Expansions of quantum group invariants

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Introduction

Knots have been used for many purposes through the ages. The first to attempt to classify knots was Peter Tait in 1876 in [34]. In his book of knots Tait succeded to classify all unique knots up to and including 9 crossings, which are eighty-four in total. Classification of bigger knots was not possible due to the size of the knots. To classify bigger knots, so called knot invariants were needed. Knot invariants are expressions that are identical for equivalent knots. Two knots are considered equivalent if they can be obtained from each other through a number of 'smooth' transformations in $\mathbb{R}^3$.

The first major development in classifying knots took place roughly 50 years after Taits first attempt. In 1927, Reidemeister (and Alexander and Briggs independently in 1926) described the equivalence of knots in terms of three moves on diagrams [28]. These moves are called the Reidemeister moves, and they enabled the definition of (classical) knot invariants. To prove that an expression was identical for equivalent knots, one only needed to prove the invariance of the expression under the Reidemeister moves.

The most important classical knot invariant is the Alexander polynomial. The Alexander polynomial can be calculated in a number of ways, and it was invented by James Wadell Alexander II shortly after the introduction of the Reidemeister moves in 1928 [1]. In his original paper, the polynomial is calculated by considering the incidence matrix of a knot diagram. The Alexander polynomial can also be calculated from the Seifert surface of the knot, or from the cyclic cover of the knot complement to name a few options.

It was proven by Alexander that the Alexander polynomial satisfied a skein relation. A skein relation is a relation between a crossing and the sum of two trivial strands horizontally and vertically positioned. To say that the Alexander polynomials satisfies a skein relation means that it is invariant under replacing a crossing with the sum of two knots with in the same place two trivial strands horizontally and vertically positioned.

In the 1960s it was discovered by Conway that the Alexander polynomial could also be computed by a skein relation and a value for the unknot, the circle embedded in $\mathbb{R}^3$. These two identities together yield a variant of the Alexander polynomial called the Conway-Alexander polynomial. In 1984 this construction was applied by Vaughan Jones to define the Jones polynomial [16].

The Jones polynomial is the first example of a new class of knot invariants called quantum invariants. Originally it was defined using Neumann algebras. It can also be obtained by coloring links with two dimensional representations of the quantum group $U_q(sl_2)$.

The class of quantum invariants consists of quantum group invariants, which are obtained as solutions of the Knizhnik-Zamolodchikov equation (KZ-equation) and by using finite dimensional representations of quantum groups. Another class of quantum invariants is obtained through solutions of the Yang-Baxter equation in statistical physics.
Shortly after the discovery of the Jones polynomial, other similar polynomials were discovered, the most important of which are the HOMFLY-polynomial and the n-colored Jones polynomial. The HOMFLY polynomial is a generalization of the Jones polynomial in the sense that it considers $U_q(sl_n)$-representations instead of $U_q(sl_2)$. The n-colored Jones invariant, as the name suggests, considers n dimensional representations of $U_q(sl_2)$.
Quantum groups like $U_q(sl_2)$ are the central object in the construction of quantum invariants. The first description of $U_q(sl_2)$ was due to Kulish and Sklyanin in 1981. The term quantum group was first presented by Drinfel'd in 1986 [7]. In his paper he defines a quantum group as the dual space of the 'quantized' algebra of functions on some (Lie) group G. Denote the commutative algebra of functions on a Lie group G with $\mathcal{F}(G)$. If G represents the state space of a system, then quantizing this system corresponds to quantizing $\mathcal{F}(G)$. This quantization can be done in various ways, and will be denoted with $\mathcal{F}_h(G)$. The quantum group corresponding to $\mathcal{F}_h(G)$ is defined by Drinfel'd as the dual space of $\mathcal{F}_h(G)$. Drinfel'd notes that the notion of a Hopf algebra and a quantum group are equivalent, however the term quantum group naturally has a geometric meaning.
When defined this way, the Hopf algebraic structure of $U_q(sl_2)$ arises naturally from the group multiplication of the Poisson Lie group $SL(2)$. Concretely, the group multiplication translates to an algebra homomorphism $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$. Quantizing $\mathcal{F}(G)$ means turning the commutative algebra structure into a non-commutative one. In the dual of $\mathcal{F}(G)$ this translates to a Hopf algebra structure.
A great discovery in 1993 was the Kontsevich universal invariant. The discovery of this universal invariant followed the construction of many new quantum invariants in the 1980s and classified these invariants as the result of a single construction. The Kontsevich invariant is derived from the universal KZ equation and uses weight systems in its construction. The Kontsevich invariant takes values in the space of Jacobi diagrams, which are uni-trivalent graphs. Jacobi diagrams are also referred to as loop diagrams, or Feynman diagrams.
Jacobi diagrams can be assigned a specific (simple) Lie algebra $\mathfrak{g}$ and an irreducible finite dimensional representation V to yield the quantum group invariant $(\mathfrak{g}, V)$. This assignment is called a weight system [25]. It was proven that this provides a solution to the Yang-Baxter equation. Kohno and Drinfel'd proved that the R-matrix obtained in this way is equivalent to the quantum R-matrix of the quantum group $U_q(\mathfrak{g})$ corresponding to the Lie algebra $\mathfrak{g}$. So any quantum invariant can be found by substituting a specific Lie algebra and representation into the Jacobi diagrams of the Kontsevich universal invariant.
Using the Kontsevich invariant, we can obtain the n-colored Jones polynomial

as an expansion of loop diagrams. In the second half of the 90s this expansion was studied by Melvin and Morton for the Jones polynomial. They conjectured that the Alexander polynomial arises in the loop expansion of the Jones polynomial [24]. This conjecture was generalized by Lev Rozansky, and later proven by Dror Bar-Natan in 1996 in [2]. It is conjectured in [35] that the polynomial time knot invariants constructed from $U_q(sl_2)$ in [35] and [36] by Dror Bar-Natan and Roland van der Veen coincide with 2-loop terms in the loop expansion of the n-colored Jones polynomial. Concretely, by introducing a parameter ϵ such that $\epsilon^2 = 0$ in the upper (or lower) triangular subalgebra of $U_q(sl_2)$, it is conjectured that one obtains the 2-loop invariant as defined in [32] and studied by Ohtsuki in [26]. One would expect for a similar conjecture to hold for higher loop polynomial [35]. Furthermore, it could be expected that similar conjectures hold for the loop expansion of the HOMFLY polynomial and the quantum group invariant constructed in this thesis.

Overview

In the first chapter of this thesis the quantization of Lie bialgebras is studied. In particular we quantize the Lie bialgebra sl_3^ϵ, which is derived from sl_3 by multiplying the comultiplication on the upper Borel subalgebra with a parameter ϵ. This provides a family of quantum groups that depend on a parameter ϵ. Expanding this quantum group to ϵ, we get an approximation of the $U_q(sl_3)$ quantum group. For clarity we work mostly in first order of ϵ. The first order expansion in ϵ is constructed over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. This construction yields several difficulties, which are the main subject of the first chapter.
We first define the concept of a Lie bialgebra, and introduce the example $b^\pm$, the lower- and upper Borel subalgebra of sl_3 with a parameter ϵ, where $\epsilon^2 = 0$. We proceed with the process of quantization of a Lie bialgebra as done in [6]. In this way we obtain the dual Hopf algebras $U_q(b_\epsilon^\pm)$. To these Hopf algebras, the Drinfel'd double construction is applied to obtain the quasitriangular Hopf algebra $U_q(sl_3^\epsilon)$. We prove that this Hopf algebra is in fact a quasitriangular ribbon Hopf algebra.
In the second chapter we provide a formalism to compute commutators of exponentials. This is done by providing a map between a Hopf algebra and a commutative ring. An Hopf algebra A over a ring R, topologically generated by a basis B, corresponds to the commutative ring $R[B]$. It turns out that maps on Hopf algebras can be translated to the 'zipping' of exponentials in B and dual generators B^*. Zipping refers to the process of substituting a differential operator for the elements of B, that act on the elements of B^*, or vice versa. However, contrary to what one would expect, this construction does not yield a functor between categories as not all morphisms can be translated to the zipping of exponentials.
In the first and second section of chapter two, the zipping-process is defined and convergence is proven for a certain subspace of Gaussian exponentials. In the

third section we prove the existence of these tensors in the case of sl_3, and an implementation of the Drinfel'd double construction is given. This process is implemented in Wolfram Mathematica, and can be found in appendix A.1.[39]
In the third chapter we construct the knot invariant Z_3^ϵ corresponding to $U_q(sl_3^\epsilon)$. This is a construction that can be done for any quasitriangular ribbon Hopf algebra, and is called the universal knot invariant for a quasitriangular ribbon Hopf algebra A. In essence we put copies of the R-matrix of A on each crossing of the knot, and we put the expression in a normal ordered form. There are some details in this construction however, these are covered in the first section.
In the second section, as an example, we compute the Alexander-polynomial of a knot by constructing its Seifert surface. In the third section, we prove the factorization of the $\epsilon = 0$ term of Z_3^ϵ into Alexander polynomials. We also prove that the calculation of the Z_3^ϵ can be done in $O(n^{10})$ calculations, where n is the number of crossings of a knot.
In chapter 4 we attempt the construction of a $U_q(sl_n^\epsilon)$ invariant using the quantum Weyl group. Since $U_q(sl_n^\epsilon)$ is not semisimple, there are some difficulties associated to this construction. The main goal of this chapter is to construct a set of algebra automorphisms that can be associated to a quantum Weyl group. In this construction, we mainly follow [29], and many details are the same in both constructions.
In the first section we define the Lie bialgebra sl_n^ϵ and its corresponding quantization $U_q(sl_n^\epsilon)$. In the second section we study the quantum Weyl group associated with $U_q(sl_2^\epsilon)$, and we give the corresponding algebra automorphism T. In the third section, a general formula for the comultiplication of (non-simple) generators is given. This formula is proven through the quantum Weyl-group construction following [29]. This construction takes the second and third section of chapter 4. In section 4.4 we provide a link with the first chapter through a Wigner contraction on the usual $U_q(sl_n)$ quantum group.
The construction of the quantum double $U_q(sl_n^\epsilon)$ is not implemented in Wolfram Mathematica, although a general formula for the pairing of monomials is proven in the first section, and we also provide a way to calculate the comultiplication of the (non-)simple generators. This finishes the construction of the $U_q(sl_n^\epsilon)$ quasitriangular Hopf algebra in theory. In practice it is necessary to calculate the Hopf algebra (co)multiplication explicitly for the non-simple generators, in order to implement the algebra in Mathematica.
A particular interesting result in this direction is given in section 4.4. We prove that to calculate the comultiplication of $U_q(sl_n^\epsilon)$ modulo ϵ^{k+1}, one needs to work modulo ϵ^{n+k-1}. This is a consequence of a particular transformation on the algebra used in 4 to obtain the $U_q(sl_n^\epsilon)$ Hopf algebra as defined in chapter 1.

Preliminaries

In this section, we mention some conventions that are used throughout this thesis. We prove most of these facts in appendix A.5. Let $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. In the first and in the second chapter, the main examples are Lie and Hopf algebras over R_ϵ. A Lie algebra over the ring R_ϵ is defined similarly as a Lie algebra over a field, with the difference that a vector space over the ring R_ϵ is defined to be a free module over R_ϵ. Any free module M over a ring is also flat, implying that exact sequences are conserved under the tensor product $\otimes_{R_\epsilon} M$. When M is free over R_ϵ, and we have an $\mathbb{R}$-basis $\{\tilde{m}_i\}_{i\in I}$ of $M/\epsilon M$, this basis can be extended to an R_ϵ-basis $\{m_i\}_{i\in I}$ of M. We refer to a free R_ϵ-module as a vector space over R_ϵ.
If M is an R_ϵ-module, then the dual of M is defined as $M^* = Hom_{R_\epsilon}(M, R_\epsilon)$. It is not in general true that the dual of a module is flat (or free). However, when M is finitely generated as a module, then the dual is free as well. For the Lie algebras we consider this will be the case. The underlying vector spaces are finite dimensional.
If M is an infinite dimensional, free module over a ring, it is not true in general that M^* is free. However, in the cases we consider it will turn out to be the case. We will have to prove this by providing a set that spans M^* explicitly. Of course, one also proves that this set is linearly independent over $\mathbb{R}$.
The basis of M^* is usually given by the dual $\mathbb{R}$-basis elements ϕ_a of $(M/\epsilon M)^*$, where $\phi_a(e^{a'}) = \delta_{a,a'}$, and e^a is the $\mathbb{R}$-basis of $M/\epsilon M$. One can extend ϕ_a to an R_ϵ basis when M^* is free.
In the case of Hopf algebras, the modules we consider are equipped with the $R_\epsilon[[h]]$-adic topology. Let h be an indeterminate, and consider a module M over $R_\epsilon[[h]]$. The open sets around $0 \in M$ are generated by the sets $U_n = \{h^n x | x \in M\}$, for $n \geq 0$. The collection $\{a + U_n | a \in M, n \geq 0\}$ is a basis for the h-adic topology on M. Continuous maps are h-linear maps.
We define an algebra on generators by considering the module of noncommutative monomials, and dividing out by the (closure of the) ideal generated by the algebra relations. Suppose we have a set $B = \{x_i\}_{i\in I}$, where I is finite. We refer to x_i as algebra generators, or generators. Consider the module $R_\epsilon[\{x_i\}]$ of non-commutative polynomials in x_i. Let I be the ideal generated by the relations $f_j(x_1, \cdots, x_n) = 0$. The algebra A generated by x_i is defined as the quotient $A = R_\epsilon[\{x_i\}]/\bar{I}$, where $\bar{I}$ is the h-adic closure of I.
An Hopf algebra H is an algebra that is a coalgebra equipped with an antipode $S : H \to H$, which is an anti-homomorphism obeying certain compatibility conditions. The Hopf structure is defined on the algebra-generators, and extending the comultiplication, (co)unit and antipode as $R_\epsilon[[h]]$ homomorphisms. Chari

and Pressley in [6] refer to the Hopf-algebra as topologically generated by the algebra generators. We use the same convention. The tensor product is assumed to be completed in the h-adic topology whenever relevant. In the Lie-algebraic setting the tensor product is the usual tensor product on finite dimensional vector spaces.
The dual of a coalgebra has a natural algebra structure. The converse is more complicated, as $(A \otimes A)^* \neq A^* \otimes A^*$, the map $m^* : A^* \to (A \otimes A)^*$ dual to the multiplication map $m : A \otimes A \to A$ does not in general map to $A^* \otimes A^*$. The Hopf dual is defined as the set $A^\circ = \{a \in A^* | m^*(a) \in A^* \otimes A^*\}$.
The Hopf dual is different from the QUE-dual of a topological Hopf algebra, as we will later see. The set A° can be interpreted as the set of all finite dimensional representations when A° is a Hopf algebra over a field [6]. This interpretation plays a role in the last chapter. When we say dual Hopf algebra, we will always mean the QUE-dual, unless stated otherwise.

1. An expansion of the $U_q(sl_3)$ quantum group

Introduction

Hopf algebras are used to produce knot invariants such as the Jones polynomial and the Alexander polynomial. In this chapter we will construct a quasitriangular Hopf algebra that is in some sense a deformed version of the quantum group $U_q(sl_3)$. The knot invariant associated with this Hopf algebra is stronger than the Alexander polynomial, and is computable in polynomial time. These facts will be proven in a later chapter.
In order to arrive at the correct quasitriangular Hopf algebra, we first cover its classical limit, the deformed sl_3 Lie bialgebra. We proceed with quantizing these Lie bialgebras. Finally we cover the quantum or Drinfel'd double construction to obtain the deformed version of $U_q(sl_3)$.
We aim to quantize the algebras over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. Usually the quantization and Drinfel'd double construction is done over a field. Most theorems also hold for coordinate rings. The subject of this chapter, besides obtaining the $U_q(sl_3^\epsilon)$ algebra, is the question 'What is ϵ?'. There is no definite answer to this question. One may choose between the possible perspectives we provide in this thesis. Some perspectives do provide more information than others, however.
Most of the constructions presented here are covered in sources like [23] and [6]. It is advisable to consult these sources on the subject. The deformed Hopf algebra presented here is based on the research by Van der Veen and Bar-Natan in [35]. While [35] covers the $U_q(sl_2^\epsilon)$-quantum group invariant, we cover the $U_q(sl_3^\epsilon)$ quantum invariant. The construction is the same in essence. However, we hope to gain insight in the problems arising in the quantization of sl_n.

1.1. Lie bialgebras

In this section we treat quasi-triangular Lie bialgebras. From a general Lie bialgebra it is possible to construct a quasitriangular Lie bialgebra through the classical double construction. Using this construction, we turn the deformed lower Borel sub Lie bialgebra of sl_3 into a quasi-triangular Lie bialgebra. It is possible to obtain the same Lie algebra relations through Wigner group contraction on the upper Borel Lie subalgebra of gl_n. See appendix A.4 for more information.
By a vector space over a ring we mean a free module over a ring. In the case of

the Lie algebras considered here, the modules are finitely generated. When we say a Lie algebra or Lie bialgebra, we will mean a Lie (bi)algebra over a ring R. R will be specified when necessary. A ring is always commutative with unit in this thesis.
We will often work over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. A specific problem that arises is the non-degeneracy of a bilinear pairing $\langle , \rangle : M^* \otimes M \to R_\epsilon$, where M is a free R_ϵ module. In this and the next chapter, whenever we say that a pairing is nondegenerate, we will mean that it is nondegenerate over $\mathbb{R} = R_\epsilon/\epsilon R_\epsilon$. In other words, the map $\langle , \rangle : M^*/\epsilon M^* \otimes M/\epsilon M \to \mathbb{R}$ is nondegenerate.
Since $\epsilon^2 = 0$, we can only pair in M^* with expressions in M as an element in $M^*/\epsilon M$ and $M/\epsilon M$. In practice, this is what we will use the pairings in this chapter for. As noted in the prelimenaries, we can extend an $\mathbb{R}$-basis of a module $M/\epsilon M$ to an R_ϵ-basis of M, where M is an R_ϵ-module. The same is true for the dual basis, since we consider finite dimensional modules in this section.

Definition 1.1.1. *(Lie bialgebra) A Lie bialgebra $(\mathfrak{g}, [,], \delta)$ is a vector space $\mathfrak{g}$ over a ring R together with a bilinear map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (the bracket) and a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ (the cobracket) satisfying the following axioms:*

1. $[X, X] = 0 \; \forall X \in \mathfrak{g}$,
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$,
3. $\tau \circ \delta(X) = -\delta(X) \; \forall X \in \mathfrak{g}$, *where* $\tau(A \otimes B) = (B \otimes A)$,
4. $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ *is a bracket on the dual Lie algebra* $\mathfrak{g}^*$,
5. $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$ *for all* $X, Y \in \mathfrak{g}$.

$X.\delta(Y) = (ad_X \otimes 1 + 1 \otimes ad_X)(\delta(Y))$, and $ad_X(Y) = [X, Y]$ is the (left-)action of the Lie algebra on itself, for all $X, Y \in \mathfrak{g}$. We introduce the Sweedler notation $\delta(a) = \sum a_1 \otimes a_2 = a_1 \otimes a_2$, where we leave out the summation symbol, but only indicate the entry in the tensor product. Let us define the Lie bialgebra cohomology.

Definition 1.1.2. *(Chevalley-Eilenburg complex) Let M be a $\mathfrak{g}$-module, where $\mathfrak{g}$ is a Lie algebra over a ring R. Set*

$$C^n(\mathfrak{g}, M) := Hom_R(\bigwedge^n \mathfrak{g}, M), \; n > 0,$$

and $C_0(g, M) := M$, where $\bigwedge^n \mathfrak{g}$ is the n-th exterior power of $\mathfrak{g}$. This is the Chevalley-Eilenberg cochain complex.
The differential d on $c \in C^n(\mathfrak{g}, M)$ is defined as

$$dc(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} x_i.c(x_1, \cdots, \hat{x}_i, \cdots, x_n) + \sum_{1 \le i < j \le n+1} (-1)^{i+j} c([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}), \tag{1.1}$$

where $x_1, \cdots, x_{n+1} \in \mathfrak{g}$, and $x.d$ is the module action of $\mathfrak{g}$ on $d \in M$.

With this complex one can now define the cocycles and coboundaries.

Definition 1.1.3. *(Lie bi algebra cohomology) Define the space of cocycles*

$$Z^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) | dc = 0\},$$

and the space of coboundaries

$$B^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) | \exists c' \in C^{p-1}(\mathfrak{g}, M) \text{ s.t. } dc' = c\}.$$

Then define the Lie algebra cohomology as $H^p(\mathfrak{g}, M) := Z^p(\mathfrak{g}, M) / B^p(\mathfrak{g}, M)$.

The condition $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$ in the definition of a Lie bialgebra states that δ is a 1-cocycle in the Lie algebra complex $C^*(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$, with the adjoint action of $\mathfrak{g}$ on the tensor product module $\mathfrak{g} \otimes \mathfrak{g}$.
According to the definition, δ is a 1-cocycle, so we can look at the cases when δ is a coboundary: $\delta(X) = X.r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$ and for all $X \in \mathfrak{g}$. A Lie bialgebra where δ is a coboundary is called a coboundary Lie bialgebra. $\mathfrak{g}$ is coboundary if and only if r obeys (let $r = \sum r_{12} = \sum r^{[1]} \otimes r^{[2]}$):

1. $2r_+ = r_{12} + r_{21}$ is a invariant under the action of $\mathfrak{g}$.

2. $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.

Here $[r_{12}, s_{13}] = \sum [r^{[1]}, s^{[1]}] \otimes r^{[2]} \otimes s^{[2]}$. See proposition 8.1.3 in [23] for the proof that $(\mathfrak{g}, [,], r)$ defines a coboundary Lie bialgebra if and only if the above conditions hold. Conditon 2 is called the classical Yang-Baxter equation, and r is called the classical r-matrix. If the Lie bialgebra structure arises from a classical r-matrix, then we call the Lie bialgebra quasitriangular. The condition that $2r_+$ is ad-invariant is usually not included in the definition of an r-matrix, but for simplicity we will do so. Usually the following definition is taken for a triangular Lie algebra. See for example [23], chapter 8.

Definition 1.1.4. *Let $\mathfrak{g}$ be a Lie algebra. Define the classical r-matrix as an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ which obeys ($r = \sum r_{12} = \sum r^{[1]} \otimes r^{[2]}$):*

1. *$r_{12} + r_{21}$ is a invariant under the action of $\mathfrak{g}$.*

2. *$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.*

Note that $[r_{12}, s_{13}] = \sum [r^{[1]}, s^{[1]}] \otimes r^{[2]} \otimes s^{[3]}$ and similarly when other indices overlap. Let us now proceed with the construction of quasitriangular Lie bialgebras through the classical double construction. We remind the reader that a Lie algebra in this thesis is always finite dimensional. When this is the case, the dual space is well defined and is again a Lie bialgebra.

Definition 1.1.5. *The dual of a Lie bialgebra $\mathfrak{g}$ over a ring R is the dual vector space $\mathfrak{g}^*$ with bracket and cobracket and a R-linear pairing $\langle , \rangle : \mathfrak{g}^* \oplus \mathfrak{g} \to R$ satisfying the axioms*

$$\langle [a,b], c\rangle := \langle a \otimes b, \delta c\rangle \tag{1.2}$$
$$\langle \delta a, c \otimes d\rangle := \langle a, [c,d]\rangle, \tag{1.3}$$

where $a, b \in \mathfrak{g}^*$, and $c, d \in \mathfrak{g}$. We extend the bracket to the tensor-product $\mathfrak{g}^* \otimes \mathfrak{g}^* \oplus \mathfrak{g} \otimes \mathfrak{g}$ by $\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle$.

It is interesting to turn a dual pairing into an inner product $(,) : (\mathfrak{g} \oplus \mathfrak{g}^*) \oplus (\mathfrak{g} \oplus \mathfrak{g}^*) \to R$ by defining $(X, \eta) = \langle X, \eta\rangle$ and $(X, X) = (\eta, \eta) = 0$ for $X \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$. We record the following fact, see lemma 1.3.5 of [6].

Lemma 1.1.1. *Let $\mathfrak{g}, \mathfrak{g}^*$ be Lie algebras with inner product $(,)$ on the space $\mathfrak{g} \oplus \mathfrak{g}^*$. Then $\mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra structure with $\mathfrak{g}$ and $\mathfrak{g}^*$ as Lie subalgebras, where the inner product $(,)$ is invariant under the adjoint action of $\mathfrak{g}$ and $\mathfrak{g}^*$ if and only if $\mathfrak{g}$ is a Lie bialgebra. Moreover, the Lie algebra structure is unique.*

There is a natural candidate for this Lie bialgebra structure when $\mathfrak{g}$ is a Lie bialgebra: the classical double. Before defining the quantum double, we cover some examples. The following examples are the classical versions of the $U_q(sl_3^\epsilon)$ lower and upper Borel subalgebras, as will become clear in the next section.

Example 1.1.1. *Consider the Lie bialgebra $(b^-, [,], \delta)$ over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$ generated by the elements $\{b, a, z, y, x\}$ as a R_ϵ-module and the relations*

$$[a,x] = -2x,\ [a,y] = y,\ [a,z] = -z \tag{1.4}$$
$$[b,x] = x,\ [b,y] = -2y,\ [b,z] = -z \tag{1.5}$$
$$[x,y] = z \tag{1.6}$$
$$\delta(x) = \epsilon(x \otimes a - a \otimes x) \tag{1.7}$$
$$\delta(y) = \epsilon(y \otimes b - b \otimes y) \tag{1.8}$$
$$\delta(z) = \epsilon(z \otimes (a+b) - (a+b) \otimes z + 2x \otimes y - 2y \otimes x), \tag{1.9}$$

and all other identies on generators zero. Concretely, we define the Lie algebra b^- as the R_ϵ-module generated by b, a, z, y, x, divided out to the ideal generated by the algebra relations stated above. The algebra-relations are equal to the relations in the lower triangular subalgebra of the Lie algebra $sl_3(\mathbb{R})$. The cobracket is the usual cobracket multiplied by ϵ. Hence it satisfies the Lie bialgebra axioms.

The b^- Lie bialgebra is a central object in this chapter. We might denote the generators $\{b, a, z, y, x\}$ of b^- as the more general

$$\{H_1^-, H_2^-, X_1^-, X_2^-, X_3^-\},$$

with $H_3^- = H_1^- + H_2^-$. Although b^- is not semisimple, as can be seen from calculating the Killing form, where the diagonal block-matrix corresponding to

(z, y, x) (using the order (b, a, z, y, x)) vanishes, the Killing form κ restricted to the maximal toral subalgebra H of b^- is nondegenerate over $\mathbb{R}$. See appendix A.3 for the definition of the Killing form.
The Killing form is not nondegenerate over R_ϵ since the Killing form is bilinear in ϵ. Formally, as remarked in the preliminaries, we can extend an $\mathbb{R}$-basis to an R_ϵ-basis. So we can consider $\{b, a, z, y, x\}$ as elements in $b^-/\epsilon b^-$. In this sense is the Killing form nondegenerate. Note that the construction of the Killing form is independent of ϵ.
In fact $\kappa(a, a) = \kappa(b, b) = 6$ and $\kappa(a, b) = \kappa(b, a) = -3$. This is due to the fact that b^- is derived from a semisimple Lie algebra. The following lemma holds. Again, the remark is that the $\mathbb{R}$-basis can be extended to an R_ϵ-basis.

Lemma 1.1.2. *Let $\phi \in H^*$ and κ be the Killing form on H, the maximal toral subalgebra of b^-. Then there exists a unique $t_\phi \in H$ such that $\phi(h) = \kappa(t_\phi, h)$.*

This lemma allows us to identify H with H^*, and also enables us to define a nondegenerate form on H^* by transferring the Killing form. Since the set of roots $\Phi \in H^*$ consists of 3 roots $\alpha, \beta, \alpha + \beta$ corresponding to the root-spaces generated by x, y and z, respectively (the root-spaces corresponding to these roots are nonzero), the set of roots Φ of b^- is the same as the set of sl_3. Moreover, the Cartan matrix is the same as the Cartan matrix for sl_3, which reads

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{1.10}$$

Another example of a Lie bialgebra is the lower Borel subalgebra of sl_2.

Example 1.1.2. *Let $\mathfrak{g}$ be the algebra over $\mathbb{R}$ generated by A and X and the relations*

$$[X, A] = X, [X, X] = 0, [A, A] = 0, \tag{1.11}$$
$$\delta(X) = X \otimes A - A \otimes X, \tag{1.12}$$
$$\delta(A) = 0. \tag{1.13}$$

Let us check the axioms explicitly for this example. Clearly the first two Lie bialgebra axioms are satisfied, as there are only two generators. By definition, the third axioms is satisfied. If we use the formula for δ, the last axiom is equivalent to

$$\delta([A, X]) = [A, X] \otimes A - A \otimes [A, X] = X \otimes A - A \otimes X.$$

Let us now calculate the dual of $\mathfrak{g}$, the generators of which are denoted by a and x and are dual to A and X respectively. Using the properties of the dual pairing ($\mathfrak{g}$ being finite dimensional) we get

$$\langle [a, x], X \rangle = \langle a \otimes x, \delta(X) \rangle = \langle a \otimes x, X \otimes A - A \otimes X \rangle \tag{1.14}$$
$$= \langle a \otimes x, -A \otimes X \rangle = -1. \tag{1.15}$$

Hence the algebra is generated by $[a, x] = -x$, and the other relations zero. In the same way we get $\delta(x) = x \otimes a - a \otimes x$. Checking the Jacobi identity, we see that δ^* is indeed a bracket on $\mathfrak{g}^*$, and hence $\mathfrak{g}$ is a Lie-bi algebra. This last exercise is left to the reader.

Let us construct the dual $(b^-)^*$ of b^-. $(b^-)^*$ is needed for the classical double construction.

Example 1.1.3. *The dual of b^-, which we will call b^+ suggestively, can be defined by using a pairing $\langle , \rangle : (b^-)^* \oplus b^- \to k$. If we extend the dual basis over $\mathbb{R}$ obtained through this pairing R_ϵ-linearly, we obtain the dual $(b^-)^*$ that is generated by the dual basis $\{X, Y, Z, A, B\} \subset b^+$. Since b^- is finite dimensional, so is its dual.*

$$\langle X, x\rangle = 1,\ \langle Y, y\rangle = 1,\ \langle Z, z\rangle = 1,\ \langle A, a\rangle = 1,\ \langle B, b\rangle = 1, \tag{1.16}$$

and relations between other generators zero. The generators of b^+ satisfy the following relations

$$[X, Y] = 2\epsilon Z, \tag{1.17}$$
$$[X, A] = \epsilon X,\ [X, B] = 0, \tag{1.18}$$
$$[Y, A] = 0,\ [Y, B] = \epsilon Y, \tag{1.19}$$
$$[Z, A] = \epsilon Z,\ [Z, B] = \epsilon Z, \tag{1.20}$$
$$\delta(A) = \delta(B) = 0 \tag{1.21}$$
$$\delta(X) = X \otimes (2A - B) - (2A - B) \otimes X \tag{1.22}$$
$$\delta(Y) = Y \otimes (2B - A) - (2B - A) \otimes Y \tag{1.23}$$
$$\delta(Z) = Z \otimes (A + B) - (A + B) \otimes Z + X \otimes Y - Y \otimes X. \tag{1.24}$$

The relations between generators that are not mentioned here are zero. It can be checked that these relations indeed satisfy the pairing axioms in a similar fashion as the previous example (in fact, there is only one relation that is different from the previous example, namely $[X, Y] = 2\epsilon Z$). It follows that b^+ is a Lie bialgebra. Note that b^+ is constructed in such a way that the generators are dual with respect to the pairing $\langle , \rangle$. It is interesting to see that this algebra is solvable.

Let us now define the classical double construction. For a proof that this structure indeed defines a quasitriangular Lie bialgebra in the sense of definition 1.1.4 and definition 1.1.1 see 8.2.1 in [23].

Definition 1.1.6. *(classical double) Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra over a ring R with Lie-dual $\mathfrak{g}^*$ (i.e. there exists a dual pairing obeying 1.16). The classical dual $D(\mathfrak{g})$ is the vector space $\mathfrak{g}^* \oplus \mathfrak{g}$ together with Lie bracket, cobracket and classical*

r-matrix

$$[a \oplus b, c \oplus d]_D = ([c,a] + \sum c_1\langle c_2, b\rangle - a_1\langle a_2, d\rangle) \oplus ([b,d] + \sum b_1\langle c, b_2\rangle - d_1\langle a, d_2\rangle) \tag{1.25}$$

$$\delta_D(a \oplus b) = \sum (a_1 \oplus 0) \otimes (a_2 \oplus 0) + \sum (0 \oplus b_1) \otimes (0 \oplus b_2), \tag{1.26}$$

$$r_D = \sum_a (f^a \oplus 0) \otimes (0 \oplus e_a). \tag{1.27}$$

Here, $a, c \in \mathfrak{g}^*$ and $b, d \in \mathfrak{g}$. The elements $f^a \in \mathfrak{g}^*$ form a basis dual to the basis $e^a \in \mathfrak{g}$. We use the Sweedler notation for $\delta(a) = \sum a_1 \otimes a_2$, where we often forget the summation symbol.

Note that $\mathfrak{g}$ and $(\mathfrak{g}^*)^{op}$ are included in $D(\mathfrak{g})$ as sub Lie bialgebras. See Lemma 1.4.2 of [6] and lemma 1.1.1 for the connection between the classical double and b^- and $(b^-)^*$.

Example 1.1.4. *Let us construct the classical double of the upper Borel of sl_2 from the previous example. From the definition we get*

$$[x \oplus 1, 1 \oplus X] = a \oplus A \tag{1.28}$$

$$[x \oplus 1, 1 \oplus A] = -x \tag{1.29}$$

$$[a \oplus 1, 1 \oplus X] = -X, \tag{1.30}$$

$$r = x \oplus X + a \oplus A. \tag{1.31}$$

The r-matrix follows from the definition of the dual generators. The algebra relations follow by direct calculation.

The classical double $D(\mathfrak{g})$ is a quasitriangular Lie bialgebra built on the vector space $(g)^* \oplus g$ with bracket, cobracket and r-matrix. Note that g^* has the negated (opposite) bracket in $D(\mathfrak{g})$. This specific Lie bialgebra structure is related to the invariance of the inner product on $\mathfrak{g}^* \oplus \mathfrak{g}$ under the adjoint action. See [23] chapter 8.2 for example. Let us briefly describe what we mean.
Let $2r_+$ be the symmetric part of $r \in D(\mathfrak{g}) \otimes D(\mathfrak{g})$ as defined before. Since $\mathfrak{g}$ is finite dimensional, $2r_+$ can be interpreted as a map $2r_+ : D(\mathfrak{g})^* \to D(\mathfrak{g})$. This map is given by $2r_+(\xi \oplus \phi) = \langle \xi \oplus \phi, r^{[1]}\rangle r^{[2]} + \langle \xi \oplus \phi, r^{[2]}\rangle r^{[1]}$. Note that since e_a and f^a are dual basis with respect to $\langle , \rangle$ we get $2r_+(\xi \oplus \phi) = \phi \oplus \xi$, so $2r_+$ is an linear isomorphism (over $\mathbb{R}$) with an inverse $2r_+^{-1} : D(\mathfrak{g}) \to D(\mathfrak{g})^*$ which is invariant under the adjoint action because r is an r-matrix, so $2r_+ = r_{12} + r_{21}$ is invariant under the adjoint action of $\mathfrak{g}$. This implies that $2r_+$ gives rise to a Lie algebra isomorphism between $D(\mathfrak{g})$ and $D(\mathfrak{g})^*$.
Because we are not working over a field but over the ring R_ϵ, the definition of $2r_+^{-1}$ does not follow automaticaly from $2r_+$. However, as noted in the prelimenaries, we can extend an $\mathbb{R}$-basis to an R_ϵ-basis. In this way we can define $2r_+^{-1}$ to be the inverse of $2r_+$ as a map of $\mathbb{R}$-modules and extend it to an R_ϵ map. This map is injective. We could also construct $2r_+^{-1}$ by introducing ϵ as an invertible

parameter. Since this changes nothing in the algebra-relations on generators, we can take $\epsilon^2 = 0$ after defining $2r^+$.
This means that we can write $2r_+^{-1}(\phi \oplus \xi) = K(\phi \oplus \xi, \cdot)$ for an adjoint-invariant element $K \in D(\mathfrak{g})^* \otimes D(\mathfrak{g})^*$. Since K is a bilinear map this will define a bilinear form. So in short, if r is quasitriangular, we can define a bilinear symmetric form on $D(\mathfrak{g}) \otimes D(\mathfrak{g})^*$ that is invariant under the adjoint action.
The question arises when is r quasitriangular. In particular, when is $2r_+$ ad-invariant. It turns out that this is the case if we use the opposite multiplication on $\mathfrak{g}^*$. Let $\phi \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. We identify $\phi \oplus 0 = \phi$ and $0 \oplus \xi = \xi$ in the notation. For the moment we denote $\delta(x) = x_{[1]} \otimes x_{[2]}$.

$$\begin{aligned} ad_\phi(r) &= [\phi, f^a]_{D(\mathfrak{g})} \otimes e_a + f^a \otimes [\phi, e_a]_{D(\mathfrak{g})} \\ &= [f^a, \phi] \otimes e_a - f^a \otimes \phi_{[1]} \langle \phi_{[2]}, e_a \rangle - f^a \otimes e_{a[1]} \langle \phi, e_{a[2]} \rangle \\ &= f^a \otimes e_{a[1]} \langle \phi, e_{a[2]} \rangle - f^a \otimes \phi_{[1]} \langle \phi_{[2]}, e_a \rangle - f^a \otimes e_{a[1]} \langle \phi, e_{a[2]} \rangle \\ &= -f^a \otimes \phi_{[1]} \langle \phi_{[2]}, e_a \rangle \\ &= -\phi_{[2]} \otimes \phi_{[1]} = \delta(\phi). \end{aligned}$$

We used the properties of a Lie bialgebra pairing 1.1.5. The same result holds for $\delta(\xi)$, only the minus signs change in + signs, since the product on $\mathfrak{g}$ is the multiplication on the double. However, this is exactly the right order to obtain $ad_\xi(r) = \delta(\xi)$. Since δ is anti-symmetric we see that $ad_\phi(2r_+) = ad_\xi(2r_+) = 0$. It follows that the opposite multiplication is essential to obtain a coboundary Lie bialgebra.
We denote this sub-Lie algebra with $\mathfrak{g}^{*op}$. One can take the dual of $D(\mathfrak{g})$, the double construction this results in is called the co-double. The resulting Lie algebras are equivalent in the sense that there exists an explicit Lie algebra isomorphism relating the two.

