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Expansions of quantum group invariants

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Introduction

Knots have been used for many purposes through the ages. The first to attempt to classify knots was Peter Tait in 1876 in [34]. In his book of knots Tait succeded to classify all unique knots up to and including 9 crossings, which are eighty-four in total. Classification of bigger knots was not possible due to the size of the knots. To classify bigger knots, so called knot invariants were needed. Knot invariants are expressions that are identical for equivalent knots. Two knots are considered equivalent if they can be obtained from each other through a number of 'smooth' transformations in $\mathbb{R}^3$.

The first major development in classifying knots took place roughly 50 years after Taits first attempt. In 1927, Reidemeister (and Alexander and Briggs independently in 1926) described the equivalence of knots in terms of three moves on diagrams [28]. These moves are called the Reidemeister moves, and they enabled the definition of (classical) knot invariants. To prove that an expression was identical for equivalent knots, one only needed to prove the invariance of the expression under the Reidemeister moves.

The most important classical knot invariant is the Alexander polynomial. The Alexander polynomial can be calculated in a number of ways, and it was invented by James Wadell Alexander II shortly after the introduction of the Reidemeister moves in 1928 [1]. In his original paper, the polynomial is calculated by considering the incidence matrix of a knot diagram. The Alexander polynomial can also be calculated from the Seifert surface of the knot, or from the cyclic cover of the knot complement to name a few options.

It was proven by Alexander that the Alexander polynomial satisfied a skein relation. A skein relation is a relation between a crossing and the sum of two trivial strands horizontally and vertically positioned. To say that the Alexander polynomials satisfies a skein relation means that it is invariant under replacing a crossing with the sum of two knots with in the same place two trivial strands horizontally and vertically positioned.

In the 1960s it was discovered by Conway that the Alexander polynomial could also be computed by a skein relation and a value for the unknot, the circle embedded in $\mathbb{R}^3$. These two identities together yield a variant of the Alexander polynomial called the Conway-Alexander polynomial. In 1984 this construction was applied by Vaughan Jones to define the Jones polynomial [16].

The Jones polynomial is the first example of a new class of knot invariants called quantum invariants. Originally it was defined using Neumann algebras. It can also be obtained by coloring links with two dimensional representations of the quantum group $U_q(sl_2)$.

The class of quantum invariants consists of quantum group invariants, which are obtained as solutions of the Knizhnik-Zamolodchikov equation (KZ-equation) and by using finite dimensional representations of quantum groups. Another class of quantum invariants is obtained through solutions of the Yang-Baxter equation in statistical physics.
Shortly after the discovery of the Jones polynomial, other similar polynomials were discovered, the most important of which are the HOMFLY-polynomial and the n-colored Jones polynomial. The HOMFLY polynomial is a generalization of the Jones polynomial in the sense that it considers $U_q(sl_n)$-representations instead of $U_q(sl_2)$. The n-colored Jones invariant, as the name suggests, considers n dimensional representations of $U_q(sl_2)$.
Quantum groups like $U_q(sl_2)$ are the central object in the construction of quantum invariants. The first description of $U_q(sl_2)$ was due to Kulish and Sklyanin in 1981. The term quantum group was first presented by Drinfel'd in 1986 [7]. In his paper he defines a quantum group as the dual space of the 'quantized' algebra of functions on some (Lie) group G. Denote the commutative algebra of functions on a Lie group G with $\mathcal{F}(G)$. If G represents the state space of a system, then quantizing this system corresponds to quantizing $\mathcal{F}(G)$. This quantization can be done in various ways, and will be denoted with $\mathcal{F}_h(G)$. The quantum group corresponding to $\mathcal{F}_h(G)$ is defined by Drinfel'd as the dual space of $\mathcal{F}_h(G)$. Drinfel'd notes that the notion of a Hopf algebra and a quantum group are equivalent, however the term quantum group naturally has a geometric meaning.
When defined this way, the Hopf algebraic structure of $U_q(sl_2)$ arises naturally from the group multiplication of the Poisson Lie group $SL(2)$. Concretely, the group multiplication translates to an algebra homomorphism $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$. Quantizing $\mathcal{F}(G)$ means turning the commutative algebra structure into a non-commutative one. In the dual of $\mathcal{F}(G)$ this translates to a Hopf algebra structure.
A great discovery in 1993 was the Kontsevich universal invariant. The discovery of this universal invariant followed the construction of many new quantum invariants in the 1980s and classified these invariants as the result of a single construction. The Kontsevich invariant is derived from the universal KZ equation and uses weight systems in its construction. The Kontsevich invariant takes values in the space of Jacobi diagrams, which are uni-trivalent graphs. Jacobi diagrams are also referred to as loop diagrams, or Feynman diagrams.
Jacobi diagrams can be assigned a specific (simple) Lie algebra $\mathfrak{g}$ and an irreducible finite dimensional representation V to yield the quantum group invariant $(\mathfrak{g}, V)$. This assignment is called a weight system [25]. It was proven that this provides a solution to the Yang-Baxter equation. Kohno and Drinfel'd proved that the R-matrix obtained in this way is equivalent to the quantum R-matrix of the quantum group $U_q(\mathfrak{g})$ corresponding to the Lie algebra $\mathfrak{g}$. So any quantum invariant can be found by substituting a specific Lie algebra and representation into the Jacobi diagrams of the Kontsevich universal invariant.
Using the Kontsevich invariant, we can obtain the n-colored Jones polynomial

