it is enough to prove that the monomials in $X^\pm$ and $H^\pm$ span $U(b^\pm)$ respectively. Let us denote $\mathfrak{g}$ as sl_n^ϵ.
First consider $U(b^-)$. The universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ has a countable basis called the Poincare-Birkhoff-Witt or PBW basis. This theorem is proved in for example [14] in the case where $\mathfrak{g}$ is a semisimple Lie algebra over a field. sl_n^ϵ contains the lower triangular matrices $b^- \subset sl_n$ as a Lie subalgebra over $\mathbb{R}$. Since the commutation relations in b^- do not contain ϵ, we can use the theorem for semisimple Lie algebras by dividing out to $\epsilon\mathfrak{g}$. We conclude that $U(b^-) \subset U(sl_3^\epsilon)$ has a PBW basis as well, and thereby also its dual has a PBW basis.
Consider $U(b^+)$. One can easily see that $U(b^+)$ is spanned by commutative monomials, since any expression can be rewritten using the Lie algebra relations in b^+. So $U(b^+)$ is free. The pairing is nondegenerate over $\mathbb{R}$, and we can extend the dual basis over $\mathbb{R}$ to an R_ϵ-basis, since $U(b^+)$ is free. The dual basis is given

by monomials in the generators dual to the generators of $U(b^-)$, which can be checked on generators. We conclude that the dual basis of b^+ forms a basis over R_ϵ that is identical to the monomials stated in the theorem. □

For completeness we mention the Chevalley basis of a general Lie algebra over a field k. A semisimple Lie algebra $\mathfrak{g}$ over a field k with Cartan matrix a_{ij} generated by a Chevalley basis X_i, H_i has the following relations.

$$[X_i^-, H_j] = a_{ij}X_i^-,\ [X_i^+, H_j] = -a_{ij}X_i^+, \tag{1.42}$$
$$[X_i^-, X_j^+] = H_i\delta_{ij},\quad \sum_{k=0}^{k=1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-a_{ij}-k} = 0.$$

The last relations are called the Serre relations. The Serre relations yield the commutation relations for non-simple generators. It can be rewritten as $ad_{X_i^\pm}^{-a_{ij}+1}(X_j^\pm) = 0$, for $i \neq j$.
The non-simple generators can be defined in a nice way using the Weyl group. The Weyl group is the space of reflections of the root-space and can be used to define a set automorphisms denoted by T_i. For a semisimple Lie algebra, the T_i are given by the adjoint action on non-Cartan elements and as the reflections on Cartan subalgebra elements. The T_i are Lie algebra automorphisms and are given by the following expressions.

$$T_i(X_i^\pm) = -X_i^\pm,\ T_i(H_j) = H_j - a_{ij}H_i, \tag{1.43}$$
$$T_i(X_j^\pm) = \frac{(\pm 1)^{a_{ij}}}{(-a_{ij})!}(ad_{X_i^\pm})^{-a_{ij}}(X_j^\pm),\ i \neq j. \tag{1.44}$$

This yields a braid group action on $\mathfrak{g}$ [6]. These relations also define a set of generators X_α for every root α with the property that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$.
We will later attempt to quantize this braid group action in the setting of sl_3^ϵ. However, the described T_i fail to be algebra automorphisms when $\epsilon^k = 0$. Non-simple generators can still be defined using the T_i, but we have to be careful when using the T_i further. See chapter 4 for a solution to this problem.

1.2. Hopf algebras

We proceed with the construction of a quasitriangular Hopf algebra based on the Lie bialgebra constructed in the previous section. Let $U(b^-)$ be the universal enveloping algebra of b^- as a Hopf algebra. This turns $U(b^-)$ into a Hopf algebra over R_ϵ. The goal is to find a quantization of the universal enveloping algebra of b^-, such that the Lie bialgebra structure is incorperated in the Hopf algebra structure. After the quantization we apply the Drinfel'd double construction to find a new solution to the Yang-Baxter equation.
The definitions below are usually given over a field, but since we work over the

ring R_ϵ in this chapter (except when stated otherwise), we state the definitions over a ring. We observe that the usual definitions hold over a ring as well [23]. As remarked in the prelimenaries, when considering a free module M over the ring R_ϵ, we can extend any $\mathbb{R}$-basis of $M/\epsilon M$ to an R_ϵ-basis of M. See [23], chapter 7, for the explicit proof of propositions 1.2.2 and 2.3.1. For the definition of the Hopf cohomology, see [6]. We remind the reader of the fact that we refer to a module over a ring R as a vector space over R. A ring is always commutative with unit in this thesis. When we say Hopf algebra, we mean a Hopf algebra over a ring R. Often, R will be implicit.

Definition 1.2.1. *((co)algebra) An algebra $(H, m, \mathbf{1})$ over a ring R is a vector space $(H, +, R)$ with a compatible multiplication m (also denoted as $\cdot$ or as the concatination of two elements) and unit map $\mathbf{1}$ with the following properties. Let $i, a \in R$.*

1. *the multiplication $m : H \otimes H \to H$ is an associative, bilinear map which preserves the unit,*
2. *the unit map $\mathbf{1} : R \to H$ is a linear map with property $\cdot \circ \mathbf{1} \otimes id(i \otimes a) = \mathbf{1}(i \cdot a)$, and $\cdot \circ id \otimes \mathbf{1}(a \otimes i) = \mathbf{1}(i \cdot a)$ for all $a \in H, i \in k$ (or $\mathbf{1}(1) = 1_H$).*

A coalgebra (H, Δ, ε) over R is a vector space $(H, +, R)$ with a compatible comultiplication Δ and counit ε with the following properties. Let $h \in H$.

1. *the comultiplication $\Delta : H \to H \otimes H$ is a linear, coassociative map, where coassociativity means $\Delta \otimes id \circ \Delta = id \otimes \Delta \circ \Delta$ and $\Delta(1_H) = 1_H \otimes 1_H$,*
2. *the counit $\varepsilon : H \to R$ has property $(id \otimes \varepsilon) \circ \Delta(h) = (\varepsilon \otimes id) \circ \Delta(h) = h$ (so $\varepsilon(1_H) = 1$).*

We define a Hopf algebra as follows.

Definition 1.2.2. *A Hopf algebra $(H, +, m, \mathbf{1}, \Delta, \varepsilon, S, R)$ over R is a vector space $(H, +, R)$ which is both an algebra $(H, m, \mathbf{1})$ and a coalgebra (H, Δ, ε), and is equipped with a linear antipode map $S : H \to H$ (which is an anti-homomorphism) obeying*

1. *$\Delta(gh) = \Delta(g)\Delta(h)$,*
2. *$\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$,*
3. *$m(S \otimes id) \circ \Delta = m(id \otimes S) \circ \Delta = \mathbf{1} \circ \varepsilon$.*

To construct a parallel between Lie bialgebras and Hopf algebras, let us define the Hopf algebra cohomology using the following cochain complex.

Definition 1.2.3. *(see p. 173 in [6]) Let H be a Hopf algebra. For $i, j \geq 1$, define $C^{i,j} := Hom_R(H^{\otimes i}, H^{\otimes j})$, and define $d'_{i,j} : C^{i,j} \to C^{i+1,j}$ and $d''_{i,j} : C^{i,j} \to C^{i,j+1}$ as* *follows (let $\gamma \in C^{i,j}$):*

$$
\begin{aligned}
&(d'\gamma)(a_1 \otimes \cdots \otimes a_{i+1}) := \Delta^{(j)}(a_1) \cdot \gamma(a_2 \otimes \cdots \otimes a_{i+1}) + \\
&\sum_{r=1}^{i} (-1)^r \gamma(a_1 \otimes \cdots \otimes a_{r-1}a_{r+1} \otimes a_{r+2} \otimes \cdots \otimes a_{i+1}) \\
&+ (-1)^{i+1} \gamma(a_1 \otimes \cdots \otimes a_i).\Delta^{(j)}(a_{i+1}), \\
&(d''\gamma)(a_1 \otimes \cdots \otimes a_i) := \\
&(m^{(i)} \otimes \gamma)(\Delta_{1,i+1}(a_1)\Delta_{2,i+2}(a_2) \cdots \Delta_{i,2i}(a_i)) \\
&+ \sum_{r=1}^{j} (-1)^r (id^{\otimes r-1} \otimes \Delta \otimes id^{\otimes j-r})(\gamma(a_1 \otimes \cdots \otimes a_i)) \\
&+ (-1)^{j+1} (\gamma \otimes m^{(i)})(\Delta_{1,i+1}(a_1)\Delta_{2,i+2}(a_2) \cdots \Delta_{i,2i}(a_i)).
\end{aligned}
$$

in this definition, $m^{(i)}$ and $\Delta^{(j)}$ are defined as follows

$$
\begin{aligned}
m^{(i)}(a_1 \otimes \cdots \otimes a_i) &= a_1 \cdot \cdots \cdot a_i \\
\Delta^{(j)}(a) &= (id \otimes \cdots \otimes id \otimes \Delta) \cdots (id \otimes \Delta)(\Delta(a)).
\end{aligned}
$$

(apply the comultiplication j times).

The $\Delta_{i,j}$ means sending the coproduct to the i-th and the j-th coordinate. The next proposition follows by direct computation and can be found on p. 175 in [6].

Theorem 1.2.1. *Let d' and d'' be as in the definition, then, $d' \circ d' = d'' \circ d'' = d' \circ d'' + d'' \circ d' = 0$.*

Finally we can define the Hopf algebra cochain complex. The previous theorem implies that the cohomologies are well defined.

Definition 1.2.4. *Let H be a Hopf algebra, and let d' and d'' be as defined previously, and set $d = d'_{ij} + (-1)^i d''_{ij}$ and $C^n = \oplus_{i+j=n+1} C^{ij}$. Then $d : C^n \to C^{n+1}$ and (C, d) is a cochain complex with cohomology groups $H^*(H, H)$.*
Define $H^*_{alg}(H, H)$ as the cohomology of the complex $(C^{\cdot 1}, d')$, and similarly define $H^*_{coalg}(H, H)$ as the cohomology of the complex $(C^{1,\cdot}, d'')$.

This cohomology will become important once we start studying deformations of Hopf algebras, for example of the universal enveloping algebra of a Lie bialgebra $\mathfrak{g}$. We can write down the cocycle conditions for this cochain complex. See example 2.3.1 in [23]. [23] uses a simpler definition of the cochain complex, which is equivalent, but is not well defined for $n > 2$.

Proposition 1.2.1. *Let H be an Hopf algebra. Then a 1-cocycle is an invertible element $\chi \in H$ such that*

$$\chi \otimes \chi = \Delta(\chi).$$

A 2-cocycle is an invertible element $\chi \in H \otimes H$ such that

$$(1 \otimes \chi)(id \otimes \Delta)\chi = (\chi \otimes 1)(\Delta \otimes id)\chi.$$

Some Hopf algebras can be equipped with an R-matrix.

Definition 1.2.5. *(Quasitriangular Hopf algebras) A Quasitriangular Hopf Algebra is a pair $(H, \mathcal{R})$, where H is a Hopf algebra and $\mathcal{R} \in H \otimes H$ is invertible and obeys*

1. $(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{23}$ *and* $(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$
2. $\tau \circ \Delta(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}$, *for all* $h \in H$.

$\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, *and* $\mathcal{R}_{ij} = \sum 1 \otimes \cdots \otimes \mathcal{R}^{(1)} \otimes 1 \cdots \otimes \mathcal{R}^{(2)} \otimes \cdots \otimes 1$, *with* $\mathcal{R}^{(1)}$ *and* $\mathcal{R}^{(2)}$ *on the i-th, resp. j-th entry.*

We see that the first condition of quasitriangularity is equivalent to the 2-cocycle condition together with the requirement that for a cocycle χ $(1 \otimes \chi)(id \otimes \Delta)\chi = \chi_{13}$.
A Hopf algebra is a bialgebra in particular, so it makes sense to look at both commutativity and cocommutativity, since algebra and coalgebra structures are dual to each other. An R-matrix measures the non cocommutativity of the comultiplication.