Definition 1.1.7. *Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be finite dimensional dual Lie algebras over a ring R. Let $a, d \in \mathfrak{g}^*$ and $b, c \in \mathfrak{g}$. Define the Lie algebra pairing $\langle , \rangle_D : D(\mathfrak{g}) \times D(\mathfrak{g})^* \to R$: $\langle a \oplus b, d \oplus c \rangle_D = \langle a, c \rangle + \langle d, b \rangle$.*

With this pairing it is possible to calculate the dual of a classical double $D(\mathfrak{g})$. To this end we state the following lemma, which simplifies this task.

Lemma 1.1.3. *Let $a, d \in \mathfrak{g}^*$ and $b, c \in \mathfrak{g}$. Then*

$$\langle [a, b]_D, c \rangle_D = \langle a, [b, c] \rangle, \langle a, [d, c]_D * \rangle_D = \langle [d, a], c \rangle.$$

The bracket [,] denotes the bracket in $\mathfrak{g}$ and $\mathfrak{g}^*$.

Proof. The proof is by direct verification, and can be found in [23] for example. We use the fact that $\langle , \rangle$ is a Lie bialgebra pairing, and the definition of $[,]_D$. □

We proceed to use the classical double construction on b^- and its dual. We are using the co-double construction, which is the dual of the classical double. Let

us first give the classical double $D(b^-)$, by combining examples 1.1.1 and 1.1.3, where we put the opposite bracket on the $b^+ = (b^-)^*$ side. Only the trivial relations and the relations of b^- and b^{-*} are left out. Remember that $(b^-)^*$ has the opposite bracket in $D(b^-)$. The bracket on b^- is the usual bracket.

$$[X,b] = X,\ [X,a] = -2X,\ [X,z] = 2\epsilon y,\ [X,x] = 2A - B + \epsilon a, \tag{1.32}$$
$$[Y,b] = -2Y,\ [Y,a] = Y,\ [Y,z] = -2\epsilon x,\ [Y,y] = 2B - A + \epsilon b, \tag{1.33}$$
$$[Z,b] = -Z,\ [Z,a] = -Z,\ [Z,z] = A + B + \epsilon a + \epsilon b,\ [Z,y] = -X,\ [Z,x] = Y \tag{1.34}$$
$$[A,z] = -\epsilon z,\ [A,x] = -\epsilon x,\ [B,z] = -\epsilon z,\ [B,y] = -\epsilon y, \tag{1.35}$$
$$r_D = A \otimes a + B \otimes b + X \otimes x + Y \otimes y + Z \otimes z, \tag{1.36}$$

Lemma 1.1.4. *$D(b^-)$ is a quasitriangular Lie bialgebra, and is the classical double of the Lie bialgebras b^- and $(b^-)^*$.*

Proof. $\mathfrak{g} = D(b^-)$ is a quasitriangular Lie bialgebra by construction. The bracket-relations follow by direct calculation from the definition, and can be checked manually. Observe that the relations follow from group contraction on the standard gl_n structure. See for example 4.4.1 in [10]. See appendix A.4 for an example of Wigner group contraction.
The axioms for the cobracket are satisfied by examples 1.1.1 and 1.1.3, together with the definition of the cobracket on the classical double. The r-matrix follows from the definition of the algebra relations of the classical double. The fact that $D(b^-)$ is the double of b^- and $(b^-)^*$ can be seen from the uniqueness of the classical double (see lemma 1.1.1) and the fact that $D(b^-)$ contains b^- and $(b^-)^*$ as Lie subalgebras. This finishes the proof. □

The relations of the algebra $D(b^-)^*$ are given by

$$[X,b] = X,\ [X,a] = -2X,\ [X,z] = -2\epsilon y,\ [X,x] = -2A + B - \epsilon a, \tag{1.37}$$
$$[Y,b] = -2Y,\ [Y,a] = Y,\ [Y,z] = 2\epsilon x,\ [Y,y] = -2B + A - \epsilon b, \tag{1.38}$$
$$[Z,b] = -Z,\ [Z,a] = -Z,\ [Z,z] = -A - B - \epsilon a - \epsilon b,\ [Z,y] = -X,\ [Z,x] = Y \tag{1.39}$$
$$[A,z] = \epsilon z,\ [A,x] = \epsilon x,\ [B,z] = \epsilon z,\ [B,y] = \epsilon y, \tag{1.40}$$
$$r_D = A \otimes a + B \otimes b + X \otimes x + Y \otimes y + Z \otimes z, \tag{1.41}$$

and the bracket as defined above on the Lie subalgebras $b^\pm$. In particular, b^+ does not have the opposite bracket in $D(\mathfrak{g})^*$. The cobracket δ is negated on $(b^-)^* \subset D(\mathfrak{g})^*$ (as Lie algebras), and stays the same on b^-. The cobracket in general is very complicated, and can be calculated by using the dual pairing. We will not describe the cobracket of $D(\mathfrak{g})^*$ explicitly.

Theorem 1.1.1. *The algebra $D(b^-)^*$ constructed above is a quasitriangular Lie bialgebra, and is the dual of $D(b^-)$. We refer to this Lie bialgebra as sl_3^ϵ.*

Proof. The bracket of $b^{\pm} \subset D(\mathfrak{g})^*$ is calculated from the cobracket of $D(\mathfrak{g})$. By construction of $b^+ = (b^-)^*$, it follows that the bracket on $b^{\pm}$ is the usual (non-opposite) bracket. The other relations follow by using the ad-invariance of $\langle , \rangle : D(\mathfrak{g})^* \times D(\mathfrak{g}) \to R_\epsilon$. This enables us to directly calculate the bracket of $D(\mathfrak{g})^*$. The fact that $D(b^-)^*$ is dual to $D(b^-)$ follows from this calculation. We will do one example with the generators X and z. In the classical double $[X, z] = 2\epsilon y$. y pairs dually with Y, hence it follows by ad-invariance of the inner product $\langle , \rangle$ that

$$\begin{aligned}\langle X \oplus 1, [z \oplus 1, 1 \oplus Y]_{D^*} \rangle_D &= \langle [z, X], Y \rangle \\ &= \langle -2\epsilon y, Y \rangle \\ &= -2\epsilon.\end{aligned}$$

So we can conclude that $[Y, z]$ at least contains a term $2\epsilon x$, since x is the only generator that pairs nonzero with X. Since there are no generators in $D(b^-)$ that commute with z to yield y except X, we can conclude that $[Y, z] = 2\epsilon x$. The other relations follow in a similar fashion. On generators, the cobracket of $D(\mathfrak{g})^*$ is negated on $(b^-)^* \subset D(\mathfrak{g})^*$. From the pairing we can define the cobracket on mixed terms. This will not be done explicitly, but it is also not necessary. We conclude that $D(\mathfrak{g})^*$ is a Lie bialgebra.
To obtain a quasitriangular Lie bialgebra we need to check that $\delta(u) = u \cdot r$ for all $u \in D(\mathfrak{g})^*$, and moreover that $2r_+ = r_{12} + r_{21}$ is a invariant under the action of $\mathfrak{g}$, and $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$. Define $\mathfrak{a} = \mathfrak{g}^{*cop}$, where $\mathfrak{g}^{*cop}$ refers to $\mathfrak{g}^*$ with the negated cobracket, then $\mathfrak{a}^{*op} = \mathfrak{g}$. This follows from the Lie bialgebra pairing axioms. One can do the usual double construction on the Lie bialgebra $\mathfrak{a}$ to obtain a quasitriangular Lie bialgebra structure on $\mathfrak{g} \oplus \mathfrak{g}^{*cop}$ with classical r-matrix r_D. By lemma 1.1.1 this Lie algebra structure is unique and coincides with the Lie algebra $D(\mathfrak{g})^*$. The adjoint action on r_D coincides in both algebras. Moreover the cobracket of $\mathfrak{a}^{*op}$ agrees with the cobracket on $\mathfrak{g}$ by definition (and similarly for the dual), and hence the cobracket of $D(\mathfrak{g})^*$ is identical to the cobracket on $D(\mathfrak{a})$. So we see that $D(\mathfrak{g})^*$ is indeed quasitriangular. □

For completeness we also mention the definition of the universal enveloping algebra of a Lie bialgebra. The universal enveloping algebra $U(\mathfrak{g})$ is the noncommutative algebra generated by 1 and the elements of $\mathfrak{g}$. Formally, we have the following definition. One can also define $U(\mathfrak{g})$ by a universal property, see for example page 90 of [14]. The concept of a Hopf algebra will be defined in the next section. In this definition we only define the relevant maps without proving that they in fact define a Hopf algebra.

Definition 1.1.8. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a ring R. Let $S^n(\mathfrak{g}) = \bigotimes_{i=1}^{n}$ the n-th tensor space, and define the tensor algebra $T(\mathfrak{g}) = \lim_{n\to\infty} \bigoplus_{i=0}^{n} S^i(\mathfrak{g})$ (with tensor products over R). Define the universal enveloping algebra $U(\mathfrak{g})$ as $T(\mathfrak{g})$ modulo the relations $[a, b] = a \otimes b - b \otimes a$ for all $a, b \in \mathfrak{g}$. Let $a \in U(\mathfrak{g})$. The coproduct*

$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, counit $\varepsilon : U(\mathfrak{g}) \to R$ and antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ are given by

$$\Delta a = a \otimes 1 + 1 \otimes a, \varepsilon a = 0, Sa = -a,$$

where Δ, ε are extended as algebra maps, and S as an antialgebra map.

Note that this bialgebra is cocommutative, so we can take the R-matrix to be trivial to make $U(\mathfrak{g})$ a quasitriangular Hopf-algebra, as we will see in the next section. On a seperate note, observe that $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \otimes U(\mathfrak{h})$. This gives a nice insight in the relation between the classical double and the Drinfel'd double, which uses the tensor product instead of the sum. We have extended the bracket of a Lie bialgebra $\mathfrak{g}$ to $U(\mathfrak{g})$, and we have equipped $U(\mathfrak{g})$ with a Hopf algebra structure, but we have not yet extended δ to $U(\mathfrak{g})$.

Definition 1.1.9. *(Co-Poisson Hopf algebras) A co-Poisson Hopf algebra over a ring R is a cocommutative Hopf algebra H with a skew symmetric R-module map $\delta : H \to H \otimes H$ (the Poisson cobracket) satisfying*

1. *$\sigma \circ \delta \otimes id \circ \delta = 0$, where σ means summing over cyclic permutations of the tensor product.*
2. *$(\Delta \otimes id)\delta = (id \otimes \delta)\Delta + \sigma_{23}(\delta \otimes id)\Delta$, where σ_{23} means switching the second and third factor.*
3. *For all $a, b \in H$, $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$.*

This definition is natural, as follows from the following proposition. This proposition can be found as proposition 6.2.3 in [6]. Although we state the proposition for a specific Lie algebra over the ring R_ϵ, it is expected to hold for a general ring of characteristic zero and a general Lie algebra, with a proof similar to the proof in [6]. Of course we have the same proposition for the Lie bialgebras constructed in example 1.1.2. The proof is identical, and we will not state this proposition here, as it is only a formality.

Proposition 1.1.1. *Let $\mathfrak{g} = b^{\pm}$ be the Lie bialgebra over the ring R_ϵ as defined above. Then the Lie cobracket extends uniquely to a Poisson cobracket δ on $U(\mathfrak{g})$, making $U(\mathfrak{g})$ a co-Poisson Hopf algebra.*
Conversely, if $U(\mathfrak{g})$ has a Poisson cobracket δ, then $\delta|_{\mathfrak{g}}$ is a Lie cobracket on $\mathfrak{g}$.

Proof. First consider $b^+/\epsilon b^+$ as a Lie bialgebra. According to proposition 6.2.3 in [6] the cobracket extends uniquely to a Poisson bracket on $U(b^+/\epsilon b^+)$. The cobracket on $b^+/\epsilon b^+$ is also a cobracket when extended to b^+. We obtain the proposition for the ring R_ϵ by considering the universal enveloping algebra of the Lie algebra $\mathfrak{g} = b^+$ over R_ϵ. One can check that this yields the correct cobracket on the Lie algebra generators, trivially. The co-Poisson bracket obeys the first axiom, as δ is antisymmetric. Since $\delta(1) = 0$, the second axiom follows straightforwardly. The last axiom follows from the fact that δ is 1-cocycle.

Secondly, consider the Lie bialgebra $b \subset sl_3$ of lower triangular matrices over $\mathbb{R}$. This extends uniquely to a co-Poisson Hopf algebra by proposition 6.2.3 in [6]. Consider the map $\phi : b^- \to b$ by taking the cobracket δ on b^- and forgetting about ϵ. This map extends to the co-Poisson Hopf algebra $U(b^-)$, since $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$, and Δ is trivial in $U(b^-)$. Suppose that δ_{b^-} cannot be extended uniquely to $U(b^-)$. The difference between the two extensions is proportional to ϵ. Taking its image under ϕ, we get a contradiction with the uniqueness of the cobracket on $U(b)$. This proves the proposition. □

In the following theorem and in later chapters we will use the language of roots and root systems. In particular it is important to know that a Lie bialgebra is characterised by its Cartan matrix (a_{ij}). The Cartan matrix has integer coefficients. For more information on this subject see for example [14]. The most important notions are collected in appendix A.3. Remember that to each simple root one can associate a simple generator, as we showed before. Non-simple roots are sums of simple roots, and to non simple roots one can associate non-simple generators. We will denote the two roots of the Lie algebra sl_3 with greek letters α and β. More generally, we may write β_i or α_i for the roots of the Lie algebra .
Root systems are interesting for the classification of semisimple Lie algebras. They also provide a way of ordering the generators of a Lie algebra. We present a general way in which these generators form a basis for the universal enveloping algebra of a Lie algebra .

Theorem 1.1.2. *Let $\mathfrak{g} = b^+ \oplus b^-$ be a Lie bialgebra over R_ϵ with simple generators $X_i^\pm, H_i^\pm$, $i = 1, \cdots, n$, where n is the rank of $\mathfrak{b}^\pm$. $\mathfrak{g}$ is defined as the classical double of the Lie bialgebras $b^\pm$. b^- is defined as the lower Borel subalgebra of sl_n, where the cobracket is multiplied by ϵ, and b^+ is defined as its dual. Then $U(\mathfrak{g})$ is spanned as a vector space by the monomials $(X_1^+)^{i_1} \cdots X_n^{(+)i_n} (H_1^+)^{l_1} \cdots (H_n^+)^{l_n} (H_1^-)^{j_1} \cdots (H_n^-)^{j_n} (X_1^-)^{k_1} \cdots (X^+)_n^{k_n}$. This basis is called the PBW basis.*

Proof. We first observe that $U(\mathfrak{g}) = U(b^+) \otimes U(b^-)$, so it is enough to prove that the monomials in $X^\pm$ and $H^\pm$ span $U(b^\pm)$ respectively. Let us denote $\mathfrak{g}$ as sl_n^ϵ.
First consider $U(b^-)$. The universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ has a countable basis called the Poincare-Birkhoff-Witt or PBW basis. This theorem is proved in for example [14] in the case where $\mathfrak{g}$ is a semisimple Lie algebra over a field. sl_n^ϵ contains the lower triangular matrices $b^- \subset sl_n$ as a Lie subalgebra over $\mathbb{R}$. Since the commutation relations in b^- do not contain ϵ, we can use the theorem for semisimple Lie algebras by dividing out to $\epsilon\mathfrak{g}$. We conclude that $U(b^-) \subset U(sl_3^\epsilon)$ has a PBW basis as well, and thereby also its dual has a PBW basis.
Consider $U(b^+)$. One can easily see that $U(b^+)$ is spanned by commutative monomials, since any expression can be rewritten using the Lie algebra relations in b^+. So $U(b^+)$ is free. The pairing is nondegenerate over $\mathbb{R}$, and we can extend the dual basis over $\mathbb{R}$ to an R_ϵ-basis, since $U(b^+)$ is free. The dual basis is given

by monomials in the generators dual to the generators of $U(b^-)$, which can be checked on generators. We conclude that the dual basis of b^+ forms a basis over R_ϵ that is identical to the monomials stated in the theorem. □

For completeness we mention the Chevalley basis of a general Lie algebra over a field k. A semisimple Lie algebra $\mathfrak{g}$ over a field k with Cartan matrix a_{ij} generated by a Chevalley basis X_i, H_i has the following relations.

$$[X_i^-, H_j] = a_{ij}X_i^-,\ [X_i^+, H_j] = -a_{ij}X_i^+, \tag{1.42}$$
$$[X_i^-, X_j^+] = H_i\delta_{ij},\quad \sum_{k=0}^{k=1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-a_{ij}-k} = 0.$$

The last relations are called the Serre relations. The Serre relations yield the commutation relations for non-simple generators. It can be rewritten as $ad_{X_i^\pm}^{-a_{ij}+1}(X_j^\pm) = 0$, for $i \neq j$.
The non-simple generators can be defined in a nice way using the Weyl group. The Weyl group is the space of reflections of the root-space and can be used to define a set automorphisms denoted by T_i. For a semisimple Lie algebra, the T_i are given by the adjoint action on non-Cartan elements and as the reflections on Cartan subalgebra elements. The T_i are Lie algebra automorphisms and are given by the following expressions.

$$T_i(X_i^\pm) = -X_i^\pm,\ T_i(H_j) = H_j - a_{ij}H_i, \tag{1.43}$$
$$T_i(X_j^\pm) = \frac{(\pm 1)^{a_{ij}}}{(-a_{ij})!}(ad_{X_i^\pm})^{-a_{ij}}(X_j^\pm),\ i \neq j. \tag{1.44}$$

This yields a braid group action on $\mathfrak{g}$ [6]. These relations also define a set of generators X_α for every root α with the property that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$.
We will later attempt to quantize this braid group action in the setting of sl_3^ϵ. However, the described T_i fail to be algebra automorphisms when $\epsilon^k = 0$. Non-simple generators can still be defined using the T_i, but we have to be careful when using the T_i further. See chapter 4 for a solution to this problem.

1.2. Hopf algebras

We proceed with the construction of a quasitriangular Hopf algebra based on the Lie bialgebra constructed in the previous section. Let $U(b^-)$ be the universal enveloping algebra of b^- as a Hopf algebra. This turns $U(b^-)$ into a Hopf algebra over R_ϵ. The goal is to find a quantization of the universal enveloping algebra of b^-, such that the Lie bialgebra structure is incorperated in the Hopf algebra structure. After the quantization we apply the Drinfel'd double construction to find a new solution to the Yang-Baxter equation.
The definitions below are usually given over a field, but since we work over the

ring R_ϵ in this chapter (except when stated otherwise), we state the definitions over a ring. We observe that the usual definitions hold over a ring as well [23]. As remarked in the prelimenaries, when considering a free module M over the ring R_ϵ, we can extend any $\mathbb{R}$-basis of $M/\epsilon M$ to an R_ϵ-basis of M. See [23], chapter 7, for the explicit proof of propositions 1.2.2 and 2.3.1. For the definition of the Hopf cohomology, see [6]. We remind the reader of the fact that we refer to a module over a ring R as a vector space over R. A ring is always commutative with unit in this thesis. When we say Hopf algebra, we mean a Hopf algebra over a ring R. Often, R will be implicit.

Definition 1.2.1. *((co)algebra) An algebra $(H, m, \mathbf{1})$ over a ring R is a vector space $(H, +, R)$ with a compatible multiplication m (also denoted as $\cdot$ or as the concatination of two elements) and unit map $\mathbf{1}$ with the following properties. Let $i, a \in R$.*

1. *the multiplication $m : H \otimes H \to H$ is an associative, bilinear map which preserves the unit,*

2. *the unit map $\mathbf{1} : R \to H$ is a linear map with property $\cdot \circ \mathbf{1} \otimes id(i \otimes a) = \mathbf{1}(i \cdot a)$, and $\cdot \circ id \otimes \mathbf{1}(a \otimes i) = \mathbf{1}(i \cdot a)$ for all $a \in H, i \in k$ (or $\mathbf{1}(1) = 1_H$).*

A coalgebra (H, Δ, ε) over R is a vector space $(H, +, R)$ with a compatible comultiplication Δ and counit ε with the following properties. Let $h \in H$.

1. *the comultiplication $\Delta : H \to H \otimes H$ is a linear, coassociative map, where coassociativity means $\Delta \otimes id \circ \Delta = id \otimes \Delta \circ \Delta$ and $\Delta(1_H) = 1_H \otimes 1_H$,*

2. *the counit $\varepsilon : H \to R$ has property $(id \otimes \varepsilon) \circ \Delta(h) = (\varepsilon \otimes id) \circ \Delta(h) = h$ (so $\varepsilon(1_H) = 1$).*

We define a Hopf algebra as follows.

Definition 1.2.2. *A Hopf algebra $(H, +, m, \mathbf{1}, \Delta, \varepsilon, S, R)$ over R is a vector space $(H, +, R)$ which is both an algebra $(H, m, \mathbf{1})$ and a coalgebra (H, Δ, ε), and is equipped with a linear antipode map $S : H \to H$ (which is an anti-homomorphism) obeying*

1. $\Delta(gh) = \Delta(g)\Delta(h)$,

2. $\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$,

3. $m(S \otimes id) \circ \Delta = m(id \otimes S) \circ \Delta = \mathbf{1} \circ \varepsilon$.

To construct a parallel between Lie bialgebras and Hopf algebras, let us define the Hopf algebra cohomology using the following cochain complex.

Definition 1.2.3. *(see p. 173 in [6]) Let H be a Hopf algebra. For $i, j \geq 1$, define $C^{i,j} := Hom_R(H^{\otimes i}, H^{\otimes j})$, and define $d'_{i,j} : C^{i,j} \to C^{i+1,j}$ and $d''_{i,j} : C^{i,j} \to C^{i,j+1}$ as*

follows (let $\gamma \in C^{i,j}$):

$$
\begin{aligned}
&(d'\gamma)(a_1 \otimes \cdots \otimes a_{i+1}) \coloneqq \Delta^{(j)}(a_1) \cdot \gamma(a_2 \otimes \cdots \otimes a_{i+1}) + \\
&\sum_{r=1}^{i} (-1)^r \gamma(a_1 \otimes \cdots \otimes a_{r-1}a_{r+1} \otimes a_{r+2} \otimes \cdots \otimes a_{i+1}) \\
&+ (-1)^{i+1} \gamma(a_1 \otimes \cdots \otimes a_i).\Delta^{(j)}(a_{i+1}), \\
&(d''\gamma)(a_1 \otimes \cdots \otimes a_i) \coloneqq \\
&(m^{(i)} \otimes \gamma)(\Delta_{1,i+1}(a_1)\Delta_{2,i+2}(a_2) \cdots \Delta_{i,2i}(a_i)) \\
&+ \sum_{r=1}^{j} (-1)^r (id^{\otimes r-1} \otimes \Delta \otimes id^{\otimes j-r})(\gamma(a_1 \otimes \cdots \otimes a_i)) \\
&+ (-1)^{j+1} (\gamma \otimes m^{(i)})(\Delta_{1,i+1}(a_1)\Delta_{2,i+2}(a_2) \cdots \Delta_{i,2i}(a_i)).
\end{aligned}
$$

in this definition, $m^{(i)}$ and $\Delta^{(j)}$ are defined as follows

$$
\begin{aligned}
m^{(i)}(a_1 \otimes \cdots \otimes a_i) &= a_1 \cdot \cdots \cdot a_i \\
\Delta^{(j)}(a) &= (id \otimes \cdots \otimes id \otimes \Delta) \cdots (id \otimes \Delta)(\Delta(a)). \\
&\qquad \text{(apply the comultiplication j times).}
\end{aligned}
$$

The $\Delta_{i,j}$ means sending the coproduct to the i-th and the j-th coordinate. The next proposition follows by direct computation and can be found on p. 175 in [6].

Theorem 1.2.1. *Let d' and d'' be as in the definition, then, $d' \circ d' = d'' \circ d'' = d' \circ d'' + d'' \circ d' = 0$.*

Finally we can define the Hopf algebra cochain complex. The previous theorem implies that the cohomologies are well defined.

Definition 1.2.4. *Let H be a Hopf algebra, and let d' and d'' be as defined previously, and set $d = d'_{ij} + (-1)^i d''_{ij}$ and $C^n = \oplus_{i+j=n+1} C^{ij}$. Then $d : C^n \to C^{n+1}$ and (C, d) is a cochain complex with cohomology groups $H^*(H, H)$.*
Define $H^*_{alg}(H, H)$ as the cohomology of the complex $(C^{\cdot 1}, d')$, and similarly define $H^*_{coalg}(H, H)$ as the cohomology of the complex $(C^{1,\cdot}, d'')$.

This cohomology will become important once we start studying deformations of Hopf algebras, for example of the universal enveloping algebra of a Lie bialgebra $\mathfrak{g}$. We can write down the cocycle conditions for this cochain complex. See example 2.3.1 in [23]. [23] uses a simpler definition of the cochain complex, which is equivalent, but is not well defined for $n > 2$.

Proposition 1.2.1. *Let H be an Hopf algebra. Then a 1-cocycle is an invertible element $\chi \in H$ such that*

$$\chi \otimes \chi = \Delta(\chi).$$

A 2-cocycle is an invertible element $\chi \in H \otimes H$ such that

$$(1 \otimes \chi)(id \otimes \Delta)\chi = (\chi \otimes 1)(\Delta \otimes id)\chi.$$

Some Hopf algebras can be equipped with an R-matrix.

Definition 1.2.5. *(Quasitriangular Hopf algebras) A Quasitriangular Hopf Algebra is a pair $(H, \mathcal{R})$, where H is a Hopf algebra and $\mathcal{R} \in H \otimes H$ is invertible and obeys*

1. *$(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{23}$ and $(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$*
2. *$\tau \circ \Delta(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}$, for all $h \in H$.*

$\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, and $\mathcal{R}_{ij} = \sum 1 \otimes \cdots \otimes \mathcal{R}^{(1)} \otimes 1 \cdots \otimes \mathcal{R}^{(2)} \otimes \cdots \otimes 1$, with $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ on the i-th, resp. j-th entry.

We see that the first condition of quasitriangularity is equivalent to the 2-cocycle condition together with the requirement that for a cocycle χ $(1 \otimes \chi)(id \otimes \Delta)\chi = \chi_{13}$.
A Hopf algebra is a bialgebra in particular, so it makes sense to look at both commutativity and cocommutativity, since algebra and coalgebra structures are dual to each other. An R-matrix measures the non cocommutativity of the comultiplication.

Definition 1.2.6. *((Co-)commutative) A Hopf algebra is said to be commutative if it is commutative as an algebra, and cocommutative if the coproduct Δ obeys $\tau \circ \Delta = \Delta$.*

The R-matrix is used to solve the Yang-Baxter equation. The Yang-Baxter equatation follows from the axioms for quasitriangularity. See chapter 2 of [23] for more information.

Proposition 1.2.2. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra, then $\mathcal{R}$ solves the equation: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, called the (quantum) Yang-Baxter equation.*

In order to use a Hopf algebra for constructing knot invariants, one needs to have a ribbon element. This will be explained in chapter 3. Let us write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. Then define $u = \sum (S\mathcal{R}^{(2)})\mathcal{R}^{(1)} \in H$, and $v = Su = \sum \mathcal{R}^{(1)} S\mathcal{R}^{(2)}$. The following proposition is proven in chapter 2 of [23].

Proposition 1.2.3. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra with antipode S. Then S is invertible and $S^2(h) = uhu^{-1}$ for all $h \in H$, and $S^{-2}(h) = vhv^{-1}$.*

We can now define the ribbon element.

Definition 1.2.7. *(Ribbon element) A quasitriangular Hopf algebra is called a ribbon Hopf algebra if the element uv has a central square root ν, called the ribbon element, such that $\nu^2 = vu$, $S\nu = \nu$, $\epsilon\nu = 1$ and $\Delta\nu = Q^{-1}(\nu \otimes \nu)$, where $Q = \mathcal{R}_{21}\mathcal{R}$.*

Let us construct the main example $U_q(sl_3^\varepsilon)$ in the next section.

1.3. Quantizing a Lie bialgebra

In this example we follow the basic example in paragraph 6.4 in [6]. After quantizing the b^- subalgebra of sl_3 (which technically is quantizing the Hopf algebra structure on the enveloping algebra of b^-), we construct an explicit set of generators.
In order to construct $U_q(sl_3)$, one usually quantizes b^+ (or b^-), and takes the Drinfel'd double of this Hopf algebra and its dual with the opposite multiplication or comultiplication. After this procedure, the generators associated to the simple roots on both sides are identified to construct $U_q(sl_3)$. With the introduction of ϵ, this identification is not possible.
Formally, when quantizing $U(\mathfrak{g})$, we introduce an indeterminate h (or $\hbar$ in some sections) to obtain $U_h(\mathfrak{g})$, which is isomorphic as an h-module to $U(\mathfrak{g})[[h]]$. However, it is possible to leave h more implicit, and introduce $q = e^h$, or in our case $q = e^{-\epsilon h}$. The Hopf algebra is denoted as $U_q(\mathfrak{g})$ in this case. The two notations will mean the same thing in this thesis. This implies that it is always possible to expand q in terms of h and ϵ.
Throughout this chapter we work over the ring $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. Since we are considering free modules over R_ϵ, we can use most of the results that hold for Hopf and Lie bialgebras over a field. For future reference, the Drinfel'd double construction yields a quasitriangular Hopf algebra for any commutative ring. [6]
Note that it is possible to do the quantization of sl_3^ϵ for ϵ in $\mathbb{R}(\epsilon)$, and afterwards take the expansion in terms of ϵ. In this case, one has to prove that taking this expansion is possible. The reason to take ϵ to be invertible is that this provides an Hopf algebra isomorphism between $U_q(sl_n^\epsilon)$ and $U_q(sl_n)$. See chapter 4 for this approach.
Let us start with defining the h-adic topology for an indeterminate h. In this section we consider Hopf algebras over a general ring R.

Definition 1.3.1. *Let h be an indeterminate, and let H be an $R[[h]]$-module. Define the basis of the neighbourhoods of $0 \in H$ as the sets $C_n = \{h^n H | n \geq 0\}$. Define the h-adic topology to be the topology such that translations are continuous. In other words, the sets $\{a + C_n\}_{a \in H}$ form a basis for the topology.*

All Hopf algebra maps are continuous, meaning they are h-linear maps, by definition. The following examples are equipped with the h-adic topology. Some caution is advisable in this subject. In particular when taking the dual of an infinite dimensional Hopf algebra we will have to pay attention to the topology. This will be addressed later in this section. Tensor products are assumed to be completed in the h-adic topology.
Let us define what a quantized universal enveloping algebra is.

Definition 1.3.2. *A deformation of a Hopf algebra $(H, \mathbf{1}, m, \varepsilon, \Delta, S)$ over a ring R is a topological Hopf algebra $(H_h, \mathbf{1}_h, m_h, \varepsilon_h, \Delta_h, S_h)$ over the ring $R[[h]]$ of formal power series in h over R, such that*

1. *H_h is isomorphic to $H[[h]]$ as a $R[[h]]$ module.*

2. *$m_h = m \bmod h$, $\Delta_h = \Delta \bmod h$.*

Two Hopf algebra deformations are said to be equivalent if there is an isomorphism f_h of Hopf algebras over $R[[h]]$ which is the identity (mod h).

Let us write $a_h = a + a_1 h + a_2 h^2 + \cdots$ for an element of H_h, where $a = 0 \bmod h$, $a_i = 0$ mod h. Here we use the isomorphism $H_h \tilde{\to} H[[h]]$. Because m_h and Δ_h are $R[[h]]$-module maps, they are determined by their values on elements of H_h for which $a_1 = a_2 = \cdots = 0$, $a_i \in H$. Write

$$m_h(a \otimes a') = m(a \otimes a') + m_1(a \otimes a')h + m_2(a \otimes a')h^2 + \cdots \tag{1.45}$$

$$\Delta_h(a) = \Delta(a) + \Delta_1(a)h + \Delta_2(a)h^2 + \cdots \tag{1.46}$$

The (co)associativity and algebra homomorphism conditions of the Hopf algebra deformation are

$$\begin{aligned} m_h(m_h(a_1 \otimes a_2) \otimes a_3) &= m_h(a_1 \otimes m_h(a_2 \otimes a_3)) \\ (\Delta_h \otimes id)\Delta_h(a) &= (id \otimes \Delta_h)\Delta_h(a) \\ \Delta_h(m_h(a_1 \otimes a_2)) &= (m_h \otimes m_h)\Delta_h^{13}(a_1)\Delta_h^{24}(a_2). \end{aligned}$$

Modulo h^2, this translates to the following proposition.

Proposition 1.3.1. *A pair of R-module map (m_1, Δ_1) is a deformation mod h^2 of a Hopf algebra H if it satisfies*

$$\begin{aligned} &m_1(a_1a_2 \otimes a_3) + m_1(a_1 \otimes a_2)a_3 = a_1 m_1(a_2 \otimes a_3) \\ &+ m_1(a_1 \otimes a_2a_3) \\ &(\Delta \otimes id)\Delta_1(a) + (\Delta_1 \otimes id)\Delta(a) = \\ &(id \otimes \Delta)\Delta_1(a) + (id \otimes \Delta_1)\Delta(a) \\ &\Delta(m_1(a_1 \otimes a_2)) + \Delta_1(a_1a_2) = (m \otimes m_1 + m_1 \otimes m)\Delta^{13}(a_1)\Delta^{24}(a_2) \\ &+ \Delta_1(a_1)\Delta(a_2) + \Delta(a_1)\Delta_1(a_2). \end{aligned}$$

More generally, a deformation mod h^{n+1} is a 2n-tuple $(m_1, \cdots, m_n, \Delta_1, \cdots, \Delta_n)$ which satisfies the (co)associativity and algebra homomorphism conditions (mod h^{n+1}). We now have the following classification of Hopf algebra deformations.

Theorem 1.3.1. *Let H be a Hopf algebra. The following relations between Hopf algebra cohomology and Hopf algebra relations hold:*

1. *there is a natural bijection between $H^2(H, H)$ and the set of equivalence classes of deformation (mod h^2) of H,*

2. *If $H^2(H, H) = 0$, every deformation of H is trivial and*

3. *If $H^3(H,H) = 0$, every deformation (mod h^2) of H extends to a genuine deformation of H.*

Using this theorem we can state an important result in Hopf algebra deformation theory called the rigidity theorem. The theorem is formulated in terms of reductive Lie algebras in [6]. Semisimple Lie algebras are reductive, see [14], and we will skip the definition alltogether. For an R_ϵ-module M, an $\mathbb{R}$-basis of the module $M/\epsilon M$ can be extended to an R_ϵ-basis of M. Using this basis we can generalize the following theorem to a Hopf algebra over the ring R_ϵ.

Theorem 1.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Then $H^{*2}_{alg}(U(\mathfrak{g}), U(\mathfrak{g})) = 0$. So every deformation of $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra.*

The example we work with is not semisimple, so we have to come up with a workaround to use the rigidity theorem. The idea is to only look at the half of the deformed sl_3^ϵ which has a Lie algebra structure that agrees with sl_3. Since we are looking for an algebra isomorphism (or even an Isomorphism of R_ϵ-modules) between $U_h(b^-)$ and $U(b^-)[[h]]$ (not an Hopf algebra isomorphism), we can simply restrict the isomorphism between $U_h(sl_3)$ and $U(sl_3)[[h]]$ to $U_h(b^-)$. Of course we will have to pay attention to R_ϵ too.

Definition 1.3.3. *(Quantized universal enveloping algebra (QUE)) A Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is called a quantized universal enveloping algebra, or QUE algebra.*

The isomorphism in the definition of a deformation of a Hopf algebra is an isomorphism of $R_\epsilon[[h]]$-modules, meaning that the isomorphism does not necessarily respect the Hopf structure. In certain cases one can prove that (if $\mathfrak{g}$ is semisimple and is associated to a reductive algebraic group) every deformation of $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra. This isomorphism is not an Hopf algebra isomorphism. See Proposition 6.3.1 in [6].
Finally, the quantization of a Hopf algebra can be defined.

Definition 1.3.4. *(Quantization of Hopf algebra) Let A be a cocommutative co-Poisson-Hopf algebra over a ring R of characteristic zero, and let δ be its Poisson cobracket. A Quantization of A is a Hopf algebra deformation A_h of A such that*

$$\delta(x) = \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \ (mod\ h),$$

where $x \in A$ and $a \in A_h$ such that $x = a$ (mod h), and $\Delta^{op} = \tau \circ \Delta$ is the opposite cobracket.
A quantization of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a quantization $U_h(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$ equipped with the co-Poisson-Hopf structure. Conversely, $(\mathfrak{g}, \delta)$ is called the classical limit of the QUE algebra $U_h(\mathfrak{g})$.

For more details see e.g. [23] and [6]. We will use both the notation $U_q(\mathfrak{g})$ with $q = e^{h\epsilon}$ and $U_h(\mathfrak{g})$ in our examples. The difference is subtle and is pointed out in for example [6]. The main difference is topological, as we pointed out before. The U_h-notation has an explicit h-adic topology, while in the U_q notation this topology is hidden. One concrete application of hiding the parameter h in q is that one can specify q to a root of unity, for example. Since we are not concerned with these properties, and always work with h explicitly present (and ϵ), we will use both notations. For the main example we will use the notation $H_{\epsilon,n}$ or $U_q(sl_3^\epsilon)$, taking the notation from [36].
Before doing the main example, we will briefly state the usual quantization of the lower Borel subalgebra of sl_3. This example can be found in many sources, for example [6]. In this example we introduce an invertible parameter γ.