as an expansion of loop diagrams. In the second half of the 90s this expansion was studied by Melvin and Morton for the Jones polynomial. They conjectured that the Alexander polynomial arises in the loop expansion of the Jones polynomial [24]. This conjecture was generalized by Lev Rozansky, and later proven by Dror Bar-Natan in 1996 in [2]. It is conjectured in [35] that the polynomial time knot invariants constructed from $U_q(sl_2)$ in [35] and [36] by Dror Bar-Natan and Roland van der Veen coincide with 2-loop terms in the loop expansion of the n-colored Jones polynomial. Concretely, by introducing a parameter ϵ such that $\epsilon^2 = 0$ in the upper (or lower) triangular subalgebra of $U_q(sl_2)$, it is conjectured that one obtains the 2-loop invariant as defined in [32] and studied by Ohtsuki in [26]. One would expect for a similar conjecture to hold for higher loop polynomial [35]. Furthermore, it could be expected that similar conjectures hold for the loop expansion of the HOMFLY polynomial and the quantum group invariant constructed in this thesis.

Overview

In the first chapter of this thesis the quantization of Lie bialgebras is studied. In particular we quantize the Lie bialgebra sl_3^ϵ, which is derived from sl_3 by multiplying the comultiplication on the upper Borel subalgebra with a parameter ϵ. This provides a family of quantum groups that depend on a parameter ϵ. Expanding this quantum group to ϵ, we get an approximation of the $U_q(sl_3)$ quantum group. For clarity we work mostly in first order of ϵ. The first order expansion in ϵ is constructed over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. This construction yields several difficulties, which are the main subject of the first chapter.
We first define the concept of a Lie bialgebra, and introduce the example $b^\pm$, the lower- and upper Borel subalgebra of sl_3 with a parameter ϵ, where $\epsilon^2 = 0$. We proceed with the process of quantization of a Lie bialgebra as done in [6]. In this way we obtain the dual Hopf algebras $U_q(b_\epsilon^\pm)$. To these Hopf algebras, the Drinfel'd double construction is applied to obtain the quasitriangular Hopf algebra $U_q(sl_3^\epsilon)$. We prove that this Hopf algebra is in fact a quasitriangular ribbon Hopf algebra.
In the second chapter we provide a formalism to compute commutators of exponentials. This is done by providing a map between a Hopf algebra and a commutative ring. An Hopf algebra A over a ring R, topologically generated by a basis B, corresponds to the commutative ring $R[B]$. It turns out that maps on Hopf algebras can be translated to the 'zipping' of exponentials in B and dual generators B^*. Zipping refers to the process of substituting a differential operator for the elements of B, that act on the elements of B^*, or vice versa. However, contrary to what one would expect, this construction does not yield a functor between categories as not all morphisms can be translated to the zipping of exponentials.
In the first and second section of chapter two, the zipping-process is defined and convergence is proven for a certain subspace of Gaussian exponentials. In the

third section we prove the existence of these tensors in the case of sl_3, and an implementation of the Drinfel'd double construction is given. This process is implemented in Wolfram Mathematica, and can be found in appendix A.1.[39]
In the third chapter we construct the knot invariant Z_3^ϵ corresponding to $U_q(sl_3^\epsilon)$. This is a construction that can be done for any quasitriangular ribbon Hopf algebra, and is called the universal knot invariant for a quasitriangular ribbon Hopf algebra A. In essence we put copies of the R-matrix of A on each crossing of the knot, and we put the expression in a normal ordered form. There are some details in this construction however, these are covered in the first section.
In the second section, as an example, we compute the Alexander-polynomial of a knot by constructing its Seifert surface. In the third section, we prove the factorization of the $\epsilon = 0$ term of Z_3^ϵ into Alexander polynomials. We also prove that the calculation of the Z_3^ϵ can be done in $O(n^{10})$ calculations, where n is the number of crossings of a knot.
In chapter 4 we attempt the construction of a $U_q(sl_n^\epsilon)$ invariant using the quantum Weyl group. Since $U_q(sl_n^\epsilon)$ is not semisimple, there are some difficulties associated to this construction. The main goal of this chapter is to construct a set of algebra automorphisms that can be associated to a quantum Weyl group. In this construction, we mainly follow [29], and many details are the same in both constructions.
In the first section we define the Lie bialgebra sl_n^ϵ and its corresponding quantization $U_q(sl_n^\epsilon)$. In the second section we study the quantum Weyl group associated with $U_q(sl_2^\epsilon)$, and we give the corresponding algebra automorphism T. In the third section, a general formula for the comultiplication of (non-simple) generators is given. This formula is proven through the quantum Weyl-group construction following [29]. This construction takes the second and third section of chapter 4. In section 4.4 we provide a link with the first chapter through a Wigner contraction on the usual $U_q(sl_n)$ quantum group.
The construction of the quantum double $U_q(sl_n^\epsilon)$ is not implemented in Wolfram Mathematica, although a general formula for the pairing of monomials is proven in the first section, and we also provide a way to calculate the comultiplication of the (non-)simple generators. This finishes the construction of the $U_q(sl_n^\epsilon)$ quasitriangular Hopf algebra in theory. In practice it is necessary to calculate the Hopf algebra (co)multiplication explicitly for the non-simple generators, in order to implement the algebra in Mathematica.
A particular interesting result in this direction is given in section 4.4. We prove that to calculate the comultiplication of $U_q(sl_n^\epsilon)$ modulo ϵ^{k+1}, one needs to work modulo ϵ^{n+k-1}. This is a consequence of a particular transformation on the algebra used in 4 to obtain the $U_q(sl_n^\epsilon)$ Hopf algebra as defined in chapter 1.