Definition 1.2.6. *((Co-)commutative) A Hopf algebra is said to be commutative if it is commutative as an algebra, and cocommutative if the coproduct Δ obeys $\tau \circ \Delta = \Delta$.*

The R-matrix is used to solve the Yang-Baxter equation. The Yang-Baxter equatation follows from the axioms for quasitriangularity. See chapter 2 of [23] for more information.

Proposition 1.2.2. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra, then $\mathcal{R}$ solves the equation: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, called the (quantum) Yang-Baxter equation.*

In order to use a Hopf algebra for constructing knot invariants, one needs to have a ribbon element. This will be explained in chapter 3. Let us write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. Then define $u = \sum (S\mathcal{R}^{(2)})\mathcal{R}^{(1)} \in H$, and $v = Su = \sum \mathcal{R}^{(1)} S\mathcal{R}^{(2)}$. The following proposition is proven in chapter 2 of [23].

Proposition 1.2.3. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra with antipode S. Then S is invertible and $S^2(h) = uhu^{-1}$ for all $h \in H$, and $S^{-2}(h) = vhv^{-1}$.*

We can now define the ribbon element.

Definition 1.2.7. *(Ribbon element) A quasitriangular Hopf algebra is called a ribbon Hopf algebra if the element uv has a central square root ν, called the ribbon element, such that $\nu^2 = vu$, $S\nu = \nu$, $\epsilon\nu = 1$ and $\Delta\nu = Q^{-1}(\nu \otimes \nu)$, where $Q = \mathcal{R}_{21}\mathcal{R}$.*

Let us construct the main example $U_q(sl_3^\epsilon)$ in the next section.

1.3. Quantizing a Lie bialgebra

In this example we follow the basic example in paragraph 6.4 in [6]. After quantizing the b^- subalgebra of sl_3 (which technically is quantizing the Hopf algebra structure on the enveloping algebra of b^-), we construct an explicit set of generators.
In order to construct $U_q(sl_3)$, one usually quantizes b^+ (or b^-), and takes the Drinfel'd double of this Hopf algebra and its dual with the opposite multiplication or comultiplication. After this procedure, the generators associated to the simple roots on both sides are identified to construct $U_q(sl_3)$. With the introduction of ϵ, this identification is not possible.
Formally, when quantizing $U(\mathfrak{g})$, we introduce an indeterminate h (or $\hbar$ in some sections) to obtain $U_h(\mathfrak{g})$, which is isomorphic as an h-module to $U(\mathfrak{g})[[h]]$. However, it is possible to leave h more implicit, and introduce $q = e^h$, or in our case $q = e^{-\epsilon h}$. The Hopf algebra is denoted as $U_q(\mathfrak{g})$ in this case. The two notations will mean the same thing in this thesis. This implies that it is always possible to expand q in terms of h and ϵ.
Throughout this chapter we work over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. Since we are considering free modules over R_ϵ, we can use most of the results that hold for Hopf and Lie bialgebras over a field. For future reference, the Drinfel'd double construction yields a quasitriangular Hopf algebra for any commutative ring. [6]
Note that it is possible to do the quantization of sl_3^ϵ for ϵ in $\mathbb{R}(\epsilon)$, and afterwards take the expansion in terms of ϵ. In this case, one has to prove that taking this expansion is possible. The reason to take ϵ to be invertible is that this provides an Hopf algebra isomorphism between $U_q(sl_n^\epsilon)$ and $U_q(sl_n)$. See chapter 4 for this approach.
Let us start with defining the h-adic topology for an indeterminate h. In this section we consider Hopf algebras over a general ring R.

Definition 1.3.1. *Let h be an indeterminate, and let H be an $R[[h]]$-module. Define the basis of the neighbourhoods of $0 \in H$ as the sets $C_n = \{h^n H | n \geq 0\}$. Define the h-adic topology to be the topology such that translations are continuous. In other words, the sets $\{a + C_n\}_{a \in H}$ form a basis for the topology.*

All Hopf algebra maps are continuous, meaning they are h-linear maps, by definition. The following examples are equipped with the h-adic topology. Some caution is advisable in this subject. In particular when taking the dual of an infinite dimensional Hopf algebra we will have to pay attention to the topology. This will be addressed later in this section. Tensor products are assumed to be completed in the h-adic topology.
Let us define what a quantized universal enveloping algebra is.

Definition 1.3.2. *A deformation of a Hopf algebra $(H, \mathbf{1}, m, \varepsilon, \Delta, S)$ over a ring R is a topological Hopf algebra $(H_h, \mathbf{1}_h, m_h, \varepsilon_h, \Delta_h, S_h)$ over the ring $R[[h]]$ of formal power series in h over R, such that*

1. *H_h is isomorphic to $H[[h]]$ as a $R[[h]]$ module.*

2. *$m_h = m$ mod h, $\Delta_h = \Delta$ mod h.*

Two Hopf algebra deformations are said to be equivalent if there is an isomorphism f_h of Hopf algebras over $R[[h]]$ which is the identity (mod h).

Let us write $a_h = a + a_1 h + a_2 h^2 + \cdots$ for an element of H_h, where $a = 0$ mod h, $a_i = 0$ mod h. Here we use the isomorphism $H_h \tilde{\to} H[[h]]$. Because m_h and Δ_h are $R[[h]]$-module maps, they are determined by their values on elements of H_h for which $a_1 = a_2 = \cdots = 0$, $a_i \in H$. Write

$$m_h(a \otimes a') = m(a \otimes a') + m_1(a \otimes a')h + m_2(a \otimes a')h^2 + \cdots \tag{1.45}$$

$$\Delta_h(a) = \Delta(a) + \Delta_1(a)h + \Delta_2(a)h^2 + \cdots \tag{1.46}$$

The (co)associativity and algebra homomorphism conditions of the Hopf algebra deformation are

$$\begin{aligned} m_h(m_h(a_1 \otimes a_2) \otimes a_3) &= m_h(a_1 \otimes m_h(a_2 \otimes a_3)) \\ (\Delta_h \otimes id)\Delta_h(a) &= (id \otimes \Delta_h)\Delta_h(a) \\ \Delta_h(m_h(a_1 \otimes a_2)) &= (m_h \otimes m_h)\Delta_h^{13}(a_1)\Delta_h^{24}(a_2). \end{aligned}$$

Modulo h^2, this translates to the following proposition.

Proposition 1.3.1. *A pair of R-module map (m_1, Δ_1) is a deformation mod h^2 of a Hopf algebra H if it satisfies*

$$\begin{aligned} & m_1(a_1 a_2 \otimes a_3) + m_1(a_1 \otimes a_2)a_3 = a_1 m_1(a_2 \otimes a_3) \\ & + m_1(a_1 \otimes a_2 a_3) \\ & (\Delta \otimes id)\Delta_1(a) + (\Delta_1 \otimes id)\Delta(a) = \\ & (id \otimes \Delta)\Delta_1(a) + (id \otimes \Delta_1)\Delta(a) \\ & \Delta(m_1(a_1 \otimes a_2)) + \Delta_1(a_1 a_2) = (m \otimes m_1 + m_1 \otimes m)\Delta^{13}(a_1)\Delta^{24}(a_2) \\ & + \Delta_1(a_1)\Delta(a_2) + \Delta(a_1)\Delta_1(a_2). \end{aligned}$$

More generally, a deformation mod h^{n+1} is a 2n-tuple $(m_1, \cdots, m_n, \Delta_1, \cdots, \Delta_n)$ which satisfies the (co)associativity and algebra homomorphism conditions (mod h^{n+1}). We now have the following classification of Hopf algebra deformations.

Theorem 1.3.1. *Let H be a Hopf algebra. The following relations between Hopf algebra cohomology and Hopf algebra relations hold:*

1. *there is a natural bijection between $H^2(H, H)$ and the set of equivalence classes of deformation (mod h^2) of H,*

2. *If $H^2(H, H) = 0$, every deformation of H is trivial and*

3. *If $H^3(H,H) = 0$, every deformation (mod h^2) of H extends to a genuine deformation of H.*

Using this theorem we can state an important result in Hopf algebra deformation theory called the rigidity theorem. The theorem is formulated in terms of reductive Lie algebras in [6]. Semisimple Lie algebras are reductive, see [14], and we will skip the definition alltogether. For an R_ϵ-module M, an $\mathbb{R}$-basis of the module $M/\epsilon M$ can be extended to an R_ϵ-basis of M. Using this basis we can generalize the following theorem to a Hopf algebra over the ring R_ϵ.

Theorem 1.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Then $H^{*2}_{alg}(U(\mathfrak{g}), U(\mathfrak{g})) = 0$. So every deformation of $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra.*

The example we work with is not semisimple, so we have to come up with a workaround to use the rigidity theorem. The idea is to only look at the half of the deformed sl_3^ϵ which has a Lie algebra structure that agrees with sl_3. Since we are looking for an algebra isomorphism (or even an Isomorphism of R_ϵ-modules) between $U_h(b^-)$ and $U(b^-)[[h]]$ (not an Hopf algebra isomorphism), we can simply restrict the isomorphism between $U_h(sl_3)$ and $U(sl_3)[[h]]$ to $U_h(b^-)$. Of course we will have to pay attention to R_ϵ too.

Definition 1.3.3. *(Quantized universal enveloping algebra (QUE)) A Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is called a quantized universal enveloping algebra, or QUE algebra.*

The isomorphism in the definition of a deformation of a Hopf algebra is an isomorphism of $R_\epsilon[[h]]$-modules, meaning that the isomorphism does not necessarily respect the Hopf structure. In certain cases one can prove that (if $\mathfrak{g}$ is semisimple and is associated to a reductive algebraic group) every deformation of $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra. This isomorphism is not an Hopf algebra isomorphism. See Proposition 6.3.1 in [6].
Finally, the quantization of a Hopf algebra can be defined.

Definition 1.3.4. *(Quantization of Hopf algebra) Let A be a cocommutative co-Poisson-Hopf algebra over a ring R of characteristic zero, and let δ be its Poisson cobracket. A Quantization of A is a Hopf algebra deformation A_h of A such that*

$$\delta(x) = \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \text{ (mod h)},$$

where $x \in A$ and $a \in A_h$ such that $x = a$ (mod h), and $\Delta^{op} = \tau \circ \Delta$ is the opposite cobracket.
A quantization of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a quantization $U_h(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$ equipped with the co-Poisson-Hopf structure. Conversely, $(\mathfrak{g}, \delta)$ is called the classical limit of the QUE algebra $U_h(\mathfrak{g})$.

For more details see e.g. [23] and [6]. We will use both the notation $U_q(\mathfrak{g})$ with $q = e^{h\epsilon}$ and $U_h(\mathfrak{g})$ in our examples. The difference is subtle and is pointed out in for example [6]. The main difference is topological, as we pointed out before. The U_h-notation has an explicit h-adic topology, while in the U_q notation this topology is hidden. One concrete application of hiding the parameter h in q is that one can specify q to a root of unity, for example. Since we are not concerned with these properties, and always work with h explicitly present (and ϵ), we will use both notations. For the main example we will use the notation $H_{\epsilon,n}$ or $U_q(sl_3^\epsilon)$, taking the notation from [36].
Before doing the main example, we will briefly state the usual quantization of the lower Borel subalgebra of sl_3. This example can be found in many sources, for example [6]. In this example we introduce an invertible parameter γ.