Example 1.3.1. *Let b^- be the Lie bialgebra as in example 1.1.1, with an invertible indeterminate γ instead of ϵ. The following relations define the Hopf algebra $U_h(b^-)$. Moreover, it is the quantization of the Lie bialgebra b^-. We use the generators $\{b, a, z, y, x\}$, and take the free noncommutative module over $\mathbb{R}(\gamma)$ in these generators. Define the quantum commutator as $[u, v]_q = uv - qvu$, and let $q = e^{-\gamma h}$ for the duration of this example. The algebra $U_h(b^-)$ is defined as the module of noncommutative polynomials in $\{b, a, z, y, x\}$ divided out to the iddeal generated by the following relations*

$$[a, x] = -2x,\ [a, y] = y,\ [a, z] = -z \tag{1.47}$$
$$[b, x] = x,\ [b, y] = -2y,\ [b, z] = -z \tag{1.48}$$
$$[x, y]_q = z,\ [x, z]_{q^{-1}} = 0,\ [y, z]_q = 0. \tag{1.49}$$

This is the standard example with a parameter introduced, so it is obvious from literature that this algebra has a basis consisting of the ordered monomials in the generators. This proves that the quotient is not empty, and that the multiplication defined here is associative. The Hopf algebra structure is defined by the following identities

$$\begin{aligned}
&\Delta(b) = b \otimes 1 + 1 \otimes b,\ \Delta(a) = a \otimes 1 + 1 \otimes a, \\
&\Delta(z) = z \otimes 1 + q^{a+b} \otimes z + (q^{-1} - q)q^b x \otimes y, \\
&\Delta(y) = y \otimes 1 + q^b \otimes y, \\
&\Delta(x) = x \otimes 1 + q^a \otimes x, \\
&S(a) = -a,\ S(b) = -b, \\
&S(z) = -q^2 q^{-a-b} z + q^2(q^{-1} - q)q^{-a-b} yx, \\
&S(y) = -q^{-b} y, \\
&S(x) = -q^{-a} x.
\end{aligned} \tag{1.50}$$

Let us check that Δ is an algebra homomorphism. In fact, the only non-trivial relations to check are $[x, y]_q$, $[x, z]_{q^{-1}}$ and $[y, z]_q$, as it is easy to see that $[\Delta(x), \Delta(a)] = [x, a] \otimes$

$1 + q^a \otimes [x, a] = \Delta([x, a])$, *and similarly for the other relations.*

$$\begin{aligned}[\Delta(x), \Delta(y)]_q &= xy \otimes 1 + xq^b \otimes y + q^a y \otimes x + q^{a+b} \otimes xy \\ &\quad - qyx \otimes 1 - qyq^a \otimes x - qq^b x \otimes y - qq^{a+b} \otimes yx \\ &= xy \otimes 1 - qyx \otimes 1 + (q^{-1} - q)q^b x \otimes y + q^{a+b} \otimes xy - qq^{a+b} \otimes yx \\ &= z \otimes 1 + q^{a+b} \otimes z + (q^{-1} - q)q^b x \otimes y.\end{aligned}$$

Let us check $[\Delta(x), \Delta(z)]_{q^{-1}} = 0$, *and leave* $[\Delta(y), \Delta(z)]_q = 0$ *to the reader, as this follows in the same way.*

$$\begin{aligned}[\Delta(x), \Delta(z)]_{q^{-1}} &= [x, z]_{q^{-1}} \otimes 1 + q^{2a+b} \otimes [x, z]_{q^{-1}} + qq^{a+b}x \otimes z - qq^{-2}q^{a+b}x \otimes z \\ &\quad + (q^{-1} - q^{-1})zq^a \otimes x + (q^{-1} - q)q^{a+b}x \otimes xy - q(q^{-1} - q)q^{a+b}x \otimes yx \\ &= (q - q^{-1})q^{a+b}x \otimes z + (q^{-1} - q)q^{a+b}x \otimes z \\ &= 0.\end{aligned}$$

Let us show that Δ *is coassociative. Coassociativity on a and b is trivial. For coassociativity of* Δ *on x and y, observe that* q^a *is grouplike, so* $\Delta(q^a) = q^a \otimes q^a$. *Let us explicitly perform the calculation*

$$\begin{aligned}Id \otimes \Delta(\Delta(z)) &= Id \otimes \Delta(z \otimes 1 + q^{a+b} \otimes z + (q^{-1} - q)q^b x \otimes y) \\ &= z \otimes 1 \otimes 1 + q^{a+b} \otimes z \otimes 1 + q^{a+b} \otimes q^{a+b} \otimes z \\ &\quad + (q^{-1} - q)q^{a+b} \otimes q^b x \otimes y + (q^{-1} - q)(q^b x \otimes y \otimes 1 + q^b x \otimes q^b \otimes y).\end{aligned}$$

Performing a similar calculation for $\Delta \otimes Id(\Delta(z))$, *we see that coassociativity holds. Note that* q^b *is grouplike (similarly for* q^{a+b}*). Plugging in* $\Delta(z)$ *and* $\Delta(x)$, *and using the fact that* Δ *is an algebra homomorphism, we obtain the desired result.*
We check if the antipode is the involution inverse of the comultiplication. Denote the multiplication as $m_{1,2}(u \otimes v) = uv$. *The indices 1 and 2 stand for the first and second tensor entry. We will use the more general version later. We only check this explicitly for z, the axiom is obvious for a and b, and is left to the reader in the case of x and y.*

$$\begin{aligned}m_{1,2}(S \otimes id(\Delta(z))) &= m_{1,2}(-q^2q^{-a-b}z \otimes 1 + q^2(q^{-1} - q)q^{-a-b}yx \otimes 1 + q^{-a-b} \otimes z \\ &\quad - (1 - q^2)q^{-a-b}x \otimes y) \\ &= -q^2q^{-a-b}z + (q - q^3)q^{-a-b}yx + q^{-a-b}z \\ &\quad - (q - q^3)q^{-a-b}yx - (1 - q^2)q^{-a-b}z \\ &= 0.\end{aligned}$$

The antipode is continued as an anti-algebra homomorphism. This is by definition. The counit axioms are also satisfied, as one can check for oneself. We can conclude that we have a Hopf algebra.
We are quick to notice that this Hopf algebra does indeed have the right classical limit,

and since this algebra is imbedded in $U_q(sl_3)$ there exists an isomorphism $U_h(b^-) \to U(b^-)[[h]]$, by the rigidity theorem [6]. Hence we have obtained a quantization of the Lie bialgebra b^-. This finishes the example.

Quantizing a sub Lie bialgebra sl_3

We now treat the quantization of the Lie bialgebra b^- defined in example 1.1.1 for non-invertible ϵ. We follow the basic example 6.4 in [6]. The results of this section are summarized in theorem 1.3.3.
Quantizing the b^- subalgebra of the sl_3 Lie algebra starts with constructing the comultiplication Δ_h which has the b^- cobracket as classical limit. For a and b, which have $\delta(a) = \delta(b) = 0$, the choice is $\Delta_h(a) = a \otimes 1 + 1 \otimes a$, and the same for b. This is the trivial Hopf algebra structure on $U(b^-)$. Note that $U(b^-)$ is a graded algebra with $deg(a, b) = 0$ and $deg(x, y) = 1$. Hence $deg(z) = 2$. The multiplication and comultiplication have to preserve the grading. We define a grading on the tensor product by adding the grading of the factors. We can guess $\Delta_h(x) = x \otimes f + g \otimes x$.
Let Δ denote the trivial comultiplication on $U(b^-)$. Since $\Delta_h \equiv \Delta(\text{mod h})$, we get $f \cong g \equiv 1$ (mod h). We want Δ_h to be an algebra homomorphism that is coassociative. Working out the condition for coassociativity forces $\Delta_h(f) = f \otimes f$, and the same relation for g. Hence f and g have to be group-like (by definition). Note that $\Delta_h : U(b^-)[[h]] \to U(b^-)[[h]] \otimes U(b^-)[[h]]$, where the tensor product is completed in the h-adic topology. This yields $(U(b^-) \otimes U(b^-))[[h]]$ as the image of Δ_h. It is a simple computation to show that all group like elements are of the form $e^{h\mu H}$, where $H \in \mathfrak{h}$, an element of the Cartan subalgebra, and $\mu \in \mathbb{R}[[h]]$ [6]. Hence $\Delta_h(x) = x \otimes e^{\nu h a} + e^{\mu h a} \otimes x$. Since Δ_h is an algebra homomorphism, we may multiply x with a grouplike element to simplify the expression to $\Delta_h(x) = x \otimes 1 + e^{\mu h a} \otimes x$. The definition of the quantization of a Hopf algebra then gives $\Delta_h(x) = x \otimes 1 + e^{-\epsilon h a} \otimes x$.
The definition of a Hopf algebra can be used to obtain the antipode of x. In the same way the comultiplication of y can be deformed. Since the cobracket for a and b is trivial we can easily quantize this cobracket with the trivial comultiplication. As a result, the multiplication relations between a, b and x and a, b and y equal the classical relations. We obtain the comultiplication and antipode relations for x, y, a and b displayed in 1.57. The multiplication between x and y needs to be altered in order for Δ_h to be an algebra homomorphism.
Let us consider the Serre relations of the Lie bialgebra b^-. In our case they need to be slightly altered in order for Δ_h to be an algebra homomorphism. The so called quantum Serre relations are obtained, and we use these to calculate the products between the non-simple algebra generators.
The classical Serre relations in the case of b^- are given by $[x, x, y] = 0$ and $[y, y, x] = 0$. This can be rewritten as $X_i^-(X_j^-)^2 - 2X_i^- X_j^- X_i^- + X_i X_j^2 = 0$, where $i \neq j, i, j \in \{1, 2\}$, or $(X_i^-)^2 X_j^- + X_j^-(X_i^-)^2 = 2X_i^- X_j^- X_i^-$. Applying the comulti-

plication

$$\Delta(y) = y \otimes 1 + e^{-\epsilon h(b)} \otimes y, \tag{1.51}$$
$$\Delta(x) = x \otimes 1 + e^{-\epsilon h(a)} \otimes x \tag{1.52}$$

to the left hand side, we get for b^- (defining $q = 1 - \epsilon = e^{-\epsilon}$):

$$\begin{aligned}\Delta_h(x^2y + yx^2) &= \Delta_h(x)^2\Delta_h(y) + \Delta_h(y)\Delta_h(x)^2 \\ &= x^2y \otimes 1 + (1+q^2)e^{-\epsilon ha}xy \otimes x + e^{-2\epsilon ha}y \otimes x^2 + q^2e^{-\epsilon hb}x^2 \otimes y \\ &+ (1+q^2)qe^{-\epsilon h(a+b)}x \otimes xy + e^{-\epsilon h(2a+b)} \otimes x^2y + e^{-\epsilon h(2a+b)} \otimes yx^2 \\ &+ yx^2 \otimes 1 + (1+q^2)qe^{-\epsilon ha}yx \otimes x + q^2e^{-2\epsilon ha}y \otimes x^2 \\ &+ e^{-\epsilon hb}x^2 \otimes y + (1+q^2)e^{-\epsilon h(b+a)}x \otimes yx.\end{aligned} \tag{1.53}$$

Now we will use the classical Serre relation as ansatz. We assume $x^2y + yx^2 = Cxyx$. We will compute C by applying Δ on both sides. We apply Δ_h to the right handside now:

$$\begin{aligned}\Delta_h(x)\Delta_h(y)\Delta_h(x) &= xyx \otimes 1 + qe^{-\epsilon hb}x^2 \otimes y + e^{-h\epsilon a}yx \otimes x + e^{-\epsilon h(a+b)}x \otimes xy \\ &+ q^{-1}e^{-ah\epsilon}xy \otimes x + q^{-1}e^{-\epsilon h(a+b)}x \otimes yx \\ &+ q^{-1}e^{-2\epsilon h(a)}y \otimes x^2 + e^{-\epsilon h(2a+b)} \otimes xyx.\end{aligned} \tag{1.54}$$

We simplify by taking the exponentials up front. We do not know what C is, but we assumed that $x^2y + yx^2 = Cxyx$ holds for some C. This simplifies the equation between 1.53 and 1.54. The terms involving triple products of x and y on one side of the tensor product cancel out. We can compare the terms term by term. Doing this, we note that $C = q + q^{-1}$, and the following relation should hold

$$x^2y + yx^2 = (q + q^{-1})xyx. \tag{1.55}$$

This relation is called the quantum Serre relation, and is also derived in [6]. We will use the convention $[n]_q = \frac{1-q^{-2n}}{1-q^{-2}}$ in this chapter, where we differ from [6] in a factor q^n, making future notations easier. Observe that $\frac{1-q^{-2n}}{1-q^{-2}} = (1 + q^{-2} + \cdots + q^{-2n+2})$, the geometric series. So the expansion in ϵ of $[n]_q$ is well defined, as the singularity is removable. Using this convention, $q + q^{-1} = q(\frac{1-q^{-4}}{1-q^{-2}}) = q[2]_q$. In fact, this is generalizable, as we will see in chapter 4. For now we remember that $\epsilon^2 = 0$, so the quantum Serre relation for us is equal to the classical Serre relation. To obtain a complete set of generators corresponding to the elements of the root system of b^-, the Weyl group action is needed. In this case, the generators corresponding to the non-simple roots can be calculated by using the Weyl group

action on $U_h(\mathfrak{g})$ (sometimes refered to as the quantum Weyl group). This action respects the algebra structure, but not the co algebra structure and the antipode, in the sense that the braid group acts via algebra automorphisms that are not coalgebra maps, and hence the comultiplication on the non-simple generators needs to be calculated differently. [6] is followed in this case (in particular chapter 8.1 and 8.2). This procedure is described in the last chapter for the b^+ subalgebra of sl_n for general $n > 0$ with Cartan-matrix a_{ij}.
Let us continue with defining the generator corresponding to the commutator of x and y. This is the only generator of $U_h(b^-)$ corresponding to a non-simple root. The fact that we can talk about root spaces and elements corresponding to roots is a consequence of the fact that b^- is a subalgebra of the semisimple Lie algebra sl_3, so we can use the sl_3 root space and basis for the Borel subalgebra b^-. This was discussed in the first section of this chapter.
Let us define the following map on $U_h(b^-)$, for $i \neq j$ $X_1^- = x$ and $X_2^- = y$. This map can be defined on $U_q(sl_n^\epsilon)$ as

$$T_i(X_j^-) = ad_{-(X_i^-)^{-a_{ij}}}(X_j^-).$$

Here, *ad* is the adjoint action of the Hopf algebra on itself, see [23].
When $\epsilon^k = 0$, T_i cannot be extended to yield a set of global automorphisms, see chapter 4. For the non-Cartan elements X_i^- of $U_q(b^-)$, T_i are automorphisms over $\mathbb{R}[[\epsilon]]$, however. So we can use the T_i to define non-simple generators.
We define the generator

$$z = T_\alpha(y) = ad_x(y) = x_{(1)}yS(x_{(2)}),$$

using the Sweedler notation. The automorphisms T_i differ from the automorphisms defined in chapter 4 by a central factor. This factor is essential for the Weyl property. We absorb this factor into z, as the Weyl property is trivial for sl_3. Note that we need the antipode of x to do this calculation, which can be computed from $\Delta_h(x)$. We obtain

$$z = xy - e^{-h\epsilon a}ye^{h\epsilon a}x = xy - (1-\epsilon)yx.$$

We used the multiplication relations between y and a and the antipode and comultiplication of x. Using the above calculated quantum Serre relation, together with the definition of z, we get the following commutation relations.

$$[z, y] = h\epsilon zy,\ [z, x] = -h\epsilon zx.$$

We can also calculate the comultiplication of z

$$\begin{aligned}\Delta_h(z) =&\Delta_h(x)\Delta_h(y) - (1-\epsilon)\Delta_h(y)\Delta_h(x) \\ =&(x\otimes 1 + e^{-\epsilon h(a)}\otimes x)(y\otimes 1 + e^{-\epsilon h(b)}\otimes y) \\ &- (1-\epsilon)(y\otimes 1 + e^{-\epsilon h(b)}\otimes y)(x\otimes 1 + e^{-\epsilon h(a)}\otimes x) \\ =&(xy-(1-\epsilon)yx)\otimes 1 + e^{-\epsilon(a+b)}\otimes(xy-(1-\epsilon)yx) \\ &+ e^{\epsilon}e^{-\epsilon b}x\otimes y + e^{-\epsilon a}y\otimes x - (1-\epsilon)(e^{\epsilon}e^{-\epsilon a}y\otimes x + e^{-\epsilon b}x\otimes y) \\ =&z\otimes 1 + e^{-\epsilon(a+b)}\otimes z + 2\epsilon e^{-\epsilon b}x\otimes y.\end{aligned} \tag{1.56}$$

This ends the construction of $U_q(b^-)$ as a quantization of the Lie bialgebra b^-. Let us summarize the construction. Consider the $R_\epsilon[[h]]$-module M of noncommutative polynomials in the generators $\{b,a,z,y,x\}$. Let I be the ideal of M generated by the following relations, where $[,]$ stands for the commutator.

$$\begin{aligned}&[b,z] = -z,\ [b,y] = -2y,\ [b,x] = x,\\ &[a,z] = -z,\ [a,y] = y,\ [a,x] = -2x,\\ &[z,y] = h\epsilon zy,\ [z,x] = -h\epsilon zx,\ [y,x] = -z + h\epsilon yx.\end{aligned}$$

We consider the closure $\bar{I}$ of I in the h-adic topology on M. Define the algebra $U_h(b^-)$ (also denoted as $U_q(b^-)$) as $M/\bar{I}$. Furthermore, there are $R_\epsilon[[h]]$-algebra homomorphisms $\Delta_h : U_h(b^-) \to U_h(b^-)\otimes U_h(b^-)$, $\varepsilon : U_h(b^-) \to R_\epsilon[[h]]$ and algebra anti-homomorphism $S : U_h(b^-) \to U_h(b^-)$ that define a Hopf algebra structure on $U_h(b^-)$. These maps are defined by the following relations.

$$\begin{aligned}\Delta_h(b) &= b\otimes 1 + 1\otimes b,\ \Delta_h(a) = a\otimes 1 + 1\otimes a,\\ \Delta_h(z) &= z\otimes 1 + e^{-\epsilon h(a+b)}\otimes z + 2\epsilon hx\otimes y,\\ \Delta_h(y) &= y\otimes 1 + e^{-\epsilon h(b)}\otimes y,\\ \Delta_h(x) &= x\otimes 1 + e^{-\epsilon h(a)}\otimes x,\\ S(a) &= -a,\ S(b) = -b,\\ S(z) &= -(1-2\epsilon h)e^{h\epsilon(a+b)}z + 2\epsilon hyx,\\ S(y) &= -e^{h\epsilon(b)}y,\\ S(x) &= -e^{h\epsilon a}x,\\ \varepsilon(u) &= 0 \text{ if } u \neq 1_{U_h(b^-)}, \varepsilon(1_{U_h(b^-)}) = 1.\end{aligned}$$

Theorem 1.3.3. *The Hopf algebra $U_q(b^-)$ is a quantized universal enveloping algebra with classical limit the Lie bialgebra b^-.*

Proof. By construction $U_q(b^-)$ we have the correct classical limits of the multiplication and comultiplication. Furthermore, the (co)multiplication obeys the Hopf algebra axioms by construction. The antipode can be easily computed from the

(co)multiplication. By theorem 1.4.2, which will be proven separately, $U_q(b^-)$ has a PBW basis. By sending monomials to monomials in $U(b^-)[[h]]$ we obtain an $R_\epsilon[[h]]$-module isomorphism between $U_q(b^-)$ and $U(b^-)[[h]]$. This finishes the proof. □

For clarification, we provide another proof of theorem 1.3.3. This proof relies on example 1.3.1, and looks at a quotient of this algebra.

Proof. Observe that the relations presented here are exactly the same as the relations in example 1.3.1 with $\gamma \to \epsilon$ and $\epsilon^2 = 0$. In particular, this yields a basis of monomials of $U_q(b^-)$, which we will refer to as a PBW basis.
To prove that $U_h(b^-) \cong U(b^-)[[h]]$ as $R_\epsilon[[h]]$-modules, consider $b^- \subset sl_3$ as a Lie algebra, ignoring the comultiplication. Observe that ϵ occurs only together with h in $U_h(b^-)$ in the quantum Serre relations. Furthermore, we know that the isomorphism between $U_h(sl_3) \cong U(sl_3)[[h]]$ is an isomorphism between $\mathbb{R}[[h]]$-modules. So we know that this isomorphism must be an isomorphism when we replace h with $h' = h\epsilon$ and putting $\epsilon^2 = 0$, given that epsilon only occurs in $q = 1 - \epsilon h \bmod \epsilon^2$, which is invertible modulo ϵ^2. In particular we know that the constructed isomorphism must be the identity modulo h, so it sends monomials of generators to monomials of generators of the Lie algebra b^-. This finishes the proof. □

We wish to do the double construction with this algebra and write down the universal R-matrix. We will do this in section 1.4.

1.4. The $U_q(sl_3^\epsilon)$ relations

In the previous section we obtained an Hopf algebra $U_h(b^-)$ or $U_q(b^-)$ that is a quantization of $U(b^-)$. The explicit check of the Hopf algebra axioms is a lengthy exercise. For this reason we present a Wolfram Mathematica implementation of the Hopf algebra in the next chapter, and set up the required formalism. The program can be found in the appendix A.1. The algebra relations are easier to implement, these can be found in a seperate program in A.1. The interesting axioms to check manually are (co)associativity, that Δ is an algebra homomorphism and that the antipode is an anti-algebra homomorphism.
We work over the ring $R_\epsilon[[h]]$ of formal power series of an indeterminate h. We repeat theorem 1.3.3 for completeness.

Theorem 1.4.1. *The following relations define a Hopf algebra $U_q(b^-)$ over $R_\epsilon[[h]]$.*

Moreover, $U_q(b^-)$ is the quantization of the Lie bialgebra b^-.

$$\begin{aligned}
&[b,z] = -z,\ [b,y] = -2y,\ [b,x] = x, \\
&[a,z] = -z,\ [a,y] = y,\ [a,x] = -2x, \\
&[z,y] = h\epsilon zy,\ [z,x] = -h\epsilon zx,\ [y,x] = -z + h\epsilon yx \\
&\Delta(b) = b\otimes 1 + 1\otimes b,\ \Delta(a) = a\otimes 1 + 1\otimes a, \\
&\Delta(z) = z\otimes 1 + e^{-\epsilon h(a+b)}\otimes z + 2\epsilon hx\otimes y, \\
&\Delta(y) = y\otimes 1 + e^{-\epsilon h(b)}\otimes y, \\
&\Delta(x) = x\otimes 1 + e^{-\epsilon h(a)}\otimes x, \\
&S(a) = -a,\ S(b) = -b, \\
&S(z) = -(1-2\epsilon h)e^{h\epsilon(a+b)}z + 2\epsilon hyx, \\
&S(y) = -e^{h\epsilon(b)}y, \\
&S(x) = -e^{h\epsilon a}x
\end{aligned} \tag{1.57}$$

To obtain a deformation we need an algebra isomorphism between $U_q(b^-)$ and $U(b^-)[[h]]$ as $R_\epsilon[[h]]$-modules, as stated in the proof of theorem 1.3.3. This isomorphism can be found by means of the rigidity theorems if ϵ is invertible. If ϵ is not invertible, it is possible to construct a basis of monomials which can be send to the classical PBW basis of $U(b^-)[[h]]$. The algebra $U(b^-)[[h]]$ has the multiplication of the universal enveloping algebra of $U(b^-)$, h-linearly extended. On the other hand, if one has such an isomorphism, it is possible to directly construct the q-PBW basis of $U_q(sl_3^\epsilon)$ by looking at the image of the classical PPBW basis. This is how [6] proves the existence. We will provide a direct proof.

Theorem 1.4.2. *The monomials $b^{n_1}a^{n_2}z^{n_3}y^{n_4}x^{n_5}$, $n_i \in \mathbb{N}$ form a basis of $U_h(b^-)$ as an $R_\epsilon[[h]]$-module.*

Proof. The proof is similar to the proof in [31], and it uses a Q-degree on $U_h(b^-)$. It is enough to prove that $b^{n_1}a^{n_2}z^{n_3}y^{n_4}x^{n_5}$, $n_i \in \mathbb{N}$ form a basis of $U_h(b^-)/\epsilon U_h(b^-)$ as an $\mathbb{R}[[h]]$-module. We can then extend the monomials to a basis of $U_h(b^-)$ as an $R_\epsilon[[h]]$ module.
Let $\mathfrak{h}$ be the Cartan subalgebra of b^-. Via quantization we can associate to each element in $\mathfrak{h}$ an element in $U_h(b^-)$. Let us call this subalgebra H for the duration of this proof. Firstly, define the elements $K_\lambda = e^{\epsilon H_\lambda}$, where $\lambda \in \Phi$ is a root of b^-, and H_λ the element in H corresponding to λ via lemma 1.1.2. Following [20], we define an action of K_λ on $U_q(b^-)$ by conjugating:

$$K_\lambda p K_\lambda^{-1} = q^{(\lambda,\rho)}p. \tag{1.58}$$

The root ρ is called the Q-degree of p. The Q-degree of p is well-defined since $(\cdot,\cdot)$ is nondegenerate on H^*, so if p has Q-degree σ and ρ, then $q^{(\lambda,\rho)} = q^{(\lambda,\sigma)}$ for all $\lambda \in H^*$, and so $\sigma = \lambda$. Now the proof consists of two parts: (a) proving that

the monomials span the vector space, and (b) proving linear independence. Part (a) is proven by choosing a normal ordering, in our case (b, a, z, y, x), on $U_h(b^-)$. It is always possible to write an expression in a normal ordered way in finitely many steps. This is left to the reader. It can be proven by induction, see for example [31] for the explicit calculations.
Part (b) is proven in 2 parts. Firstly, one can prove that a, b are linearly independent in the same way lemma 12 is proven in chapter 6.1 of [20]. Let us scetch the proof. The details can be found in [20]. As a first step we prove that $x^{n_x}y^{n_y}z^{n_z} \neq 0$ for $\mathbf{n} = (n_x, n_y, n_z) \in \mathbb{N}^3$. This is done by constructing an algebra homomorphism between $U_q(sl_2^\epsilon)$ to the ring of power series $R_\epsilon[[h]][e, l, f, f^{-1}]$, and mapping x and y to a copy of $U_q(sl_2^\epsilon)$ in which they are nonzero. See proposition 3.1 in [20]. The factor of $\frac{1}{q-q^{-1}}$ is only cosmetic in [20].
Suppose now $\sum_\gamma a_\gamma K_\gamma = 0$. We apply the adjoint action $ad(\sum_\gamma a_\gamma K_\gamma)$ to a monomial $z^{n_z}y^{n_y}x^{n_x}$, $n_i \in \mathbb{N}$. We obtain $\sum_\gamma a_\gamma + a_\gamma \epsilon(\gamma, \sum n_i\rho_i) = 0$ for any $\sum n_i\rho_i$, as $z^{n_z}y^{n_y}x^{n_x} \neq 0$. Since ρ_i span the root space of sl_3, as noted earlier in this chapter, we obtain that $a_\gamma = 0$ for all $\gamma \in \Phi$. Since $\epsilon^2 = 0$, this implies that monomials in a and b are also linearly independent for different exponents of a and b.
The independence of x, y and z is proven by following [31], with induction to the Q-degree. We know that monomials in a, b, x, y, z are nonzero. Assume that we have a relation between monomials in x, y, z. By applying Δ, which conserves Q-degree by construction, and using the linear independence of monomials in a and b, we see that the terms in the relation have the same Q-degree. The case where the Q-degree is equal to one of the simple roots is equivalent to the $U_q(sl_2^\epsilon)$ case, for which we refer to the proof of proposition 6.4.7 in [6].
For the case where the Q-degree is a sum of roots α_i, we can look at the Q-degree in both factors after applying Δ. Consider the biggest i such that $E_{\alpha_i+\dots}$ (taking the convention that $x = E_{\alpha_1}$, $y = E_{\alpha_2}$, $z = E_{\alpha_1+\alpha_2}$) occurs with a nonzero exponent. Let n be the biggest common exponent of $E_{\alpha_i+\dots}$. After applying Δ, consider the terms with Q-degree $n\alpha_i$ left of $\otimes$. The relation obtained on the right of $\otimes$ in this way is a (nonzero) multiple of the original relation, as is clear by a calculation similar to the one in [31], and is of a strictly lower degree, hence the coefficients of these terms are zero by the induction hypothesis. So ordered monomials in x, y, z are linearly independent.
Finally, the linear independence of ordered monomials in b, a, z, y, x is proven by following the proof of lemma 13 in chapter 6.1 of [20]. This is a similar argument as before, by applying Δ and combining the linear independence of monomials in a, b and monomials in x, y, z. For the explicit proof of this lemma we refer to [20]. □

Duality

In order to apply the Drinfel'd double construction we need the dual of $U_h(b^-)$. To obtain the correct dual of $U_h(b^-)$, one has to take the dual of a smaller subalgebra called a quantized formal series Hopf algebra (QFSH-algebra for short).

[6] The dual of a quantum formal series Hopf algebra is a quantized universal enveloping algebra (QUE-algebra). Notice that it is also possible to define the QUE-dual of a QUE algebra the other way around, by taking the QUE-algebra corresponding to the Hopf algebra-dual of a QUE-algebra, which is a QFSH algebra [7].
Before we introduce the notion of a quantized formal series Hopf algebra, let us consider the dual of the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$. The following can be found as example 4.1.16 in [6].
Let $\mathfrak{g}$ be a Lie algebra over a ring R. Concretely, we have the ring R_ϵ in mind. We need R to have certain nice properties, such that R is obtained by extending $\mathbb{R}$ with a finite number of algebraic elements. In this way, we can extend any $\mathbb{R}$-basis of a quotient module to an R-basis of the entire module, as is discussed for R_ϵ in the appendix A.5. We do not go into details here, as we will only be concerned with Hopf algebras over a field or Hopf algebras over the ring R_ϵ.
The Lie algebra $\mathfrak{g}$ has a basis $\{x_1, \cdots x_d\}$, so $U(\mathfrak{g})$ has a PBW basis consisting of ordered monomials in x_i. We number this basis by $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_d) \in \mathbb{N}^d$, and denote $x_\lambda = \frac{x_1^{\lambda_1} \cdots x_d^{\lambda_d}}{\lambda_1! \cdots \lambda_d!}$. The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has the structure of a Hopf algebra if we take the trivial coproduct, and in this case we obtain

$$\Delta(x_\lambda) = \sum_{\mu,\nu} \delta_{\mu+\nu,\lambda} x_\mu \otimes x_\nu.$$

Consider $\xi^\lambda \in U(\mathfrak{g})^*$ defined by $\xi^\lambda(x_\mu) = \delta_{\lambda\mu}$, then the multiplication $m : U(\mathfrak{g})^* \otimes U(\mathfrak{g})^* \to U(\mathfrak{g})^*$ is defined by $m(\xi^\mu \otimes \xi^\nu)(x_\lambda) := \xi^\mu \otimes \xi^\nu(\Delta(x_\lambda))$. From the definition we obtain the relation $\xi^\mu \xi^\nu = \xi^{\mu+\nu}$ (apply both sides to $x_\lambda \in U(\mathfrak{g})$). Let $R_\epsilon[[\xi_1, \cdots, \xi_d]]$ be the algebra of formal power series in indeterminates ξ_i. By sending $\xi^\lambda \to \xi_1^{\lambda_1} \cdots \xi_d^{\lambda_d}$ we obtain an isomorphism between $U(\mathfrak{g})^* \to R[[\xi_1, \cdots, \xi_d]]$.
With the dual of $U(\mathfrak{g})$ in mind we state the following definition. We follow paragraph 7 of [7]. The condition of a field, as is used in [7], is not essential. It is essential that the ring R has the property that any $\mathbb{R}$-basis can be extended to an R-basis, together with other properties discussed in appendix A.5. We are concerned with the ring $R_\epsilon[[h]]$ mainly in this paragraph. For this reason, we do not state the precise conditions on the ring R.

Definition 1.4.1. *(QFSH-algebra) A quantum formal series Hopf algebra is a topological Hopf algebra B_h over the ring R[[h]], where R is a ring, such that B_h is isomorphic as a R[[h]]-module to $R[[h]]^I$ (equipped with the product topology) for some set I, and $B_h/hB_h \cong R[[\xi_1, \xi_2, \cdots]]$ as a topological algebra.*

The dual of the universal enveloping algebra is equipped with the weak topology. An isomorphism of topological algebras should be continuous and have a continuous inverse. To illustrate this definition, let us consider the following example.

Example 1.4.1. *We start with the lower Borel subalgebra A_h of $U_q(sl_2)$ over $\mathbb{R}$, generated by $\{x, a\}$ and the relations $[a, x] = -2x$, $\Delta(a) = a \otimes 1 + 1 \otimes a$, $\Delta(x) =$*

$x \otimes 1 + e^{-ha} \otimes x$. If we define $\tilde{a} = ha$ and $\tilde{x} = hx$, we obtain the algebra B_h with relations $[\tilde{a}, \tilde{x}] = -2h\tilde{x}$, $\Delta(\tilde{a}) = \tilde{a} \otimes 1 + 1 \otimes \tilde{a}$, $\Delta(\tilde{x}) = \tilde{x} \otimes 1 + e^{-\tilde{a}} \otimes \tilde{x}$.
As a $\mathbb{R}[[h]]$-module, B_h is isomorphic to $\mathbb{R}[[h]]^I = Map(I, \mathbb{R}[[h]])$, where $I = \mathbb{N} \times \mathbb{N} = \mathbb{N}$ enumerates the PBW basis of B_h. The PBW basis of B_h is given by ordered monomials in $\tilde{a}$ and $\tilde{x}$. Concretely, we send $\tilde{x}^{n_1}\tilde{a}^{n_2} \mapsto \phi_{(n_1,n_2)}$, where $\phi_{n,m}(x^{n'}a^{m'}) = \delta_{nn'}\delta_{mm'}h^{n+m}$. Furthermore, B_h/hB_h is commutative as an algebra, and hence B_h/hB_h is isomorphic to $\mathbb{R}[[\tilde{x}, \tilde{a}]]$, as required. So B_h is a QFSH-algebra. See also [7] for this example.

Generalizing the previous example, one can define a QFSH algebra inside any QUE algebra [6].

Definition 1.4.2. *Let A_h be any QUE algebra with cobracket Δ_h. Define $\Delta_n(a) : A_h \to A_h^{\otimes n}$ as $\Delta_n(a) = (id - \mu E)^{\otimes n}\Delta_h^{(n)}(a)$. Here $\Delta^{(n)} = \cdots(\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)\cdots$ is the iterated cobracket with n-1 Δs. Then define $B_h = \{a \in A_h | \Delta_n(a) = 0$ (Mod h^n) for all $n \geq 1\}$.*

The statement is that B_h is a QFSH subalgebra of A_h. For a proof, see proposition 8.3.3 in [6]. To prove the proposition, one proves that an element is in B_h if and only if a monomial of total degree n has a prefactor that is divisable by h^n. As we have seen from the previous example, this proves that B_h is a QFSH-algebra.

Proposition 1.4.1. *Let $B_h \in A_h$ be as defined above, and let A_h be a QUE Hopf-algebra. Then B_h is a QFSH-subalgebra of A_h.*

We can prove the following proposition (see chapter 10 of [9] for the proof). It is essentially a generalization from the classical case, where we could calculate the dual of $U(\mathfrak{g})$ explicitly.

Proposition 1.4.2. *Let H be a quantized universal enveloping algebra over a ring R. Then the dual $H^* = Hom_R(H, R)$ of H is a QFSH algebra. Conversely, the dual of a QFSH algebra is a QUE algebra.*

With the dual of any QUE algebra H, we will mean the dual of the QFSH subalgebra $B_h \subset H$ as definied in 1.4.2, which is a QUE algebra, and we will denote this in the usual way, as H^*. We refer to this space as the QUE-dual of H. Sometimes literature uses the reduced dual, or the Hopf dual H° of any finite or infinite dimensional Hopf algebra, meaning they take the subset of the dual for which the comultiplication lands in the usual tensor product. Since this does not always happen in the infinite dimensial case, this is a useful definition. We will not use this definition here. We use the completed tensor product.
Let us describe the QFSH subalgebra of $U_h(b^-)$, in terms of its basis.

Proposition 1.4.3. *Let a,b,x,y,z be the generators of $U_h(b^-)$ with relations 1.57. Then the QFSH subalgebra of $U_h(b^-)$ as defined in 1.4.2 is topologically generated by ha, hb, hx, hy and hz.*

Proof. The proof is a straightforward repetition of the proof of proposition 8.3.3 in [6].

□

We calculate the dual of the QFSH subalgebra of $U_h(b^-)$ by introducing X, Y, Z, A, B as the linear functionals equal to one on $hx = \tilde{x}, hy = \tilde{y}, hz = \tilde{z}, ha = \tilde{a}, hb = \tilde{b}$ respectively, and zero on all other monomials of the form $\tilde{x}^{n_1}\tilde{y}^{n_2}\tilde{z}^{n_3}\tilde{a}^{n_4}\tilde{b}^{n_5}$. We denote this evaluation as follows:

$$\langle X, \tilde{x}\rangle = 1,\ \langle Y, \tilde{y}\rangle = 1,\ \langle Z, \tilde{z}\rangle = 1,\ \langle A, \tilde{a}\rangle = 1,\ \langle B, \tilde{b}\rangle = 1. \tag{1.59}$$

The pairing is extended as a Hopf algebra pairing, according to the following definition. This defines a Hopf algebra structure on $U_h(b^-)^*$. We will later show that if $U_h(b^-)$ has an R_ϵ, then so does $U_h(b^-)^*$. This basis is given by noncommutative monomials in $\{X, Y, Z, A, B\}$.

Definition 1.4.3. *Let $(H, \cdot, \Delta, \varepsilon, \mathbf{1})$ be a QUE Hopf algebra over the ring R_ϵ with dual $(H^*, \cdot, \Delta, \varepsilon, \mu)$. Let $a, b \in H^*$ and $c, d \in H$. Denote by $\langle , \rangle$ a bilinear map $\langle , \rangle : H^* \otimes H \to R_\epsilon$. The map $\langle , \rangle$ is called a Hopf algebra pairing if it obeys*

$$\begin{aligned} \langle ab, c\rangle &= \langle a \otimes b, \Delta c\rangle \\ \langle \Delta a, c \otimes d\rangle &= \langle a, cd\rangle \\ \langle 1, c\rangle &= \epsilon(c) \\ \langle a, 1\rangle &= \epsilon(a) \\ \langle Sa, c\rangle &= \langle a, Sc\rangle. \end{aligned} \tag{1.60}$$

We will refer to $\langle , \rangle$ as nondegenerate if it is nondegenerate over $\mathbb{R}$, interpreted as a pairing on $H^*/\epsilon H^* \otimes H/\epsilon H \to \mathbb{R}$.