Preliminaries

In this section, we mention some conventions that are used throughout this thesis. We prove most of these facts in appendix A.5. Let $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. In the first and in the second chapter, the main examples are Lie and Hopf algebras over R_ϵ. A Lie algebra over the ring R_ϵ is defined similarly as a Lie algebra over a field, with the difference that a vector space over the ring R_ϵ is defined to be a free module over R_ϵ. Any free module M over a ring is also flat, implying that exact sequences are conserved under the tensor product $\otimes_{R_\epsilon} M$. When M is free over R_ϵ, and we have an $\mathbb{R}$-basis $\{\tilde{m}_i\}_{i\in I}$ of $M/\epsilon M$, this basis can be extended to an R_ϵ-basis $\{m_i\}_{i\in I}$ of M. We refer to a free R_ϵ-module as a vector space over R_ϵ.
If M is an R_ϵ-module, then the dual of M is defined as $M^* = Hom_{R_\epsilon}(M, R_\epsilon)$. It is not in general true that the dual of a module is flat (or free). However, when M is finitely generated as a module, then the dual is free as well. For the Lie algebras we consider this will be the case. The underlying vector spaces are finite dimensional.
If M is an infinite dimensional, free module over a ring, it is not true in general that M^* is free. However, in the cases we consider it will turn out to be the case. We will have to prove this by providing a set that spans M^* explicitly. Of course, one also proves that this set is linearly independent over $\mathbb{R}$.
The basis of M^* is usually given by the dual $\mathbb{R}$-basis elements ϕ_a of $(M/\epsilon M)^*$, where $\phi_a(e^{a'}) = \delta_{a,a'}$, and e^a is the $\mathbb{R}$-basis of $M/\epsilon M$. One can extend ϕ_a to an R_ϵ basis when M^* is free.
In the case of Hopf algebras, the modules we consider are equipped with the $R_\epsilon[[h]]$-adic topology. Let h be an indeterminate, and consider a module M over $R_\epsilon[[h]]$. The open sets around $0 \in M$ are generated by the sets $U_n = \{h^n x | x \in M\}$, for $n \geq 0$. The collection $\{a + U_n | a \in M, n \geq 0\}$ is a basis for the h-adic topology on M. Continuous maps are h-linear maps.
We define an algebra on generators by considering the module of noncommutative monomials, and dividing out by the (closure of the) ideal generated by the algebra relations. Suppose we have a set $B = \{x_i\}_{i\in I}$, where I is finite. We refer to x_i as algebra generators, or generators. Consider the module $R_\epsilon[\{x_i\}]$ of non-commutative polynomials in x_i. Let I be the ideal generated by the relations $f_j(x_1, \cdots, x_n) = 0$. The algebra A generated by x_i is defined as the quotient $A = R_\epsilon[\{x_i\}]/\bar{I}$, where $\bar{I}$ is the h-adic closure of I.
An Hopf algebra H is an algebra that is a coalgebra equipped with an antipode $S : H \to H$, which is an anti-homomorphism obeying certain compatibility conditions. The Hopf structure is defined on the algebra-generators, and extending the comultiplication, (co)unit and antipode as $R_\epsilon[[h]]$ homomorphisms. Chari

and Pressley in [6] refer to the Hopf-algebra as topologically generated by the algebra generators. We use the same convention. The tensor product is assumed to be completed in the h-adic topology whenever relevant. In the Lie-algebraic setting the tensor product is the usual tensor product on finite dimensional vector spaces.
The dual of a coalgebra has a natural algebra structure. The converse is more complicated, as $(A \otimes A)^* \neq A^* \otimes A^*$, the map $m^* : A^* \to (A \otimes A)^*$ dual to the multiplication map $m : A \otimes A \to A$ does not in general map to $A^* \otimes A^*$. The Hopf dual is defined as the set $A^\circ = \{a \in A^* | m^*(a) \in A^* \otimes A^*\}$.
The Hopf dual is different from the QUE-dual of a topological Hopf algebra, as we will later see. The set A° can be interpreted as the set of all finite dimensional representations when A° is a Hopf algebra over a field [6]. This interpretation plays a role in the last chapter. When we say dual Hopf algebra, we will always mean the QUE-dual, unless stated otherwise.