Example 1.3.1. *Let b^- be the Lie bialgebra as in example 1.1.1, with an invertible indeterminate γ instead of ϵ. The following relations define the Hopf algebra $U_h(b^-)$. Moreover, it is the quantization of the Lie bialgebra b^-. We use the generators $\{b, a, z, y, x\}$, and take the free noncommutative module over $\mathbb{R}(\gamma)$ in these generators. Define the quantum commutator as $[u, v]_q = uv - qvu$, and let $q = e^{-\gamma h}$ for the duration of this example. The algebra $U_h(b^-)$ is defined as the module of noncommutative polynomials in $\{b, a, z, y, x\}$ divided out to the iddeal generated by the following relations*

$$[a, x] = -2x,\ [a, y] = y,\ [a, z] = -z \tag{1.47}$$
$$[b, x] = x,\ [b, y] = -2y,\ [b, z] = -z \tag{1.48}$$
$$[x, y]_q = z,\ [x, z]_{q^{-1}} = 0,\ [y, z]_q = 0. \tag{1.49}$$

This is the standard example with a parameter introduced, so it is obvious from literature that this algebra has a basis consisting of the ordered monomials in the generators. This proves that the quotient is not empty, and that the multiplication defined here is associative. The Hopf algebra structure is defined by the following identities

$$\begin{aligned}
\Delta(b) &= b \otimes 1 + 1 \otimes b,\ \Delta(a) = a \otimes 1 + 1 \otimes a, \\
\Delta(z) &= z \otimes 1 + q^{a+b} \otimes z + (q^{-1} - q)q^b x \otimes y, \\
\Delta(y) &= y \otimes 1 + q^b \otimes y, \\
\Delta(x) &= x \otimes 1 + q^a \otimes x, \\
S(a) &= -a,\ S(b) = -b, \\
S(z) &= -q^2 q^{-a-b} z + q^2(q^{-1} - q)q^{-a-b} yx, \\
S(y) &= -q^{-b} y, \\
S(x) &= -q^{-a} x.
\end{aligned} \tag{1.50}$$

Let us check that Δ is an algebra homomorphism. In fact, the only non-trivial relations to check are $[x, y]_q$, $[x, z]_{q^{-1}}$ and $[y, z]_q$, as it is easy to see that $[\Delta(x), \Delta(a)] = [x, a] \otimes$

$1 + q^a \otimes [x,a] = \Delta([x,a])$, *and similarly for the other relations.*

$$\begin{aligned}[\Delta(x),\Delta(y)]_q &= xy\otimes 1 + xq^b \otimes y + q^a y \otimes x + q^{a+b}\otimes xy \\ &\quad - qyx \otimes 1 - qyq^a \otimes x - qq^b x \otimes y - qq^{a+b}\otimes yx \\ &= xy \otimes 1 - qyx\otimes 1 + (q^{-1}-q)q^b x \otimes y + q^{a+b}\otimes xy - qq^{a+b}\otimes yx \\ &= z\otimes 1 + q^{a+b}\otimes z + (q^{-1}-q)q^b x\otimes y.\end{aligned}$$

Let us check $[\Delta(x),\Delta(z)]_{q^{-1}} = 0$*, and leave* $[\Delta(y),\Delta(z)]_q = 0$ *to the reader, as this follows in the same way.*

$$\begin{aligned}[\Delta(x),\Delta(z)]_{q^{-1}} &= [x,z]_{q^{-1}}\otimes 1 + q^{2a+b}\otimes [x,z]_{q^{-1}} + qq^{a+b}x\otimes z - qq^{-2}q^{a+b}x\otimes z \\ &\quad + (q^{-1}-q^{-1})zq^a \otimes x + (q^{-1}-q)q^{a+b}x\otimes xy - q(q^{-1}-q)q^{a+b}x\otimes yx \\ &= (q-q^{-1})q^{a+b}x\otimes z + (q^{-1}-q)q^{a+b}x \otimes z \\ &= 0.\end{aligned}$$

Let us show that Δ *is coassociative. Coassociativity on a and b is trivial. For coassociativity of* Δ *on x and y, observe that* q^a *is grouplike, so* $\Delta(q^a) = q^a\otimes q^a$*. Let us explicitly perform the calculation*

$$\begin{aligned}Id\otimes\Delta(\Delta(z)) &= Id\otimes\Delta(z\otimes 1 + q^{a+b}\otimes z + (q^{-1}-q)q^b x\otimes y) \\ &= z\otimes 1\otimes 1 + q^{a+b}\otimes z\otimes 1 + q^{a+b}\otimes q^{a+b}\otimes z \\ &\quad + (q^{-1}-q)q^{a+b}\otimes q^b x\otimes y + (q^{-1}-q)(q^b x\otimes y\otimes 1 + q^b x\otimes q^b\otimes y).\end{aligned}$$

Performing a similar calculation for $\Delta\otimes Id(\Delta(z))$*, we see that coassociativity holds. Note that* q^b *is grouplike (similarly for* q^{a+b}*). Plugging in* $\Delta(z)$ *and* $\Delta(x)$*, and using the fact that* Δ *is an algebra homomorphism, we obtain the desired result.*
We check if the antipode is the involution inverse of the comultiplication. Denote the multiplication as $m_{1,2}(u\otimes v) = uv$*. The indices 1 and 2 stand for the first and second tensor entry. We will use the more general version later. We only check this explicitly for z, the axiom is obvious for a and b, and is left to the reader in the case of x and y.*

$$\begin{aligned}m_{1,2}(S\otimes id(\Delta(z))) &= m_{1,2}(-q^2q^{-a-b}z\otimes 1 + q^2(q^{-1}-q)q^{-a-b}yx\otimes 1 + q^{-a-b}\otimes z \\ &\quad - (1-q^2)q^{-a-b}x\otimes y) \\ &= -q^2q^{-a-b}z + (q-q^3)q^{-a-b}yx + q^{-a-b}z \\ &\quad - (q-q^3)q^{-a-b}yx - (1-q^2)q^{-a-b}z \\ &= 0.\end{aligned}$$

The antipode is continued as an anti-algebra homomorphism. This is by definition. The counit axioms are also satisfied, as one can check for oneself. We can conclude that we have a Hopf algebra.
We are quick to notice that this Hopf algebra does indeed have the right classical limit,

and since this algebra is imbedded in $U_q(sl_3)$ there exists an isomorphism $U_h(b^-) \rightarrow U(b^-)[[h]]$, by the rigidity theorem [6]. Hence we have obtained a quantization of the Lie bialgebra b^-. This finishes the example.

Quantizing a sub Lie bialgebra sl_3

We now treat the quantization of the Lie bialgebra b^- defined in example 1.1.1 for non-invertible ϵ. We follow the basic example 6.4 in [6]. The results of this section are summarized in theorem 1.3.3.

Quantizing the b^- subalgebra of the sl_3 Lie algebra starts with constructing the comultiplication Δ_h which has the b^- cobracket as classical limit. For a and b, which have $\delta(a) = \delta(b) = 0$, the choice is $\Delta_h(a) = a \otimes 1 + 1 \otimes a$, and the same for b. This is the trivial Hopf algebra structure on $U(b^-)$. Note that $U(b^-)$ is a graded algebra with $deg(a,b) = 0$ and $deg(x,y) = 1$. Hence $deg(z) = 2$. The multiplication and comultiplication have to preserve the grading. We define a grading on the tensor product by adding the grading of the factors. We can guess $\Delta_h(x) = x \otimes f + g \otimes x$.

Let Δ denote the trivial comultiplication on $U(b^-)$. Since $\Delta_h \equiv \Delta(\text{mod h})$, we get $f \cong g \equiv 1$ (mod h). We want Δ_h to be an algebra homomorphism that is coassociative. Working out the condition for coassociativity forces $\Delta_h(f) = f \otimes f$, and the same relation for g. Hence f and g have to be group-like (by definition). Note that $\Delta_h : U(b^-)[[h]] \rightarrow U(b^-)[[h]] \otimes U(b^-)[[h]]$, where the tensor product is completed in the h-adic topology. This yields $(U(b^-) \otimes U(b^-))[[h]]$ as the image of Δ_h. It is a simple computation to show that all group like elements are of the form $e^{h\mu H}$, where $H \in \mathfrak{h}$, an element of the Cartan subalgebra, and $\mu \in \mathbb{R}[[h]]$ [6]. Hence $\Delta_h(x) = x \otimes e^{\nu ha} + e^{\mu ha} \otimes x$. Since Δ_h is an algebra homomorphism, we may multiply x with a grouplike element to simplify the expression to $\Delta_h(x) = x \otimes 1 + e^{\mu ha} \otimes x$. The definition of the quantization of a Hopf algebra then gives $\Delta_h(x) = x \otimes 1 + e^{-\epsilon ha} \otimes x$.

The definition of a Hopf algebra can be used to obtain the antipode of x. In the same way the comultiplication of y can be deformed. Since the cobracket for a and b is trivial we can easily quantize this cobracket with the trivial comultiplication. As a result, the multiplication relations between a, b and x and a, b and y equal the classical relations. We obtain the comultiplication and antipode relations for x, y, a and b displayed in 1.57. The multiplication between x and y needs to be altered in order for Δ_h to be an algebra homomorphism.

Let us consider the Serre relations of the Lie bialgebra b^-. In our case they need to be slightly altered in order for Δ_h to be an algebra homomorphism. The so called quantum Serre relations are obtained, and we use these to calculate the products between the non-simple algebra generators.

The classical Serre relations in the case of b^- are given by $[x,x,y] = 0$ and $[y,y,x] = 0$. This can be rewritten as $X_i^-(X_j^-)^2 - 2X_i^-X_j^-X_i^- + X_iX_j^2 = 0$, where $i \neq j$, $i, j \in \{1,2\}$, or $(X_i^-)^2X_j^- + X_j^-(X_i^-)^2 = 2X_i^-X_j^-X_i^-$. Applying the comultiplication

$$\Delta(y) = y \otimes 1 + e^{-\epsilon h(b)} \otimes y, \tag{1.51}$$
$$\Delta(x) = x \otimes 1 + e^{-\epsilon h(a)} \otimes x \tag{1.52}$$

to the left hand side, we get for b^- (defining $q = 1 - \epsilon = e^{-\epsilon}$):

$$\begin{aligned}\Delta_h(x^2y + yx^2) &= \Delta_h(x)^2\Delta_h(y) + \Delta_h(y)\Delta_h(x)^2 \\ &= x^2y \otimes 1 + (1+q^2)e^{-\epsilon ha}xy \otimes x + e^{-2\epsilon ha}y \otimes x^2 + q^2e^{-\epsilon hb}x^2 \otimes y \\ &+ (1+q^2)qe^{-\epsilon h(a+b)}x \otimes xy + e^{-\epsilon h(2a+b)} \otimes x^2y + e^{-\epsilon h(2a+b)} \otimes yx^2 \\ &+ yx^2 \otimes 1 + (1+q^2)qe^{-\epsilon ha}yx \otimes x + q^2e^{-2\epsilon ha}y \otimes x^2 \\ &+ e^{-\epsilon hb}x^2 \otimes y + (1+q^2)e^{-\epsilon h(b+a)}x \otimes yx.\end{aligned} \tag{1.53}$$

Now we will use the classical Serre relation as ansatz. We assume $x^2y + yx^2 = Cxyx$. We will compute C by applying Δ on both sides. We apply Δ_h to the right handside now:

$$\begin{aligned}\Delta_h(x)\Delta_h(y)\Delta_h(x) &= xyx \otimes 1 + qe^{-\epsilon hb}x^2 \otimes y + e^{-h\epsilon a}yx \otimes x + e^{-\epsilon h(a+b)}x \otimes xy \\ &+ q^{-1}e^{-ah\epsilon}xy \otimes x + q^{-1}e^{-\epsilon h(a+b)}x \otimes yx \\ &+ q^{-1}e^{-2\epsilon h(a)}y \otimes x^2 + e^{-\epsilon h(2a+b)} \otimes xyx.\end{aligned} \tag{1.54}$$