The space we use as the dual of $U_h(b^-)$ is the QUE-dual of $U_h(b^-)$. We use the notation $U_h(b^-)^*$. We write $\frac{1}{h}$, but this is informal notation for the topology on the dual module. For example, if ξ is the dual basis element of $\tilde{x}$, then we may write $\frac{\xi}{h}$ as the dual element of x, informally. For this reason it is important to keep track of the factors of h.
Only when applying the pairing to $H^* \otimes H$ one has to be careful with the factors of $\frac{1}{h}$, since the pairing is only defined on the subspace $H^* \otimes B_h \subset H^* \hat{\otimes} H$. We defined the QUE-dual of $U_q(b^-)$ (which is $U_q(b^+)$) to be the dual of the QFSH-subalgebra of $U_q(b^-)$. So for an element that is not part of the QFSH-subalgebra of $U_q(b^-)$ the pairing will not be defined. This problem is resolved in the Drinfel'd double by applying the antipode to one side of the pairing, cancelling the $\frac{1}{h}$ term in the final expression for the product of the Drinfel'd double. Constructing the Drinfel'd double will be the only application of the pairing on $U_h(b^-)^* \otimes U_h(b^-)$.
Another point of care arises when applying the pairing on elements in $U_h(b^-)^* \hat{\otimes} U_h(b^-)$. The R-matrix we will construct later for example, can be written as an element in

the completed tensor product of these algebras, when considered as R_ϵ-modules. Applying the pairing to the R-matrix diverges, as we will see later. Furthermore define $\langle a \otimes b, c \otimes d\rangle := \langle a, c\rangle\langle b, d\rangle$.

Lemma 1.4.1. *The following relations define a Hopf algebra $U_h(b^+)$ that is dual to the Hopf algebra 1.57. Moreover, it is the quantization of the Lie bialgebra b^+.*

$$
\begin{aligned}
&[Y, X] = -2\epsilon Z + \epsilon h XY,\ [Z, X] = -\epsilon h XZ,\ [A, X] = -\epsilon X,\ [B, X] = 0 \qquad (1.61)\\
&[Z, Y] = \epsilon h YZ,\ [Y, A] = 0,\ [B, Y] = -\epsilon Y,\\
&[A, Z] = -\epsilon Z,\ [B, Z] = -\epsilon Z,\\
&\Delta(X) = X \otimes e^{h(2A-B)} + 1 \otimes X,\\
&\Delta(Y) = Y \otimes e^{h(2B-A)} + 1 \otimes Y,\\
&\Delta(Z) = Z \otimes e^{h(A+B)} + 1 \otimes Z + h(X \otimes Y e^{h(2A-B)})\\
&\Delta(A) = A \otimes 1 + 1 \otimes A,\ \Delta(B) = B \otimes 1 + 1 \otimes B,\\
&S(X) = -Xe^{-h(2A-B)},\ S(Y) = -Ye^{-h(2B-A)},\\
&S(Z) = -(1 - 2h\epsilon)(Z - XYh)e^{-h(A+B)},\ S(A) = -A,\ S(B) = -B.
\end{aligned}
$$

Proof. By theorem 1.4.3, the module of noncommutative polynomials divided out by the algebra relations has a basis of ordered polynomials, so the quotient is nontrivial. To prove coassociativity, one repeats the calculation in the case of $U_q(b^-)$. It is obvious that coassociativity holds. It also follows straightforwardly that Δ is an homomorphism. In this case we only need to check three relations, effectively. The antipode and (co)unit axioms are straightforward to check on generators. We leave this to the reader.
To prove duality, let $u, u' \in U_q(b^+)$. We have to check that for all $v, w \in U_q(b^-)$, $\langle \Delta(u), v \otimes w\rangle = \langle u, vw\rangle$. We assume normal ordering $\{b, a, z, y, x\}$ on $U_q(b^-)$. Let $n \geq 0$ be an integer.

$$
\begin{aligned}
\langle X, xa^n\rangle &= \langle X, (a+2)^n x\rangle\\
&= \langle X, 2^n x\rangle = \frac{2^n}{h}.
\end{aligned}
$$

Similarly, $\langle X, b^n x\rangle = \frac{(-1)^n}{h}$. By duality, we observe that these expressions are the only terms that pair nonzero with X. On the other hand we have

$$
\begin{aligned}
\langle \Delta(X), x \otimes a^n\rangle &= \langle X \otimes e^{h(2A-B)}, x \otimes a^n\rangle\\
&= \langle X \otimes \frac{(2hA)^n}{n!}, x \otimes a^n\rangle\\
&= \frac{2^n}{h}.
\end{aligned}
$$

We obtain $\langle \Delta(X), x \otimes b^m a^n\rangle = \langle X, xb^m a^n\rangle$ for all positive m and n. As observed before, these are the only monomials that pair nonzero with $\Delta(X)$, so we obtain

$\langle \Delta(X), v \otimes w \rangle = \langle X, vw \rangle$ for all v, w. By the same argument we obtain $\langle \Delta(Y), v \otimes w \rangle = \langle Y, vw \rangle$. The argument for Z is a little bit more involved, as we have to check the monomials containing x and y too. However, we observe

$$\begin{aligned}\langle Z, xy \rangle &= \langle Z, z - (\epsilon - 1)xy \rangle = \frac{1}{h} \\ &= \langle hX \otimes Y, x \otimes y \rangle = \langle \Delta(Z), x \otimes y \rangle.\end{aligned}$$

We have to prove that the same axiom holds for the comultiplication on $U_q(b^-)$. This can be checked on generators in a similar way. We leave this to the reader, as well as the counit and antipode pairing axioms. This proves that $U_q(b^+)$ is the Hopf algebra dual of $U_q(b^-)$.
The fact that $U_q(b^+)$ is the quantization of b^+ follows in the same way as 1.3.3, by checking the classical limit of $U_h(b^+)$. By duality we have a PBW basis of $U_q(b^+)$ (we will prove this explicitly in the next theorem), and hence we have an isomorphism between $U_q(b^+)$ and $U(b^+)[[h]]$. This proves the lemma. □

The Hopf algebra obtained is the quantization of b^+ (the dual of b^- in the Lie bialgebraic sense), which is why we call it $U_h(b^+)$. To proof the existence of an algebra isomorphism between $U_h(b+)$ and $U(b^+)[[h]]$ is difficult to do explicitly. A proof for the case where ϵ is invertible that evades this problem can be seen in prop. 4.8 to 4.11 in [9], however one should beware of the different conventions used when computing the double. Roughly speaking, [9] first takes the dual and then makes the space smaller, while we do the opposite. This is the same in the end [7].
As we noted before, the usual finite dimensional highest weight representations of $U_q(sl_n^\epsilon)$ do not exist if ϵ is not invertible. So the usual geometrical interpretation of the Hopf-dual of $U_q(sl_n^\epsilon)$ does not apply here. The geometrical interpretation as functions on a Poisson Lie group is probably lost. See [7] for a discussion on this subject. Some definitions can be found in appendix A.2.
On $U_h(b^+)$ we choose the order $\{X, Y, Z, A, B\}$. By the definition of the pairing, it is nondegenerate over $\mathbb{R}$, so we have a PBW basis of monomials of generators.

Theorem 1.4.3. *Let X, Y, Z, A, B be the elements of $U_h(b^-)^*$ dual to the generators b, a, z, y, x of $U_h(b^-)$. Then the monomials $X^{n_1}Y^{n_2}Z^{n_3}A^{n_4}B^{n_5}$ form a basis of $U_h(b^-)^*$.*

Proof. The fact that they span the space is easy, since we can rewrite any expression in a normal ordered way. This implies that $U_q(b^+)$ is free as a $R_\epsilon[[h]]$-module. The pairing is nondegenerate over $\mathbb{R}$, proving that the monomials are linearly independent over $\mathbb{R}$. As we prove in the appendix A.5, we can extend a basis over $\mathbb{R}$ to a basis over R_ϵ if the module is free, and hence we obtain an R_ϵ basis of $U_q(b^-)^*$. □

For future reference, let us calculate a basis for $U_h(b^-)^*$, the elements of which we require to pair to one with the basis elements of $U_h(b^-)$. We already have a

basis for $U_h(b^-)$ and $U_h(b^-)^*$, which consists of monomials of dual generators. These monomials form a dual basis when normalized.

Proposition 1.4.4.

$$\langle X^l Y^m Z^n A^o B^p, b^{p'} a^{o'} z^{n'} y^{m'} x^{l'} \rangle = \delta_{l,l'}\delta_{m,m'}\delta_{n,n'}\delta_{m,m'}\delta_{l,l'} h^{-o-p-l-m-n} o!p![n]_q![m]_q![l]_q!,$$

where $[n]_q = \frac{1-q^{2n}}{1-q^2}$.

Proof. We use the comultiplication to prove the proposition, following the proof of lemma 8.3.4 in [6]. Consider $\langle X^n, \tilde{x}^{n'} \rangle$. Applying Δ to the non-capital side for $n = 2$ yields $\langle X \otimes X, (\tilde{x} \otimes 1 + e^{-\epsilon h a} \otimes \tilde{x})^2 \rangle = \langle X \otimes X, \tilde{x}e^{-\epsilon h a} \otimes \tilde{x} + e^{-\epsilon h a}\tilde{x} \otimes \tilde{x} \rangle$. The other terms pair to zero. Commuting $\tilde{x}e^{-2\epsilon h a} = q^2 e^{-\epsilon h a}\tilde{x}$ with $q = e^{-\epsilon h}$ yields $\langle X^2, \tilde{x}^2 \rangle = \frac{1+q^2}{h} = \frac{1-q^4}{h(1-q^2)}$.
Now consider $\langle X^n, \tilde{x}^{n'} \rangle$. We observe that $n = n'$. Applying $\Delta^{(n)}$ to the non-capital side gives $\Delta^{(n)}(\tilde{x})^n = (\tilde{x} \otimes e^{-\epsilon h a} \otimes \cdots \otimes e^{-\epsilon h a} + \cdots + 1 \otimes 1 \otimes \cdots \otimes \tilde{x})^n$. Let us, like [6], denote this expression with $(a_1 + \cdots + a_n)^n$. Each term a_i in this expression has a commutator q^2 with a term a_j for $i < j$, since in each term there is precisely on $\tilde{x}$. In the final expression, terms that contain a quadratic factor a_i^2 can be dropped from the expression, since this will pair to zero with $X \otimes X \otimes \cdots \otimes X$. So we only consider permutations of $a_{i_1}a_{i_2}\cdots a_{i_n}$, where $a_{i_k} \neq a_{i_l}$ if $k \neq l$, and $i_l = 1, 2, \cdots n$. Let us now perform an induction argument on n. For $n = 2$ we saw that the coefficient c_2 of a_1a_2 equals $1 + q^2 = \frac{1-q^4}{h(1-q^2)}$. Now assume that $c_n = [n]_q!$. We consider the coefficient of the term $a_1a_2\cdots a_{n+1}$ in the expression $(a_1 + \cdots + a_{n+1})^{n+1}$. We obtain the $n+1$ case from the n case by adding a tensor factor $\otimes e^{-\epsilon h a}$ to a_i for $i < n+1$, and taking $a_{n+1} = 1 \otimes \cdots \otimes 1 \otimes \tilde{x}$. The argument now follows from counting the factors of q. If in the first factor $(a_1 + \cdots + a_{n+1})$ in $(a_1 + \cdots + a_{n+1})^{n+1}$ we pick a_{n+1}, then this term will contribute q^{2n} to coefficient of $a_1 \cdots a_{n+1}$ since we have to commute n factors of $e^{-\epsilon a}$. In a similar way we obtain a contribution of q^{2i-2} by choosing in the i-th factor a_{n+1}. Hence $c_{n+1} = (1 + q^2 + \cdots + q^{2n})c_n = \frac{1-q^{2n+2}}{1-q^2}c_n$. From this we obtain $\langle X^n, \tilde{x}^{n'} \rangle = \delta_{n,n'}[n]_q!$.
By performing a similar induction argument we get the desired results for a, b and y. For $\langle X^l Y^m A^o B^p, b^{p'} a^{o'} y^{m'} x^{l'} \rangle$ we now obtain that

$$\langle X^l Y^m A^o B^p, b^{p'} a^{o'} y^{m'} x^{l'} \rangle = \delta_{l,l'}\delta_{m,m'}\delta_{m,m'}\delta_{l,l'} h^{-o-p-l-m} o!p![m]_q![l]_q!,$$

by duality of x, y, a, b and X, Y, A, B. Observe that we apply $\Delta^{(l+m+o+p)}$ to the non capital side, and that the only terms that pair nonzero are of the form $b \otimes \cdots \otimes b \otimes a \otimes \cdots \otimes x$, this allows for no mixing between the terms.
The only possibly troublesome generator is z, since $\Delta(z) = z \otimes 1 + e^{-\epsilon h(a+b)} \otimes z + 2\epsilon h x \otimes y$. We scetch the argument to prove that there occurs no mixing here. One can prove this relation in general in a much more elegant way in the manner of proposition 8.3.7 in [6]. This is done in the last chapter.
To prove that there occurs no mixing of terms, we observe that the number of x's

and y's can only increase after applying Δ to $b^{p'}a^{o'}z^{n'}y^{m'}x^{l'}$ through a contribution of $\Delta(z)$. To see this observe that the $x \otimes y$ term in $\Delta(z)$ has a factor of ϵ, and $\epsilon[z,x] = \epsilon[z,y] = 0$. The only way to increase the number of z's is by a term $\epsilon xy \otimes y = \epsilon z \otimes y$ occuring. For this we need a contribution from $\Delta(y)$. This implies that an entry that pairs with a Y does not contain a y. The only way to increase the number of ys is by the $x \otimes y$ term in $\Delta(z)$. Since we apply Δ to one entry only (it does not matter which order we take, due to coassociativity), we observe that creating a z in one entry annihilates an y or an x in another entry. So the term which originates from $2\epsilon hx \otimes y$ in $\Delta(z)$ will necessarily pair to zero with $X \otimes \cdots \otimes X \otimes Y \otimes \cdots \otimes B$, as it can never yield a contribution to $b \otimes \cdots \otimes a \otimes \cdots \otimes z \otimes y \otimes \cdots \otimes x$. This implies the result. □

The Drinfel'd double

For the Drinfel'd double $D(H)$, let H be any Hopf algebra with dual H^*. In the infinite dimensional case, let H^* be the QUE-dual of H. Consider the vector space $H^* \otimes H$. Note that the tensor product is the completed tensor product, since the comultiplication doesn't map to the $H^* \otimes H \otimes H^* \otimes H$ in the infinite dimensial case in general. See for example 4.1.16 of [6] for a discussion on this subject.

Definition 1.4.4. *Let H be a Hopf algebra with QUE-dual H^*. The Drinfel'd double $D(H)$ (also called quantum double) is a quasitriangular Hopf algebra generated by H, H^{*op} as Hopf subalgebras with the quasitriangular structure $\mathcal{R} = \sum_a f^a \otimes e_a$, where $\{e_a\}$ is the basis of H and $\{f^a\}$ its dual basis. D(H) is realised on the vector space $H^* \otimes H$ with product $(a \otimes h)(b \otimes g) = \sum b_2 a \otimes h_2 g \langle Sh_1, b_1\rangle \langle h_3, b_3\rangle$, and the tensor-product unit, counit and coproduct:*

$$\Delta(h \otimes a) = \sum a_{(1)} \otimes h_{(1)} \otimes a_{(2)} \otimes h_{(2)}$$

That this definition yields a quasitriangular Hopf algebra is proven in the first paragraph of chapter 7 of [23]. The proof over rings is exactly the same and is not repeated here.
One can show that the antipode S provides an isomorphism between H^{*op} and H^{*cop}, where *cop* stands for the opposite coproduct. See [23] or page 253 of [20]. Using this isomorphism, we get another version of $D(H)$. We will use the multiplication more often than the comultiplication. So the latter definition is the definition we will use, although both constructions are equivalent[6]. Again, the definition is the same as in [23].

Definition 1.4.5. *The quantum double D(H) in a form containing H, H^{*cop} as subalgebras, is a quasitriangular Hopf algebra generated by these subalgebras on the vector*

space $H^* \otimes H$ together with the relations

$$\mathcal{R} = \sum_a f^a \otimes e_a \tag{1.62}$$

$$(h \otimes a)(g \otimes b) = \sum a_{(2)} b \otimes h_{(2)} g \langle h_{(3)}, S^{-1}(b_{(1)}) \rangle \langle h_{(1)}, b_{(3)} \rangle \tag{1.63}$$

$$\Delta(h \otimes a) = \sum a_{(2)} \otimes h_{(1)} \otimes a_{(1)} \otimes h_{(2)}. \tag{1.64}$$

Here the antipode on H^{*cop} is the inverse of the antipode of H^*.

That this construction works was first proven by Drinfel'd. The theorem is that the relations of $D(H)$ define a quasitriangular Hopf algebra. For the proof we refer to [23]. Let $D(U_h(b^-))$ be the quantum double on the space $U_q(b^+) \otimes U_q(b^-)$. We have the following theorem.

Theorem 1.4.4. *Let $U_q(b^\pm)$ be the Hopf algebras as defined in the previous section, with pairing $\langle , \rangle$. Let $D(U_h(b^-))$ be the quantum double on the space $U_q(b^+) \otimes U_q(b^-)$ with (co)multiplication, antipode and (co)unit as defined above. Then $D(U_h(b^-))$ is a quasitriangular Hopf algebra with R-matrix*

$$\mathcal{R} = \sum_{n,m,l,o,p} \frac{X^l Y^m Z^n A^o B^p b^p a^o z^n y^m x^l}{h^{-o-p-l-m-n} o! p! [n]_q! [m]_q! [l]_q!}.$$

We might write $U_q(sl_3^\epsilon)$ instead of $D(U_q(b^-))$.

Proof. We calculated the dual of $U_h(b^-)$ before, and proved that $U_h(b^-)$ is indeed a Hopf algebra. One can explicitly calculate the Drinfel'd double of $U_q(b^\pm)$. This will be done in the next chapter. Observe that Drinfel'ds theorem remains true in the case where a ring is used instead of a field [23].
It is a trivial matter to prove quasitriangularity. From theorem 1.4.3 it follows that the monomials $X^{n_1} Y^{n_2} Z^{n_3} A^{n_4} B^{n_5}$ and $b^{n_5} a^{n_4} z^{n_3} y^{n_2} x^{n_1}$ form a basis of respectively $U_q(b^+)$ and $U_q(b^-)$. By construction the monomials are dual to each other, up to a factor. This factor was computed in proposition 1.4.4. Hence by Drinfel'ds theorem it follows that $\mathcal{R}$ is an R-matrix. □

For completeness we state the explicit algebra relations. We will calculate the explicit algebra relations on generators in the next chapter, when we have developed the necessary tools. These relations might be calculated explicitly by hand. However, since there are a lot of relations, and the possibility for errors is high, it is better to use the computer.

Theorem 1.4.5. *The following relations define, together with the antipode and the cobracket as defined on $U_q(b^-)$ and the opposite coproduct on $U_q(b^+)$ with the inverse*

antipode, the quasitriangular Hopf algebra $U_q(sl_3^\epsilon)$ with R-matrix R.

$$
\begin{aligned}
&[X,Y] = 2\epsilon Z - \epsilon h XY,\ [Z,X] = -\epsilon h XZ,\ [X,A] = \epsilon X,\ [X,B] = 0, \\
&[X,b] = X,\ [X,a] = -2X,\ [X,z] = -2\epsilon y + \epsilon h Xz, \\
&[X,y] = -\epsilon h Xy,\ ,[Y,x] = -\epsilon h Yx, \\
&[X,x] = \frac{e^{-h(2A-B)}-1}{h} - \epsilon a e^{-h(2A-B)} + 2\epsilon h Xx, \\
&[Y,Z] = -\epsilon h YZ,\ [Y,A] = 0,\ [Y,B] = \epsilon Y,\ [Y,b] = -2Y,\ [Y,a] = Y, \\
&[A,x] = \epsilon x,\ [A,y] = 0,\ [B,x] = 0,\ [B,y] = y, [A,z] = [B,z] = \epsilon z, \\
&[Y,z] = 2\epsilon x e^{-h(2B-A)} + \epsilon h Yz,\ [Y,y] = \frac{e^{-h(2B-A)}-1}{h} - \epsilon b e^{-h(2B-A)} + 2\epsilon h Yy, \\
&[Z,A] = \epsilon Z,\ [Z,B] = \epsilon Z,\ [Z,b] = -Z,\ [Z,a] = -Z, \\
&[Z,z] = \frac{-1+e^{-h(A+B)}}{h} - \epsilon e^{-h(A+B)}(a+b) + 2\epsilon h z Z, \\
&[Z,y] = -X + h\epsilon Z y, \\
&[Z,x] = Y e^{-h(2A-B)} + h\epsilon(Zx - (-1+a)Y e^{-h(2A-B)}), \\
&[b,z] = -z,\ [b,y] = -2y,\ [b,x] = x,\ [a,z] = -z,\ [a,y] = y,\ [a,x] = -2x, \\
&[y,z] = -h\epsilon z y,\ [x,z] = h\epsilon z x,\ [x,y] = zh - \epsilon h y x, \\
&\mathcal{R} = \sum_{n,m,l,o,p} \frac{X^l Y^m Z^n A^o B^p b^p a^o z^n y^m x^l}{h^{-o-p-l-m-n} o!p![n]_q![m]_q![l]_q!}.
\end{aligned}
\tag{1.65}
$$

Here the antipode is as defined on the generators before, and is extended as an antihomomorphism. The comultiplication is reverted on the 'capital' side, and is extended as an algebra homomorphism.

We need to be careful when doing calculations with the pairing. The pairing axioms will not hold in general for the comultiplication of the double, since it has the opposite order. The relations agree nicely with the commutation relations found in [35]. This was to be expected, given that they used the same construction, namely the *cop*-construction for the quantum double. It is possible to do the *op*-construction of course, yielding a different set of commutation relations. The classical limit of the relations here agrees nicely with the classical co-double calculated in theorem 1.1.1.
It is more difficult to do the quantization of sl_3^ϵ without treating the two Borel subalgebras separately. The problem lies with the fact that sl_3^ϵ is not semisimple, so it is not possible to use the rigidity theorems as we did in theorem 1.3.3. This seems the biggest issue, although other problems might arrise in the definition of the R-matrix and finding the algebra relations in general. The basic example in [6], chapter 6.4 gives insight in how to perform this quantization for sl_2. This is very tedious to do in our case.
We interpret the tensor product in the h-adically completed sense. The Hopf-

algebra $U_h(b^-)$ is graded, where the grading is inherited from $U(b^-)$. The grading on the b^+ side is inhereted from the grading on the lowercase side, since the coproduct by construction respects the grading.

1.5. R-matrix and ribbon element

As noted in the previous section, we have a basis of the both Borel subalgebras which consists of monomials in the generators. To make these bases dual we correct with the pairing of the monomials, calculated in the previous section. Remember that we use the convention $[n]_q = \frac{1-q^{-2n}}{1-q^{-2}}$.

$$\langle X^l Y^m Z^n A^o B^p, b^p a^o z^n y^m x^l \rangle = h^{-o-p-l-m-n} o! p! [n]_q! [m]_q! [l]_q!. \tag{1.66}$$

With this identity it is easy to write down the formal R-matrix. The real trouble is working with the R-matrix to calculate the knot invariant itself, and in this activity we will find use for the ϵ introduced in the algebra. Without ϵ, so with the ordinary sl_3 quantum invariant, or even with the sl_2 invariant this procedure is exponential in the number of crossings of a knot. When $\epsilon^k = 0$, the procedure is polynomial time, which we will prove in the next chapter. We introduce a trick for working with quantum exponentials, of which the R-matrix is an example.

$$\mathcal{R} = \sum_{n,m,l,o,p} \frac{X^l Y^m Z^n A^o B^p b^p a^o z^n y^m x^l}{h^{-o-p-l-m-n} o! p! [n]_q! [m]_q! [l]_q!} \tag{1.67}$$

In this identity, $q = 1 - h\epsilon$, $[n]_q = \frac{1-q^{-2n}}{1-q^{-2}}$ and $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$. The fact that this is an R-matrix follows from the Drinfel'd double construction.
The R-matrix can also be written with quantum exponentials, which are defined as follows.

$$e_q^d = e_q(d) = \sum_n \frac{d^n}{[n]_q!}. \tag{1.68}$$

This expression is a formal power series in h. However, we observe that $[n]_q! = n! \bmod h$, giving the connection to the usual exponential. The R-matrix can be written with ordered polynomials. In order to rewrite the R-matrix, we map the expression to a commutative ring generated by the generators of the Hopf algebra over the ring $R_\epsilon[[h]]$. The ordening is indicated, so that one can give the inverse of this map to ordered monomials. In this ring, expressions become much more compact.
Let us call such a map $\mathbb{O}(\cdot|p) : O \to H$, where $O = R_\epsilon[[h]][X, Y, Z, A, B, b, a, z, y, x]$ is the ring of (commutative) power series over $R_\epsilon[[h]]$ with generators

$$\{X, Y, Z, A, B, b, a, z, y, x\}$$

, and $H = D(U_q(b^-)) = U_q(sl_3^\epsilon)$. p is a specific ordering. In our case $p = XYZABbazyx$. $\mathbb{O}(T|p)$ sends an unordered expression $T \in O$ to an ordered expression with ordering p in all monomials. With this map we can rewrite for example

$$\sum_n \frac{x^n a^n}{n!} = \mathbb{O}(e^{xa}|xa). \tag{1.69}$$

Likewise, we can rewrite $\mathcal{R}$ in terms of quantum exponentials

$$\mathcal{R} = \mathbb{O}(e^{Aa} e^{Bb} e_q^{Xx} e_q^{Yy} e_q^{Zz} | XYZABbazyx). \tag{1.70}$$

This notation will become important in the following chapter, where commutative rings will provide a nice way of calculating commutation relations. We prove the following lemma for the implementation of the R-matrix in Mathematica. The formula is called the Faddeev-Quesne formula. The proof is due to D. Zagier.

Lemma 1.5.1. $e_q(x) = e^{\sum_{n=1}^{\infty} \frac{(q^{-2}-1)^n x^n}{n(1-q^{-2n})}}$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define the operator $D_{q^{-2}}(f)(x) = \frac{f(q^{-2}x)-f(x)}{q^{-2}x-x}$. Note that $D_{q^{-2}} e_q^x = e_q^x$, since

$$D_{q^{-2}} e_q^x = \sum_{n=1} \frac{q^{-2n}x^n - x^n}{[n]_q!(q^{-2}-1)x} = \sum_{n=1} \frac{x^{n-1}}{[n-1]_q!} = e_q^x.$$

The second last equality follows by definition of $[n]_q$.
Now suppose that a function f has $D_{q^{-2}}(f) = f$, then $f(q^{-2}x) = (q^{-2}x - x + 1)f(x)$, or in other words $log(f(q^{-2}x)) = log(1 - x(1 - q^{-2})) + log(f(x))$. Let us assume that $log(f(x))$ can be expressed as a power series $log(f(x)) = \sum a_n x^n$, then, using the expansion of $log(1 - x)$, we get $q^{-2n} a_n = -\frac{(1-q^{-2})^n}{n} + a_n$. This gives the desired result. □

Lemma 1.5.2. *If $q = e^{-\gamma h}$, and $\gamma^k = 0$, then $e_q^x = e^{\sum_{n=1}^{k} \frac{(q^{-2}-1)^n x^n}{n(1-q^{-2n})}}$.*

Proof. For the proof let us look at the n-th term $\frac{(q^{-2}-1)^n x^n}{n(1-q^{-2n})}$. Observe that $(q^{-2} - 1)^n = (-2h)^n \gamma^n + O(\gamma^{n+1})$. Also, $(1 - q^{-2n}) \sim \gamma + O(\gamma^2)$. This proves the lemma. □

The Ribbon-element is calculated from the R-matrix. Let us write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. Then $u = \sum \mathcal{R}^{(2)} S(\mathcal{R}^{(1)})$, and $v = S(u)$, where S is the antipode. uv is a central element [23]. The Ribbon element ν is defined as the square root of the product vu. If we assume that v is of the form uw^2, for some $w \in H$, then

$v^2 = uuw^2$, and, since uv is central, $v = uw$. So we can calculate the square root of $u^{-1}v = w^2$ and multiply by u to obtain ν.

$$\begin{aligned}\nu = \mathcal{O}(&e^{-hAa-hBb-e^{h(-2A+B)}hXx-e^{h(A-2B)}Yy+e^{-h(A+B)}h^2XYz-e^{h(A+B)}hZz}(e^{h(A+B)}-\\
&axX\epsilon\hbar^2e^{2B\hbar-A\hbar}-a\epsilon\hbar e^{A\hbar+B\hbar}+aXYz\epsilon\hbar^3-\\
&azZ\epsilon\hbar^2-byY\epsilon\hbar^2e^{2A\hbar-B\hbar}-b\epsilon\hbar e^{A\hbar+B\hbar}+\\
&\frac{3}{2}x^2X^2\epsilon\hbar^3e^{3B\hbar-3A\hbar}-2xX^2Yz\epsilon\hbar^4e^{B\hbar-2A\hbar}+\\
&2xX\epsilon\hbar^2e^{2B\hbar-A\hbar}+2X^2Y^2z^2\epsilon\hbar^5e^{-A\hbar-B\hbar}-\\
&XyY^2z\epsilon\hbar^4e^{A\hbar-2B\hbar}-2XYz^2Z\epsilon\hbar^4e^{-A\hbar-B\hbar}+\\
&\frac{3}{2}y^2Y^2\epsilon\hbar^3e^{3A\hbar-3B\hbar}+2yY\epsilon\hbar^2e^{2A\hbar-B\hbar}+\\
&\frac{3}{2}z^2Z^2\epsilon\hbar^3e^{-A\hbar-B\hbar}+e^{A\hbar+B\hbar}+bXYz\epsilon\hbar^3-\\
&bzZ\epsilon\hbar^2-2xXyY\epsilon\hbar^3+2xyZ\epsilon\hbar^2-5XYz\epsilon\hbar^3+6zZ\epsilon\hbar^2)|p)\end{aligned}$$

Remember the following notation: $\mathcal{R}_{ij} = \sum 1 \otimes \cdots \otimes \mathcal{R}^{(1)} \otimes \cdots \otimes \mathcal{R}^{(2)} \otimes 1 \cdots \otimes 1$, where the $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ are on the i-th and the j-th position respectively. The ribbon element is central and invertible, and it has the following properties.

$$\begin{aligned}\varepsilon(\nu) = 1,\ \nu^2 = uS(u),\ S(\nu) = \nu,\\
\Delta(\nu) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(\nu \otimes \nu).\end{aligned} \tag{1.71}$$

Remember that ε is the counit, and here u is as defined previously. Combining the various results of the previous sections we now have the main theorem of this chapter.

Theorem 1.5.1. *The Hopf algebra $U_q(sl_3^\epsilon)$ together with R-matrix $\mathcal{R}$ and ribbon element ν is a quasitriangular ribbon Hopf algebras that is the quantization of the quasitriangular Lie bialgebra sl_3^ϵ.*

Proof. The only thing left to prove is that ν is the ribbon element corresponding the the R-matrix $\mathcal{R}$. This check is performed in Mathematica in the next chapter. □

In the next chapter we will proceed with the implementation of this algebra in Wolfram Mathematica. The main problem is of course commuting normal ordered exponentials.

Conclusion

In this chapter we started with constructing a quasitriangular Lie bialgebra sl_3^ϵ through the classical double. We quantized this Lie algebra by quantizing the

lower Borel subalgebra $b^- \subset sl_3^\epsilon$ and taking the Drinfel'd double of the resulting Hopf algebra $U_q(b^-)$ to obtain the quasitriangular ribbon Hopf algebra $U_q(sl_3^\epsilon)$. We succeeded in working over R_ϵ, where $\epsilon^2 = 0$. We could also have quantized this algebra starting from the b^+ side. It is possible to do the same procedure explicitly for $\epsilon^k = 0$, for any positive k. We calculated an example for invertible ϵ, and it is clear that this algebra can be turned into a quasitriangular Hopf algebra. In this algebra one can take the expansion up to any order of ϵ. This will be done in chapter 4.
Since $U_q(b^-)$ has the same structure as the quantization of the lower Borel subalgebra in the usual sl_3, we could prove a number of results, including the existence of a PBW basis and the fact that $U_q(b^-)$ is a quantization of b^-. As for the question 'what is ϵ?', we showed that it can be viewed as part of the underlying ring. This introduces a number of difficulties which we could work around. Overall it seems a better strategy to work with an invertible epsilon, and afterwards prove that one can take the expansion in ϵ up to any order in the calculations. In fact, this is how we will approach the problem for constructing $U_q(sl_n^\epsilon)$ in chapter 4.
Interesting variations for future research would be to introduce a second parameter γ dual to ϵ. The knot invariant of this algebra is expected to yield a finite type invariant, which is in some sense an expansion of the $U_q(sl_3^\epsilon)$ invariant. An advantage to this knot invariant is that although it will be much weaker, it will also be much faster to compute. One might even prove certain properties of $U_q(sl_3^\epsilon)$, such as detection of mutants, this way.

2. Perturbed Gaussians and their applications

Introduction

In this chapter we describe an approach to do calculations in the quantum group described in the previous chapter. The idea is to describe Hopf algebra maps such as (co)multiplication as a single object, instead of a number of seperate relations. This method has an advantage when computing the knot invariant, it provides a formalism through which we can do computations in Wolfram Mathematica. Also, since the information of the Hopf algebra is contained in only a few objects that can be checked with the computer, there are less errors to be made. The goal is to prove that this construction is isomorphic to the construction given in the previous chapter. A key ingredient is the map $\mathbb{O} : O \to H$ introduced in the previous chapter for some ring O of commutative power series and a Hopf algebra H. This operator gives a vector space isomorphism between the commutative ring O and the Hopf algebra H. The essential theorem is the PBW theorem that is proven in the previous chapter.

2.1. The tensor formalism

Let A and A' be Hopf algebras over $R_\epsilon[[h]]$, where $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$. We write $Hom(A, A') := Hom_{R_\epsilon[[h]]}(A, A')$ for the algebra homomorphisms between A and A'. Let $B \subset A$ be a finite subset of A that generates A. In other words, we assume that A has a basis of ordered monomials in the elements of B. In the previous chapter we saw that $B = \{X, Y, Z, A, B, b, a, z, y, x\}$ generates $U_q(sl_3^\epsilon)$ topologically.
Let V be the vector space over $R_\epsilon[[h]]$ generated the (finite) basis B. Let V^* be the linear dual of V. As B is finite, this is well defined. We will refer to the basis of V^* dual to z_i as z_i^*. $z_i^* \in V^*$ are not to be confused with the generators of the dual Hopf algebra A^*. To make sure this confusion does not take place, when we refer to the generators of the dual Hopf algebra A^*, we will write them with capital letters $Z_i \in A^*$. This will become important mainly in the second section, where we will work with the Hopf algebra constructed in the previous chapter.
Consider the tensor algebra $T(V)$ associated with V. $T(V)$ is equipped with the trivial product, which concatenates tensor products of elements of V. To make this product commutative, let us divide out to the relation $u \otimes v - v \otimes u$, $u, v \in$

$T(V)$. This new space is denoted as $S(V)$, the symmetric tensor algebra of V. In the same way one can construct $S(V^*)$. Note that $S(V)$ is isomorphic to the ring of polynomials in the elements of $S(V) \cong R_\epsilon[[h]][B]$. If $B = \{z_1, \cdots, z_n\}$, then the ring of polynomials in z_i is also written as $R_\epsilon[[h]][z_1, \cdots, z_n] = P[z_1, \cdots, z_n] =: P[B]$. Here $P = R_\epsilon[[h]]$.
$S(V) = P[B]$ can be equipped with the J-adic topology, where J is the ideal $(z_1, \cdots, z_n)$ generated by the elements $z_i \in B$. Then the J-adic topology has a basis consisting of the sets $x + J^n$, $x \in P[B], n \in \mathbb{N}$. The sets J^n are the polynomials of which the monomials have minimal degree n. In this topology, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a cauchy sequence if for all n, there exists $N > 0$ such that for all $i, j > N$, $x_i - x_j \in J^n$. This is equivalent to saying that x_i and x_j differ by a polynomial which has monimials of minimal degree n, in the ring $R_\epsilon[[h]][z_1, \cdots, z_n]$. To take the completion in the J-adic topology amounts to adding the limits of Cauchy sequences to the space. These limits can have nonzero powers in an infinite number of terms. So the completion of $S(V) = P[B]$ is isomorphic to the ring of formal power series in the generators B. We use the notation $\hat{S}(B^*) = P[[B^*]]$ for the completion $\hat{S}(V^*)$ of $S(V^*)$.
We will mainly use the ring of power series in $z_i \in B$, and leave the symmetric tensor algebra on the background. Instead of $R_\epsilon[[h]][z_1, \cdots, z_n]$ and $R_\epsilon[[h]][[z_1, \cdots, z_n]]$ we may also write $P[B]$ and $P[[B]]$ respectively. R_ϵ in $P = R_\epsilon[[h]]$ has the discrete topology, so exponentials in h exist only as formal power series in the ring $R_\epsilon[[h]]$, since $R_\epsilon[[h]]$ has the h-adic topology.
On $P[B]$, one can define the map $\mathbb{O}(\cdot|p) : P[B] \to A$ for an ordering p on A. We state the definition of $\mathbb{O}$ for a general Hopf algebra A with PBW basis consisting of monomials in $z \in B$. Whenever it is obvious what ordering p we use we will leave it out of the notation, writing simply $\mathbb{O}(\cdots)$. When we write a general Hopf algebra A, we have $U_q(sl_3^\epsilon) = D(U_h(b^-))$ in mind, which was constructed in the previous chapter, with $B = \{X, Y, Z, A, B, b, a, z, y, x\}$. By the PBW theorem, we have an isomorphism of $D(U_q(b^-)) = U_q(sl_3^\epsilon) \cong P[B]$, as was mentioned in the previous chapter.

Definition 2.1.1. *Let B be a finite set with ordering p. Let A be a Hopf algebra with a basis consisting of ordered monomials in elements of B. Define the map $\mathbb{O}(\cdot|p) : P[B] \to A$ as the map that sends an unordered expression $T \in P[B]$ to an expression $T \in A$ in which all monomials are ordered via ordering p on B.*

We have the following proposition.