We simplify by taking the exponentials up front. We do not know what C is, but we assumed that $x^2y + yx^2 = Cxyx$ holds for some C. This simplifies the equation between 1.53 and 1.54. The terms involving triple products of x and y on one side of the tensor product cancel out. We can compare the terms term by term. Doing this, we note that $C = q + q^{-1}$, and the following relation should hold

$$x^2y + yx^2 = (q + q^{-1})xyx. \tag{1.55}$$

This relation is called the quantum Serre relation, and is also derived in [6]. We will use the convention $[n]_q = \frac{1-q^{-2n}}{1-q^{-2}}$ in this chapter, where we differ from [6] in a factor q^n, making future notations easier. Observe that $\frac{1-q^{-2n}}{1-q^{-2}} = (1 + q^{-2} + \cdots + q^{-2n+2})$, the geometric series. So the expansion in ϵ of $[n]_q$ is well defined, as the singularity is removable. Using this convention, $q + q^{-1} = q(\frac{1-q^{-4}}{1-q^{-2}}) = q[2]_q$. In fact, this is generalizable, as we will see in chapter 4. For now we remember that $\epsilon^2 = 0$, so the quantum Serre relation for us is equal to the classical Serre relation. To obtain a complete set of generators corresponding to the elements of the root system of b^-, the Weyl group action is needed. In this case, the generators corresponding to the non-simple roots can be calculated by using the Weyl group

action on $U_h(\mathfrak{g})$ (sometimes refered to as the quantum Weyl group). This action respects the algebra structure, but not the co algebra structure and the antipode, in the sense that the braid group acts via algebra automorphisms that are not coalgebra maps, and hence the comultiplication on the non-simple generators needs to be calculated differently. [6] is followed in this case (in particular chapter 8.1 and 8.2). This procedure is described in the last chapter for the b^+ subalgebra of sl_n for general $n > 0$ with Cartan-matrix a_{ij}.
Let us continue with defining the generator corresponding to the commutator of x and y. This is the only generator of $U_h(b^-)$ corresponding to a non-simple root. The fact that we can talk about root spaces and elements corresponding to roots is a consequence of the fact that b^- is a subalgebra of the semisimple Lie algebra sl_3, so we can use the sl_3 root space and basis for the Borel subalgebra b^-. This was discussed in the first section of this chapter.
Let us define the following map on $U_h(b^-)$, for $i \neq j$ $X_1^- = x$ and $X_2^- = y$. This map can be defined on $U_q(sl_n^\epsilon)$ as

$$T_i(X_j^-) = ad_{-(X_i^-)^{-a_{ij}}}(X_j^-).$$

Here, ad is the adjoint action of the Hopf algebra on itself, see [23].
When $\epsilon^k = 0$, T_i cannot be extended to yield a set of global automorphisms, see chapter 4. For the non-Cartan elements X_i^- of $U_q(b^-)$, T_i are automorphisms over $\mathbb{R}[[\epsilon]]$, however. So we can use the T_i to define non-simple generators.
We define the generator

$$z = T_\alpha(y) = ad_x(y) = x_{(1)} y S(x_{(2)}),$$

using the Sweedler notation. The automorphisms T_i differ from the automorphisms defined in chapter 4 by a central factor. This factor is essential for the Weyl property. We absorb this factor into z, as the Weyl property is trivial for sl_3. Note that we need the antipode of x to do this calculation, which can be computed from $\Delta_h(x)$. We obtain

$$z = xy - e^{-h\epsilon a} y e^{h\epsilon a} x = xy - (1-\epsilon)yx.$$

We used the multiplication relations between y and a and the antipode and comultiplication of x. Using the above calculated quantum Serre relation, together with the definition of z, we get the following commutation relations.

$$[z,y] = h\epsilon zy, \; [z,x] = -h\epsilon zx.$$

We can also calculate the comultiplication of z

$$\begin{aligned}\Delta_h(z) =&\Delta_h(x)\Delta_h(y) - (1-\epsilon)\Delta_h(y)\Delta_h(x) \\ =&(x\otimes 1 + e^{-\epsilon h(a)}\otimes x)(y\otimes 1 + e^{-\epsilon h(b)}\otimes y) \\ &-(1-\epsilon)(y\otimes 1 + e^{-\epsilon h(b)}\otimes y)(x\otimes 1 + e^{-\epsilon h(a)}\otimes x) \\ =&(xy-(1-\epsilon)yx)\otimes 1 + e^{-\epsilon(a+b)}\otimes(xy-(1-\epsilon)yx) \\ &+e^{\epsilon}e^{-\epsilon b}x\otimes y + e^{-\epsilon a}y\otimes x - (1-\epsilon)(e^{\epsilon}e^{-\epsilon a}y\otimes x + e^{-\epsilon b}x\otimes y) \\ =&z\otimes 1 + e^{-\epsilon(a+b)}\otimes z + 2\epsilon e^{-\epsilon b}x\otimes y.\end{aligned} \tag{1.56}$$

This ends the construction of $U_q(b^-)$ as a quantization of the Lie bialgebra b^-. Let us summarize the construction. Consider the $R_\epsilon[[h]]$-module M of noncommutative polynomials in the generators $\{b,a,z,y,x\}$. Let I be the ideal of M generated by the following relations, where $[,]$ stands for the commutator.

$$\begin{aligned}&[b,z]=-z,\ [b,y]=-2y,\ [b,x]=x,\\ &[a,z]=-z,\ [a,y]=y,\ [a,x]=-2x,\\ &[z,y]=h\epsilon zy,\ [z,x]=-h\epsilon zx,\ [y,x]=-z+h\epsilon yx.\end{aligned}$$

We consider the closure $\bar{I}$ of I in the h-adic topology on M. Define the algebra $U_h(b^-)$ (also denoted as $U_q(b^-)$) as $M/\bar{I}$. Furthermore, there are $R_\epsilon[[h]]$-algebra homomorphisms $\Delta_h : U_h(b^-) \to U_h(b^-)\otimes U_h(b^-)$, $\varepsilon : U_h(b^-) \to R_\epsilon[[h]]$ and algebra anti-homomorphism $S : U_h(b^-) \to U_h(b^-)$ that define a Hopf algebra structure on $U_h(b^-)$. These maps are defined by the following relations.

$$\begin{aligned}\Delta_h(b) &= b\otimes 1 + 1\otimes b,\ \Delta_h(a) = a\otimes 1 + 1\otimes a,\\ \Delta_h(z) &= z\otimes 1 + e^{-\epsilon h(a+b)}\otimes z + 2\epsilon h x\otimes y,\\ \Delta_h(y) &= y\otimes 1 + e^{-\epsilon h(b)}\otimes y,\\ \Delta_h(x) &= x\otimes 1 + e^{-\epsilon h(a)}\otimes x,\\ S(a) &= -a,\ S(b) = -b,\\ S(z) &= -(1-2\epsilon h)e^{h\epsilon(a+b)}z + 2\epsilon h yx,\\ S(y) &= -e^{h\epsilon(b)}y,\\ S(x) &= -e^{h\epsilon a}x,\\ \varepsilon(u) &= 0 \text{ if } u \neq 1_{U_h(b^-)}, \varepsilon(1_{U_h(b^-)}) = 1.\end{aligned}$$

Theorem 1.3.3. *The Hopf algebra $U_q(b^-)$ is a quantized universal enveloping algebra with classical limit the Lie bialgebra b^-.*

Proof. By construction $U_q(b^-)$ we have the correct classical limits of the multiplication and comultiplication. Furthermore, the (co)multiplication obeys the Hopf algebra axioms by construction. The antipode can be easily computed from the (co)multiplication. By theorem 1.4.2, which will be proven separately, $U_q(b^-)$ has a PBW basis. By sending monomials to monomials in $U(b^-)[[h]]$ we obtain an $R_\epsilon[[h]]$-module isomorphism between $U_q(b^-)$ and $U(b^-)[[h]]$. This finishes the proof. □

For clarification, we provide another proof of theorem 1.3.3. This proof relies on example 1.3.1, and looks at a quotient of this algebra.

Proof. Observe that the relations presented here are exactly the same as the relations in example 1.3.1 with $\gamma \to \epsilon$ and $\epsilon^2 = 0$. In particular, this yields a basis of monomials of $U_q(b^-)$, which we will refer to as a PBW basis.
To prove that $U_h(b^-) \cong U(b^-)[[h]]$ as $R_\epsilon[[h]]$-modules, consider $b^- \subset sl_3$ as a Lie algebra, ignoring the comultiplication. Observe that ϵ occurs only together with h in $U_h(b^-)$ in the quantum Serre relations. Furthermore, we know that the isomorphism between $U_h(sl_3) \cong U(sl_3)[[h]]$ is an isomorphism between $\mathbb{R}[[h]]$-modules. So we know that this isomorphism must be an isomorphism when we replace h with $h' = h\epsilon$ and putting $\epsilon^2 = 0$, given that epsilon only occurs in $q = 1 - \epsilon h \bmod \epsilon^2$, which is invertible modulo ϵ^2. In particular we know that the constructed isomorphism must be the identity modulo h, so it sends monomials of generators to monomials of generators of the Lie algebra b^-. This finishes the proof. □

We wish to do the double construction with this algebra and write down the universal R-matrix. We will do this in section 1.4.

1.4. The $U_q(sl_3^\epsilon)$ relations

In the previous section we obtained an Hopf algebra $U_h(b^-)$ or $U_q(b^-)$ that is a quantization of $U(b^-)$. The explicit check of the Hopf algebra axioms is a lengthy exercise. For this reason we present a Wolfram Mathematica implementation of the Hopf algebra in the next chapter, and set up the required formalism. The program can be found in the appendix A.1. The algebra relations are easier to implement, these can be found in a seperate program in A.1. The interesting axioms to check manually are (co)associativity, that Δ is an algebra homomorphism and that the antipode is an anti-algebra homomorphism.
We work over the ring $R_\epsilon[[h]]$ of formal power series of an indeterminate h. We repeat theorem 1.3.3 for completeness.

Theorem 1.4.1. *The following relations define a Hopf algebra $U_q(b^-)$ over $R_\epsilon[[h]]$.*

Moreover, $U_q(b^-)$ is the quantization of the Lie bialgebra b^-.

$$\begin{aligned}
[b,z] &= -z,\ [b,y] = -2y,\ [b,x] = x, \\
[a,z] &= -z,\ [a,y] = y,\ [a,x] = -2x, \\
[z,y] &= h\epsilon zy,\ [z,x] = -h\epsilon zx,\ [y,x] = -z + h\epsilon yx \\
\Delta(b) &= b\otimes 1 + 1\otimes b,\ \Delta(a) = a\otimes 1 + 1\otimes a, \\
\Delta(z) &= z\otimes 1 + e^{-\epsilon h(a+b)}\otimes z + 2\epsilon hx\otimes y, \\
\Delta(y) &= y\otimes 1 + e^{-\epsilon h(b)}\otimes y, \\
\Delta(x) &= x\otimes 1 + e^{-\epsilon h(a)}\otimes x, \\
S(a) &= -a,\ S(b) = -b, \\
S(z) &= -(1-2\epsilon h)e^{h\epsilon(a+b)}z + 2\epsilon hyx, \\
S(y) &= -e^{h\epsilon(b)}y, \\
S(x) &= -e^{h\epsilon a}x
\end{aligned} \tag{1.57}$$

To obtain a deformation we need an algebra isomorphism between $U_q(b^-)$ and $U(b^-)[[h]]$ as $R_\epsilon[[h]]$-modules, as stated in the proof of theorem 1.3.3. This isomorphism can be found by means of the rigidity theorems if ϵ is invertible. If ϵ is not invertible, it is possible to construct a basis of monomials which can be send to the classical PBW basis of $U(b^-)[[h]]$. The algebra $U(b^-)[[h]]$ has the multiplication of the universal enveloping algebra of $U(b^-)$, h-linearly extended. On the other hand, if one has such an isomorphism, it is possible to directly construct the q-PBW basis of $U_q(sl_3^\epsilon)$ by looking at the image of the classical PPBW basis. This is how [6] proves the existence. We will provide a direct proof.

Theorem 1.4.2. *The monomials $b^{n_1}a^{n_2}z^{n_3}y^{n_4}x^{n_5}$, $n_i\in\mathbb{N}$ form a basis of $U_h(b^-)$ as an $R_\epsilon[[h]]$-module.*

Proof. The proof is similar to the proof in [31], and it uses a Q-degree on $U_h(b^-)$. It is enough to prove that $b^{n_1}a^{n_2}z^{n_3}y^{n_4}x^{n_5}$, $n_i\in\mathbb{N}$ form a basis of $U_h(b^-)/\epsilon U_h(b^-)$ as an $\mathbb{R}[[h]]$-module. We can then extend the monomials to a basis of $U_h(b^-)$ as an $R_\epsilon[[h]]$ module.
Let $\mathfrak{h}$ be the Cartan subalgebra of b^-. Via quantization we can associate to each element in $\mathfrak{h}$ an element in $U_h(b^-)$. Let us call this subalgebra H for the duration of this proof. Firstly, define the elements $K_\lambda = e^{\epsilon H_\lambda}$, where $\lambda\in\Phi$ is a root of b^-, and H_λ the element in H corresponding to λ via lemma 1.1.2. Following [20], we define an action of K_λ on $U_q(b^-)$ by conjugating:

$$K_\lambda p K_\lambda^{-1} = q^{(\lambda,\rho)}p. \tag{1.58}$$

The root ρ is called the Q-degree of p. The Q-degree of p is well-defined since $(\cdot,\cdot)$ is nondegenerate on H^*, so if p has Q-degree σ and ρ, then $q^{(\lambda,\rho)} = q^{(\lambda,\sigma)}$ for all $\lambda\in H^*$, and so $\sigma=\lambda$. Now the proof consists of two parts: (a) proving that

the monomials span the vector space, and (b) proving linear independence. Part (a) is proven by choosing a normal ordering, in our case (b, a, z, y, x), on $U_h(b^-)$. It is always possible to write an expression in a normal ordered way in finitely many steps. This is left to the reader. It can be proven by induction, see for example [31] for the explicit calculations.
Part (b) is proven in 2 parts. Firstly, one can prove that a, b are linearly independent in the same way lemma 12 is proven in chapter 6.1 of [20]. Let us scetch the proof. The details can be found in [20]. As a first step we prove that $x^{n_x}y^{n_y}z^{n_z} \neq 0$ for $\mathbf{n} = (n_x, n_y, n_z) \in \mathbb{N}^3$. This is done by constructing an algebra homomorphism between $U_q(sl_2^\epsilon)$ to the ring of power series $R_\epsilon[[h]][e, l, f, f^{-1}]$, and mapping x and y to a copy of $U_q(sl_2^\epsilon)$ in which they are nonzero. See proposition 3.1 in [20]. The factor of $\frac{1}{q-q^{-1}}$ is only cosmetic in [20].
Suppose now $\sum_\gamma a_\gamma K_\gamma = 0$. We apply the adjoint action $ad(\sum_\gamma a_\gamma K_\gamma)$ to a monomial $z^{n_z}y^{n_y}x^{n_x}$, $n_i \in \mathbb{N}$. We obtain $\sum_\gamma a_\gamma + a_\gamma\epsilon(\gamma, \sum n_i\rho_i) = 0$ for any $\sum n_i\rho_i$, as $z^{n_z}y^{n_y}x^{n_x} \neq 0$. Since ρ_i span the root space of sl_3, as noted earlier in this chapter, we obtain that $a_\gamma = 0$ for all $\gamma \in \Phi$. Since $\epsilon^2 = 0$, this implies that monomials in a and b are also linearly independent for different exponents of a and b.
The independence of x, y and z is proven by following [31], with induction to the Q-degree. We know that monomials in a, b, x, y, z are nonzero. Assume that we have a relation between monomials in x, y, z. By applying Δ, which conserves Q-degree by construction, and using the linear independence of monomials in a and b, we see that the terms in the relation have the same Q-degree. The case where the Q-degree is equal to one of the simple roots is equivalent to the $U_q(sl_2^\epsilon)$ case, for which we refer to the proof of proposition 6.4.7 in [6].
For the case where the Q-degree is a sum of roots α_i, we can look at the Q-degree in both factors after applying Δ. Consider the biggest i such that $E_{\alpha_i+\cdots}$ (taking the convention that $x = E_{\alpha_1}$, $y = E_{\alpha_2}$, $z = E_{\alpha_1+\alpha_2}$) occurs with a nonzero exponent. Let n be the biggest common exponent of $E_{\alpha_i+\cdots}$. After applying Δ, consider the terms with Q-degree $n\alpha_i$ left of $\otimes$. The relation obtained on the right of $\otimes$ in this way is a (nonzero) multiple of the original relation, as is clear by a calculation similar to the one in [31], and is of a strictly lower degree, hence the coefficients of these terms are zero by the induction hypothesis. So ordered monomials in x, y, z are linearly independent.
Finally, the linear independence of ordered monomials in b, a, z, y, x is proven by following the proof of lemma 13 in chapter 6.1 of [20]. This is a similar argument as before, by applying Δ and combining the linear independence of monomials in a, b and monomials in x, y, z. For the explicit proof of this lemma we refer to [20]. □

Duality

In order to apply the Drinfel'd double construction we need the dual of $U_h(b^-)$. To obtain the correct dual of $U_h(b^-)$, one has to take the dual of a smaller subalgebra called a quantized formal series Hopf algebra (QFSH-algebra for short).

[6] The dual of a quantum formal series Hopf algebra is a quantized universal enveloping algebra (QUE-algebra). Notice that it is also possible to define the QUE-dual of a QUE algebra the other way around, by taking the QUE-algebra corresponding to the Hopf algebra-dual of a QUE-algebra, which is a QFSH algebra [7].
Before we introduce the notion of a quantized formal series Hopf algebra, let us consider the dual of the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$. The following can be found as example 4.1.16 in [6].
Let $\mathfrak{g}$ be a Lie algebra over a ring R. Concretely, we have the ring R_ϵ in mind. We need R to have certain nice properties, such that R is obtained by extending $\mathbb{R}$ with a finite number of algebraic elements. In this way, we can extend any $\mathbb{R}$-basis of a quotient module to an R-basis of the entire module, as is discussed for R_ϵ in the appendix A.5. We do not go into details here, as we will only be concerned with Hopf algebras over a field or Hopf algebras over the ring R_ϵ.
The Lie algebra $\mathfrak{g}$ has a basis $\{x_1, \cdots x_d\}$, so $U(\mathfrak{g})$ has a PBW basis consisting of ordered monomials in x_i. We number this basis by $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_d) \in \mathbb{N}^d$, and denote $x_{\boldsymbol{\lambda}} = \frac{x_1^{\lambda_1} \cdots x_d^{\lambda_d}}{\lambda_1! \cdots \lambda_d!}$. The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has the structure of a Hopf algebra if we take the trivial coproduct, and in this case we obtain

$$\Delta(x_{\boldsymbol{\lambda}}) = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \delta_{\boldsymbol{\mu}+\boldsymbol{\nu},\boldsymbol{\lambda}} x_{\boldsymbol{\mu}} \otimes x_{\boldsymbol{\nu}}.$$

Consider $\xi^{\boldsymbol{\lambda}} \in U(\mathfrak{g})^*$ defined by $\xi^{\boldsymbol{\lambda}}(x_{\boldsymbol{\mu}}) = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$, then the multiplication $m : U(\mathfrak{g})^* \otimes U(\mathfrak{g})^* \to U(\mathfrak{g})^*$ is defined by $m(\xi^{\boldsymbol{\mu}} \otimes \xi^{\boldsymbol{\nu}})(x_{\boldsymbol{\lambda}}) := \xi^{\boldsymbol{\mu}} \otimes \xi^{\boldsymbol{\nu}}(\Delta(x_{\boldsymbol{\lambda}}))$. From the definition we obtain the relation $\xi^{\boldsymbol{\mu}}\xi^{\boldsymbol{\nu}} = \xi^{\boldsymbol{\mu}+\boldsymbol{\nu}}$ (apply both sides to $x_{\boldsymbol{\lambda}} \in U(\mathfrak{g})$).
Let $R_\epsilon[[\xi_1, \cdots, \xi_d]]$ be the algebra of formal power series in indeterminates ξ_i. By sending $\xi^{\boldsymbol{\lambda}} \to \xi_1^{\lambda_1} \cdots \xi_d^{\lambda_d}$ we obtain an isomorphism between $U(\mathfrak{g})^* \to R[[\xi_1, \cdots, \xi_d]]$.
With the dual of $U(\mathfrak{g})$ in mind we state the following definition. We follow paragraph 7 of [7]. The condition of a field, as is used in [7], is not essential. It is essential that the ring R has the property that any $\mathbb{R}$-basis can be extended to an R-basis, together with other properties discussed in appendix A.5. We are concerned with the ring $R_\epsilon[[h]]$ mainly in this paragraph. For this reason, we do not state the precise conditions on the ring R.

Definition 1.4.1. *(QFSH-algebra) A quantum formal series Hopf algebra is a topological Hopf algebra B_h over the ring R[[h]], where R is a ring, such that B_h is isomorphic as a R[[h]]-module to $R[[h]]^I$ (equipped with the product topology) for some set I, and $B_h/hB_h \hat{=} R[[\xi_1, \xi_2, \cdots]]$ as a topological algebra.*

The dual of the universal enveloping algebra is equipped with the weak topology. An isomorphism of topological algebras should be continuous and have a continuous inverse. To illustrate this definition, let us consider the following example.

Example 1.4.1. *We start with the lower Borel subalgebra A_h of $U_q(sl_2)$ over $\mathbb{R}$, generated by $\{x, a\}$ and the relations $[a, x] = -2x$, $\Delta(a) = a \otimes 1 + 1 \otimes a$, $\Delta(x) =$*

$x \otimes 1 + e^{-ha} \otimes x$. If we define $\tilde{a} = ha$ and $\tilde{x} = hx$, we obtain the algebra B_h with relations $[\tilde{a}, \tilde{x}] = -2h\tilde{x}$, $\Delta(\tilde{a}) = \tilde{a} \otimes 1 + 1 \otimes \tilde{a}$, $\Delta(\tilde{x}) = \tilde{x} \otimes 1 + e^{-\tilde{a}} \otimes \tilde{x}$.
As a $\mathbb{R}[[h]]$-module, B_h is isomorphic to $\mathbb{R}[[h]]^I = Map(I, \mathbb{R}[[h]])$, where $I = \mathbb{N} \times \mathbb{N} = \mathbb{N}$ enumerates the PBW basis of B_h. The PBW basis of B_h is given by ordered monomials in $\tilde{a}$ and $\tilde{x}$. Concretely, we send $\tilde{x}^{n_1}\tilde{a}^{n_2} \mapsto \phi_{(n_1,n_2)}$, where $\phi_{n,m}(x^{n'}a^{m'}) = \delta_{nn'}\delta_{mm'}h^{n+m}$. Furthermore, B_h/hB_h is commutative as an algebra, and hence B_h/hB_h is isomorphic to $\mathbb{R}[[\tilde{x}, \tilde{a}]]$, as required. So B_h is a QFSH-algebra. See also [7] for this example.

Generalizing the previous example, one can define a QFSH algebra inside any QUE algebra [6].

Definition 1.4.2. *Let A_h be any QUE algebra with cobracket Δ_h. Define $\Delta_n(a) : A_h \to A_h^{\otimes n}$ as $\Delta_n(a) = (id - \mu E)^{\otimes n} \Delta_h^{(n)}(a)$. Here $\Delta^{(n)} = \cdots (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1) \cdots$ is the iterated cobracket with n-1 Δs. Then define $B_h = \{a \in A_h | \Delta_n(a) = 0 \text{ (Mod } h^n \text{) for all } n \geq 1\}$.*

The statement is that B_h is a QFSH subalgebra of A_h. For a proof, see proposition 8.3.3 in [6]. To prove the proposition, one proves that an element is in B_h if and only if a monomial of total degree n has a prefactor that is divisable by h^n. As we have seen from the previous example, this proves that B_h is a QFSH-algebra.

Proposition 1.4.1. *Let $B_h \in A_h$ be as defined above, and let A_h be a QUE Hopf-algebra. Then B_h is a QFSH-subalgebra of A_h.*

We can prove the following proposition (see chapter 10 of [9] for the proof). It is essentially a generalization from the classical case, where we could calculate the dual of $U(\mathfrak{g})$ explicitly.

Proposition 1.4.2. *Let H be a quantized universal enveloping algebra over a ring R. Then the dual $H^* = Hom_R(H, R)$ of H is a QFSH algebra. Conversely, the dual of a QFSH algebra is a QUE algebra.*

With the dual of any QUE algebra H, we will mean the dual of the QFSH subalgebra $B_h \subset H$ as definied in 1.4.2, which is a QUE algebra, and we will denote this in the usual way, as H^*. We refer to this space as the QUE-dual of H. Sometimes literature uses the reduced dual, or the Hopf dual H° of any finite or infinite dimensional Hopf algebra, meaning they take the subset of the dual for which the comultiplication lands in the usual tensor product. Since this does not always happen in the infinite dimensional case, this is a useful definition. We will not use this definition here. We use the completed tensor product.
Let us describe the QFSH subalgebra of $U_h(b^-)$, in terms of its basis.