Proposition 2.1.1. *$\mathbb{O} : P[B] \to D(U_h(b^-))$ is an isomorphism of $R_\epsilon[[h]]$ modules, with $B = \{X, Y, Z, A, B, b, a, z, y, x\}$.*

Proof. The proposition is obvious from the PBW theorem in chapter 1. □

This proposition holds for $\mathbb{O}(\cdot|p) : P[B] \to U_q(sl_3^\epsilon)$, and for any Hopf algebra A with 'nice' properties in general. If we wish to extend $\mathbb{O}$ to multiple copies of $P[B]$, as we would for example do for the multiplication $m : P[B_1, B_2] \to$

$P[B']$, we have to specify on which indices $\mathbb{O}$ takes its input. Especially when there are more copies than the algebra map takes input on, for example. This is dependent on which entries we specify the algebra maps to take their input. As a consequence, when writing down a map for which this could be ambiguous, we have to specify the entries of its input.
Let $J \subset \mathbb{N}$ be a finite subset of $\mathbb{N}$. We define the map $\mathbb{O}_J$ as the map $\mathbb{O}$ extended to $A^{\otimes J}$ and $P[B_j]_{j\in J}$. We can define an inverse of $\mathbb{O}$ by forgetting the ordering of a normal ordered expression $Q \in A$. For multiple indices, this is only well-defined if we specify on which entries (index $j \in J$) the expression Q is put by $\mathbb{O}^{-1}$, so we write $\mathbb{O}_I^{-1}$, for some finite set indices I.
We want to use $\mathbb{O}$ to be able to calculate $F : P[B] \to P[B']$ for any Hopf algebra map $F : A \to A'$ or on $A^{\otimes J}$ in general. This happens by taking the pullback of F under $\mathbb{O}$. The notation for $\mathbb{O}_J^{-1}$ is a specification of the input of F when F acts on $A^{\otimes J}$. In particular when composing multiple maps it is important to keep track of the indices. We denote $A_i := 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1 \subset A^I$, with a copy of A only on entry $i \in I$.

Definition 2.1.2. *Let B_i for each $i \in J$, where J is a finite index set, be a finite set of generators of the Hopf algebra $A_i \subset A^{\otimes J}$. Denote an element in $P[B_i]$ as $\boldsymbol{z}_i^{\boldsymbol{n}_i} = (z_1)_i^{n_1}(z_2)_i^{n_2}\cdots(z_m)_i^{n_m}$, where $(z_j)_i \in B_i$. For a subset $J' \subset J$, define $\mathbb{O}_{J'} : P[B_i]_{i\in J} \to A^{\otimes J'} \otimes P[B_i]_{i\in J-J'}$ as $\mathbb{O}$ on each index $j \in J'$ and as the identity on $j \in J - J'$. Conversely, define $\mathbb{O}_{J'}^{-1}$ as the map that is equal to $\mathbb{O}^{-1}$ on $A^{\otimes J}$ for each index $j \in J'$ and acts as the identity on the other indices.*

Concretely, for any map F we will have to specify its input data in terms of the indices or tensor factor on which F takes its input. So instead of writing F, we will write F_J^I, where I and J are a finite set of entries corresponding to the domain and codomain of $F : A^{\otimes I} \to A^{\otimes J}$.

Definition 2.1.3. *Let $F : A^{\otimes I} \to A^{\otimes J}$ be a map between tensor products of the algebra A. Define the map $F_I^J : P[B_i]_{i\in I} \to P[B_j]_{j\in J}$ as $\mathbb{O}_J^{-1} \circ F \circ \mathbb{O}_I : P[B_i]_{i\in I} \to P[B_j]_{j\in J}$.*

There are no requirements for a general map F on multiple copies of A. We distinguish cases. First we cover the one-dimensional case, where there is only one copy on both sides: $F : A \to A'$. This may be an homomorphism or an antihomomorphism like the antipode. Later we will cover the cases where F has $A^{\otimes J}$ as a domain and A as codomain and vice versa.
Let $a \in A$, then we define $F : P[B] \to P[B']$ using the PBW ordering on A. So $F : P[B] \to P[B']$ is denoted in the same way as $F : A \to A'$, and is defined as $\mathbb{O}^{-1} \circ F \circ \mathbb{O}$. Since the input of F is assumed to be normal ordered, $F : P[B] \to P[B']$ corresponding to an algebra map F is automatically a ring-homomorphism on $P[B]$. So $F \in Hom(P[B], P[B'])$. Hence we can define the following.

Definition 2.1.4. *Let $F \in Hom(P[B], P[B'])$ and let $B = \{z_i\}_{i\in J'}$, and $B^* = \{z_i^*\}_{i\in J'}$, where $z_i^*(z_j) = \delta_{ij}$. Define a map ϕ between $Hom(P[B], P[B'])$ and $P[B'][[B^*]]$ which*

maps $F \in Hom(P[B], P[B'])$ to ${}^{\mathbf{t}}F := F(\exp[\sum_{i\in J'} z_i^* z_i]) =$, $z_i^* \in B^*, z_i \in B$. We say that ${}^{\mathbf{t}}F$ is the tensor corresponding to F.

In this definition we denote elements of $P[B'][[B^*]]$ as ${}^{\mathbf{t}}F$ with a bold **t**. This is not to be confused with the transposed matrix, which we will denote with the usual t, and we write it as A^t for a matrix A. The map that one should have in mind is the antipode.

Proposition 2.1.2. *ϕ is well-defined.*

Proof. Note that $\exp[\sum z_i^* z_i] \in S(B) \otimes \hat{S}(B^*) = P[B][[B^*]]$. A homomorphism $F : P[B] \to P[B']$ yields a well defined map $F : P[[B^*]][B] \to P[[B^*]][B']$, which is the identity on $P[[B^*]]$. Since F is an homomorphism on $P[B]$, F is specified by its image on B. For any generator $z \in B$, $F(z)$ is an element of $P[B']$. F is linear in the generators z_i, and by definition acts trivially on B^*, so ${}^{\mathbf{t}}F = \exp[\sum z_i^* F(z_i)] \in P[B'][[B^*]]$. In particular, the tensor $\exp[\sum z_i^* z_i] \in P[B'][[B^*]]$ corresponds to the identity in $Hom(P[B], P[B'])$.

□

A more general version that is defined in the same way is where $F : A \to A'^{\otimes I}$ is an algebra homomorphsim. Consider $\mathbb{O}_I^{-1} \circ F \circ \mathbb{O} : P[B] \to P[B'_i]_{i\in I} = P[B'_I]$. We introduce the notation $B_I = \bigcup_{i\in I} B_i$, where $B_i = \{z_j\}_{j\in J_i}$ for all $i \in I$, for some finite index set I. Then we can extend ϕ by the same definition.

Definition 2.1.5. *Let $F \in Hom(P[B], P[B'_I])$ and let $B = \{z_i\}_{i\in J'}$, and $B^* = \{z_i^*\}_{i\in J'}$, where $z_i^*(z_j) = \delta_{ij}$. Define $\phi : Hom(P[B], P[B'_I]) \to P[B'_I][[B^*]]$ by sending $F \in Hom(P[B], P[B'_I])$ to ${}^{\mathbf{t}}F^I := F(\exp[\sum_{i\in J'} z_i^* z_i])$.*

Where F is an algebra map and is extended B^*-linearly to $P[B_I][[B^*]]$, so ${}^{\mathbf{t}}F^I = \exp[\sum_{i\in J'} z_i^* F(z_i)] \in P[B'_I][[B^*]]$. The map that one should have in mind is the comultiplication in a Hopf algebra.
For a map $F : A^{\otimes I} \to A$ the calculation of ${}^{\mathbf{t}}F$ is slightly more difficult, as there are more generators to check. We explicitly use $\mathbb{O}$ in the definition of ${}^{\mathbf{t}}F$ in this case.

Definition 2.1.6. *Consider the isomorphism $\mathbb{O} : P[B_I] \to A^{\otimes I}$ for finite I. Let $B_i = \{z_j\}_{j\in J'_i}$, J'_i finite for all $i \in I$, and let $B_i^* = \{z_j^*\}_{j\in J'_i}$, where $z_l^*(z_k) = \delta_{lk}$. For $F : A^{\otimes I} \to A'$, define $F_I := \mathbb{O}^{-1} \circ F \circ \mathbb{O}_I : P[B_I] \to P[B']$. Define $\phi : Hom(P[B_I], P[B']) \to P[B'][[B_I^*]]$ which maps $F \in Hom(P[B], P[B'_I])$ to ${}^{\mathbf{t}}F_I := F(\exp[\sum_{j\in\cup_{i\in I} J'_i} z_j^* z_j]) =$*

$$\mathbb{O}^{-1} \circ F \circ \mathbb{O}_I(\exp[\sum_{j\in\cup_{i\in I} J'_i} z_j^* z_j]).$$

Here $P[[B]]\hat{\otimes}P[[B']] \cong P[[B, B']]$, and in this sense we interpret $P[[B_I]]$ for finite I. We note that $\mathbb{O}_I : \bigotimes_{i\in I} P[B_i] = P[B_I] \to A^{\otimes I}$ is an isomorphism. The tensor

products are over $P = R_\epsilon[[h]]$. Usually the calculation of ${}^{\mathbf{t}}F_I$ something complicated. The typical example is when F is the multiplication map. This will be the most important instant of this construction. To illustrate the definition we will cover two examples. Through these examples, one can develop some intuition for what we are calculating later in the case of $U_q(sl_3^\epsilon)$.

Example 2.1.1. *Let A be the coalgebra $\mathbb{R}[[z]]$ generated by the element z, and the map $\Delta : A \to A \otimes A$, with $\Delta(z) = z \otimes 1 + 1 \otimes z$, and the usual counit sending z to 0 and 1 to $1 \in \mathbb{R}$. It is clear that this obeys the coalgebra axioms. We define the pullback of Δ under $\mathbb{O}$ as $\Delta_1^{2,3} = \mathbb{O}_{2,3}^{-1} \circ \Delta \circ \mathbb{O}_1$, where $\mathbb{O}_{2,3}^{-1}(a \otimes b) = a_2 b_3$, and 1 is not written. So $\Delta_1^{2,3}(z_1) = z_2 + z_3$.*
Following the above definitions we obtain ${}^{\mathbf{t}}\Delta_1^{2,3} = e^{z_1^* \Delta_1^{2,3}(z_1)} = e^{z_1^*(z_2+z_3)} \in \mathbb{R}[[z_1^*]][z_2, z_3]$. Conversely, we can obtain $\Delta_1^{2,3}(z_1)$ from ${}^{\mathbf{t}}\Delta_1^{2,3}$ by substituting $z_1 \mapsto \partial_{z_1^*}$ in the expression $z_1 {}^{\mathbf{t}}\Delta_1^{2,3} = z_1 e^{z_1^*(z_2+z_3)}$ and putting z_1^* to zero after differentiating. We denote this process as $\langle z_1 {}^{\mathbf{t}}\Delta_1^{2,3} \rangle_{z_1}$, or $\langle z_1 {}^{\mathbf{t}}\Delta_1^{2,3} \rangle_1$ for short. The subscript indicates that the variables z_i with index 1 are substituted for a derivative $\partial_{z_1^*}$, and that z_i^* with subscript 1 are put to zero. We obtain

$$\langle z_1 {}^{\mathbf{t}}\Delta_1^{2,3} \rangle_1 = (\partial_{z_1^*} e^{z_1^*(z_2+z_3)})|_{z_1^*=0} = z_2 + z_3.$$

This is the so called Feynman-trick, and can be generalized $\Delta_1^{2,3}(z_1^n)$, or even to exponentials of z. Since z is grouplike, this is an easy exercise. When z is not grouplike, the process will become more complicated.

Let us consider a more difficult example, the $U_q(b^+) \subset U_q(sl_2^\epsilon)$. Instead of checking the relations in Mathematica and using other tricks, as we will do in the next section, one can see the correctness of the tensors and other identities used in the implementation by directly performing the multiplication of $\mathbb{O}_{i,j}({}^{\mathbf{t}}\mathrm{Id}_i {}^{\mathbf{t}}\mathrm{Id}_j)$, for example. The following example is taken from [36].
Observe that it is (in theory) a straightforward but lengthy exercise to generalize this example to $U_q(sl_3^\epsilon)$. In practice one runs into the problem of calculating the commutator between ordered exponentials. This would correspond to the naive way of multiplying R-matrices. For $\epsilon = 0$ this can be done explicitly, but for $\epsilon \neq 0$ trails so far have been unsuccesful. One has to use some kind of Feynman-trick, using differential operators to obtain these commutators. See [36] for more information. In fact, this is where the zipping theorem originates.

Example 2.1.2. *Consider the universal enveloping algebra A of the Lie algebra generated by x and a and the relation $[x, a] = -x$. This becomes an algebra when we take the universal enveloping algebra, and we choose to ignore the underlying tensor-algebra structure by just writing $u \otimes v = uv \in \mathbb{R}[x, a]$. Consider multiplication m in A. We wish to consider the description of m as a tensor ${}^{\mathbf{t}}m$. Let L be finite. Remember that in $\mathbb{R}[x, a]^{\otimes \{i,j\} \cup L}$ we label the variables with i and j to indicate in which tensor factor they are. We write $x_i = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1$, with the x on the i-th tensor factor. We have to add indices to ${}^{\mathbf{t}}m$ to indicate on which tensor factor m is acting. We denote ${}^{\mathbf{t}}m_{i,j}^k$ for the*

multiplication tensor of the multiplication map $m : \mathbb{R}[x,a]^{\otimes\{i,j\}\cup L} \to \mathbb{R}[x,a]^{\otimes\{k\}\cup L}$. *In order to find* ${}^{\mathbf{t}}m$, *observe that A has a basis consisting of ordered monomials in x and a, by the PBW theorem, so that we can define* $\mathbb{O}^{-1}$. *Now we can define* ${}^{\mathbf{t}}m$ *as the pullback under* $\mathbb{O}$, *as we did before. We use order ax in this example.*

$$\begin{aligned}
{}^{\mathbf{t}}m^k_{i,j} &= \mathbb{O}_k^{-1}(m(\mathbb{O}_{i,j}(exp(x_i x_i^* + a_i a_i^* + x_j x_j^* + a_j a_j^*)|p))) \\
&= \mathbb{O}_k^{-1}(\sum \frac{(x_i^*)^q (a_j^*)^r}{q!r!}(e^{a_i^* a} x^q a^r e^{x_j^* x})) \\
&= \mathbb{O}_k^{-1}(\sum \frac{(x_i^*)^q (a_j^*)^r}{q!r!}(e^{a_i^* a}(a-q)^r x^q e^{x_j^* x})) \\
&= \sum \frac{(x_i^*)^q}{q!} e^{a_i^* a_k} e^{a_j^*(a_k - q)} (x_k)^q e^{x_j^* x_k} \\
&= exp((a_i^* + a_j^*)a_k + (e^{-a_j^*} x_i^* + x_j^*)x_k).
\end{aligned}$$

We can now do some basic calculations with ${}^{\mathbf{t}}m$, *like checking associativity, or calculating commutation relations. For checking associativity in tensor-language, one needs to check that* ${}^{\mathbf{t}}m^k_{1,2} // {}^{\mathbf{t}}m^l_{k,3} = {}^{\mathbf{t}}m^k_{2,3} // {}^{\mathbf{t}}m^l_{1,k}$, *for example. The concatination operation // is defined as* ${}^{\mathbf{t}}F // {}^{\mathbf{t}}G := {}^{\mathbf{t}}(G \circ F)$. *This is the main subject of the next section. For calculating [x,a] using* ${}^{\mathbf{t}}m$, *we proceed as follows. We start with the expression* $x \otimes a \in A \otimes A$, *which is in the "wrong" order. We now map this to* $\mathbb{O}^{-1}_{1,2}(x \otimes a) = a_2 x_1 \in \mathbb{R}[x_1, a_1, x_2, a_2]$, *which is a choice of convention for* $\mathbb{O}^{-1}_{1,2}$.

$$\begin{aligned}
{}^{\mathbf{t}}m^k_{1,2}(a_2 x_1) &= \langle a_2 x_1 exp((a_1^* + a_2^*)a_k + (e^{-a_2^*} x_1^* + x_2^*)x_k)\rangle_{1,2} \\
&= \langle a_2 exp(((a_1^* + a_2^*)a_k)e^{-a_2^*} x_k\rangle_2 = -x_k + a_k x_k = (a_k - 1)x_k.
\end{aligned}$$

2.2. The ZIP-function

We proceed with translating the Hopf algebra structure to the tensor formalism, including the (co)multiplication maps and the antipode. Composition of tensors is the most important tool that is developed in this section. Furthermore, we will prove that a certain space of perturbed Gaussian exponentials is closed under composition. Care will have to be taken when attempting to interpret general exponentials as maps on Hopf algebras. Some conditions will have to be put in place.
We want to be able to calculate the composition tensor ${}^{\mathbf{t}}(G \circ F) = {}^{\mathbf{t}}F \,//\, {}^{\mathbf{t}}G = \exp[\sum z_i^* G(F(z_i))]$ from the two seperate tensors ${}^{\mathbf{t}}F, {}^{\mathbf{t}}G$. In this section we will define the concatination // of two tensors using the ZIP function. To do calculations we wish to specify how to compute ${}^{\mathbf{t}}F^I_{I'} // {}^{\mathbf{t}}G^J_I$ for maps F and G on $U_q(sl_3^\epsilon)^{\otimes I}$.

In this section we consider an Hopf algebra A over the ring R_ϵ, except when noted otherwise, with topological generators $B = \{z_1, \cdots z_n\}$. The ordered mono-

mials in the elements of B form an $R_\epsilon[[h]]$-basis of A by definition. Furthermore we consider the vector space V linearly generated by the elements of B, which is a finite dimensional vector space. We introduced the notation $P[B] = R_\epsilon[[h]][z_1, \cdots, z_n]$ in the previous section. By the ordering operator we obtained an isomorphism $\mathbb{O} : P[B] \to A$, which uses the PBW basis of A of ordered monomials in the elements of B. The algebra $A = U_q(sl_3^\epsilon)$ obeys the conditions. In particular it has a PBW basis, and a set of generators $B = \{X, Y, Z, A, B, b, a, z, y, x\}$, with an ordering.
This was generalized in the previous section to the case where multiple copies of A are present by introducing indices through the map $\mathbb{O}_I : P[B_I] \to A^{\otimes I}$. This map provides an isomorphism of R_ϵ-modules. By specifying on which tensor factors a map acts it is possible to define the same maps on the space $P[B_I]$. Given a map F on $P[B_I]$, we defined its corresponding tensor ${}^tF_I$. In this section we continue with the concatination of such tensors.

Definition 2.2.1. *Let V and V^* be the vectorspaces generated by the finite sets B and its dual B^* respectively. Denote with $\langle , \rangle$ the natural pairing on V and V^* which evaluates the functionals of V^* on elements of V. Consider the space $S(V^*) \otimes S(V)$. Using the leibniz rule one can use this pairing to define a map $\langle \cdot \rangle : S(V^*) \otimes S(V) \to R_\epsilon[[h]]$ where we define $\langle \phi_1 \otimes \cdots \otimes \phi_m \otimes f_1 \otimes \cdots \otimes f_n \rangle = \langle \phi_1, \langle \phi_2, \cdots \langle \phi_m, f_1 \otimes \cdots \otimes f_n \rangle \cdots \rangle$, and extend the map linearly. Here $\phi_i \in V^*$ and $f_i \in V$.*

Note that $V^{**} = V$. Using the isomorphism between $S(V)$ and $R_\epsilon[[h]][B]$ we see that this pairing acts the same way as substituting the formal derivative ∂_{v^*} for $v \in V$ since the pairing by definition obeys the Leibniz rule. To obtain a pairing, we put v^* to zero after differentiating. We see that the pairing is well defined whenever the expression converges in $R_\epsilon[[h]]$. Since $S(V^*) \otimes S(V) \cong R_\epsilon[[h]][B, B^*]$, and this ring consists of finite polynomials, we see that this is always the case. We present the following alternative definition, which we will use throughout this chapter. It is a more specific version of $\langle \rangle$ in which we can pair specific generators of B. We refer to this function as the ZIP-function.

Definition 2.2.2. *Consider the finite sets B and its dual B^* respectively, and let $\phi_i \in B^*$ and $f_i \in B$. Define the ZIP-function as*

$$\langle \cdot \rangle_{\phi_i} : P[B^*, B] \to R_\epsilon[[h]] : \phi_1 \cdots \phi_i \cdots \phi_m f_1 \cdots f_n \mapsto \phi_1 \cdots \partial_{f_i} \cdots \phi_n f_1 \cdots f_n|_{\phi_i = 0},$$

sending $\phi_i \mapsto \partial_{f_i}$, and acting as the identity on ϕ_j, for $j \neq i$ and for all $f_i \in B$, and putting $f_i \mapsto 0$ after differentiating.
Let $zs \subset B$, let $\zeta s = \{x^* | x \in zs\}$, and let $Q \in P[B^*, B]$. To implicate dependence on $z \in B, \zeta \in B^*$ we write $Q(z^*, z)$. The ZIP-function $\langle \rangle_{\zeta s} : P[B^*, B] \to P[B^*, B]$ is written as $\langle \rangle_{\zeta_i} : Q(\zeta_i, z_i) \mapsto Q(\partial_{z_i}, z_i)|_{z_i = 0}$. If $\zeta s = \{\zeta_i\}_{i=1,\cdots,m}$, then we then define $\langle \rangle_{\zeta s} = \langle \cdots \langle \cdot \rangle_{\zeta_1} \cdots \rangle_{\zeta_m}$.

This function is called the ZIP-function, and we may refer to applying the ZIP function as zipping, or more general with the verb to zip. In this case, it is usually

clear from the context on which variables the ZIP-function is applied. Observe that the pairing $\langle , \rangle$ is symmetric in the definition of V and V^*, so we can choose to substitute $b \in B$ or $b^* \in B^*$ with a partial derivative. It is not clear that the definition is independent of the order in which the zipping occurs. This will be proven later in this section.
Consider the ring $P[B][[B^*]]$, which is isomorphic to the subspace $S(V) \otimes \hat{S}(V^*)$. An element $w \in P[B][[B^*]]$ can be written as the limit of a sequence $\{w_n\}$ of elements $w_n \in P[B][B^*]$. In this way, the definition of $\langle , \rangle$ can be extended to $P[B][[B^*]]$. Note that the extension of the ZIP-function to $P[B][[B^*]]$ is not always well-defined.
For example we have for a polynomial $P(x^*, x) = \sum a_{nm} x^{*n} x^m$ the formula

$$\langle P(x^*, x) \rangle_x = \sum n! a_{nn},$$

by simple calculation. This shows that $\langle e^{zz^*} \rangle_z$ diverges as R_ϵ is equipped with the discrete topology in $R_\epsilon[[h]]$. Observe furthermore that we cannot complete both $P[B]$ and $P[B^*]$, since for example $e^{\partial_x \partial_y} e^{xy}$ does not converge. Since R_ϵ has the discrete topology, $\langle e^{\partial_x} e^x \rangle_x$ does not converge either as a series in R_ϵ.
However, $R_\epsilon[[h]]$ is completed in the h-adic topology. So $\langle e^{\partial_x} e^{hx} \rangle_x = e^h$ does converge in the h-adic topology. This trick allows us to extend $\langle \rangle$ to exponentials of the form $e^{z_i + z_i^* + z_j z_i^*} \in P[B][[B^*]]$, while assuring convergence. While $\langle e^{z_i + z_i^* + z_i z_i^*} \rangle$ does not converge in $P[B][[B^*]]$, $\langle e^{hz_i + hz_i^* + hz_i z_i^*} \rangle$ does converge in $P[B][[B^*]]$. The convergence follows in essence by the observation that

$$\langle e^{hzz^*} \rangle = e^{h \partial_{z^* z^*}} = \sum_n \frac{h^n}{n!} \partial_x^n x^n = \sum_n \frac{h^n n!}{n!} = \frac{1}{1-h}.$$

More generally, we introduce formal parameters $\eta_i, y_i, \Theta_{ij} \in P[[B']]$, where $i, j \in I$, the index set labeling elements in B. B' is some finite set, and $P[[B']]$ equipped with the the J'-adic topology, where $J' = (B')$ is an ideal generated by the elements $J' = (z'_1, \cdots, z'_m)$ of B'. For convergence we require that $\eta_i, y_i, \Theta_{ij} \in (z'_1, \cdots, z'_m, h) \subset P[[B']]$. In practice, η_i,y_i and Θ_{ij} will be other variables, part of a bigger space on which we can zip later. We now prove convergence of the ZIP-function for a specific domain.

Proposition 2.2.1. *Consider the ring $P[[B', B^*]][B]$, where $B = \{z_1, \cdots, z_n\}$ and B^* its dual, and B' is some finite set. $P[[B']]$ is equipped with the (B')-adic topology, where (B') is the ideal generated by the elements of B'.Let (h, B') be the ideal generated by B' and h. Consider the subspace of exponentials*

$$\begin{aligned} W_B = &\{Q(B, B^*) \exp[c + \sum \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] | Q \in P[B', B, B^*], \\ &c, \eta_1, \cdots, \eta_n, y_1, \cdots, y_n, \Theta_{11}, \cdots, \Theta_{nn} \in (h, B') \subset P[[B']], \ det(1 - \Theta) \neq 0\}. \end{aligned}$$

Summation over indices appearing twice is assumed. Extend $\langle \rangle$ linearly in $P[[B']]$ to the ring $P[[B', B^*]][B]$. Let $w \in W_B$, and let $\zeta s = \{z_1^*, \cdots, z_m^*\} \subset B^*$. Then $\langle w \rangle_{\zeta s}$ is

convergent in $P[[B', B^*]][B]$.

Proof. Let $w = Q(B, B^*)\exp[c + \sum \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \in W_B$. Without loss of generality we can take $\zeta s = B$, and $Q = 1$, since we can obtain Q by differentiating the exponential a finite number of times to y_i and η_i. We will prove the proposition by induction to the number of elements of B. We have to prove that after one zip, the result is a power series in W_B. Then the result will follow by induction. Let us specify an order when zipping to B for now, as we do not know yet if zipping is independent of the order of zipping.
The case where $B = \{z\}$ will be proven in a seperate lemma. Let us state the result here. The idea is to reduce the general case to the case where $B = \{z\}$. Let us zip to $z_n^* \in \zeta s = \{z_1^*, \cdots, z_n^*\} = B^*$, so we substitute $z_n^* \mapsto \partial_{z_n}$. The one dimensional result is given by

$$\langle \exp[c + \eta z + y z^* + \Theta z^* z] \rangle_{\{z\}} = \det((1-\Theta)^{-1}) \langle \exp[c + \eta(1-\Theta)^{-1}(z+y)] \rangle_{\{z\}}.$$

The last expression is clearly convergent in $P[[B', B^*]][B]$. We now proceed with the general case, where sums over i and j are from 1 to n, unless stated otherwise. Let us take $c = 0$ without loss of generality.

$$\begin{aligned}
&\langle \exp[\sum_{i,j} \eta_i z_i + y_j z_j^* + \Theta_{ji} z_j^* z_i] \rangle_{z_n} = \\
&\langle exp[\sum_{i<n} \eta_i z_i + \sum_{j<n} y_j z_j^* + \sum_{i,j<n} \Theta_{ij} z_j^* z_i + (\eta_n + \sum_{j<n} \Theta_{nj} z_j^*) z_n \\
&+ (y_n + \sum_{i<n} \Theta_{in} z_i) z_n^* + \Theta_{nn} z_n^* z_n] \rangle_{z_n}.
\end{aligned}$$

Since we only zip to z_n^*, the variables z_i^*, $i < n$ are unaffected by the zipping. Hence we can take $c = \sum_{i<n} \eta_i z_i + \sum_{j<n} y_j z_j^* + \sum_{i,j<n} \Theta_{ji} z_j^* z_i$, $\eta = \eta_n + \sum_{j<n} \Theta_{nj} z_j^*$, $y = y_n + \sum_{i<n} \Theta_{in} z_i$, and $\Theta = \Theta_{nn}$ and apply the one dimensional formula. This yields

$$\langle \exp[\sum_{i,j} \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_n} = \frac{1}{1-\Theta_{nn}} \langle exp[c + \frac{1}{1-\Theta_{nn}} \eta(z_n + y)] \rangle_{z_n}.$$

To check that the result is an element of W_B, we note that Θ_{ij} is of a positive degree in $P[[B']]$, and likewise for y_n and η_n. Furthermore,

$$\frac{1}{1-\Theta_{nn}} \eta y = \frac{1}{1-\Theta_{nn}} (\sum_{i,j<n} \Theta_{jn}\Theta_{ni} z_j^* z_i + \eta_n y_n + \eta_n (\sum_{i<n} \Theta_{in} z_i) + y_n (\sum_{j<n} \Theta_{nj} z_j^*)).$$

As a coefficient of $z_j^* z_i$ we now get $(\Theta_{nj}\Theta_{in} + \Theta_{ij})$. Since $\Theta_{ij} \in (B', h) \neq P[[B']]$, and the ideal does not include 1 since none of the generators is invertible in $P[[B']]$, we know that the matrix $\mathbf{1} - \Theta_{ij} - \Theta_{jn}\Theta_{ni}$ is invertible. Note that $\mathbf{1} - \Theta$ is invertible for any matrix Θ with $\Theta_{ij} \in (B', h) \subset P[[B']]$. We conclude that the

result is indeed an element of W_B. This concludes the proof of the proposition. □

From now on, we restrict ourselfves to the space of formal power series specified in the proposition. We state a formula for the case where $B = \{z\}$. The derivatives in this section should be interpreted as formal derivatives. This interpretation makes sense as long as the variables are part of a formal power series ring. In W_B, the parameters and the elements of B only occur in the exponentials in pairs or triples. This means that when taking a derivative of the exponential to either an element of B or a parameter, the result will be an element of the formal series ring. We use this property in the proofs in this section.

Lemma 2.2.1. *Consider the ring $P[[B', B^*]][B]$, where $B = \{z\}$ and B^* its dual, and B' is some finite set. $P[[B']]$ is equipped with the (B')-adic topology, where (B') is the ideal generated by the elements of B'. Let (h, B') be the ideal generated by B' and h. Consider the subspace of exponentials*

$$\begin{aligned} W_z =&\{Q(z,z^*)\exp[c+\sum \eta z + yz^* + \Theta z^* z] | Q \in P[B', z, z^*], \\ &c, \eta, y, \Theta \in (h, B') \subset P[[B']], (1-\Theta) \neq 0\}. \end{aligned}$$

Let $w \in W_z$. Then

$$\begin{aligned} &\langle Q(z,z^*)\exp[c+\eta z + yz^* + \Theta z^* z]\rangle_{\{z^*\}} = \\ &\qquad (1-\Theta)^{-1}\langle Q((1-\Theta)^{-1}(z+y), z^*)\exp[c+\eta(1-\Theta)^{-1}(z+y)]\rangle_{\{z^*\}}. \end{aligned}$$

Proof. Let $w = \exp[\sum \eta z + yz^* + \Theta z^* z] \in W_z$. The proof of the lemma is by explicitly expanding $\langle w \rangle_{z^*}$. By differentiating to η, and $z \mapsto \partial_\eta$, we obtain the case where Q is not equal to 1. We can put $c = 0$ as zipping acts trivially on e^c. We now proceed with the actual proof.

$$\begin{aligned} \langle \exp[\eta z + yz^* + \Theta z^* z]\rangle_{\{z^*\}} &= \sum_{k,l,m} \frac{y^m \eta^l \Theta^k}{l!m!k!} \partial_z^{m+k} z^{l+k}|_{z=0} \\ &= \sum_{l,m} \frac{y^m \eta^l}{l!m!} \partial_z^m z^l \sum_k \binom{k+l}{k} \Theta^k|_{z=0} \\ &= \sum_{l,m} \frac{y^m \eta^l}{l!m!} \partial_z^m z^l (\frac{1}{1-\Theta})^{l+1}|_{z=0} \\ &= \frac{1}{1-\Theta} \sum_{p,m} \frac{y^m}{(p+m)!m!} \partial_z^m (\frac{\eta z}{1-\Theta})^{p+m}|_{z=0} \\ &= \frac{1}{1-\Theta} \sum_m \frac{y^m}{m!} (\frac{\eta}{1-\Theta})^m \sum_p \frac{(\frac{\eta z}{1-\Theta})^p}{p!}|_{z=0} \\ &= \frac{1}{1-\Theta} \langle \exp[(1-\Theta)^{-1}\eta(z+y)]\rangle_{z^*}. \end{aligned}$$

This finishes the proof. □

Consider an Hopf algebra A over $R_\epsilon[[h]]$ and a finite set I. As noted in the introduction, $A \cong P[B]$, which is completed in the h-adi topology. One can generalize the zip function to a finite tensor product $P[[B', B^*]][B]^{\otimes I} = P[[B'_I, B^*_I]][B_I]$, for a finite index set I and the completed tensor product. Since I is finite, convergence is clear by the previous proposition. In fact it is essentially identical to the case where we extend B. Only the map $\mathbb{O}$ is more complicated, as one has to keep track of the multiple tensor factors by introducing additional indices. This has been covered in the previous section.
For simplicity we will not write the tensor products explicitly, as $P[[B]] \otimes P[[B']] \cong P[[B, B']]$ when B and B' are finite and the tensor product is completed. We label elementary elements in $B^{\otimes I}$ with an index $i \in I$ to indicate the tensor factor on which they live. This is not to be confused with the labeling used earlier for the elements of B. Implicit is the isomorphism between $S(V)^{\otimes I}$ and $R_\epsilon[[h]][(z_j)_i | i \in I, (z_j)_i \in B_i]$. In practice we will denote the topological generators of A without an index, but as $B = \{X, Y, Z, A, B, b, a, z, y, x\}$ in the case of $U_q(sl_3^\epsilon)$.
We proceed with the general case of the previous lemma, which is what is used to calculate ${}^{\mathbf{t}}(G \circ F)$. We aim to calculate the ZIP-function explicitly, so we will restrict the definition of $\langle\rangle$ to the space where we can explicitly calculate the concatination ${}^{\mathbf{t}}F // {}^{\mathbf{t}}G = {}^{\mathbf{t}}(G \circ F)$ of two tensors, as we did in the case where $I = \{1\}$. The method we use to calculate explicit zippings on W_B is called the zipping formula. The theorem is true for any finite set B, given the conditions on W_B, independently of the underlying Hopf algebra.

Theorem 2.2.1. *(The zipping-formula) Let $B = \{z_1, \cdots, z_n\}$, and its dual B^*. Let*

$$Q(z_1^*, \cdots, z_n^*, z_1, \cdots, z_n) \exp[c + \sum_{i,j} (\eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i)] \in W_B.$$

Then

$$\begin{aligned} &\langle Q(z_1^*, \cdots, z_n^*, z_1, \cdots, z_n) \exp[c + \sum_{i,j} \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{B^*} \\ &= det(\tilde{\Theta}) \langle Q(z_1^*, \cdots, z_n^*, \sum_k (\tilde{\Theta}_{1k}(z_k + y_k)), \cdots, \sum_k (\tilde{\Theta}_{nk}(z_k + y_k))) \\ &\quad \exp[c + \sum_{i,k} \eta_i \tilde{\Theta}_{ik}(z_k + y_k)] \rangle_{B^*}. \end{aligned} \tag{2.1}$$

Denote by $\tilde{\Theta}$ the inverse of the matrix $1 - \Theta$. Sums are from 1 to n unless stated otherwise.

Proof. Let $w = Q \exp[\cdots] \in W_B$. Since Q is only a polynomial of finite degree, we can reduce the theorem to the case where $Q = 1$. Like before, we also can take $c = 0$ without loss of generality.

Before attempting the general case, let us prove the simpler case

$$\langle Q(z_j^*, z_i) \exp[\eta_i z_i + y_j z_j^*] \rangle_{z_j} = \langle Q(z_j^*, (z_i + y_i)) \exp[\eta_i (z_i + y_i)] \rangle_{z_j}.$$

We will prove this formula by replacing y_i by $\hbar y_i$. Introducing $\hbar$ yields

$$\langle Q(z_j^*, z_i) \exp[\eta_i z_i + \hbar y_j z_j^*] \rangle_{z_j} = \langle Q(z_j^*, (z_i + \hbar y_i)) \exp[\eta_i (z_i + \hbar y_i)] \rangle_{z_j}. \quad (2.2)$$

Evaluating at $\hbar = 0$ gives an identity that is obviously true. We will now prove that the identity holds when both sides are differentiated with respect to $\hbar$. Integrating to $\hbar$ will then prove the identity.
First, for any power series $Q(z)$, the operator $e^{\hbar y \partial_z}$ has the effect of shifting the argument of Q by $\hbar y$:

$$e^{\hbar y \partial_z} Q(z) = Q(z + \hbar y). \quad (2.3)$$

One proves the statement for any polynomial of finite degree by a straightforward induction argument on the degree. Then the statement is also clear for power series.
Now we can prove 2.2. Evaluating the left and right hand side of 2.2 at $\hbar = 0$, the equality holds clearly, as observed before. Let us differentiate the left hand side with respect to $\hbar$:

$$\begin{aligned} \partial_\hbar \langle Q(z_j^*, z_i) \exp[\eta_i z_i + \hbar y_j z_j^*] \rangle_{z_j} &= \langle y_j z_j^* Q(z_j^*, z_i) \exp[\eta_i z_i + \hbar y_j z_j^*] \rangle_{z_j} && (2.4) \\ &= \langle y_j z_j^* Q(z_j^*, z_i + \hbar y_i) \exp[\eta_i (z_i + \hbar y_i)] \rangle_{z_j} && (2.5) \\ &= \partial_\hbar \langle Q(z_j^*, z_i + \hbar y_i) \exp[\eta_i (z_i + \hbar y_i)] \rangle_{z_j}. && (2.6) \end{aligned}$$

Where we use the definition of the ZIP-function in the second and third equality, and 2.3 in the second equality. If we integrate 2.4 and 2.6 with respect to $\hbar$ we get 2.2, since we already know that 2.2 holds at $\hbar = 0$.
Let us proceed with proving 2.1. We will prove it by differentiating to Θ_{ij}, and show that both sides obey the same differential equiations

$$\partial_{\Theta_{ij}} \Psi = \partial_{\eta_i} \partial_{y_j} \Psi, \quad (2.7)$$

and that both sides agree on $\Theta_{ij} = 0$. If we can prove this, then both sides are actually the same, since this set of differential equations fully determine a solution. To see this, note that Ψ is a power series in $\Theta_{i,j}$ and η_i and y_j, so that both the right and the left hand side is a power series. The set of equation becomes a set of ordinary differential equations in the variables $\Theta_{i,j}$.
One can also try to prove a more general shift lemma, however to prove this lemma is much more cumbersome. It is insightful to prove the general shift lemma for the case where Θ is a 1×1 or a 2×2 matrix and P is a degree 1 polynomial. This gives a nice insight in where the determinant originates. The 1-dimensional case has been done in the previous lemma. The higher dimensional

calculations are left as an exercise for the curious reader.
Firstly, for $\Theta_{ij} = 0$, 2.1 reduces to 2.2, and hence holds true. Let Θ_{ij} be nonzero such that the matrix $1 - \Theta$ is invertible. We obtain on the left hand side

$$\partial_{\Theta_{ij}} \langle Q(z_j^*, z_i) \exp[c + \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_j} \tag{2.8}$$
$$= \langle z_i z_j^* Q(z_j^*, z_i) \exp[c + \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_j}$$
$$= \partial_{\eta_i} \partial_{y_j} \langle Q(z_j^*, z_i) \exp[c + \eta_i z_i + y_j z_j^* + \Theta_{ij} z_j^* z_i] \rangle_{z_j}. \tag{2.9}$$

Let $A_{ij} = (1 - \Theta)_{ij}$ be an invertible matrix. Since $A_{ij}^{-1} = \frac{A_{ji}^{adj}}{det(A)}$, $\partial_{\Theta_{ij}}(det(A)) I_{nn} = \partial_{\Theta_{ij}}(A_{nk} A_{nk}^{adj}) = \partial_{\Theta_{ij}}(A_{nk}) A_{nk}^{adj} = -I_{in} I_{jk} A_{nk}^{adj} = -I_{in} A_{nj}^{adj} = -A_{ij}^{adj}$, where A^{adj} is the adjugate matrix, or the matrix with the determinants of the minors of A (A_{ij}^{adj} is the determinant of the minor of A that arises by deleting the i-th row and the j-th column and multiplying with a sign). We use the einstein summation convention if the same index appears twice in different matrices. However, I_{nn} means the n, n-th entry of the identity matrix. Hence, if A_{nk} is dependent on Θ_{ij}, then A_{nk}^{adj} is independent of Θ_{ij}, and so by the product rule, $\partial_{\Theta_{ij}}(det(A)) = -A_{ij}^{adj}$.

Using this identity, on the right hand side of 2.1 we get using the product rule,

$$\partial_{\Theta_{ij}}(\det((1-\Theta))^{-1}\cdot \tag{2.10}$$
$$\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))\exp[c+\eta_{i'}((1-\Theta)^{-1})_{ik}(z_k+y_k)]\rangle_{z_j})$$
$$=\partial_{\Theta_{ij}}(\det((1-\Theta))^{-1})\cdot$$
$$\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))\exp[c+\eta_i((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j})$$
$$+\frac{\partial_{\Theta_{ij}}((1-\Theta)^{-1}_{i'k})}{\det((1-\Theta))}\langle(z_k+y_k)\partial_{x_{i'}}(Q(z_j^*,x_{i'}))|_{x_{i'}=((1-\Theta)^{-1})_{i'k}(z_k+y_k)}$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}+$$
$$\frac{\partial_{\Theta_{ij}}((1-\Theta)^{-1}_{i'k})}{\det((1-\Theta))}\langle\eta_{i'}(z_k+y_k)Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}$$
$$=\frac{(1-\Theta)^{-1}_{ij}}{\det(1-\Theta)}\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j} \tag{2.11}$$
$$+\frac{(1-\Theta)^{-1}_{ik}(1-\Theta)^{-1}_{i'j}}{\det((1-\Theta))}\langle(z_k+y_k)\partial_{x_{i'}}(Q(z_j^*,x_{i'}))|_{x_{i'}=((1-\Theta)^{-1})_{i'k}(z_k+y_k)}$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}+$$
$$\frac{(1-\Theta)^{-1}_{ik}(1-\Theta)^{-1}_{i'j}}{\det((1-\Theta))}\langle\eta_{i'}(z_k+y_k)Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k))$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}$$
$$=\frac{1}{\det((1-\Theta))}\partial_{\eta_i}\partial_{y_j}\langle Q(z_j^*,((1-\Theta)^{-1})_{i'k}(z_k+y_k)) \tag{2.12}$$
$$\exp[c+\eta_{i'}((1-\Theta)^{-1})_{i'k}(z_k+y_k)]\rangle_{z_j}.$$

To calculate the derivative of $(1-\Theta)^{-1}$ we used the identity $A^{-1}_{ij}=\frac{A^{adj}_{ji}}{det(A)}$. The third equality follows from the product rule. This finishes the proof of 2.1. □

The theorem implies that the ZIP-function as defined above is actually independent of the order in which we zip, so we can define the ZIP function for any set $\zeta s \subset B^*$ as substituting and differentiating to the corresponding elements of B.

Corollary 2.2.1. *Let B be a finite set of generators of a Hopf algebra A as above. Let W_B be the corresponding space of Gaussian exponentials in $P[[B',B^*]][B]$ for some finite set B'. Let $\zeta s \subset B^*$, and let $w \in W_B$. The ZIP-function $\langle w\rangle_{\zeta s}$ is independend of the order on ζs.*

We consistently chose to zip on elements of B^* in this section, and completed B^*. This was only a convention, as noted before. One might equally well choose to

substitute partial derivatives for elements of B, and complete the ring of power series in elements of B. When zipping to one set of variables this does not make a difference. However, when one chooses to zip to elements of B' afterwards for example, convergence can be an issue. One has to make sure that the exponentials that are being zipped on are convergent in the appropriate formal power series ring. Especially in maps on more than one copy of the Hopf algebra, this choice might make a difference.
In the next chapter we will have to specify which variables are completed as formal power series. This will be indicated by the phrase 'zipping to variables $\cdots$', where the polynomials in the variables $\cdots$ are then completed as power series. It is then a matter of checking that the ZIP-function is defined in each case. This mostly boils down to verification that the matrix Θ is invertible. Whenever this will come up, it will be clearly stated.
Let B and B' be finite as before, and let $F \in hom(P[B], P[B'])$. Let ${}^{\mathfrak{t}}F = \phi(F)$ as defined in the previous section. Define the map $\psi : im(\phi) \subset P[[B^*]][B'] \to Hom(P[B], P[B'])$ by sending ${}^{\mathfrak{t}}F \in im(\phi)$ to the map $\psi(F) : z \mapsto \langle zF \rangle_{B^*}$ where $z \in P[B]$. Since ${}^{\mathfrak{t}}F \in im(\phi) \subset P[[B^*]][B']$ is of the form $e^{\sum_i z_i^* F(z_i)}$, we can apply the zip-formula to see that $\langle z^{\mathfrak{t}}F \rangle_B$ converges when $z \in B$. Note here that $F(z_i) \in P[B']$, and that we only zip on the elements of B. When $z \in P[B]$ convergence is only clear when $z = Q \exp[c + \sum_i y_i z_i] \in P[B]$, where Q is a finite polynomial. In this case $z^{\mathfrak{t}}F \in W_B$, and we can use the above theorem.
More generally, when $F : A \to A^{\otimes I}$ is a map of algebras, ϕ is defined in the same way. Convergence follows in the same way as the elementary case where $F : A \to A$. The most complicated case is when $F : A^{\otimes I} \to A$. There is no way to define a universal inverse of ϕ, since it depends on the conventions chosen in $\mathbb{O}_I$ which indices have to be zipped on. The process works in the same way, but one has to consider the specific case at hand. This is what one needs to keep track of when working with tensors in practice. For the following propositions we restrict ourselves to the elementary case where $F : A \to A$ for simplicity. In this case we label the generators of the image of F differently, and denote the basis of A with B' instead of B. This is to make sure that we do not zip on the wrong set of generators.

Proposition 2.2.2. *The composition $\psi \circ \phi : Hom(P[B], P[B']) \to Hom(P[B], P[B'])$ is the identity.*

Proof. Both ψ and ϕ are continuous functions in the h-adic topology. The first because the pairing is continuous, and the second map is applying the homomorphism, which is clearly continuous. From the zipping formula and the definition of the zip formula, we can write down the following formula for monomials of generators. Since the composition $\psi \circ \phi$ is continuous, and the ZIP function is linear, we can then take the limit, so the formula will apply to elements in the completion. One can apply the ZIP function on each generator z_i sperately, and since F is a homomorphism, without loss of generality we can prove the formula for monomials of the form z_i^n, where $z_i \in B$. Note further that $F(z_i) \in P[B']$ is

independent of $z_i \in B$.

$$\begin{aligned}\psi(\phi(F))(z_i^n) &= \langle z_i^n \exp[\sum_j z_j^* F(z_j)]\rangle_{B^*} \\ &= (\frac{F(z_i)^n}{n!}(\partial_{z_i})^n z_i^n)|_{z_i=0} \\ &= F(z_i)^n = F(z_i^n).\end{aligned}$$

□

The following result follows by similar reasoning.

Proposition 2.2.3. *(composition law) Let $F : P[B] \to P[B']$ and $G : P[B'] \to P[B'']$ be homomorphisms, and let the sets B,B' and B'' be finite. Then whenever $\langle \phi(F)\phi(G)\rangle_{B'^*}$ is convergent,*

$$\langle \phi(F)\phi(G)\rangle_{B'^*} = \phi(G \circ F). \tag{2.13}$$

Proof. Because the ZIP function is a linear function on monomials of generators, we can take the limit of a sequence of monomials to obtain the result in the completion in the h-adic topology. Let J be the set of indices of the elements in B', and I the index set for B. Since F and G are functions on power series, we can take without loss of generality $F(z_k) = \prod_{j\in J}(z'_j)^{n_j}$, as we can write a general homomorphism as a sum of these elementary functions. An essential observation here is that we can indeed take the limit when considering infinite power series such as the exponential.
Let us take $F(z_i) = f_i(z'_i)^{m_i}$, where f_i is a constant, for all $i \in I$. The general case follows in the same way, one only has to pick out the right term in the exponential. We can write $\phi(F) = {}^{\mathbf{t}}F = \exp[\sum_{i\in I} z_i^* f_i (z'_i)_i^m]$, and similarly ${}^{\mathbf{t}}G = \exp[\sum_{i\in I}(z'_i)^* G(z'_i)]$. Since we can zip on one element of B' at the time, it is enough to check

$$\langle((z_i^*)^n F(z_i)^n)^{\mathbf{t}}G\rangle_{(z'_i)^*} = (z_i^*)^n G \circ F(z_i^n).$$

This follows by a simple calculation.

$$\begin{aligned}\langle(z_i^*)^n F(z_i)^n \phi(G)\rangle_{(z'_i)^*} &= \langle(z_i^*)^n F(z_i)^n \exp[\sum_j z_j^* G(z_j)]\rangle_{(z'_i)^*} \\ &= \frac{G((z'_i)^{m_i n})(\partial_{z'_i})^{nm_i}}{(m_i n)!}(z_i^*)^n f_i^n (z'_i)^{nm_i}|_{z'_i=0} \\ &= (z_i^*)^n G(z_i)^{nm_i} f_i^n \\ &= (z_i^*)^n G(F(z_i^n)).\end{aligned}$$

This finishes the proof. □

We should note that when composing maps, one has to be careful with conver-

gence of the zip-function. The composition of tensors does not always converge. In particular, the spaces of Gaussian exponentials do not form a category. Whenever care should be taken, we will clearly mention this. In most cases that we encounter it is clear that the composition of tensors converges. We will however have to prove that ${}^tF \in W_B$ explicitly in these cases. When ${}^tF \in W_B$, it is clear that composing tensors converges. This tactic will be used in the next section. We cover a simple example of this method here, to introduce the reader to the general setting in the next chapter. This example continues the examples of the previous section.
We write ${}^tF_I^J //{}^tG_J^K$ for the composition of tensors ${}^tF$ and ${}^tG$. When writing $//$, it is implied that zipping takes place on the input J of G and the output J of F. Convergence will have to be proven in this case, and the zipping takes place on the corresponding elements of B_I. The variables on which is being zipped are left implicit, and will be clear from the tensors that are composed.

Example 2.2.1. *Consider the set $B = \{x, y, z\}$ and its dual $B^* = \{x^*, y^*, z^*\}$, and let $\mathbb{R}[[h]][[B_1^*, B_2^*]][B_1, B_2, B_3]$ be the polynomial ring in variables*

$$\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$$

over the formal power series in an indeterminate h and the variables of B_1^*, B_2^*. In this ring consider the element ${}^tZ = \mathbb{E}[x_1^* y_2^* z_3]$, where we introduce the notation $e^x = \mathbb{E}[x]$. In general ${}^tZ$ occurs in a sl_3-like multiplication tensor such as the following (simplified) tensor

$${}^tm_{1,2}^3 = \mathbb{E}[x_3({x^*}_1 + {x^*}_2) + y_3({y^*}_1 + {y^*}_2) + z_3({x^*}_1 {y^*}_2 + {z^*}_1 + {z^*}_2)].$$

Let $\zeta s = \{x_1^*, y_2^*\}$, naively. We wish to perform the following zip

$$\langle (x_1 y_2)\mathbb{E}[z_3 {x^*}_1 {y^*}_2]\rangle_{\zeta s}.$$

However ${}^tZ$ is not of the form $e^{\Theta z z^*}$, so we cannot apply the zipping theorem to this expression at once. We are forced to zip to x^* and y^* separately. First we calculate

$$\begin{aligned}\langle (x_1 y_2)\mathbb{E}[z_3 {x^*}_1 {y^*}_2]\rangle_{x_1^*} &= (x_1 y_2)\mathbb{E}[z_3 \partial_{x_1} {y^*}_2]|_{x_1=0} \\ &= (x_1 + y_2^* z_3) y_2|_{x_1=0} \\ &= y_2^* z_3 y_2.\end{aligned}$$

Zipping to y_2^* now yields z_3. In this calculation we used the zipping theorem on the second line. For bigger expressions this takes a lot of computations. With a smart choice of the zipping variables we might be able to zip a cubic term like ${}^tZ$ at once, using the zipping theorem.
Observe that ${}^tZ \in \mathbb{R}[[h]][[y_1, y_2, y_3, x_1^*, x_2^*, z_1^*, z_2^*, z_3]][y_1^*, y_2^*, x_1, x_2, x_3, z_1, z_2]$. In this ring, let us take $\zeta s = \{y_2, x_1^*\} \subset \{y_1, y_2, y_3, x_1^*, x_2^*, z_1^*, z_2^*\}$. Now ${}^tZ$ is of the form $e^{\sum \Theta_{ij} z_j^* z_i}$. Concretely, we can take $\Theta_{12} = z_3$, and all other entries of Θ zero. Note that

$x_1^* \in \zeta s$ plays the role of the 'starred' variable and y_2^* plays the role of the 'z'.
Now $1-\Theta = \begin{pmatrix} 1 & -z_3 \\ 0 & 1 \end{pmatrix}$, with inverse $(1-\Theta)^{-1} = \begin{pmatrix} 1 & z_3 \\ 0 & 1 \end{pmatrix}$ and determinant 1. We can apply the zipping theorem to the formula, where we have to substitute $\sum_k (1-\Theta)^{-1}_{1k}(x_k) = (x_1 + y_2^* z_3)$ for x_1 in the polynomial in front of the exponential. This yields

$$\begin{aligned} \langle (x_1 y_2) \mathbb{E}[z_3 {x^*}_1 {y^*}_2] \rangle_{\zeta s} &= 1 \cdot \langle ((x_1 + z_3 y_2^*) y_2) \rangle_{\zeta s} \\ &= z_3 \end{aligned}$$

We are now ready to calculate more complicated examples, such as the multiplication of e^x and e^y by using ${}^t m$ and zipping $e^{x_1+y_2} {}^t m$ to $\zeta s = \{y_2, x_1^*\}$. This is now a simple application of the zipping theorem. This is left to the curious reader.

There is not much difference between the two ways of zipping in this example. However, when the multiplication becomes more difficult, when we wish to implement the R-matrix for example, it does matter much what choice of variables we make. There will be only a few correct choices in starred and unstarred variables to make that will lead to convergent expressions. Moreover, as we illustrated in the first half of the example, sometimes one can only zip in steps.
In the next section we will encounter a combination of these problems. As the number of variables is increased, we will make a specific choice to which variables we zip, and we will have to zip to those variables in three steps. This way of zipping is referred to as the three-way or three-stage zip. It is this method that will enable us to implement the $U_q(sl_3^\epsilon)$ Hopf algebra in Mathematica. One must not forget the essential role ϵ plays here. ϵ cuts of higher order terms that would occur in the exponential otherwise. This allows us to use the zipping theorem in calculating these zips.

2.3. The tensor formalism for $U_q(sl_3^\epsilon)$

In this section we give the explicit tensors for the $U_q(sl_3^\epsilon)$ quantum group, and prove that the formulas we use in the implementation in Mathematica are correct. We have proven that the $U_q(b^-)$ is a Hopf algebra in the previous chapter. In the previous section we have proven the essential zip-theorem. In the implementation of this theorem in Mathematica, we check the axioms for the tensors associated with the $U_q(b^-)$ algebra. We prove in this section that the given tensors indeed correspond to the $U_q(b^-)$ algebra.
The tensors and algebra relations are calculated by implementing the Drinfel'd-double construction in the tensor formalism. We define the tensors up to order ϵ. We can obtain the order ϵ^k term from the usual $U_q(b^-) \subset U_q(sl_n)$ QUE algebra. See chapter 1 and chapter 4.
We denote $T \in W_B$, for $B = \{X, Y, Z, A, B, b, a, z, y, x\}$ as $\mathbb{E}[L, Q, P]$, where P is the term proportional to ϵ, which is finite, since we work over $R_\epsilon[[h]]$. L is the ϵ independent part that only depends on A, B, b and a. Q is the ϵ independent part

of T in the exponential that is not in L.
Instead of h, we may use $\hbar$. To simplify the notation, we introduce $\mathbb{A} = exp[-hA]$, and similarly for B. For the explicit tensor ${}^{\mathbf{t}}dm_{ij}^{k}$ for the multiplication in the quantum double we refer to the appendix.
We should warn the reader that the starred variables in this section correspond to the dual variables of elements of $U_q(sl_3^\epsilon)$, as to indicate on which entry a tensor ${}^{\mathbf{t}}F$ takes its input and output. This should not be confused with a choice for the set of variables on which we zip.
In this section we use the three stage ZIP function. We first zip to the variables $\{A^*, B^*, b, a\}$, then to $\{X, Y^*, y, x^*\}$ and at last to $\{Z^*, z\}$. In this section we prove that this choice, together with the stated expressions leads to a convergent implementation of $U_q(sl_3^\epsilon)$ in the tensor-formalism.

Theorem 2.3.1. *The quantum double multiplication dm of the Hopf algebra $D(U_q(b^-))$, where $b^- \subset sl_3^\epsilon$ is the lower borel sub Lie bialgebra, can be constructed as a tensor ${}^{\mathbf{t}}dm$ by composing ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$, ${}^{\mathbf{t}}\overline{\mathcal{R}}$ and ${}^{\mathbf{t}}P$ as*

$$\begin{aligned}
{}^{\mathbf{t}}dm_{i,j}^{k} = (\mathbb{E}_{\{i,j\}}^{\{i,j\}}[&a_j a^*{}_j + A_i A^*{}_i + b_j b^*{}_j + B_i B^*{}_i, \\
&X_i X^*{}_i + Y_i Y^*{}_i + Z_i Z^*{}_i + x_j x^*{}_j + y_j y^*{}_j + z_j z^*{}_j, 1] \\
&\left({}^{\mathbf{t}}a\Delta_i^{\{1,2\}} //{}^{\mathbf{t}}a\Delta_2^{\{2,3\}} //{}^{\mathbf{t}}\overline{aS}_3\right)\left({}^{\mathbf{t}}b\Delta_j^{\{-1,-2\}} //{}^{\mathbf{t}}b\Delta_{-2}^{\{-2,-3\}}\right)) \\
&// \left({}^{\mathbf{t}}P_{-1,3}{}^{\mathbf{t}}P_{-3,1}{}^{\mathbf{t}}am_{2,j}^{k}{}^{\mathbf{t}}bm_{i,-2}^{k}\right).
\end{aligned}$$

Moreover, the composition of tensors converges. In particular, the tensor ${}^{\mathbf{t}}am$ corresponds to multiplication on $U_q(b^-)$, ${}^{\mathbf{t}}bm$ corresponds to multiplication on $U_q(b^-)^*$, ${}^{\mathbf{t}}\mathcal{R}$ corresponds to the R-matrix of $U_q(sl_3^\epsilon)$, ${}^{\mathbf{t}}\overline{\mathcal{R}}$ to its inverse and ${}^{\mathbf{t}}P$ to the pairing between $U_q(b^-)$ and $U_q(b^-)^*$. ${}^{\mathbf{t}}\overline{aS}$ is the inverse of the antipode on $U_q(b^-)$ and ${}^{\mathbf{t}}a\Delta$ and ${}^{\mathbf{t}}b\Delta$ are the comultiplication on $U_q(b^-)$ and $U_q(b^+)$ respectively.

${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t P$ have the following explicit form

$$
\begin{aligned}
{}^t am_{ij}^k &= \mathbb{E}\left[a_k\left(a^*{}_i + a^*{}_j\right) + b_k\left(b^*{}_i + b^*{}_j\right),\right.\\
&\quad x_k\left(e^{2a^*{}_j - b^*{}_j}x^*{}_i + x^*{}_j\right) + y_k\left(e^{-a^*{}_j + 2b^*{}_j}y^*{}_i + y^*{}_j\right) +\\
&\quad z_k\left(e^{2a^*{}_j - b^*{}_j}x^*{}_i y^*{}_j + e^{a^*{}_j + b^*{}_j}z^*{}_i + z^*{}_j\right),\\
&\quad 1 - e^{-a^*{}_j - b^*{}_j}\hbar\left(e^{3a^*{}_j}x_k y_k x^*{}_i y^*{}_j + e^{2a^*{}_j + 2b^*{}_j}y_k z_k x^*{}_i y^*{}_i y^*{}_j + e^{3a^*{}_j}y_k z_k x^*{}_i\left(y^*\right)_j^2 -\right.\\
&\quad \left.\left.e^{3a^*{}_j}x_k z_k x^*{}_i z^*{}_j + e^{3b^*{}_j}y_k z_k y^*{}_i z^*{}_j - e^{3a^*{}_j}z_k^2 x^*{}_i y^*{}_j z^*{}_j\right)\epsilon + O[\epsilon]^2\right]\\
{}^t bm_{ij}^k &= \mathbb{E}\left[A_k A^*{}_i + A_k A^*{}_j + B_k B^*{}_i + B_k B^*{}_j,\right.\\
&\quad X_k X^*{}_i + X_k X^*{}_j + Y_k Y^*{}_i + Y_k Y^*{}_j + Z_k Z^*{}_i + Z_k Z^*{}_j,\\
&\quad 1 + \left(-X_k A^*{}_i X^*{}_j + \hbar X_k Y_k X^*{}_j Y^*{}_i - Y_k B^*{}_i Y^*{}_j + 2Z_k X^*{}_i Y^*{}_j - \hbar X_k Z_k X^*{}_j Z^*{}_i +\right.\\
&\quad \left.\left.\hbar Y_k Z_k Y^*{}_j Z^*{}_i - Z_k A^*{}_i Z^*{}_j - Z_k B^*{}_i Z^*{}_j\right)\epsilon + O[\epsilon]^2\right]\\
{}^t\mathcal{R}_{ij} &= \mathbb{E}\left[\hbar a_j A_i + \hbar b_j B_i, \hbar x_j X_i + \hbar y_j Y_i + \hbar z_j Z_i,\right.\\
&\quad \left.1 + \frac{1}{2}\left(\hbar^3 x_j^2 X_i^2 + \hbar^3 y_j^2 Y_i^2 + \hbar^3 z_j^2 Z_i^2\right)\epsilon + O[\epsilon]^2\right]\\
{}^t\overline{\mathcal{R}}_{ij} &= \mathbb{E}\left[-\hbar a_j A_i - \hbar b_j B_i, -\hbar x_j X_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{-\hbar} + \hbar^2 X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} -\right.\\
&\quad \hbar y_j Y_i \mathbb{A}_i^{-\hbar}\mathbb{B}_i^{2\hbar},\\
&\quad 1 +\\
&\quad \left(-2\hbar^4 x_j X_i^2 Y_i z_j \mathbb{A}_i^{3\hbar} + \frac{3}{2}\hbar^3 x_j^2 X_i^2 \mathbb{A}_i^{4\hbar}\mathbb{B}_i^{-2\hbar} - \hbar^2 a_j x_j X_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{-\hbar} -\right.\\
&\quad 2\hbar^3 x_j X_i y_j Y_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar^3 X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + \hbar^3 a_j X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} +\\
&\quad \hbar^3 b_j X_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + 2\hbar^2 x_j y_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} + 2\hbar^2 z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar^2 a_j z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} -\\
&\quad \hbar^2 b_j z_j Z_i \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar^2 b_j y_j Y_i \mathbb{A}_i^{-\hbar}\mathbb{B}_i^{2\hbar} + 2\hbar^5 X_i^2 Y_i^2 z_j^2 \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} -\\
&\quad 2\hbar^4 X_i Y_i z_j^2 Z_i \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} + \frac{3}{2}\hbar^3 z_j^2 Z_i^2 \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{2\hbar} - \hbar^4 X_i y_j Y_i^2 z_j \mathbb{B}_i^{3\hbar} +\\
&\quad \left.\left.\frac{3}{2}\hbar^3 y_j^2 Y_i^2 \mathbb{A}_i^{-2\hbar}\mathbb{B}_i^{4\hbar}\right)\epsilon + O[\epsilon]^2\right]\\
{}^t P^{ij} &= \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{a^*{}_j A^*{}_i}{\hbar} + \frac{b^*{}_j B^*{}_i}{\hbar}, \frac{x^*{}_j X^*{}_i}{\hbar} + \frac{y^*{}_j Y^*{}_i}{\hbar} + \frac{z^*{}_j Z^*{}_i}{\hbar},\right.\\
&\quad \left.1 + \left(-\frac{\left(x^*\right)_j^2\left(X^*\right)_i^2}{2\hbar} - \frac{\left(y^*\right)_j^2\left(Y^*\right)_i^2}{2\hbar} - \frac{\left(z^*\right)_j^2\left(Z^*\right)_i^2}{2\hbar}\right)\epsilon + O[\epsilon]^2\right].
\end{aligned}
$$

The proof of this theorem consists of multiple steps, and will be the subject of this section. We prove that the comutiplication and the antipode of the quantum double can be obtained from the multiplication on $U_q(b^-)$ and $U_q(b^-)^*$, the R-matrix $\mathcal{R}$, its inverse $\mathcal{R}^{-1}$ (also denoted as $\overline{\mathcal{R}}$ to avoid confusion in the indices)

and the pairing P in proposition 2.3.1.
We then provide an iterative formula (in orders of ϵ) to obtain the pairing and the inverse R-matrix for any order of ϵ, in propositions 2.3.2 and 2.3.3. Note that although we work modulo ϵ^2, we will provide proofs for general order of ϵ, whenever possible.
For the R-matrix we have an explicit formula due to the Faddeev-Quesne formula, which was proven in the first chapter. We then check that the stated tensors correspond to the pullback of the Hopf algebra maps, and that zipping on the tensors converges. In particular proving that zipping on the pairing converges is an important step in the proof of theorem 2.3.1.
Convergence of the above stated zip follows by explicitly checking that the tensors are of the correct format. When we have proven that the zips converge in the necessary compositions of tensors, we use the composition lemma in the previous section to compose tensors.
We start with providing formulas for the comultiplication and the antipode. In lemma, we will provide the corresponding tensor identities, and prove that the zipping of these identities indeed converges. This way we obtain ${}^{t}a\Delta$, ${}^{t}b\Delta$, ${}^{t}aS$ and ${}^{t}bS$. The inverse of the antipode on $U_q(b^-)$ is computed in the same way as the inverse R-matrix in a seperate proposition.

Proposition 2.3.1. *(Comultiplication and antipode) Let $u \in U_q(b^-)$, $v \in U_q(b^+)$ and $w \in U_q(sl_3^\epsilon)$. It is understood that $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ can be written without summation sign. The following relations hold for $U_q(sl_n^\epsilon)$*

$$\begin{aligned}
S_a(u) &= \overline{\mathcal{R}}^{(2)} \langle \overline{\mathcal{R}}^{(1)}, u \rangle, \\
S_b(v) &= \mathcal{R}^{(1)} \langle v, aS(\mathcal{R}^{(2)}) \rangle \\
\Delta(u)_a &= \mathcal{R}^{(2)} \otimes \mathcal{R}^{(2)} \langle \mathcal{R}^{(1)} \mathcal{R}^{(1)}, u \rangle \\
\Delta(v)_b &= \mathcal{R}^{(1)} \otimes \mathcal{R}^{(1)} \langle v, \mathcal{R}^{(2)} \mathcal{R}^{(2)} \rangle
\end{aligned}$$

Proof. Let S be the antipode in the double $U_q(sl_3^\epsilon)$. For the first identity, we have $\overline{\mathcal{R}}_{1,2} = \overline{S} \otimes Id(\mathcal{R})$. This result follows from the double construction with the opposite comultiplication and the quasitriangularity axioms. Now we get $\overline{\mathcal{R}}^{(2)} \langle \overline{\mathcal{R}}^{(1)}, u \rangle = \mathcal{R}^{(2)} \langle bS(\mathcal{R}^{(1)}), u \rangle$. $S \otimes S(\mathcal{R}) = \mathcal{R}$, so

$$\mathcal{R}^{(2)} \langle bS(\mathcal{R}^{(1)}), u \rangle = aS(\mathcal{R}^{(2)}) \langle \overline{bS}(bS(\mathcal{R}^{(1)})), u \rangle = aS(\mathcal{R}^{(2)} \langle \mathcal{R}^{(1)}, u \rangle.$$

Since $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ are the sums of dual bases with respect to the pairing by construction of the R-matrix, we get the desired result. The results for bS follows similarly.
For the formulas for $a\Delta$ and $b\Delta$ we use the quasitriangular property of the comultiplication Δ of $U_q(sl_3^\epsilon)$ $\Delta(\mathcal{R}^{(1)}) \otimes \mathcal{R}(2) = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(1)} \otimes (\mathcal{R}^{(2)} \mathcal{R}(2))$. Applying $\langle \cdot, u \rangle$ on both sides to the third factor yields the desired result. Again we use that

the R-matrix is constructed by summing over dual bases of $U_q(b^\pm)$. □

The inverse R-matrix, the pairing and the inverse of the antipode aS can be recursively calculated from the zeroth order operation in ϵ, the R-matrix, the multiplication and the antipode on both sides of the double. In the case of the pairing there is also a recursive formula, which we will prove here.
For a generator $u \in U_q(b^\pm)$, define Δ_k as the expansion of Δ in ϵ up to and including the k-th order term. If $v = u_1 \cdots u_n$ is a product of generators, not necessarily in the PBW ordering, define $\Delta_k(v) = \Delta_k(u_1) \cdots \Delta_k(u_n)$. Define $\Delta_k^{(n)}$ similarly ($\Delta^{(n)}$ stands for Δ applied n times).
Remember that $\langle UV, u'v' \rangle = \langle \Delta(UV), u' \otimes v' \rangle$. Define the k-th order pairing as

$$\begin{aligned}\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_k &= \langle \Delta_k^{(n)}(U_1 \cdots U_n), u_1 \otimes \cdots \otimes u_n \rangle \\ &= \langle U_1 \otimes \cdots \otimes U_n, \Delta_k^{(n)}(u_1 \cdots u_n) \rangle.\end{aligned}$$

For $n \neq n'$, $\langle U_1 \cdots U_{n'}, u_1 \cdots u_n \rangle = 0$ by duality. Further observe that

$$\langle \Delta^{(n)}(U_1 \cdots U_n), \Delta^{(n)}(u_1 \cdots u_n) \rangle = \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle,$$

as there is only one term in $m^{(n)}_{1,\cdots n}(\Delta^{(n)}(U_1 \cdots U_n))$ that has the correct order.

Proposition 2.3.2. *Let $U_i \in U_q(b^+)$ and $u_i \in U_q(b^-)$ for $i = 1, \cdots n$ and where we take ϵ invertible for the moment (See chapter 1 or 4). Then*

$$\begin{aligned}&\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle \ mod\ \epsilon^{k+1} \\ =&\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_0 + \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_{k-1} \\ &- \langle U_1 \cdots U_n, \mathcal{R}^{(2)} \rangle_0 \langle \mathcal{R}^{(1)}, u_1 \cdots u_n \rangle_{k-1} \ mod\ \epsilon^{k+1}.\end{aligned}$$

Proof. If we prove that $\langle U_1 \cdots U_n, \mathcal{R}^{(2)} \rangle_0 \langle \mathcal{R}^{(1)}, u_1 \cdots u_n \rangle_{k-1}$ equals

$$\langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_0 + \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_k - \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle_{k-1},$$

then we are finished. To prove this, observe that since $\mathcal{R}$ is the summation of dual basis, i.e. it is the canonical R-matrix, we have $\langle \cdot, \cdot \rangle = \langle \cdot, \mathcal{R}^{(2)} \rangle \langle \mathcal{R}^{(1)}, \cdot \rangle = \langle \Delta^{(n)}(\cdot), \mathcal{R}^{(2)}_{\otimes n} \rangle \langle \mathcal{R}^{(1)}_{\otimes n}, \Delta^{(n)}(\cdot) \rangle$, for any order of k. Here $V_{\otimes n}$ stands for the expression V split into n tensor factors. By coassociativity this is well-defined inside the pairing.
To prove the formula, we look at the expression

$$\begin{aligned}&\langle U_1 \cdots U_n, \mathcal{R}^{(2)} \rangle_0 \langle \mathcal{R}^{(1)}, u_1 \cdots u_n \rangle_{k-1} \\ &= \langle \Delta_0^{(n)}(U_1 \cdots U_n), \mathcal{R}^{(2)}_{\otimes n} \rangle \langle \mathcal{R}^{(1)}_{\otimes n}, \Delta_{k-1}^{(n)}(u_1 \cdots u_n) \rangle = \langle \Delta_0^{(n)}(U_1 \cdots U_n), \Delta_{k-1}^{(n)}(u_1 \cdots u_n) \rangle.\end{aligned}$$

We notice that for $n = 1$ the relation holds trivially, since $\Delta_0(U_i) = \Delta(U_i)$, because in $U_q(b^+)$, there is no factor of ϵ present in Δ. For $n > 1$, we look at the

ordering of $\Delta_0^{(n)}(U_1 \cdots U_n) = \Delta^{(n)}(U_1 \cdots U_n)$ and we observe that the prefactor of this expression mod ϵ^k equals the inverse-prefactor of $\Delta_{k-1}^{(n)}(u_1 \cdots u_n) = \Delta_{k-1}^{(n)}(u_1 \cdots u_n)$ mod ϵ^k, since

$$\langle \Delta^{(n)}(U_1 \cdots U_n), \Delta_{k-1}^{(n)}(u_1 \cdots u_n) \rangle = \langle U_1 \cdots U_n, u_1 \cdots u_n \rangle \bmod \epsilon^k.$$

So we are finished if we can prove that the prefactor of the ϵ^k term equals minus the k-th term of the prefactor of $\Delta_k^{(n)}(u_1 \cdots u_n)$. This follows by induction. Assume this is true for $n-1$ generators. Adding one generator on both sides, which has to be dual on both sides, only is non-trivial in the case of $U_n = X,Y$ or Z. Since there is only one correct order to put them, calculating the prefactor of $\Delta(n)(U_1 \cdots U_n)$ (and similarly that of the u_i side) now follows by commuting exponentials q^H past a number of U_i in other tensorfactors. Now observe for any exponent p, where $q = e^{-\epsilon h}$, we have (denoting q_k^p for the expansion of q^p in ϵ up to and including the k-th order term) $q^{-p} q_{k-1}^p = -q_k^p + q_{k-1}^p$. To see this, observe that $q^{-1} q_{k-1} = -q_k + q_{k-1}$. This formula follows from the expression $\sum_{l=1}^{k} \frac{(-1)^{k-l}}{(k-l)!l!} = -\frac{(-1)^k}{k!}$, which follows from Newtons binomial expansion of $(1-1)^k = 0$. Furthermore observe that the commutation relations in $U_q(b^{\pm})$ between Cartan generators and X, Y, Z (respectively x, y, z) have a minus sign compared to each other. By induction the result follows. □

It is possible to prove this proposition more directly using the generating function of the pairing, which we will write as P_{ij}. When writing multiplication, we mean applying the zip function on the appropriate indices. One should take the following proof only as a suggestion, since we are not allowed to zip the R-matrix and the pairing on both indices.
The canonical R-matrix sums over the q-PBW basis of $U_q(b^-)$ and the dual basis as $\mathcal{R} = \sum f^a \otimes e_a$. In chapter 1 the R-matrix was constructed in such a way that the monomials are normalized to pair to one: $\langle f^a, e_a \rangle = 1$. Pairing each term in $\mathcal{R}$ then yields $\sum_{\mathbb{N}} 1$. This is remeniscent of certain quantum field theoretical techniques. The difference between the k-th order and the $k-1$-th order of $(P_{ij})(\mathcal{R}_{ij})$ then is not equal to 0 as the following proof suggests, but it is presumably equal to the zipping of the zeroth order in P_{ij}, to match the expression in the proposition.
The zipping in the zeroth order of P_{ij} is left unevaluated in the Mathematica implementation, since this would diverge for the reasons mentioned above. We could try to do the computation of $(P_{ij})(\mathcal{R}_{ij})$ here explicitly, but this would amount to the proof given above. For this reason the following proof is only a suggestion, meant to give an intuition for the result.

Proof. Suppose $(P_{ij})_{k-1} = a^{-1}$ is the generating function of the pairing up to the $k-1$ th order in h. Let $(\mathcal{R}_{ij}^{-1})_{k-1} = a$ be the R-matrix up to the $k-1$-th order in h.

Define the k-th order term of the R-matrix respectively the pairing as $x = (\mathcal{R}_{ij})_k$ and $y = (P_{ij})_k$. Then mod h^{k+1}, we have $1 = (a + x)(a^{-1} + y) = 1 + ay + xa^{-1}$, since we work mod h^{k+1}. So $y = -a^{-1}xa^{-1}$. Since x is divisable by h^k, $y = -a_0^{-1}xa_0^{-1}$, where a_0 stands for the zeroth order in h of a. □

For the following proposition, which proves an iterative formula for the inverse R-matrix, we use the technique of tensor generating functions legally, contrary to the previous 'proof'. The proof is taken from [36].