Proposition 1.4.3. *Let a,b,x,y,z be the generators of $U_h(b^-)$ with relations 1.57. Then the QFSH subalgebra of $U_h(b^-)$ as defined in 1.4.2 is topologically generated by ha, hb, hx, hy and hz.*

Proof. The proof is a straightforward repetition of the proof of proposition 8.3.3 in [6].

□

We calculate the dual of the QFSH subalgebra of $U_h(b^-)$ by introducing X, Y, Z, A, B as the linear functionals equal to one on $hx = \tilde{x}, hy = \tilde{y}, hz = \tilde{z}, ha = \tilde{a}, hb = \tilde{b}$ respectively, and zero on all other monomials of the form $\tilde{x}^{n_1}\tilde{y}^{n_2}\tilde{z}^{n_3}\tilde{a}^{n_4}\tilde{b}^{n_5}$. We denote this evaluation as follows:

$$\langle X, \tilde{x}\rangle = 1,\ \langle Y, \tilde{y}\rangle = 1,\ \langle Z, \tilde{z}\rangle = 1,\ \langle A, \tilde{a}\rangle = 1,\ \langle B, \tilde{b}\rangle = 1. \tag{1.59}$$

The pairing is extended as a Hopf algebra pairing, according to the following definition. This defines a Hopf algebra structure on $U_h(b^-)^*$. We will later show that if $U_h(b^-)$ has an R_ϵ, then so does $U_h(b^-)^*$. This basis is given by noncommutative monomials in $\{X, Y, Z, A, B\}$.

Definition 1.4.3. *Let $(H, \cdot, \Delta, \varepsilon, \mathbf{1})$ be a QUE Hopf algebra over the ring R_ϵ with dual $(H^*, \cdot, \Delta, \varepsilon, \mu)$. Let $a, b \in H^*$ and $c, d \in H$. Denote by $\langle , \rangle$ a bilinear map $\langle , \rangle : H^* \otimes H \to R_\epsilon$. The map $\langle , \rangle$ is called a Hopf algebra pairing if it obeys*

$$\begin{aligned} \langle ab, c\rangle &= \langle a \otimes b, \Delta c\rangle \\ \langle \Delta a, c \otimes d\rangle &= \langle a, cd\rangle \\ \langle 1, c\rangle &= \epsilon(c) \\ \langle a, 1\rangle &= \epsilon(a) \\ \langle Sa, c\rangle &= \langle a, Sc\rangle. \end{aligned} \tag{1.60}$$

We will refer to $\langle , \rangle$ as nondegenerate if it is nondegenerate over $\mathbb{R}$, interpreted as a pairing on $H^*/\epsilon H^* \otimes H/\epsilon H \to \mathbb{R}$.

The space we use as the dual of $U_h(b^-)$ is the QUE-dual of $U_h(b^-)$. We use the notation $U_h(b^-)^*$. We write $\frac{1}{h}$, but this is informal notation for the topology on the dual module. For example, if ξ is the dual basis element of $\tilde{x}$, then we may write $\frac{\xi}{h}$ as the dual element of x, informally. For this reason it is important to keep track of the factors of h.
Only when applying the pairing to $H^* \otimes H$ one has to be careful with the factors of $\frac{1}{h}$, since the pairing is only defined on the subspace $H^* \otimes B_h \subset H^* \hat{\otimes} H$. We defined the QUE-dual of $U_q(b^-)$ (which is $U_q(b^+)$) to be the dual of the QFSH-subalgebra of $U_q(b^-)$. So for an element that is not part of the QFSH-subalgebra of $U_q(b^-)$ the pairing will not be defined. This problem is resolved in the Drinfel'd double by applying the antipode to one side of the pairing, cancelling the $\frac{1}{h}$ term in the final expression for the product of the Drinfel'd double. Constructing the Drinfel'd double will be the only application of the pairing on $U_h(b^-)^* \otimes U_h(b^-)$.
Another point of care arises when applying the pairing on elements in $U_h(b^-)^* \hat{\otimes} U_h(b^-)$. The R-matrix we will construct later for example, can be written as an element in

the completed tensor product of these algebras, when considered as R_ϵ-modules. Applying the pairing to the R-matrix diverges, as we will see later. Furthermore define $\langle a \otimes b, c \otimes d \rangle := \langle a, c \rangle \langle b, d \rangle$.

Lemma 1.4.1. *The following relations define a Hopf algebra $U_h(b^+)$ that is dual to the Hopf algebra 1.57. Moreover, it is the quantization of the Lie bialgebra b^+.*

$$\begin{aligned}
&[Y, X] = -2\epsilon Z + \epsilon h XY,\ [Z, X] = -\epsilon h XZ,\ [A, X] = -\epsilon X,\ [B, X] = 0 \qquad (1.61)\\
&[Z, Y] = \epsilon h YZ,\ [Y, A] = 0,\ [B, Y] = -\epsilon Y,\\
&[A, Z] = -\epsilon Z,\ [B, Z] = -\epsilon Z,\\
&\Delta(X) = X \otimes e^{h(2A-B)} + 1 \otimes X,\\
&\Delta(Y) = Y \otimes e^{h(2B-A)} + 1 \otimes Y,\\
&\Delta(Z) = Z \otimes e^{h(A+B)} + 1 \otimes Z + h(X \otimes Ye^{h(2A-B)})\\
&\Delta(A) = A \otimes 1 + 1 \otimes A,\ \Delta(B) = B \otimes 1 + 1 \otimes B,\\
&S(X) = -Xe^{-h(2A-B)},\ S(Y) = -Ye^{-h(2B-A)},\\
&S(Z) = -(1 - 2h\epsilon)(Z - XYh)e^{-h(A+B)},\ S(A) = -A,\ S(B) = -B.
\end{aligned}$$

Proof. By theorem 1.4.3, the module of noncommutative polynomials divided out by the algebra relations has a basis of ordered polynomials, so the quotient is nontrivial. To prove coassociativity, one repeats the calculation in the case of $U_q(b^-)$. It is obvious that coassociativity holds. It also follows straightforwardly that Δ is an homomorphism. In this case we only need to check three relations, effectively. The antipode and (co)unit axioms are straightforward to check on generators. We leave this to the reader.
To prove duality, let $u, u' \in U_q(b^+)$. We have to check that for all $v, w \in U_q(b^-)$, $\langle \Delta(u), v \otimes w \rangle = \langle u, vw \rangle$. We assume normal ordering $\{b, a, z, y, x\}$ on $U_q(b^-)$. Let $n \geq 0$ be an integer.

$$\begin{aligned}
\langle X, xa^n \rangle &= \langle X, (a+2)^n x \rangle \\
&= \langle X, 2^n x \rangle = \frac{2^n}{h}.
\end{aligned}$$

Similarly, $\langle X, b^n x \rangle = \frac{(-1)^n}{h}$. By duality, we observe that these expressions are the only terms that pair nonzero with X. On the other hand we have

$$\begin{aligned}
\langle \Delta(X), x \otimes a^n \rangle &= \langle X \otimes e^{h(2A-B)}, x \otimes a^n \rangle \\
&= \langle X \otimes \frac{(2hA)^n}{n!}, x \otimes a^n \rangle \\
&= \frac{2^n}{h}.
\end{aligned}$$

We obtain $\langle \Delta(X), x \otimes b^m a^n \rangle = \langle X, xb^m a^n \rangle$ for all positive m and n. As observed before, these are the only monomials that pair nonzero with $\Delta(X)$, so we obtain

$\langle \Delta(X), v \otimes w \rangle = \langle X, vw \rangle$ for all v, w. By the same argument we obtain $\langle \Delta(Y), v \otimes w \rangle = \langle Y, vw \rangle$. The argument for Z is a little bit more involved, as we have to check the monomials containing x and y too. However, we observe

$$\begin{aligned} \langle Z, xy \rangle &= \langle Z, z - (\epsilon - 1)xy \rangle = \frac{1}{h} \\ &= \langle hX \otimes Y, x \otimes y \rangle = \langle \Delta(Z), x \otimes y \rangle. \end{aligned}$$

We have to prove that the same axiom holds for the comultiplication on $U_q(b^-)$. This can be checked on generators in a similar way. We leave this to the reader, as well as the counit and antipode pairing axioms. This proves that $U_q(b^+)$ is the Hopf algebra dual of $U_q(b^-)$.
The fact that $U_q(b^+)$ is the quantization of b^+ follows in the same way as 1.3.3, by checking the classical limit of $U_h(b^+)$. By duality we have a PBW basis of $U_q(b^+)$ (we will prove this explicitly in the next theorem), and hence we have an isomorphism between $U_q(b^+)$ and $U(b^+)[[h]]$. This proves the lemma. □

The Hopf algebra obtained is the quantization of b^+ (the dual of b^- in the Lie bialgebraic sense), which is why we call it $U_h(b^+)$. To proof the existence of an algebra isomorphism between $U_h(b+)$ and $U(b^+)[[h]]$ is difficult to do explicitly. A proof for the case where ϵ is invertible that evades this problem can be seen in prop. 4.8 to 4.11 in [9], however one should beware of the different conventions used when computing the double. Roughly speaking, [9] first takes the dual and then makes the space smaller, while we do the opposite. This is the same in the end [7].
As we noted before, the usual finite dimensional highest weight representations of $U_q(sl_n^\epsilon)$ do not exist if ϵ is not invertible. So the usual geometrical interpretation of the Hopf-dual of $U_q(sl_n^\epsilon)$ does not apply here. The geometrical interpretation as functions on a Poisson Lie group is probably lost. See [7] for a discussion on this subject. Some definitions can be found in appendix A.2.
On $U_h(b^+)$ we choose the order $\{X, Y, Z, A, B\}$. By the definition of the pairing, it is nondegenerate over $\mathbb{R}$, so we have a PBW basis of monomials of generators.

Theorem 1.4.3. *Let X, Y, Z, A, B be the elements of $U_h(b^-)^*$ dual to the generators b, a, z, y, x of $U_h(b^-)$. Then the monomials $X^{n_1}Y^{n_2}Z^{n_3}A^{n_4}B^{n_5}$ form a basis of $U_h(b^-)^*$.*

Proof. The fact that they span the space is easy, since we can rewrite any expression in a normal ordered way. This implies that $U_q(b^+)$ is free as a $R_\epsilon[[h]]$-module. The pairing is nondegenerate over $\mathbb{R}$, proving that the monomials are linearly independent over $\mathbb{R}$. As we prove in the appendix A.5, we can extend a basis over $\mathbb{R}$ to a basis over R_ϵ if the module is free, and hence we obtain an R_ϵ basis of $U_q(b^-)^*$. □

For future reference, let us calculate a basis for $U_h(b^-)^*$, the elements of which we require to pair to one with the basis elements of $U_h(b^-)$. We already have a basis for $U_h(b^-)$ and $U_h(b^-)^*$, which consists of monomials of dual generators. These monomials form a dual basis when normalized.

Proposition 1.4.4.

$$\langle X^l Y^m Z^n A^o B^p, b^{p'} a^{o'} z^{n'} y^{m'} x^{l'} \rangle = \delta_{l,l'}\delta_{m,m'}\delta_{n,n'}\delta_{m,m'}\delta_{l,l'} h^{-o-p-l-m-n} o!p![n]_q![m]_q![l]_q!,$$

where $[n]_q = \frac{1-q^{2n}}{1-q^2}$.

Proof. We use the comultiplication to prove the proposition, following the proof of lemma 8.3.4 in [6]. Consider $\langle X^n, \tilde{x}^{n'} \rangle$. Applying Δ to the non-capital side for $n = 2$ yields $\langle X \otimes X, (\tilde{x} \otimes 1 + e^{-\epsilon ha} \otimes \tilde{x})^2 \rangle = \langle X \otimes X, \tilde{x} e^{-\epsilon ha} \otimes \tilde{x} + e^{-\epsilon ha}\tilde{x} \otimes \tilde{x} \rangle$. The other terms pair to zero. Commuting $\tilde{x} e^{-2\epsilon ha} = q^2 e^{-\epsilon ha}\tilde{x}$ with $q = e^{-\epsilon h}$ yields $\langle X^2, \tilde{x}^2 \rangle = \frac{1+q^2}{h} = \frac{1-q^4}{h(1-q^2)}$.
Now consider $\langle X^n, \tilde{x}^{n'} \rangle$. We observe that $n = n'$. Applying $\Delta^{(n)}$ to the non-capital side gives $\Delta^{(n)}(\tilde{x})^n = (\tilde{x} \otimes e^{-\epsilon ha} \otimes \cdots \otimes e^{-\epsilon ha} + \cdots + 1 \otimes 1 \otimes \cdots \otimes \tilde{x})^n$. Let us, like [6], denote this expression with $(a_1 + \cdots + a_n)^n$. Each term a_i in this expression has a commutator q^2 with a term a_j for $i < j$, since in each term there is precisely on $\tilde{x}$. In the final expression, terms that contain a quadratic factor a_i^2 can be dropped from the expression, since this will pair to zero with $X \otimes X \otimes \cdots \otimes X$. So we only consider permutations of $a_{i_1} a_{i_2} \cdots a_{i_n}$, where $a_{i_k} \neq a_{i_l}$ if $k \neq l$, and $i_l = 1, 2, \cdots n$. Let us now perform an induction argument on n. For $n = 2$ we saw that the coefficient c_2 of $a_1 a_2$ equals $1 + q^2 = \frac{1-q^4}{h(1-q^2)}$. Now assume that $c_n = [n]_q!$. We consider the coefficient of the term $a_1 a_2 \cdots a_{n+1}$ in the expression $(a_1 + \cdots + a_{n+1})^{n+1}$. We obtain the $n+1$ case from the n case by adding a tensor factor $\otimes e^{-\epsilon ha}$ to a_i for $i < n+1$, and taking $a_{n+1} = 1 \otimes \cdots \otimes 1 \otimes \tilde{x}$. The argument now follows from counting the factors of q. If in the first factor $(a_1 + \cdots + a_{n+1})$ in $(a_1 + \cdots + a_{n+1})^{n+1}$ we pick a_{n+1}, then this term will contribute q^{2n} to coefficient of $a_1 \cdots a_{n+1}$ since we have to commute n factors of $e^{-\epsilon a}$. In a similar way we obtain a contribution of q^{2i-2} by choosing in the i-th factor a_{n+1}. Hence $c_{n+1} = (1 + q^2 + \cdots + q^{2n})c_n = \frac{1-q^{2n+2}}{1-q^2} c_n$. From this we obtain $\langle X^n, \tilde{x}^{n'} \rangle = \delta_{n,n'}[n]_q!$.
By performing a similar induction argument we get the desired results for a, b and y. For $\langle X^l Y^m A^o B^p, b^{p'} a^{o'} y^{m'} x^{l'} \rangle$ we now obtain that

$$\langle X^l Y^m A^o B^p, b^{p'} a^{o'} y^{m'} x^{l'} \rangle = \delta_{l,l'}\delta_{m,m'}\delta_{m,m'}\delta_{l,l'} h^{-o-p-l-m} o!p![m]_q![l]_q!,$$

by duality of x, y, a, b and X, Y, A, B. Observe that we apply $\Delta^{(l+m+o+p)}$ to the non capital side, and that the only terms that pair nonzero are of the form $b \otimes \cdots \otimes b \otimes a \otimes \cdots \otimes x$, this allows for no mixing between the terms.
The only possibly troublesome generator is z, since $\Delta(z) = z \otimes 1 + e^{-\epsilon h(a+b)} \otimes z + 2\epsilon hx \otimes y$. We scetch the argument to prove that there occurs no mixing here. One can prove this relation in general in a much more elegant way in the manner of proposition 8.3.7 in [6]. This is done in the last chapter.
To prove that there occurs no mixing of terms, we observe that the number of x's

and y's can only increase after applying Δ to $b^{p'}a^{o'}z^{n'}y^{m'}x^{l'}$ through a contribution of $\Delta(z)$. To see this observe that the $x \otimes y$ term in $\Delta(z)$ has a factor of ϵ, and $\epsilon[z,x] = \epsilon[z,y] = 0$. The only way to increase the number of z's is by a term $\epsilon xy \otimes y = \epsilon z \otimes y$ occuring. For this we need a contribution from $\Delta(y)$. This implies that an entry that pairs with a Y does not contain a y. The only way to increase the number of ys is by the $x \otimes y$ term in $\Delta(z)$. Since we apply Δ to one entry only (it does not matter which order we take, due to coassociativity), we observe that creating a z in one entry annihilates an y or an x in another entry. So the term which originates from $2\epsilon hx \otimes y$ in $\Delta(z)$ will necessarily pair to zero with $X \otimes \cdots \otimes X \otimes Y \otimes \cdots \otimes B$, as it can never yield a contribution to $b \otimes \cdots \otimes a \otimes \cdots \otimes z \otimes y \otimes \cdots \otimes x$. This implies the result. $\square$

The Drinfel'd double

For the Drinfel'd double $D(H)$, let H be any Hopf algebra with dual H^*. In the infinite dimensional case, let H^* be the QUE-dual of H. Consider the vector space $H^* \otimes H$. Note that the tensor product is the completed tensor product, since the comultiplication doesn't map to the $H^* \otimes H \otimes H^* \otimes H$ in the infinite dimensial case in general. See for example 4.1.16 of [6] for a discussion on this subject.

Definition 1.4.4. *Let H be a Hopf algebra with QUE-dual H^*. The Drinfel'd double $D(H)$ (also called quantum double) is a quasitriangular Hopf algebra generated by H, H^{*op} as Hopf subalgebras with the quasitriangular structure $\mathcal{R} = \sum_a f^a \otimes e_a$, where $\{e_a\}$ is the basis of H and $\{f^a\}$ its dual basis. D(H) is realised on the vector space $H^* \otimes H$ with product $(a \otimes h)(b \otimes g) = \sum b_2 a \otimes h_2 g \langle Sh_1, b_1\rangle\langle h_3, b_3\rangle$, and the tensor-product unit, counit and coproduct:*

$$\Delta(h \otimes a) = \sum a_{(1)} \otimes h_{(1)} \otimes a_{(2)} \otimes h_{(2)}$$

That this definition yields a quasitriangular Hopf algebra is proven in the first paragraph of chapter 7 of [23]. The proof over rings is exactly the same and is not repeated here.
One can show that the antipode S provides an isomorphism between H^{*op} and H^{*cop}, where cop stands for the opposite coproduct. See [23] or page 253 of [20]. Using this isomorphism, we get another version of $D(H)$. We will use the multiplication more often than the comultiplication. So the latter definition is the definition we will use, although both constructions are equivalent[6]. Again, the definition is the same as in [23].

Definition 1.4.5. *The quantum double D(H) in a form containing H, H^{*cop} as subalgebras, is a quasitriangular Hopf algebra generated by these subalgebras on the vector* *space $H^* \otimes H$ together with the relations*

$$\mathcal{R} = \sum_a f^a \otimes e_a \tag{1.62}$$

$$(h \otimes a)(g \otimes b) = \sum a_{(2)} b \otimes h_{(2)} g \langle h_{(3)}, S^{-1}(b_{(1)}) \rangle \langle h_{(1)}, b_{(3)} \rangle \tag{1.63}$$

$$\Delta(h \otimes a) = \sum a_{(2)} \otimes h_{(1)} \otimes a_{(1)} \otimes h_{(2)}. \tag{1.64}$$

Here the antipode on H^{*cop} is the inverse of the antipode of H^*.

That this construction works was first proven by Drinfel'd. The theorem is that the relations of $D(H)$ define a quasitriangular Hopf algebra. For the proof we refer to [23]. Let $D(U_h(b^-))$ be the quantum double on the space $U_q(b^+) \otimes U_q(b^-)$. We have the following theorem.

Theorem 1.4.4. *Let $U_q(b^\pm)$ be the Hopf algebras as defined in the previous section, with pairing $\langle , \rangle$. Let $D(U_h(b^-))$ be the quantum double on the space $U_q(b^+) \otimes U_q(b^-)$ with (co)multiplication, antipode and (co)unit as defined above. Then $D(U_h(b^-))$ is a quasitriangular Hopf algebra with R-matrix*

$$\mathcal{R} = \sum_{n,m,l,o,p} \frac{X^l Y^m Z^n A^o B^p b^p a^o z^n y^m x^l}{h^{-o-p-l-m-n} o! p! [n]_q! [m]_q! [l]_q!}.$$

We might write $U_q(sl_3^\epsilon)$ instead of $D(U_q(b^-))$.

Proof. We calculated the dual of $U_h(b^-)$ before, and proved that $U_h(b^-)$ is indeed a Hopf algebra. One can explicitly calculate the Drinfel'd double of $U_q(b^\pm)$. This will be done in the next chapter. Observe that Drinfel'ds theorem remains true in the case where a ring is used instead of a field [23].
It is a trivial matter to prove quasitriangularity. From theorem 1.4.3 it follows that the monomials $X^{n_1} Y^{n_2} Z^{n_3} A^{n_4} B^{n_5}$ and $b^{n_5} a^{n_4} z^{n_3} y^{n_2} x^{n_1}$ form a basis of respectively $U_q(b^+)$ and $U_q(b^-)$. By construction the monomials are dual to each other, up to a factor. This factor was computed in proposition 1.4.4. Hence by Drinfel'ds theorem it follows that $\mathcal{R}$ is an R-matrix. □

For completeness we state the explicit algebra relations. We will calculate the explicit algebra relations on generators in the next chapter, when we have developed the necessary tools. These relations might be calculated explicitly by hand. However, since there are a lot of relations, and the possibility for errors is high, it is better to use the computer.

Theorem 1.4.5. *The following relations define, together with the antipode and the cobracket as defined on $U_q(b^-)$ and the opposite coproduct on $U_q(b^+)$ with the inverse* *antipode, the quasitriangular Hopf algebra $U_q(sl_3^\epsilon)$ with R-matrix R.*

$$\begin{aligned}
&[X,Y] = 2\epsilon Z - \epsilon hXY,\ [Z,X] = -\epsilon hXZ,\ [X,A] = \epsilon X,\ [X,B] = 0, \\
&[X,b] = X,\ [X,a] = -2X,\ [X,z] = -2\epsilon y + \epsilon hXz, \\
&[X,y] = -\epsilon hXy,\ ,[Y,x] = -\epsilon hYx, \\
&[X,x] = \frac{e^{-h(2A-B)} - 1}{h} - \epsilon ae^{-h(2A-B)} + 2\epsilon hXx, \\
&[Y,Z] = -\epsilon hYZ,\ [Y,A] = 0,\ [Y,B] = \epsilon Y,\ [Y,b] = -2Y,\ [Y,a] = Y, \\
&[A,x] = \epsilon x,\ [A,y] = 0,\ [B,x] = 0,\ [B,y] = y, [A,z] = [B,z] = \epsilon z, \\
&[Y,z] = 2\epsilon xe^{-h(2B-A)} + \epsilon hYz,\ [Y,y] = \frac{e^{-h(2B-A)} - 1}{h} - \epsilon be^{-h(2B-A)} + 2\epsilon hYy, \\
&[Z,A] = \epsilon Z,\ [Z,B] = \epsilon Z,\ [Z,b] = -Z,\ [Z,a] = -Z, \\
&[Z,z] = \frac{-1 + e^{-h(A+B)}}{h} - \epsilon e^{-h(A+B)}(a+b) + 2\epsilon hzZ, \\
&[Z,y] = -X + h\epsilon Zy, \\
&[Z,x] = Ye^{-h(2A-B)} + h\epsilon(Zx - (-1+a)Ye^{-h(2A-B)}), \\
&[b,z] = -z,\ [b,y] = -2y,\ [b,x] = x,\ [a,z] = -z,\ [a,y] = y,\ [a,x] = -2x, \\
&[y,z] = -h\epsilon zy,\ [x,z] = h\epsilon zx,\ [x,y] = zh - \epsilon hyx, \\
&\mathcal{R} = \sum_{n,m,l,o,p} \frac{X^l Y^m Z^n A^o B^p b^p a^o z^n y^m x^l}{h^{-o-p-l-m-n} o!p![n]_q![m]_q![l]_q!}.
\end{aligned} \tag{1.65}$$

Here the antipode is as defined on the generators before, and is extended as an antihomomorphism. The comultiplication is reverted on the 'capital' side, and is extended as an algebra homomorphism.