Proposition 2.3.3. *Define $Y_n = \mathcal{R}^{-1}$ mod h^{n+1}. Then $Y_{n+1} = 1 - \mathcal{R}_{ij}Y_n + Y_n$.*

Proof. The proof is by induction. Suppose $\mathcal{R}_{ij}^{-1} = Y_n$ mod h^{n+1}, and that $Y_{n+1} = Y_n + Zh^{n+1}$ mod h^{n+2}. Let $\mathcal{R}_{ij}Y_n = 1 + Eh^{n+1}$ mod h^{n+2}. We know that $\mathcal{R}_{ij} = 1$ mod h, so the zeroth order in h does not contribute to the product $\mathcal{R}_{ij}\mathcal{R}_{ij}^{-1} = 1 = \mathcal{R}_{ij}(Y_n + Zh^{n+1})$ mod $h^{n+2} = (E + Z)h^{n+1} + 1$ mod h^{n+2}. So we conclude that $Z = -E$, and $Y_{n+1} = Y_n - Eh^{n+1}$. This finishes the proof. Since ϵ occurs only together with h in the expression for the R-matrix, it is clear that the proposition also holds when working mod ϵ. □

We can now proceed with the implementation of the zipping formula, and proving that it converges for the specific implementation of the quantum double we use. This implementation uses the previous two propositions. See the appendix for the implementation.
In the Mathematica file we use a specific implementation of the zipping theorem, namely we zip in three stages. First we zip the Cartan Lie subalgebra, then we zip the simple generators x, y, Y, X and at last we zip the non-simple z and Z. We prove that this way of zipping called the three-stage zip converges in each step.

Theorem 2.3.2. *(Miracle-theorem) The three-stage zip converges for all finite combinations of the tensors ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t dm$.*

Proof. The perturbation is finite, so the zipping of these terms converges by the zipping theorem. To see that the zipping of exponentials without the perturbation converges we inspect the exponentials of ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t dm$ term by term. The only terms for which it is not clear that they yield a result in W_B for $B = \{X, Y^*, y, x^*\}$ or $B = \{Z^*, z\}$ are the cubic terms.
The only non-perturbative cubic term arises in the ${}^t am$ tensor and the ${}^t\overline{R}_{i,j}$ tensor. Both are essentially dual to each other. Note at first that in every non-perturbed exponential of $U_q(sl_3^\epsilon)$ the x and the y are only present with an X and an Y in front, respectively. Zipping the $x_i^* y_j^* z_k$ term to the variables x and y first yields terms like $CX_{i'}Y_{j'}$, for some i' and j', with a term C. Zipping $e^{x_i y_j z_k}$ is well-defined in the three-stage zip because we choose $B = \{X, Y^*, y, x^*\}$, and so $e^{x_i y_i z_k}$ is of the form $e^{bc^* z_k}$, $b, c \in B$.
Let us be more concrete, and look at the structure of the matrix Θ. We observe that with the choice $B = \{X, Y^*, y, x^*\}$, there are never any diagonal terms in Θ

when zipping any of the tensors ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\overline{\mathcal{R}}$ and ${}^t dm$. It is important to note here that in one application of the zipping theorem, we only zip any tensor on either its input or on its output. Moreover, remember that the lowercase generators on the input-entries are only present in the exponential with an uppercase generator in front. An exception is z, but this results in another cubic term, as mentioned before.
Observe that $1 - \Theta$ is always upper-diagonal (or lower-diagonal, depending on the conventions and assuming normal ordering). This follows since the lowercase multiplication tensor has a cubic term in the exponential, but the uppercase multiplication tensor has not. Applying the zipping theorem for an uppertriangular matrix $1 - \Theta$ with only 1's on the diagonal yields a prefactor of $det(1 - \Theta) = 1$, which is non-singular. As mentioned before, the other cubic term which is present in the inverse R-matrix has the same property that Θ is upper-diagonal.
When combining both terms, it is important to mention that the upper- and lowercase starred variables are chosen such that for example X is a 'starred' variable, while x is a non-starred variable. With this choice, $1 - \Theta$ remains of uppertriangular shape.
So far, we have not mentioned the cubic terms present in ${}^t dm$, but it can be seen that these terms are in fact equivalent to the above case by zipping once. In this case, we are again saved by the xX terms in the R-matrix, and the fact that these are the only occurances of the non-starred lowercase variables. This finishes the proof.

□

We have proven that zipping on the space spanned by the above tensors and all finite zips is well-defined. The subspace of W_B formed by these exponentials is closed under zipping.

Corollary 2.3.1. *If ${}^t T$ and ${}^t S$ are two tensors in the ring $R_\epsilon[z_1, \cdots, z_n][[h, z_1^*, \cdots, z*_n]]$ obtained from the tensors ${}^t am$, ${}^t bm$, ${}^t\mathcal{R}$, ${}^t\mathcal{R}^{-1}$ and ${}^t dm$ by a finite number of zips, then $ZIP_{B'}({}^t T {}^t S)$ converges.*

Proof. We know by the previous proposition that zipping converges on all tensors arising from finite zips of these tensors, this implies the above result. □

In the implementation of the quantum double, we need the pairing for the calculation of the multiplication in the double and for checking the pairing-axioms. It is also used in the computation of the antipode and the comultiplication. After the calculation of the double, the pairing is not used anymore.
To prove that the zipping of the pairing tensor with the appropriate tensors converges, we explicitly check convergence for these relations. The double multiplication is obtained from the multiplication on the the lower- and uppercase Borel subalgebra, together with the pairing and the (inverse-)R-matrix. In this case we will prove convergence as well.

To circumvent the troubles with the pairing, it turrns out to be possible to define the pairing from the R-matrix. See [36] for details. In particular theorem 8 of [36]. In this method, one implicitly defines the multiplication on the Drinfel'd double by proving the existence of a dual element. Although this this would be a way to circumvent the zipping of the pairing, it does not show that we may explicitly compute the double multiplication in our implementation. As this is what we are aiming for, we will explicitly prove convergence.

Proposition 2.3.4. *(Pairing-zip) Let ${}^{\mathbf{t}}P_{i,j}$ be the tensor corresponding to the pairing on the entries i and j. The zip of ${}^{\mathbf{t}}P_{i,j}$ along one of the indices i or j but not both i and j with the tensors ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$, ${}^{\mathbf{t}}\mathcal{R}^{-1}$ is well defined.*

Proof. We are zipping the pairing along the i or the j, but never along both indices. Both multiplication tensors have two input strands and one output strand. The input-variables correspond to starred variables, the output variables are the non-starred variables in our convention. In the notation of the tensors, the starred variables are not necessarily the variables on which is zipped. The non starred variables correspond to elements of $U_q(sl_3^\epsilon)$. The R-matrix and its inverse are elements of $U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$, where the exponent of the exponential is proportional to h.
Any term in the perturbation is finite and the zip of this term with ${}^{\mathbf{t}}P$ converges. Looking at non-perturbative $\epsilon = 0$ part, we note that we need to make a consistent choice for which variables to zip to. Observe that the pairing tensor does not contain non-perturbative cubic terms, so if we can make a consistent choice such that we only have $z_i z_j^*$ terms, we only need to prove that the given exponential is a power series in $R_\epsilon[[\frac{1}{h}]][[B', B^*]][B]$.
We know by the previous proposition that the zips excluding the pairing converge, and that a consistent choice is made so that cubical terms in the exponentials can also be zipped. To this end we choose $\{X, Y^*, y, x^*\}$ in our implementation, as noted before. the tensors ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$ or ${}^{\mathbf{t}}\mathcal{R}^{-1}$.
The convergence of a zip including the pairing and the tensor ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}\mathcal{R}$ or ${}^{\mathbf{t}}\mathcal{R}^{-1}$, is clear from the fact that we only zip on one of the two indices in ${}^{\mathbf{t}}P_{i,j}$. In the case of the multiplication, it is clear that the exponents are not proportional to h. In the case of the R-matrix we have to be more careful, as the exponents are proportional to h. However, when zipping the pairing and the R-matrix on one index, the result includes a term BB^* or bb^*, where $b, B \in B$, depending on which index is zipped. Independently of the choice of starred and non starred variables this is proportional to an element of the formal power series ring $R_\epsilon[[h]][[B^*]]$. So this choice is consistent with zipping the pairing on one index. This proves the proposition. □

Before providing the tensors for the comultiplication and the antipode, we have to know that we are working with the correct multiplication tensors.

Proposition 2.3.5. *${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ are the pullback under $\mathbb{O}$ of multiplication on $U_q(b^-)$ and $U_q(b^+)$ respectively.*

Proof. We verified both the associativity of the multiplication $am^k_{i,j}$ and $bm^k_{i,j}$ in $U_q(b^-)$ and $U_q(b^+)$ respectively. We verify the associativity of ${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ tensors separately in Mathematica (see appendix A.1). The proposition can be seen to be true for generators by explicitly using the zipping theorem in Mathematica. Note that $B = \{b, a, z, y, x\}$ for $U_q(b^-)$ and $B = \{X, Y, Z, A, B\}$ for $U_q(b^+)$. We get

$$\mathbb{O}_k(am^k_{i,j}) = {}^{\mathbf{t}}\mathrm{Id}_i\,{}^{\mathbf{t}}\mathrm{Id}_j // \mathbb{O}_{i,j} // am^k_{i,j},$$

and the same for $bm^k_{i,j}$. But this implies the result.
Note that it is enough to check that ${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ agree on generators. Since sl_3 has only one non-simple generator, z in our case, this follows directly from the relation $[x, y] = z$ and its q-equivalent. This finishes the proof. □

For higher n it is necessary perform the check on monomials of higher order. The same proposition is true for the R-matrix and its inverse.

Proposition 2.3.6. *${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\overline{\mathcal{R}}$ are the pullback under $\mathbb{O}$ of the R-matrix and its inverse of the quasitriangular Hopf algebra $D(U_q(b^-))$ respectively.*

Proof. The proof of the statement for the R-matrix is easy, as $\mathcal{R}$ is an element of $U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$, and by the Fadeev-Quesne formula proven in the previous chapter we obtain the given formula.
The expression for ${}^{\mathbf{t}}\overline{\mathcal{R}}$ can be obtained by applying the iterative formula for $Y_n = \mathcal{R}^{-1} \bmod \epsilon^{n+1}$, $Y_{n+1} = 1 - \mathcal{R}_{ij}Y_n + Y_n$. The $\epsilon^2 = 0$ case we are working with then corresponds to $Y_1 = 1 - \mathcal{R}_{ij}Y_0 + Y_0$. We claim that $Y_0 \in U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$ is given by

$$\begin{aligned} Y_0 = \mathbb{O}_{i,j}(\mathbb{E}^{\{i,j\}}[&-\hbar a_j A_i - \hbar b_j B_i, -\hbar \mathrm{X}_i x_j \mathbb{A}_i^{2\hbar}\mathbb{B}_i^{-\hbar} \\ &+ \hbar^2 \mathrm{X}_i Y_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar} - \hbar Y_i y_j \mathbb{A}_i^{-\hbar}\mathbb{B}_i^{2\hbar} - \hbar Z_i z_j \mathbb{A}_i^{\hbar}\mathbb{B}_i^{\hbar}]). \end{aligned}$$

Observe that in $\mathcal{R}_{ij}Y_0$, entry-wise multiplication is implied. Multiplication here is interpreted in the Hopf-algebraic sense. The multiplication of the R-matrix and Y_0 takes place entirely in $U_q(b^+) \otimes U_q(b^-) \subset U_q(sl_3^\epsilon) \otimes U_q(sl_3^\epsilon)$, as we constructed the Drinfel'd double with the opposite comultiplication. We already proved that the zipping of the corresponding tensors ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$ and ${}^{\mathbf{t}}\mathcal{R}$ converges. We now observe that $Y_0 \in W_B$ as a tensor. To see this, first observe that $\mathbb{A} = e^{-hA}$ and $1 - hA$ act as the same element of $U_q(b^+)$. So we can replace the two in the expression for $\mathbb{O}(Y_0)$, and consequently in the corresponding tensor Y_0. The terms seem like higher order products of generators. However, when we first zip to $\zeta s = \{A, B^*, b, a^*\}$, secondly to $\zeta s = \{X^*, Y, y^*, x\}$ and at last to $\zeta s = \{Z^*, z\}$ we see that in each stage Y_0 is of the appropriate form.
We check by zipping the tensors that Y_0 is the zeroth order in ϵ of the inverse R-matrix. We use the fact that ${}^{\mathbf{t}}bm$ and ${}^{\mathbf{t}}am$ are the pullback under $\mathbb{O}$ of the multiplication in $U_q(b^\pm$ respectively. It then follows from the implementation in Mathematica that ${}^{\mathbf{t}}\overline{\mathcal{R}}$ is indeed the pullback of $\overline{\mathcal{R}}$. This ends the proof. □

We can now construct the comultiplication and antipode tensors from the multiplication and R-matrix, and the pairing.

Proposition 2.3.7. *${}^t a\Delta$, ${}^t b\Delta$, ${}^t aS$ and ${}^t\overline{aS}$ are the tensors corrsponding to respectively the comultiplication on $U_q(b^-)$, $U_q(b^+)$, the antipode on $U_q(b^-)$ and the inverse of the antipode on $U_q(b^-)$. They can be constructed as a composition of tensors as follows.*

$$
\begin{aligned}
{}^t a\Delta_i^{j,k} &= ({}^t R_{1,j}\, {}^t R_{2,k}) // {}^t b m_{1,2}^3 // {}^t P_{3,i} \\
{}^t b\Delta_i^{j,k} &= \left({}^t R_{j,1}\, {}^t R_{k,2}\right) // \left({}^t a m_{1,2\to 3}\right) // {}^t P_{i,3} \\
{}^t\overline{aS}_i &= {}^t aS|_{\epsilon=0} - {}^t\overline{aS}_i|_{\epsilon=0} // {}^t aS_i // {}^t\overline{aS}_i|_{\epsilon=0} \\
{}^t aS_i &= \langle {}^t\overline{R}_{i,j}\, {}^t P_{i,j}\rangle_j,
\end{aligned}
$$

and have the following explicit form. We introduce the notation $\boldsymbol{a} = e^{-a^*}$, and similarly

for b.

$$
\begin{aligned}
{}^{t}a\Delta_i^{j,k} = \mathbb{E}_{\{i\}}^{\{j,k\}} \Big[& a_j a^*{}_i + a_k a^*{}_i + b_j b^*{}_i + b_k b^*{}_i, x_j x^*{}_i + x_k x^*{}_i + y_j y^*{}_i + y_k y^*{}_i + z_j z^*{}_i \\
& + z_k z^*{}_i, 1 + \Big(-\hbar a_j x_k x^*{}_i - \hbar x_j x_k \left(x^*\right)_i^2 - \hbar b_j y_k y^*{}_i + \hbar x_k y_j x^*{}_i y^*{}_i - \hbar y_j y_k \left(y^*\right)_i^2 + \\
& 2\hbar x_j y_k z^*{}_i - \hbar a_j z_k z^*{}_i - \hbar b_j z_k z^*{}_i - \hbar x_k z_j x^*{}_i z^*{}_i + \hbar y_k z_j y^*{}_i z^*{}_i - \hbar z_j z_k \left(z^*\right)_i^2 \Big) \epsilon \Big] \\
{}^{t}b\Delta_i^{j,k} = \mathbb{E}_{\{i\}}^{\{j,k\}} \Big[& A_j A^*{}_i + A_k A^*{}_i + B_j B^*{}_i + B_k B^*{}_i, \\
& X_k X^*{}_i + X_j \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} X^*{}_i + Y_k Y^*{}_i + Y_j \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} Y^*{}_i + Z_k Z^*{}_i + Z_j \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} Z^*{}_i + \\
& \hbar X_j Y_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} Z^*{}_i, \\
& 1 + \Big(-\hbar X_j X_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} \left(X^*\right)_i^2 - \hbar X_j Y_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} X^*{}_i Y^*{}_i - \hbar Y_j Y_k \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} \left(Y^*\right)_i^2 + \\
& \hbar X_j Z_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} X^*{}_i Z^*{}_i + \hbar^2 X_j^2 Y_k \mathbb{A}_k^{-4\hbar} \mathbb{B}_k^{2\hbar} X^*{}_i Z^*{}_i - \hbar Y_j Z_k \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} Y^*{}_i Z^*{}_i - \\
& \hbar^2 X_j Y_j Y_k \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} Y^*{}_i Z^*{}_i - \hbar^2 X_j Y_k Z_j \mathbb{A}_k^{-3\hbar} \left(Z^*\right)_i^2 - \hbar Z_j Z_k \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} \left(Z^*\right)_i^2 + \\
& \frac{1}{2} \hbar^3 X_j^2 Y_k^2 \mathbb{A}_k^{-4\hbar} \mathbb{B}_k^{2\hbar} \left(Z^*\right)_i^2 \Big) \epsilon + O[\epsilon]^2 \Big] \\
{}^{t}\overline{aS}_i = \mathbb{E}_{\{i\}}^{\{i\}} \Big[& -a_i a^*{}_i - b_i b^*{}_i, -e^{-2a^*{}_i + b^*{}_i} x_i x^*{}_i - e^{a^*{}_i - 2b^*{}_i} y_i y^*{}_i + e^{-a^*{}_i - b^*{}_i} z_i x^*{}_i y^*{}_i - \\
& e^{-a^*{}_i - b^*{}_i} z_i z^*{}_i, \\
& 1 + \Bigg(-\frac{2\hbar x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} - \frac{\hbar a_i x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} + \frac{\hbar x_i^2 \boldsymbol{b}_i^2 \left(x^*\right)_i^2}{\boldsymbol{a}_i^4} - \frac{2\hbar y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} - \frac{\hbar b_i y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} - \\
& \frac{2\hbar x_i y_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{3\hbar z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{\hbar a_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{\hbar b_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \\
& \frac{\hbar x_i z_i \left(x^*\right)_i^2 y^*{}_i}{\boldsymbol{a}_i^3} + \frac{\hbar y_i^2 \boldsymbol{a}_i^2 \left(y^*\right)_i^2}{\boldsymbol{b}_i^4} + \frac{\hbar z_i^2 \left(x^*\right)_i^2 \left(y^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{2\hbar x_i y_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \\
& \frac{2\hbar z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar a_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar b_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{2\hbar z_i^2 x^*{}_i y^*{}_i z^*{}_i}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{\hbar z_i^2 \left(z^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} \Bigg) \epsilon + O[\epsilon]^2 \Big] \\
{}^{t}aS_i = \mathbb{E}_{\{i\}}^{\{i\}} \Big[& -a_i a^*{}_i - b_i b^*{}_i, -\frac{x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} - \frac{y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} + \frac{z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i}, \\
& 1 + \\
& \Bigg(-\frac{\hbar a_i x_i \boldsymbol{b}_i x^*{}_i}{\boldsymbol{a}_i^2} + \frac{\hbar x_i^2 \boldsymbol{b}_i^2 \left(x^*\right)_i^2}{\boldsymbol{a}_i^4} - \frac{\hbar b_i y_i \boldsymbol{a}_i y^*{}_i}{\boldsymbol{b}_i^2} - \frac{2\hbar x_i y_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \\
& \frac{\hbar a_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{\hbar b_i z_i x^*{}_i y^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar x_i z_i \left(x^*\right)_i^2 y^*{}_i}{\boldsymbol{a}_i^3} + \frac{\hbar y_i^2 \boldsymbol{a}_i^2 \left(y^*\right)_i^2}{\boldsymbol{b}_i^4} + \\
& \frac{\hbar z_i^2 \left(x^*\right)_i^2 \left(y^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{2\hbar x_i y_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} + \frac{2\hbar z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar a_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \frac{\hbar b_i z_i z^*{}_i}{\boldsymbol{a}_i \boldsymbol{b}_i} - \\
& \frac{2\hbar z_i^2 x^*{}_i y^*{}_i z^*{}_i}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} + \frac{\hbar z_i^2 \left(z^*\right)_i^2}{\boldsymbol{a}_i^2 \boldsymbol{b}_i^2} \Bigg) \epsilon + O[\epsilon]^2 \Big] .
\end{aligned}
$$

Proof. For the tensors corresponding to the comultiplication, the pairing is only zipped on one index. By the previous proposition this is well-defined. As for the antipode, the same holds true. From the explicit form of ${}^{\mathbf{t}}aS$ we see that ${}^{\mathbf{t}}\overline{aS_i}|_{\epsilon=0} // {}^{\mathbf{t}}aS_i // {}^{\mathbf{t}}\overline{aS_i}|_{\epsilon=0}$ is well-defined, and that the zipping converges.
The fact that the given tensors are the pullback of the comultiplication and the (inverse) antipode follows by explicitly checking the Hopf algebra axioms in Mathematica. We refer to the implementation $U_q(sl_3^\epsilon)$ in the appendix for this calculation. We already proved that the multiplication tensors ${}^{\mathbf{t}}am$ and ${}^{\mathbf{t}}bm$ correspond to multiplication in the Hopf algebras $U_q(b^\pm)$. Hence it follows that the tensors given in this proposition must correspond to the comultiplication and antipode in $U_q(b^\pm)$, as they agree on generators. □

We now proceed with the pairing, the most problematic part of this section.

Proposition 2.3.8. *${}^{\mathbf{t}}P$ is the pullback under $\mathbb{O}$ of the pairing between $U_q(b^\pm)$.*

Proof. When pairing on generators, which is a straightforward application of the zipping formula, we see that the pairing agrees on generators, and makes $Z_i \in U_q(b^+)$ and $z_i \in U_q(b^-)$ dual.
What is left now is to prove that ${}^{\mathbf{t}}P$ obeys the pairing axioms. This means proving that the zips necessary to check the pairing axioms in the tensor formalism converge. Then we are finished, for the explicit check we refer to Mathematica.
The proof that the checking of the axioms converges in the tensor formalism boils down to proving that for the expressions in W_B for the appropriate set B have an invertible $1 - \Theta$. This is an issue here as we are dealing with a formal power series in $R_\epsilon[[\frac{1}{h}]][B]$, the inverse of h.
The check that the zipping formula converges is equivalent to checking if the prefactor in the final result is still dependent on $\frac{1}{h}$. This isequivalent to checking that the exponents of the tensors that the pairing is zipped with are not proportional to h, but to a constant in $\mathbb{R} \subset R_\epsilon$. When the result of an (intermediate) zip is not dependent on h, the power series are still well defined in $W_B \subset R_\epsilon[[\frac{1}{h}]][[B^*]][B]$ since the exponent consists of pairs zz^*, where $z^* \in B^*$. However, the prefactor Θ might yield an uninvertible matrix $1 - \Theta$ in the following zips. This leads to divergences.
To show that the result of the zip in both inputs of the pairing tensor ${}^{\mathbf{t}}P$ has invertible $1 - \Theta$, we do not have to zip, we only need to write down the tensors, which is a product in the formal power series ring. By the previous propositions we do have the explicit form of ${}^{\mathbf{t}}am$, ${}^{\mathbf{t}}bm$, ${}^{\mathbf{t}}a\Delta$, ${}^{\mathbf{t}}b\Delta$, ${}^{\mathbf{t}}aS$ and ${}^{\mathbf{t}}bS$ and the inverse antipodes. Moreover, we know that they are the pullback of the (co)multiplication and antipodes on both halves of the Drinfel'd double $U_q(sl_3^\epsilon)$, remembering that $U_q(b^+)$ has the opposite comultiplication. So if we prove convergence of the zip we know that ${}^{\mathbf{t}}$ is the pullback of the pairing, as it agrees on generators and it obeys the pairing axioms. In this last observation we use $\mathbb{O}$.
The pairing axioms that need to be checked are stated in the first chapter, and are

as follows

$$\begin{aligned}\langle ab, c\rangle &= \langle a \otimes b, \Delta c\rangle\\ \langle \Delta a, c \otimes d\rangle &= \langle a, cd\rangle\\ \langle 1, c\rangle &= \epsilon(c)\\ \langle a, 1\rangle &= \epsilon(a)\\ \langle Sa, c\rangle &= \langle a, Sc\rangle.\end{aligned}$$

We write down the corresponding tensor identity with the appropriate indices of the first relation $\langle ab, c\rangle = \langle a \otimes b, \Delta c\rangle$. The relations are checked in the Mathematica implementation. The check that the zip converges is then trivial, and consists of checking the tensors stated above on their dependence of h. The other relations are similar to the first one, and are left to the reader. The concrete expressions can be found in the Mathematica implementation in appendix A.1.
The terms we need to check are the cubic terms in ${}^{\mathbf{t}}b\Delta$ and ${}^{\mathbf{t}}am$ of the form $\hbar X_j Y_k \cdots$. Terms like this are zipped twice (or three times in the case of the term $\hbar^2 X_i Y_i z_j$ in $\mathcal{R}^{-1}$), and hence get a factor of $\frac{1}{h}^2$ and $\frac{1}{h}^3$ respectively from the pairing. We never encounter a term like $X_i X_j$ in the exponent, so there are always multiple terms in the pairing involved in zipping cubic terms.
For example for the term $\hbar^2 X_i Y_i z_j$ combines with the terms $\frac{1}{h} X_i^* x_j^*$ and $\frac{1}{h} Y_i^* y_j^*$ and $\frac{1}{h} Z_i^* z_j^*$ in the pairing to $\frac{1}{h} x_j^* y_j^* Z_i^*$. So we obtain a prefactor of $\frac{1}{h}$ in both cases. In particular, $1 - \Theta$ is invertible for a suitable choice of variables in the three-stage zip.
To check the pairing axioms we apply ${}^{\mathbf{t}}P$ to the identity tensor

$$\mathbb{E}_{\{3\}}^{\{3\}}\left[a_3 a^*{}_3 + b_3 b^*{}_3, x_3 x^*{}_3 + y_3 y^*{}_3 + z_3 z^*{}_3, 1\right]$$

. We obtain that zipping with ${}^{\mathbf{t}}$ obeys the pairing axioms for a general expression in $U_q(sl_3^\epsilon)$ by differentiating both sides, as usual.
We now give the first pairing axiom in tensor form.

$$\begin{aligned}&\langle {}^{\mathbf{t}}\mathrm{bm}_{1,2}^4 \mathbb{E}_{\{3\}}^{\{3\}}\left[a_3 a^*{}_3 + b_3 b^*{}_3, x_3 x^*{}_3 + y_3 y^*{}_3 + z_3 z^*{}_3, 1\right] {}^{\mathbf{t}}P_{4,3}\rangle_{3,4}\\ &= \langle {}^{\mathbf{t}}\mathrm{a}\Delta_3^{4,5} \mathbb{E}_{\{1\}}^{\{1\}}\left[A_1 A^*{}_1 + B_1 B^*{}_1, \mathrm{X}_1 \mathrm{X}^*{}_1 + Y_1 Y^*{}_1 + Z_1 Z^*{}_1, 1\right]\\ &\mathbb{E}_{\{2\}}^{\{2\}}\left[A_2 A^*{}_2 + B_2 B^*{}_2, \mathrm{X}_2 \mathrm{X}^*{}_2 + Y_2 Y^*{}_2 + Z_2 Z^*{}_2, 1\right] {}^{\mathbf{t}}P_{1,4} {}^{\mathbf{t}}P_{2,5}\rangle_{\{1,2,4,5\}}.\end{aligned}$$

The other expressions are left to the reader to check. This finishes the proof. □

We are now able to prove the main theorem of this chapter, theorem 2.3.1. The proof combines the lemmas and propositions in this section.

Proof. (Theorem 2.3.1) The main problem is to check convergence of the zipping of the pairing ${}^{\mathbf{t}}P$ on both indices in the formula stated in the theorem. To prove

convergence we once again look at the tensors that are taken as input. Since ${}^{\mathsf{t}}P$ is the pullback of the pairing, when the zip converges we immediately know that the corresponding tensor ${}^{\mathsf{t}}dm$ is the multiplication tensor on the Drinfel'd double.
By Drinfel'ds theorem, we know that the multiplication on $D(U_q(b^-))$ converges in the h-adic completion. This implies that the $\frac{1}{h}$ cancels. For the explicit proof we refer to [23] for example, and we invite the reader to check this for themselves. The observation is that the antipode plays an essential role in canceling the factors of $\frac{1}{h}$ by introducing a minus sign.
We are looking for a similar fact here. A straightforward way out of this situation is to calculate the explicit form of the input of the pairing in the tensor formalism, and check the h-dependence, to make sure that the exponents in both inputs are not proportional to h. We do not need to calculate this explicitly. We apply a similar tactic as the previous proposition. Let us consider the term

$$\left({}^{\mathsf{t}}\mathrm{a}\Delta_i^{\{1,2\}} /\!/ {}^{\mathsf{t}}\mathrm{a}\Delta_2^{\{2,3\}} /\!/ {}^{\mathsf{t}}\overline{\mathrm{aS}}_3\right)\left({}^{\mathsf{t}}\mathrm{b}\Delta_j^{\{-1,-2\}} /\!/ {}^{\mathsf{t}}\mathrm{b}\Delta_{-2}^{\{-2,-3\}}\right)$$

in the construction of ${}^{\mathsf{t}}dm$. We observe that the exponentials ${}^{\mathsf{t}}a\Delta$, ${}^{\mathsf{t}}b\Delta$ and ${}^{\mathsf{t}}\overline{aS}$ do not have a factor of h in the exponential apart from the terms involving more than one 'unstarred' variable. These are zipped on more than once in the three-way zip with the pairing, resulting in the appropriate factors of $\frac{1}{h}$. The essential observation here is that we never encounter a term like X_iX_j in the exponent. By applying the zipping formula we see that the resulting terms $\left({}^{\mathsf{t}}\mathrm{a}\Delta_i^{\{1,2\}} /\!/ {}^{\mathsf{t}}\mathrm{a}\Delta_2^{\{2,3\}} /\!/ {}^{\mathsf{t}}\overline{\mathrm{aS}}_3\right)\left({}^{\mathsf{t}}\mathrm{b}\Delta_j^{\{-1,-2\}} /\!/ {}^{\mathsf{t}}\mathrm{b}\Delta_{-2}^{\{-2,-3\}}\right)$ also do not have a factor of h in the exponentials. We provided the explicit form of the tensors corresponding to the (co)multiplication, antipode, pairing and the (inverse) R-matrix by calculations in Mathematica in appendix A.1. We also proved that these tensors correspond to the pullback under $\mathbb{O}$. This finishes the proof of the main theorem. □

As a corollary of the main theorem we obtain the multiplication relations of $U_q(sl_3^\epsilon)$ stated at the end of the first chapter. These relations are obtained in the Mathematica implementation by explicitly zipping ${}^{\mathsf{t}}dm$ to generators. See the appendix for details.

Conclusion

We defined and proved the convergence of a formalism that implements Feynmans trick. In other words, following [36], we calculated the commutator between exponentials in an indirect way using the zipping theorem. This formalism gives a correspondence between exponentials in the ring of formal power series and the Hopf algebra maps. The zipping theorem enables us to implement the quasitriangular Hopf algebra $U_q(sl_3^\epsilon)$ in an efficient way. We proved that that the given tensors correspond under to the multiplication maps in $U_q(b^\pm)$,

the pairing P and the R-matrix and its inverse. We proved furthermore that the quantum double $U_q(sl_3^\epsilon)$ can be constructed using only these maps. We noted that one should be careful when working with the pairing in the tensor formalism, as this can lead to divergence. When one restricts to zipping P on one index with any other tensor however, we proved that we are safe. When the pairing is zipped to on both indices one has to check convergence explicitly.
A way to circumvent this problem is by defining the pairing P indirectly by solving a differential equation. See [36] for the argument. We noted however that we still need to prove convergence of the zip function on single index zipping of P. The observation that P zipped on both indices with $\mathcal{R}$ gives a divergence means also that it is not directly possible to generalize the zipping formula to the zipping of quadratic exponentials. It might be possible to find such a formula for a subspace of these exponentials, but this is subject to future research.
A particular case of cubic zipping that is used in this chapter is the so called three-stage zip. By splitting the zipping of the variables X and Y the zipping of Z into two different stages, we managed to prove convergence for the zipping of certain exponentials associated to the (co)multiplication and antipode in the Hopf algebra $U_q(sl_3^\epsilon)$. In particular the cubic terms xyZ and XYz required a special choice of the variables on which was to be zipped. The main reason for the convergence of the three-stage zip on these terms is the fact that X and Y do not occur together in one term in $\mathcal{R}$ and P. In other words, no mixing occurs. It is interesting to see if we can define a specific subclass of tensors for which the three stage zip converges, or in general understand the zipping of cubic terms better.
It is possible that the cubic terms can be put in the perturbation P of a tensor in the case of certain solvable or nilpotent Lie algebras. A particular example of this is of course the factor of ϵ, which is a way to construct a nilpotent subalgebra. There might by a way to be able to zip cubic terms with losing slightly less information by introducing another such factor. In particular this will be helpful for reducing the computation time, but also for theoretical purposes it is valuable to understand the cubic terms.

3. A polynomial time sl_3-knot invariant

Introduction

In this chapter we explain how to turn the quasi-triangular Hopf algebra $U_q(sl_3^\epsilon)$ constructed in the first chapter into a knot invariant. We conclude with the proof that the constructed knot invariant can be calculated in polynomial time using the tensor formalism. The factorization of the knot invariant in Alexander polynomials in zeroth order of epsilon is proven.
For the general form of the invariant in the sl_2 case we refer to [35]. For $U_q(sl_3^\epsilon)$, it remains an important open question if we can find a general expression for the knot invariant. This means that one extracts the invariant part of the output of the calculation of the knot invariant. This greatly reduces the length of the polynomial for a knot K. This in turn will enable us to beter recognize the structure of the knot invariant. Another important factor is the potential reduction of the calculation time.

3.1. Knots diagrams and the Reidemeister theorem

We define the knot diagrams and we consider the usual embedddings of a knot K in $\mathbb{R}^3$. We restrict ourselves to the class of framed knots. For an unframed knot, one can always choose it to have writhe 0 and rotation number 0. If we use this normalization it is possible to choose a canonical snarl diagram, as defined in [35]. In this section we use the concept of framed knots, which enables us to define the rotation number. A knot diagram in our convention is an rv-tangle diagram in the language of [36]. We state the definition of (framed) knots.

Definition 3.1.1. *A knot K is an equivalence class of a continuous (C^∞) embedding of $i : S^1 \to \mathbb{R}^3$. The equivalence class on the space of continuous embeddings is defined by isotopies in $\mathbb{R}^3$.*

This definition uses C^∞-embeddings to exclude unrealistic posibilities such as wild knots from the space of knots. We will not go into details, see for example [4]. A knot is usually defined to be a piecewise linear embedding of the circle. This provides a way to formalize operations on knot diagrams on an elementary level. We skip this step and refer to [4] for an elementary treatment of the subject. Let $I = [0, 1]$ the unit interval. Instead of the embedding of S^1, one can embed $S^1 \times I$ into $\mathbb{R}^3$. Note that it is not essential to take the interval I as $[0, 1]$. Instead, one could also take this interval to be infinitesimally small. We wil consider I to be infinitesimally small, say $I = [0, \delta]$, for $\delta > 0$.

Definition 3.1.2. *A framed knot K is an equivalence class of a continuous embedding of $i : S^1 \times I \to \mathbb{R}^3$. The equivalence class on the space of continuous embeddings is defined by isotopies in $\mathbb{R}^3$.*

Let us define the framing of a framed knot.

Definition 3.1.3. *(Framing) Let K be a framed knot with boundary components $K^{\pm}$. The framing of K is defined as the linking number of the curves $K^{\pm}$.*

If one considers embeddings of S^1 (or $S^1 \times I$) with an orientation along S^1, one obtains an oriented knot. In this chapter we use framed oriented knots.

Definition 3.1.4. *Let $J = \{1, \cdots, n\}$ be a finite discrete index set, and fix $2n$ distinct points $x_i, y_i \in \mathbb{R}^3$. A link is the equivalence class under isotopies of a continuous embedding of the disjoint union $\phi : \amalg_{i \in J} I_i \to \mathbb{R}^3$ such that $x_i = \phi(0) \in \phi([0,1]_i)$ and $y_i = \phi(1) \in \phi([0,1]_i)$.*

Two knots or links K and K' are called isotopic if there exists a smooth family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0,1]$ such that h_0 is the identity on K and $h_1(K) = K'$.
The space of tangles will be most important in our discussion in this chapter, and is closely related to the concept of long knots. The following definition is taken from [25].

Definition 3.1.5. *(Tangle) An (m,n)-tangle is a compact 1-manifold properly embedded in $\mathbb{R} \times \mathbb{R} \times I$ such that the boundary of the embedded 1-manifold is a set of m disinct points in $\{0\} \times \mathbb{R} \times \{0\}$ and a set of n disinct points in $\{0\} \times \mathbb{R} \times \{1\}$. To (m,n)-tangles are said to be isotopic if there is an isotopy between the tangles that fixes the boundary points. A framed tangle is a tangle with a framing on each component, idem for an oriented tangle. A long knot is a $(1,1)$ tangle, where we exclude closed components.*

A knot can be obtained from a long knot by closing its endpoints. Conversely, we can obtain a long knot from a knot by cutting S^1 to obtain the interval I. This is independend from the cutting point of the knot. A knot invariant is an invariant if and only if it is an invariant of long knots, and the two invariants coincide when the knot is cut, or the long knot is closed respectively. See [18], for the context of classical knots, as we consider here.
More generally, it is not true that the closure of an (m,m) tangle without loops is in one to one correspondence with the links of m components. There are $n!$ options to close an (n,n)-tangle. This is the reason that it is usually easier to construct a knot invariant than a link invariant. In this chapter we construct an invariant of framed oriented long knots. See also paragraph X.5 of [17], or chapter 3 of [25] for more information.

Definition 3.1.6. *A pre-knot diagram is defined to be a finite oriented four-valent graph where each vertex is denoted as an over-crossing or an under crossing respectively. A*

Figure 3.1.: The possible crossings in a knot, up to rotation.

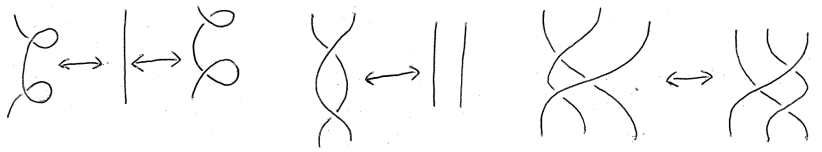

Figure 3.2.: The Reidemeister moves for a framed knot. From left to right the Reidemeister I, II and III move.

labeling of a vertex as an over- and undercrossing is a labeling of the vertex with a ± 1 *and the labeling of two opposite opposite edges ending at the vertex as the underpass. The remaining two edges are labeled as the overpass. We refer to two edges labeled as an over- or underpass as on the same strand, or as a strand.*

See figure 3.1 for the notation of an over and under-crossing.
The space of pre-knot diagrams will be subject to an equivalence relation. Two pre-knot diagrams are equivalent if they can be obtained from each other by applying a finite number of Reidemeister moves. The Reidemeister moves are depicted in figure 3.2. We are using framed knots in our theory, so the Reidemeister I move is different from the usual Reidemeister I move, in order to keep the rotation-number of the knot-diagram constant. We refer to the usual (unframed) RI move as the Reidemeister I' move. This move is depicted in figure 3.3.