We need to be careful when doing calculations with the pairing. The pairing axioms will not hold in general for the comultiplication of the double, since it has the opposite order. The relations agree nicely with the commutation relations found in [35]. This was to be expected, given that they used the same construction, namely the cop-construction for the quantum double. It is possible to do the op-construction of course, yielding a different set of commutation relations. The classical limit of the relations here agrees nicely with the classical co-double calculated in theorem 1.1.1.

It is more difficult to do the quantization of sl_3^ϵ without treating the two Borel subalgebras separately. The problem lies with the fact that sl_3^ϵ is not semisimple, so it is not possible to use the rigidity theorems as we did in theorem 1.3.3. This seems the biggest issue, although other problems might arrise in the definition of the R-matrix and finding the algebra relations in general. The basic example in [6], chapter 6.4 gives insight in how to perform this quantization for sl_2. This is very tedious to do in our case.

We interpret the tensor product in the h-adically completed sense. The Hopf-algebra $U_h(b^-)$ is graded, where the grading is inherited from $U(b^-)$. The grading on the b^+ side is inhereted from the grading on the lowercase side, since the coproduct by construction respects the grading.

1.5. R-matrix and ribbon element

As noted in the previous section, we have a basis of the both Borel subalgebras which consists of monomials in the generators. To make these bases dual we correct with the pairing of the monomials, calculated in the previous section. Remember that we use the convention $[n]_q = \frac{1-q^{-2n}}{1-q^{-2}}$.

$$\langle X^l Y^m Z^n A^o B^p, b^p a^o z^n y^m x^l \rangle = h^{-o-p-l-m-n} o!p![n]_q![m]_q![l]_q!. \tag{1.66}$$

With this identity it is easy to write down the formal R-matrix. The real trouble is working with the R-matrix to calculate the knot invariant itself, and in this activity we will find use for the ϵ introduced in the algebra. Without ϵ, so with the ordinary sl_3 quantum invariant, or even with the sl_2 invariant this procedure is exponential in the number of crossings of a knot. When $\epsilon^k = 0$, the procedure is polynomial time, which we will prove in the next chapter. We introduce a trick for working with quantum exponentials, of which the R-matrix is an example.

$$\mathcal{R} = \sum_{n,m,l,o,p} \frac{X^l Y^m Z^n A^o B^p b^p a^o z^n y^m x^l}{h^{-o-p-l-m-n} o!p![n]_q![m]_q![l]_q!} \tag{1.67}$$

In this identity, $q = 1 - h\epsilon$, $[n]_q = \frac{1-q^{-2n}}{1-q^{-2}}$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. The fact that this is an R-matrix follows from the Drinfel'd double construction.
The R-matrix can also be written with quantum exponentials, which are defined as follows.

$$e_q^d = e_q(d) = \sum_n \frac{d^n}{[n]_q!}. \tag{1.68}$$

This expression is a formal power series in h. However, we observe that $[n]_q! = n! \bmod h$, giving the connection to the usual exponential. The R-matrix can be written with ordered polynomials. In order to rewrite the R-matrix, we map the expression to a commutative ring generated by the generators of the Hopf algebra over the ring $R_\epsilon[[h]]$. The ordening is indicated, so that one can give the inverse of this map to ordered monomials. In this ring, expressions become much more compact.
Let us call such a map $\mathbb{O}(\cdot|p) : O \to H$, where $O = R_\epsilon[[h]][X, Y, Z, A, B, b, a, z, y, x]$ is the ring of (commutative) power series over $R_\epsilon[[h]]$ with generators

$$\{X, Y, Z, A, B, b, a, z, y, x\}$$

, and $H = D(U_q(b^-)) = U_q(sl_3^\epsilon)$. p is a specific ordering. In our case $p = XYZABbazyx$. $\mathbb{O}(T|p)$ sends an unordered expression $T \in O$ to an ordered expression with ordering p in all monomials. With this map we can rewrite for example

$$\sum_n \frac{x^n a^n}{n!} = \mathbb{O}(e^{xa}|xa). \tag{1.69}$$

Likewise, we can rewrite $\mathcal{R}$ in terms of quantum exponentials

$$\mathcal{R} = \mathbb{O}(e^{Aa}e^{Bb}e_q^{Xx}e_q^{Yy}e_q^{Zz}|XYZABbazyx). \tag{1.70}$$

This notation will become important in the following chapter, where commutative rings will provide a nice way of calculating commutation relations. We prove the following lemma for the implementation of the R-matrix in Mathematica. The formula is called the Faddeev-Quesne formula. The proof is due to D. Zagier.

Lemma 1.5.1. $e_q(x) = e^{\sum_{n=1}^{\infty} \frac{(q^{-2}-1)^n x^n}{n(1-q^{-2n})}}$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define the operator $D_{q^{-2}}(f)(x) = \frac{f(q^{-2}x)-f(x)}{q^{-2}x-x}$. Note that $D_{q^{-2}}e_q^x = e_q^x$, since

$$D_{q^{-2}}e_q^x = \sum_{n=1} \frac{q^{-2n}x^n - x^n}{[n]_q!(q^{-2}-1)x} = \sum_{n=1} \frac{x^{n-1}}{[n-1]_q!} = e_q^x.$$

The second last equality follows by definition of $[n]_q$.
Now suppose that a function f has $D_{q^{-2}}(f) = f$, then $f(q^{-2}x) = (q^{-2}x - x + 1)f(x)$, or in other words $log(f(q^{-2}x)) = log(1 - x(1-q^{-2})) + log(f(x))$. Let us assume that $log(f(x))$ can be expressed as a power series $log(f(x)) = \sum a_n x^n$, then, using the expansion of $log(1-x)$, we get $q^{-2n}a_n = -\frac{(1-q^{-2})^n}{n} + a_n$. This gives the desired result. □

Lemma 1.5.2. *If $q = e^{-\gamma h}$, and $\gamma^k = 0$, then $e_q^x = e^{\sum_{n=1}^{k} \frac{(q^{-2}-1)^n x^n}{n(1-q^{-2n})}}$.*

Proof. For the proof let us look at the n-th term $\frac{(q^{-2}-1)^n x^n}{n(1-q^{-2n})}$. Observe that $(q^{-2} - 1)^n = (-2h)^n\gamma^n + O(\gamma^{n+1})$. Also, $(1-q^{-2n}) \sim \gamma + O(\gamma^2)$. This proves the lemma. □

The Ribbon-element is calculated from the R-matrix. Let us write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. Then $u = \sum \mathcal{R}^{(2)}S(\mathcal{R}^{(1)})$, and $v = S(u)$, where S is the antipode. uv is a central element [23]. The Ribbon element ν is defined as the square root of the product vu. If we assume that v is of the form uw^2, for some $w \in H$, then

$v^2 = uuw^2$, and, since uv is central, $v = uw$. So we can calculate the square root of $u^{-1}v = w^2$ and multiply by u to obtain v.

$$\begin{aligned}
v = \mathbb{O}(&e^{-hAa-hBb-e^{\hbar(-2A+B)}hXx-e^{\hbar(A-2B)}Yy+e^{-\hbar(A+B)}h^2XYz-e^{\hbar(A+B)}hZz}(e^{\hbar(A+B)}- \\
&axX\epsilon\hbar^2e^{2B\hbar-A\hbar} - a\epsilon\hbar e^{A\hbar+B\hbar} + aXYz\epsilon\hbar^3- \\
&azZ\epsilon\hbar^2 - byY\epsilon\hbar^2e^{2A\hbar-B\hbar} - b\epsilon\hbar e^{A\hbar+B\hbar}+ \\
&\frac{3}{2}x^2X^2\epsilon\hbar^3e^{3B\hbar-3A\hbar} - 2xX^2Yz\epsilon\hbar^4e^{B\hbar-2A\hbar}+ \\
&2xX\epsilon\hbar^2e^{2B\hbar-A\hbar} + 2X^2Y^2z^2\epsilon\hbar^5e^{-A\hbar-B\hbar}- \\
&XyY^2z\epsilon\hbar^4e^{A\hbar-2B\hbar} - 2XYz^2Z\epsilon\hbar^4e^{-A\hbar-B\hbar}+ \\
&\frac{3}{2}y^2Y^2\epsilon\hbar^3e^{3A\hbar-3B\hbar} + 2yY\epsilon\hbar^2e^{2A\hbar-B\hbar}+ \\
&\frac{3}{2}z^2Z^2\epsilon\hbar^3e^{-A\hbar-B\hbar} + e^{A\hbar+B\hbar} + bXYz\epsilon\hbar^3- \\
&bzZ\epsilon\hbar^2 - 2xXyY\epsilon\hbar^3 + 2xyZ\epsilon\hbar^2 - 5XYz\epsilon\hbar^3 + 6zZ\epsilon\hbar^2)|p)
\end{aligned}$$

Remember the following notation: $\mathcal{R}_{ij} = \sum 1 \otimes \cdots \otimes \mathcal{R}^{(1)} \otimes \cdots \otimes \mathcal{R}^{(2)} \otimes 1 \cdots \otimes 1$, where the $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ are on the i-th and the j-th position respectively. The ribbon element is central and invertible, and it has the following properties.

$$\begin{aligned}
\varepsilon(v) = 1,\ v^2 = uS(u),\ S(v) = v, \\
\Delta(v) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(v \otimes v).
\end{aligned} \tag{1.71}$$

Remember that ε is the counit, and here u is as defined previously. Combining the various results of the previous sections we now have the main theorem of this chapter.

Theorem 1.5.1. *The Hopf algebra $U_q(sl_3^\epsilon)$ together with R-matrix $\mathcal{R}$ and ribbon element v is a quasitriangular ribbon Hopf algebra that is the quantization of the quasitriangular Lie bialgebra sl_3^ϵ.*

Proof. The only thing left to prove is that v is the ribbon element corresponding the the R-matrix $\mathcal{R}$. This check is performed in Mathematica in the next chapter. □

In the next chapter we will proceed with the implementation of this algebra in Wolfram Mathematica. The main problem is of course commuting normal ordered exponentials.

Conclusion

In this chapter we started with constructing a quasitriangular Lie bialgebra sl_3^ϵ through the classical double. We quantized this Lie algebra by quantizing the

lower Borel subalgebra $b^- \subset sl_3^\epsilon$ and taking the Drinfel'd double of the resulting Hopf algebra $U_q(b^-)$ to obtain the quasitriangular ribbon Hopf algebra $U_q(sl_3^\epsilon)$. We succeeded in working over R_ϵ, where $\epsilon^2 = 0$. We could also have quantized this algebra starting from the b^+ side. It is possible to do the same procedure explicitly for $\epsilon^k = 0$, for any positive k. We calculated an example for invertible ϵ, and it is clear that this algebra can be turned into a quasitriangular Hopf algebra. In this algebra one can take the expansion up to any order of ϵ. This will be done in chapter 4.
Since $U_q(b^-)$ has the same structure as the quantization of the lower Borel subalgebra in the usual sl_3, we could prove a number of results, including the existence of a PBW basis and the fact that $U_q(b^-)$ is a quantization of b^-. As for the question 'what is ϵ?', we showed that it can be viewed as part of the underlying ring. This introduces a number of difficulties which we could work around. Overall it seems a better strategy to work with an invertible epsilon, and afterwards prove that one can take the expansion in ϵ up to any order in the calculations. In fact, this is how we will approach the problem for constructing $U_q(sl_n^\epsilon)$ in chapter 4.
Interesting variations for future research would be to introduce a second parameter γ dual to ϵ. The knot invariant of this algebra is expected to yield a finite type invariant, which is in some sense an expansion of the $U_q(sl_3^\epsilon)$ invariant. An advantage to this knot invariant is that although it will be much weaker, it will also be much faster to compute. One might even prove certain properties of $U_q(sl_3^\epsilon)$, such as detection of mutants, this way.