Definition 3.1.7. *(knot diagrams) A knot diagram is defined as the equivalence class of a labeled pre-knot diagram under the Reidemeister moves. With labeled we mean each edge is* $\mathbb{N}$*-labeled. The number* $\mathbb{N}$ *of an edge E is referred to as the rotation number of E.*

There is a similar definition of tangle diagrams, but we consider tangles without framing, so there is no rotation number indicated on the strands. We fix the ending and starting points in $\mathbb{R}^2$ as the projections of $x_i, y_i \in \mathbb{R}^3$. Two distinct vertices in tangle diagrams are required to have a distinct height. The height of a vertex is defined through the second projection $\pi : \mathbb{R}^2 \to \mathbb{R}$. This will be implicit in our definition of tangle and knot diagrams. One can allways present a knot or tangle in this way, see for example [17].

Definition 3.1.8. *(tangle diagrams) A tangle diagram is defined as the equivalence class*

of a pre-knot diagram under the Reidemeister moves, where the Reidemeister I move is replaced by the Reidemeister I′ move.

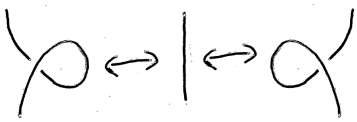

Figure 3.3.: The Reidemeister move I′

Given a framed knot K, we consider its projection on $\mathbb{R}^2$. We assume that the knot is embedded in such a way that no two crossings align with respect to the projection. The result can be presented as a knot diagram. For clarity, I is taken to be infinitesimally small. In this case, we denote the framing of the knot as an integer on the edges of the knot diagram of K, as the rotation number.
On the other hand, it is clear that a knot diagram F can be turned into a three dimensional framed knot $K(F)$. Given a knot diagram, we wish to prove two knot diagrams are equivalent if and only if the corresponding knots are equivalent. The Reidemeister theorem asserts this fact. A proof can be found in [25] for example. As a result of this theorem, we can work with knot diagrams instead of knots.
The same conventions hold in the case of a tangle, except that tangles do not have a framing, and hence there is no need to label the edges.

Theorem 3.1.1. *Two knot or tangle diagrams F and F′ are equivalent if and only if the knots (or tangles) corresponding to F and F′ are equivalent under isotopies.*

Let us be more clear about the about applying the Reidemeister moves to a knot (or tangle) diagram. Any knot diagram can be obtained from elementary tangle diagrams. These elementary diagrams are shown in figure 3.4. The fifth and sixth diagrams are refered to as caps, and the last two diagrams in figure 3.4 are referred to as cups. Since we consider finite diagrams, we can put an or-

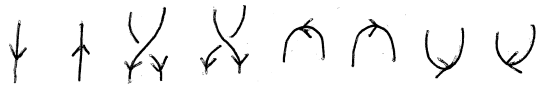

Figure 3.4.: The elementary tangles

dening on the nontrivial elementary tangle diagrams where a knot diagram K is constructed from. Concretely, we draw the diagrams in a way such that the crossings, cups and caps are ordered vertically. This way of drawing a knot diagram is referred to as a sliced knot diagram. See chapter 3 of [25]. On sliced diagrams, the Reidemeister moves take a slightly different form. These moves are referred to as the Turaev moves for oriented sliced diagrams, and are depicted in figure

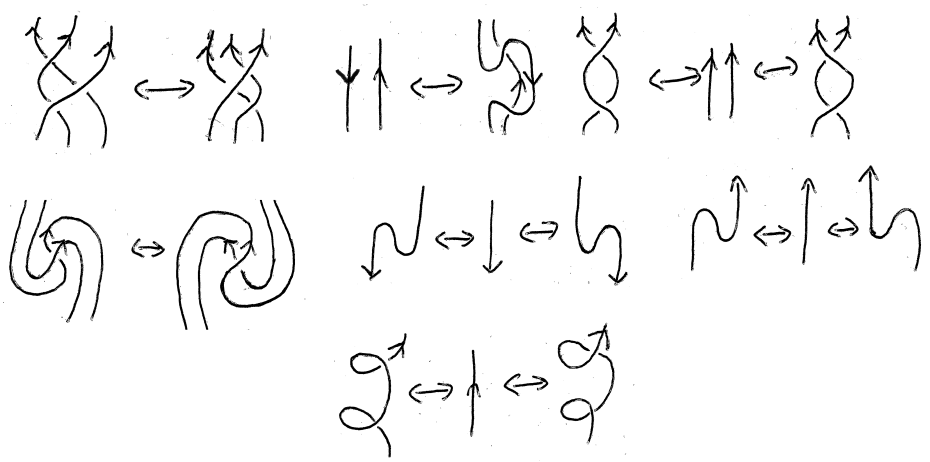

Figure 3.5.: The Turaev moves for a framed knot. The removal of trivial knots from the diagram is excluded from the pictures, but is formally a Turaev move. The last move is replaced by the Reidemeister I' move in case of a non-framed knot. The Turaev moves are numbered in left to right, downward order as $T-1, \cdots T-7$.

3.5. We have the following theorem for sliced diagrams. For the proof we refer to [25].

Theorem 3.1.2. *The knots are isotopic if and only if their corresponding two oriented sliced knot diagrams are equivalent under the Turaev moves.*

We wish to label the crossings in a knot-diagram with $i \in \{\pm 1\}$. Let K be an oriented framed long knot, so the diagram of K is oriented as a graph, where the crossings are seen as vertices. To determine the sign of the crossing, we use the convention of the left hand rule.
We draw a crossing as having a ninety degree angles between both strands. Place the left hand thumb on the upper strand in the direction of the orientation, with the palm of the hand pointed towards the paper. Align the index finger along the lower strand of the crossing. If the index finger points in the direction of the oriented strand, the crossing has sign $+1$, if it points in the opposite direction, it has sign -1. Crossings with sign $+1$ are called over-crossings, -1 crossings are refered to as under crossings. We can define the following.

Definition 3.1.9. *(Writhe) We define the writhe of an oriented knot diagram of a knot K as the difference between the number of over-crossings and the number of under-crossings. We denote the writhe of K by writhe(K).*

Lemma 3.1.1. *writhe(K) is well-defined for a framed knot.*

To prove this, we look at the admissable moves on knot diagrams, the Reidemeister moves. We observe that the number of positive and negative crossings is conserved under these moves for a framed knot.

Definition 3.1.10. *(mutants) Consider a knot diagram K, and consider a disc D in K such that there are exactly four edges crossing the boundary of the disc. Let T be the tangle in the disc D. Consider the operations $\mathcal{S}$ on the disc D where the four crossings (equally spaced on the circle, w.l.o.g.) are mapped to each other. By applying these operations to T we obtain a different tangle T'. A mutant of the knot K is a knot K' where the tangle T is replaced by a tangle T' obtained from T by any of the operations $\mathcal{S}$ on D. We call K' a mutation of K.*

Of course it does not matter which definition one takes in rotating the crossing. We are now in a position to define a knot invariant for a knot K.

Definition 3.1.11. *(knot invariant) Let S be a set. Let $Z(K) \in S$ be an expression in S corresponding to any knot K. Then Z(K) is called a knot invariant if for any two isotopic knots K and K', Z(K) = Z(K').*

Note that for a framed knot K, $writhe(K)$ is a knot invariant where $S = \mathbb{Z}$. Let us now define the invariant corresponding to $U_q(sl_3)$. Defining a knot invariant is equivalent to defining how to compute it. For the following considerations, we follow chapter 4 of [25]. For a more concise treatment of knot invariants coming from ribbon Hopf algebras we refer to this source, although there exist many other books that treat the subject as well.
A quasitriangular ribbon Hopf algebra A is equipped with an R-matrix $\mathcal{R}$, its inverse $\mathcal{R}^{-1}$, multiplication m, comultiplication Δ, unit $\mathbf{1}$ and counit ε. Note that ε is different from the parameter ϵ introduced in chapter 1. As introduced in chapter 1, we have $u = \sum S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}$, and $v = S(u)$. Since A is a ribbon Hopf algebra we have the square root ν of the element uv which is called the ribbon element.
We introduce the graphical calculus for a ribbon Hopf algebra A. Consider a framed oriented tangle diagram T. The graphical calculus is a way to denote operations in A. For a rigorous introduction of the graphical calculus, see [25]. The idea is to label the strands of T expressions in $A^{\otimes S}$, where each strand stands for a tensor factor. The concatination of strands as taking the product of the boxed quantities in the order of the orientation of the strand.
The labels of the strands are written in coupons. An (n, m)-coupon (or box) is a rectangle with n inputs and m outputs, and corresponds to a map $A^{\otimes\{1,\cdots,n\}} \to A^{\otimes\{1,\cdots,m\}}$. Multiplication with an element $w \in A^{\otimes S}$ is considered as a map $A^{\otimes S} \to A^{\otimes S}$, and is denoted as a coupon with $|S|$ in- and outputs. A strand with no coupon (or box) is the unit of A.

Definition 3.1.12. *An expression in the graphical calculus of A is a collection of (n, n)-coupons, where n may be any positive integer, where the in- and outputs of the coupon are connected by elementary oriented framed tangle diagrams. A graphical expression with n strands is referred to as an n-ribbon graph.*

Note that a ribbon graph without coupons is an oriented framed tangle diagram. An example is depicted in figure 3.6. The graphical calculus is a way to denote

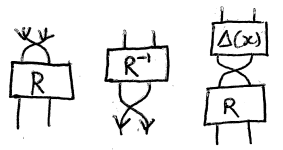

Figure 3.6.: Graphical calculus for A. From left to right: $\mathcal{R}$, $\mathcal{R}^{-1}$ and $\Delta^{op}(x)\mathcal{R}$.

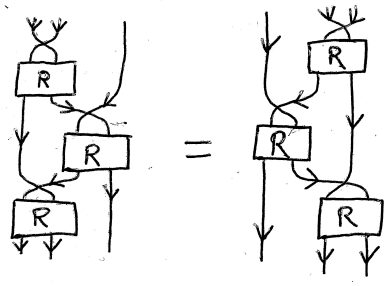

Figure 3.7.: Graphical Yang-Baxter equation.

multiplication on $A^{\otimes S}$. We denote the R-matrix of A as a crossing with sign ± 1 for $\mathcal{R}^{\pm 1}$ respectively. By the Reidemeister theorem, this is in fact well defined on the equivalence classes of tangle diagrams, since $\mathcal{R}$ obeys the Yang-Baxter equation. For clarity, we may add a coupon containing $\mathcal{R}^{\pm 1}$, to indicate multiplication with the R-matrix.

The graphical language is useful to prove properties of A. One can for example prove nicely that the Drinfel'd double is a quasi-triangular Hopf algebra. See chapter 4.1 in [25]. We state the Yang-Baxter equation in this graphical language. For more elaborate examples we refer to [25]. We may define operations on a ribbon graph. We only state the most important operations. Obviously one may multiply two n-ribbon graphs by putting the two diagrams together. To avoid confusion, we label each of the $2n$ strands with a different integer, to indicate which entries are multiplied. The multiplication of two n-ribbon graphs is referred to as the concatination of the strands.

The comultiplication in the Hopf algebra A is a map $\Delta : A \to A \otimes A$. Its operation on an unlabeled strand, or a strand labeled with a grouplike element such as uv^{-1}, is doubling the strand. By the quasitriangularity axioms, when a crossing occurs, doubling a strand results in two of the same crossings. This follows from $\Delta \otimes id(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$.

Since the antipode $id \otimes S$ inverts the R-matrix, we may define the action of S on a strand as inverting the orientation of a strand. This is also well defined when the strands are labeled with grouplike elements in A. Later in this chapter we will use these operations to prove some properties of the knot invariant we are about to define. We also note that it is customary in Hopf theory to denote calculations in a graphical way, using these oberations. See for example [10].

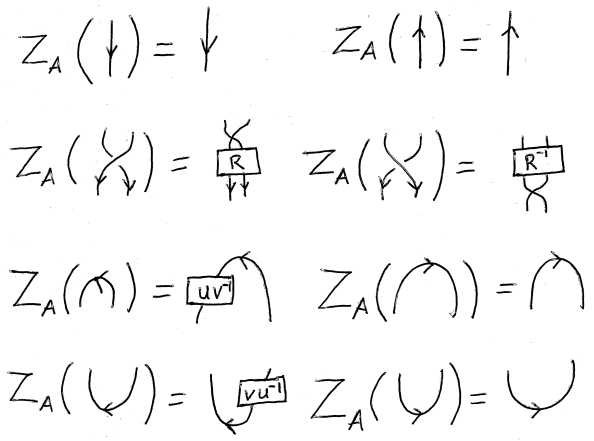

Figure 3.8.: Graphical depiction of Z_A for a ribbon Hopf algebra A.

Let us consider a ribbon Hopf algebra A and a framed sliced oriented tangle T. We will define the tangle invariant $Z_A(K)$ for any ribbon Hopf algebra A. This yields a knot invariant. This is independent from the cutting of the knot, as the knot consists of one strand. In general, this invariant is ill defined on links, as it depends on the cutting point on the embedding of the copies of S^1. We will not attempt to construct a link invariant, but this an interesting topic for future research.
Let S be the set labeling the strands of T. We define

$$Z_A : \{tangle\ diagrams\}/\sim \to A^{\otimes S}$$

as the map that takes tangle diagrams and assigns $\mathcal{R}$ to a positive crossing and $\mathcal{R}^{-1}$ to a negative crossing. To a left oriented cap we assign the element $C = uv^{-1}$, and to the left oriented cup we assign multiplication with $\overline{C} = vu^{-1}$, the product of the ribbon element v with the inverse of u. The right oriented cup and cap are left as they are, as are the single strands. Multiplication now takes place according to the graphical calculus by concatinating (or glueing) the elementary diagrams to each other in the order as they appeared in T. In algebraic terms, when two strands are concatinated, multiplication takes place on the same copy of A labeled according to the label of the strand in K. For clarity we may assign seperate labels to each side of two concatinated strands. Z_A is depicted in figure 3.8.

When considering a map $F : \{tangle\ diagrams\}/\sim \to \{tangle\ diagrams\}/\sim$ of tangle diagrams in $|S|$ strands, we introduce $Z_A(F) : A^{\otimes S} \to A^{\otimes S}$ as the corresponding map on Hopf algebras. $Z_A(F)(Z_A(T)) := Z_A(F(T))$. By the Reidemeister theorem (or Turaev's theorem, depending on the diagrams under consideration) this is well defined. We note furthermore that we may identify $Z_A(T)$ with ${}^{\mathbb{t}}Z_A(T)$ for any tangle T using the map $\mathbb{O}$, since $Z_A(T) \in A^{\otimes S}$. In what fol-

lows, we leave the $\mathbb{O}$ out of the notation.
Usually, for a closed knot diagram we have to take the quotient of $A^{\otimes S}$ with the space of commutators in A. So for a framed oriented sliced knot diagram K, $Z_A(K) \in A/J$. Here $J = \{xy - yx | x, y \in A\}$ is the vector space of all commutators in A. We quotient out to J since there is a choice how to map each elementary tangle to an element in A. In other words, if $Z_A(K)$ were not commutative, this construction would be ill-defined. For a proof of this fact we refer to [25], paragraph 4.2. However, we do not close the knot so we are safe.

Theorem 3.1.3. *Let K be a framed oriented sliced knot, and let A be a ribbon Hopf algebra. Then $Z_A(K) \in A$ is an isotopy invariant of the knot K. We refer to this invariant as the universal A invariant of K, and write it as $Z_A(K)$.*

We are now in a position to define the knot invariant for the case $A = U_q(sl_3^\epsilon)$.

Definition 3.1.13. *Let K be a sliced framed oriented knot diagram. We define $Z_3^\epsilon(K) = Z_{U_q(sl_3^\epsilon)}(K)$.*

We now state the main theorem of this section.

Theorem 3.1.4. *For any framed oriented long knot K, $Z_3^\epsilon(K)$ is invariant under the isotopies of K.*

Proof. To prove this theorem, it is enough to prove invariance of $Z_3^\epsilon(K)$ under the Turaev moves, by the above discussion. These moves are checked explicitly in Mathematica, and we refer to the implementation in appendix A.1.
We state the appropriate equations here for clarity. The equations are matched to the diagrams in figure 3.5 by reading from left to right. We use the notation introduced in earlier chapters, where $\mathcal{R}_{ij}$ is the R-matrix $\mathcal{R} = \sum R^{(1)} \otimes R^{(2)}$ acting on the i-th and j-th tensor factor by $1 \otimes \cdots \otimes R^{(1)} \otimes 1 \otimes \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$. Similarly we write C_i for a copy of C on the i-th tensor factor ('strand'). We define $\mathcal{K} = R^{(2)} C R^{(1)}$ and $\overline{\mathcal{K}} = \overline{R}^{(2)} \overline{C} \overline{R}^{(1)}$.

$$\begin{aligned}
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} &= \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \\
\overline{\mathcal{R}}_1^{(1)}\overline{C}_1\mathcal{R}_{12}\overline{\mathcal{R}}_2^{(2)} &= \overline{C}_2, \\
\mathcal{R}_{12}\overline{\mathcal{R}}_{12} = 1 &= \overline{\mathcal{R}}_{12}\mathcal{R}_{12}, \\
\overline{C}_1\overline{C}_2\mathcal{R}_{12}C_1C_2 &= \mathcal{R}_{12}, \\
\overline{C}C &= 1, \\
C\overline{C} &= 1, \\
\overline{\mathcal{K}}\mathcal{K} = 1 &= \mathcal{K}\overline{\mathcal{K}}.
\end{aligned}$$

The second equality can also be written as $\overline{R}_1^{(1)}\overline{C}_1\mathcal{R}_{12}C_1\overline{R}_2^{(2)} = 1 \otimes 1$, which is equal to the corresponding Turaev move by the using graphical calculus. The fourth identity can be rewritten in the same way.

In appendix A.1 we number the Turaev moves as $T-1, \cdots, T-7$, as indicated in the picture. The equations checked there are the exact same equations written down in this proof. Since we already know that the Hopf algebra structure of $U_q(sl_3^\epsilon)$ is implemented in Mathematica in the program *sl3invariant.nb* as shown in appendix A.1, we can directly conclude that $Z_3^\epsilon(K)$ is indeed invariant under the Turaev moves. This ends the proof. □

3.2. Computing the Alexander polynomial

In this section, the Seifert surface $S(K)$ of a knot diagram K is constructed, and we compute its Alexander polynomial. The Alexander polynomial is computed from the band representation of the Seifert surface. The idea is that we compute the linking matrix of the generators of the fundamental group of $S(K)$. For details we refer to [25] and [4]. We will follow page 17-22 of [25] and [4], page 107.
Let K be a framed oriented knot. To construct its Seifert surface, consider the planar representation of K. In the knot diagram, we replace a crossing with two untangled strands in the same direction. The result is a disjoint union of S^1. We consider the discs in $\mathbb{R}^3$ bound by these discs.
When a disc within a disc occurs, we elevate one of the two discs in the direction perpendicular to the disc. Finally, connect the discs according to the crossings present in the diagram of K. For a positive crossing we attach a band with a positive half twist, and for a negative crossing in the diagram of K we attach the discs with a negative half twist. The surface we obtain is the Seifert surface $S(K)$ of K.
Note that any Seifert surface can be expressed in a band form. This is called the

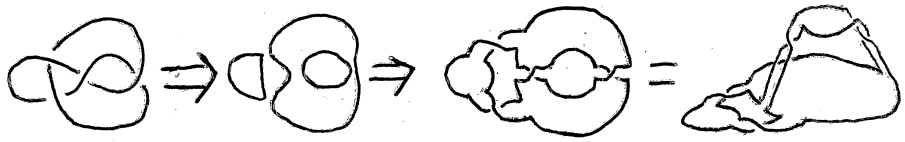

Figure 3.9.: The Seifert surface for the Trefoil knot.

band representation of the Seifert surface, see figure 3.10. This fact is proven by considering one of the discs as the base of the band representation, and contracting other discs to bands. Twists occuring will be denoted as curls in the band representation. For a proof of this elementary fact, see [4], page 105.
We observe that the boundary of the Seifert surfact of a knot K is equal to K. The boundary of the bands of the Seifert surface consists of two strands that are linked together with linking number +1 or -1, depending on the convention chosen. Moreover, the two strands always have opposite orientation, if we compare them to the orientation of the band. This motivates the definition of the map B in the next section.

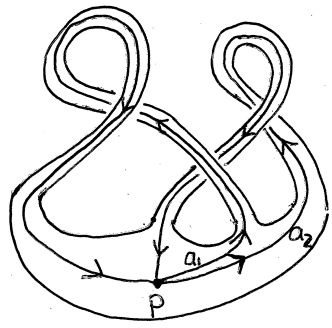

Figure 3.10.: Band representation for the Trefoil knot.

To calculate the Alexander polynomial Δ_K of K, consider the fundamental group $\pi^1(S)$ of $S = S(K)$. We wish to compute the linking number of the generators of $\pi^1(S)$. For simplicity we chose a basebpoint p of $\pi^1(S)$. Since $S(K)$ is connected the final result is independent of the coice of p. Let p be in the 'base' of $S(K)$, i.e. the rectengular part on which the bands are attached. Consider an orientation of each generator $a_i \in \pi^1(S)$. Here, i runs through the number of bands attached to the base. For the Trefoil for example, $i = 1, 2$.
Considering this orientation, we define the numbers l_{ij} and r_{ij}. Consider two bands B_i and B_j with orientation $a_i, a_j \in \pi^1(S)$, respectively. When B_i overcrosses B_j from left to right, define $l_{ij} = 1$, else $l_{ij} = 0$. Similarly, when B_i overcrosses B_j from right to left, define $r_{ij} = 1$. Then define the Seifert matrix V of S by $V_{ij} = l_{ij} - r_{ij}$. Finally, we define the Alexander polynomial $\Delta_K(t)$ in the indeterminate t as $det(t^{-1/2}V^T - t^{1/2}V)$. This normalization forces $\Delta_K(t)$ to be symmetric under $t \mapsto t^{-1}$. See [4] for details.
We calculate the Alexander polynomial for the Trefoil knot with the loops in figure 3.10. We see that a_1 overcrosses a_2 from right to left, a_1 overcrosses a_1 (itself) from right to left and similarly, a_2 overcorsses itself from right to left. We have $V_{11} = V_{22} = V_{12} = -1$ and $V_{21} = 0$. Computing Alexander's polynomial we get

$$\begin{aligned} det((t^{-1/2}V^T - t^{1/2}V) &= det(\begin{pmatrix} -t^{-1/2} & 0 \\ -t^{-1/2} & -t^{-1/2} \end{pmatrix} + \begin{pmatrix} t^{1/2} & t^{1/2} \\ 0 & t^{1/2} \end{pmatrix}) \\ &= t^{-1} - 1 + t. \end{aligned}$$

3.3. Multiplying R-matrices

When $\epsilon = 0$, we know that the invariant connected to $U_q(sl_2)$ is the Alexander polynomial [35]. The $U_q(sl^\epsilon_{2,i})$ algebra relations are identical to the algebra in [35], so we can connect the $U_q(sl^\epsilon_3)$ invariant to the Alexander polynomial as well. Note that for $\epsilon = 0$, and a knot K, $Z^0_3(K)$ is a polynomial in the variables

S, T, ST [35].
Let K be a knot, and let G be the tangle associated to the band representation of the Seifert surface. The tangle G is obtained from the Seifert surface, disconnecting the bands from the central disc, and labeling them with indices $1 \cdots 2g$, where g is the genus of K. Let us define the operation $B = \prod_{n=0}^{g} B^{k_n}_{i_n,j_n}$ as the operation that turns G into the knot K. We can describe this operation as doubling the strands $1, \cdots, 2g$, reversing the orientation on one of the strands and then connecting the strands as they fit on the Seifert-surface to match the orientation of K. We have the following lemma.

Lemma 3.3.1. ${}^{t}Z^{\epsilon}_{3}(B)^{k}_{i,j} = {}^{t}\Delta^{l1,r1}_{i}\,{}^{t}\Delta^{l2,r2}_{j} //{}^{t}\bar{S}_{r1} //{}^{t}S_{r2} //{}^{t}m^{k}_{l1,r1,r2,l2}$.

Proof. The proof follows from considering the action of Δ, S and m on tangle diagrams. The action of multiplication on two given strands, as we already saw, is that of concatinating the two strands. The comultiplication doubles a given strand without changing its orientation. The antipode reverses the orientation of a given strand. From the definition of a quasitriangular Hopf algebra, we know that $\mathcal{R} \in U_q(sl^{\epsilon}_3) \otimes U_q(sl^{\epsilon}_3)$ obeys the Reidemeister moves. On knot diagrams we define the action of multiplication with the R-matrix as a crossing.
Now we easily see that the action of B is equivalent to

$$ {}^{\mathbf{t}}\Delta^{i}_{l1,r1}\,{}^{\mathbf{t}}\Delta^{l2,r2}_{j} //{}^{\mathbf{t}}\bar{S}_{r1} //{}^{\mathbf{t}}S_{r2} //{}^{\mathbf{t}}m^{k}_{l1,r1,r2,l2} $$

in $U_q(sl^{\epsilon}_3)$. This finishes the proof. □

In what follows, we denote ${}^{\mathbf{t}}Z^{\epsilon}_{3}(B)^{k}_{i,j}$ as ${}^{\mathbf{t}}B^{k}_{i,j}$ for short, sometimes leaving out the indices. We write down the following explicit form of ${}^{\mathbf{t}}B^{k}_{ij}$. For convenience, a^* and b^* have been put to zero. We will see that they do not play a role. We used

Mathematica to calculate ${}^{\mathbf{t}}B^k_{ij}$, see the appendix for more information.

$$
\begin{aligned}
{}^{\mathbf{t}}B^k_{ij} = \mathbb{E}\Bigg[0, &\frac{\mathbb{A}_k^{-2\hbar}\left(-\mathbb{A}_k^{2\hbar}+\mathbb{B}_k^{\hbar}\right)x^*{}_j X^*{}_i}{\hbar} + \frac{\mathbb{A}_k^{-2\hbar}\left(\mathbb{A}_k^{2\hbar}-\mathbb{B}_k^{\hbar}\right)x^*{}_i X^*{}_j}{\hbar} - z_k x^*{}_j y^*{}_i + \\
&z_k x^*{}_i y^*{}_j + \frac{\mathbb{B}_k^{-2\hbar}\left(\mathbb{A}_k^{\hbar}-\mathbb{B}_k^{2\hbar}\right)y^*{}_j Y^*{}_i}{\hbar} + \frac{\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{\hbar}+\mathbb{B}_k^{2\hbar}\right)y^*{}_i Y^*{}_j}{\hbar} + \\
&Y_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_j Z^*{}_i + \frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbb{A}_k^{\hbar}-\mathbb{B}_k^{2\hbar}\right)x^*{}_j y^*{}_i Z^*{}_i}{\hbar} - X_k\mathbb{A}_k^{\hbar}\mathbb{B}_k^{-2\hbar}y^*{}_j Z^*{}_i + \\
&\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(\mathbb{A}_k^{3\hbar}-\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i y^*{}_j Z^*{}_i}{\hbar} + \frac{\mathbb{A}_k^{-2\hbar}\left(\mathbb{A}_k^{2\hbar}-\mathbb{B}_k^{\hbar}\right)x^*{}_j y^*{}_j Z^*{}_i}{\hbar} + \\
&\frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(1-\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)z^*{}_j Z^*{}_i}{\hbar} - Y_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i Z^*{}_j + X_k\mathbb{A}_k^{\hbar}\mathbb{B}_k^{-2\hbar}y^*{}_i Z^*{}_j + \\
&\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{3\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i y^*{}_i Z^*{}_j}{\hbar} + \frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{3\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_j y^*{}_i Z^*{}_j}{\hbar} + \\
&\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}+\mathbb{B}_k^{3\hbar}\right)x^*{}_i y^*{}_j Z^*{}_j}{\hbar} + \frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(-1+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)z^*{}_i Z^*{}_j}{\hbar}, 1\Bigg]
\end{aligned}
$$

Note that in particular we get

$$Z_3^0(K) = Z_3^0(G)//{}^{\mathbf{t}}B.$$

Now we can prove the following. The proof is based on the proof in [36]. We use the term mixing, which refers to the $U_q(sl^{\epsilon}_{2,i})$ subalgebras of $U_q(sl^{\epsilon}_3)$ that are invariant under certain algebra maps. This translates to the tensor-formalism by looking at specific terms in the tensors that are used. To see this, we invite the reader to inspect the zipping-formula closely. In particular, we observe that terms like e^{uu^*} leave the $U_q(sl^{\epsilon}_{2,i})$ subalgebras invariant.

Theorem 3.3.1. *Let K be an oriented framed long knot. The knot invariant $Z_3^0(K)$ is the product of inverse of the Alexander polynomial of K in the variables S, T and ST.*

Proof. Let K be a knot, and let G be the tangle associated to the band representation of $S(K)$. We have $Z_3^{\epsilon}(K) = Z_3^{\epsilon}(G)//{}^{\mathbf{t}}B$. We first prove that the invariant factorizes into sl_2 parts by showing that the only terms that contribute to $Z_3^0(G)//{}^{\mathbf{t}}B$ are the terms of the form $\mathbb{E}[0, uU, 1]$, where $u = x, y, z$, and U stands for the dual (capital) generator.
By symmetry of the x, y, z terms occurring in ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}dm$ and the absence of mixing terms like $x_i Y_j$ in the non-perturbative part of the exponentials we will obtain the factorization. Note that we connect all strands of G to one strand, since K is a knot. Observe furthermore that we only need to consider $Z_3^0(G)//{}^{\mathbf{t}}B^k_{i,j}$, the case where G consists of two strands. By induction to the number of strands the theorem will follow for general G, with the same argument.
The second part is proving that each of the sl_2 terms zips to the Alexander poly-

nomial. This is done in a similar fashion, and is done explicitly in [36]. We will not do this here explicitly, but refer to [36] for the argument.
We start with rewriting the multiplication tensor ${}^{\mathbf{t}}dm$ and R-matrices with (note $\epsilon = 0$) $s = 2A - B$ and $t = 2B - A$, where we observe that in ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1} = {}^{\mathbf{t}}\bar{\mathcal{R}}$ the s and t only occur with an a or a b in front. Note that since the antipode S is the convolution inverse of Δ, and $\bar{S}(p) = S(p) = -p$ for p an element of the Cartan subalgebra, ${}^{\mathbf{t}}B^k_{i,j}$ has no terms that consist only of elements of the Cartan-subalgebra. This follows from lemma 3.3.1, since S and $\bar{S}$ are applied on the same index of Δ. After multiplication the indices are changed to k, and the a and b drop out. Observe that $Z(G)$ only consists of products of R-matrices and trivial curls (which are central elements).
In ${}^{\mathbf{t}}\mathcal{R}$, we see that s and t only occur in combination with a and b respectively. Hence we can set s_n and t_n for $n = i, j$ to zero in ${}^{\mathbf{t}}B^k_{i,j}$. That is, we put s and t with the'incoming' indices i and j in ${}^{\mathbf{t}}B^k_{i,j}$ to zero. This is equivalent to setting S and T to 1, since s and t will never occur from zipping a and b. This is because the a and b dependence in the Cartan part of the exponential ${}^{\mathbf{t}}Z_3^{\epsilon=0}(B)^k_{i,j}$ drops out. Since we take $\epsilon = 0$, we can put a^* and b^* to zero in $Z_3^0(G)$ (which follows from the format of ${}^{\mathbf{t}}dm$).
Now we look at the non-square terms in the R-matrices and the multiplication tensor ${}^{\mathbf{t}}dm$. A calculation in Mathematica shows that before the zipping the only cubic terms are (looking at ${}^{\mathbf{t}}dm^k_{i,j}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1}$):

$$X_i Y_i z_j, \tag{3.1}$$

$$x_i^* y_j^* z_k - Y_k x_i^* Z_j^* + X_k y_i^* Z_j^*. \tag{3.2}$$

3.2 arises in ${}^{\mathbf{t}}dm$, and 3.1 arrises in ${}^{\mathbf{t}}\mathcal{R}^{-1}$. If we look at the last two terms 3.2, we see that by symmetry of x and y occurring (and X and Y terms, as a result, since these only occur together) in ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1}$ the two will cancel out. Here we use that in the end we are left with one strand, i.e. one index. The first term of 3.2 and 3.1 are similar in the sense that after zipping, $x_i^* y_j^* z_k \mapsto X_{i'} Y_{j'} z_k$.
From ${}^{\mathbf{t}}dm$ we can see that the cubic terms are unchanged by zipping with ${}^{\mathbf{t}}B$, since X^* only occurs with an x in ${}^{\mathbf{t}}B$, and similarly for Y. The only way these terms could contribute is after zipping of ${}^{\mathbf{t}}Z_3^0(B)^k_{i,j}$ (without loss of generality we can take $i' = i$ and $j' = j$).
In ${}^{\mathbf{t}}B$, ${}^{\mathbf{t}}\mathcal{R}$ and ${}^{\mathbf{t}}\mathcal{R}^{-1}$, z^* does not occur in combination with x and y. We once again remind the reader that $\epsilon = 0$ for the duration of this proof, so we only consider the non-perturbative part of the exponentials. Similarly, x_i does not occur in combination with y_j. So there is no mixing of the variables. The reader is encuraged to check this for themselves using the Mathematica implementation in the appendix. This finishes the proof. □

The following theorem uses a rough upper bound for the number of variables zipped. The proof follows [35] by looking the sizes of the matrices in the zipping

theorem. We are using the three-stage zip and the zippinng theorem on 8 generators in total, in the most optimal implementation of $U_q(sl_3^\epsilon)$. The computational complexity of calculating an m by m determinant is assumed to be m^3. Inverting the matrix is assumed to be of complexity m^3. Differentiating a monomial of degree m times is of complexity c^m for some constant c. So differentiating a polynomial of m' terms will take $O(m')$ computations. There exist faster algorithms, but we calculate an upper-bound. It is possible to generalize this argument to the case $\epsilon^k = 0$, see [36] for an argument in the sl_2 case.

Theorem 3.3.2. *The calculation of $Z_3^\epsilon(K)$ for a knot K with n crossings has a computational complexity of at most $O(n^{10})$.*

Proof. Let K be a knot with n crossings. We will assume that it has rotation-number 0 and writhe 0. This can be done by inserting curls after the last crossing, which does not infect the final result, apart from a normalization of the Alexander polynomial. Note that conjugating an element x in the Hopf algebra $U_q(sl_3^\epsilon)$ with C equals $S^2(x)$. So putting the curls that occur in K after the final crossing will only change the normalization with factors of q. Moreover, the curls cancel out. Note that S^2 multiplies the non-Cartan generators with a factor of q, so this operation does not contribute to the upper bound. So we can assume that the knot invariant is calculated by zipping over and under crossings and multiplication tensors.
We assume that the zipping is done at once for K, meaning there are n crossings where each crossing has 2 indices to be concatinated. This follows from the relation between edges and vertices in any 4-valent graph. Furthermore we observe that s and t are central, so we can take them to be coefficients, and leave them out of the zipping. As a consequence we drop the index from t_i, s_i and s_i^*, t_i^*.
The first step is zipping a, a^*, b^*, b. Consider a tensor of the form $E = e^{L+Q}P$, where L stands for the Cartan-part, Q stands for rest of terms in the ϵ-independent part. As we put s and t constant, the only contribution to L comes from ${}^{\mathbf{t}}dm$, and is of the form uu^*. This is a diagonal matrix, so differentiating and computing the inverse and determinant takes $O(n^2)$ operations. The contraction of the perturbation and differentiating takes $O(n)$ operations, as in each monomial of P, a, b have a degree of at most 2, and replacement of a with ∂_{a^*} also takes $O(n)$ computations.
The contraction of $\{X, Y, y^*, x^*\}$ takes the most computations, as it has the most generators that need to be contracted. Note that it is of the same complexity as zipping Z, z^*. Only the prefactor differs because of the number of variables and the number of monomials in P with the corresponding variables differ. Zipping E to the variables $\{X, Y, x^*, y^*\}$ computes the matrix q of Q by differentiating Q with ∂_{u,u^*} and taking its inverse. This takes $O(n^4)$ computations, as there are $\sim n$ generators in Q.
To compute the contraction of P, we count the number of monomials in P with the variables X, Y, y and x. Observe that there are $32n$ variables (counting the indices and the fact that we are always contracting two different indices) and

the degree of each monomial is 8 at most, looking at the multiplication tensor in the quantum double. Now we get a total of at most $\binom{32n+8}{8} \leq (32n+8)^8$ monomials. Differentiating each monomial is of a constant complexity. However, for contracting P we first need to substitute the variables by applying the zipping theorem. As this is a vector of length $16n$ at worst, this takes $O(n^2)$ operations at worst, since the substitution could depend on $\sim n$ variables. So we find that zipping to $\{X, Y, x^*, y^*\}$ takes $O(n^{10})$ operations at most.
We observe that applying the zipping theorem multiple times is more efficient than performing the zip for all variables at once (in the absence of cubic terms of course). In particular since this reduces the size of the matrices we work with. In this case we have 3 implementations of the zipping theorem, the contribution of each we can add together. This finishes the proof. □

Although in theory our computation is very efficient in terms of the number of crossings, the biggest problem occurs when scaling the sl_3 invariant to an sl_N invariant. The number of variables m per strand (currently 8) will increase with $m = (N-1)(N+1)$, if we count the number of generators in sl_N, compensating for the central elements in the algebra, of which there are $N-1$. Since we have to compute the inverse of a matrix of size m times m by differentiating, this takes $\sim m^4$ operations. Adding in the substitution in the zipping theorem which takes m^2 operations we get a complexity of $O(m^{10}) = O(N^{20})$ for an sl_N invariant where all the variables are zipped in one implementation (in the assumed absence of troublesome terms), and $O(N^{22})$ if the zips are splitted into N different zips (one zip for each group of generators associated to a root-vector of a particular length $< N$). So in terms of the number of crossings, the sl_N invariant is less efficient as say the sl_2 invariant, although not much less. But the problem arises from the constant prefactor.
In the case of sl_3 the prefactor arising from the number of generators in the $\{X, Y, y^*, x^*\}$-zip can be estimated to be approximately $\sim 8(2 \cdot 16)^{10}$. The factor of 4 arises from the number of operations it takes to differentiate each term (with a degree of at most 2 in each generator). The factor $2 \cdot 16$ counts the number of generators. If we start with $\{X, Y, y, x\}$ and the dual variables, we should add in a factor of 4 since we are always contracting 2 indices, and each crossing has two strands. For a small number of crossings we see that the contribution from the number of generators per crossing is very big in comparison to the contribution of n^{10} from the number of crossings. In total, we see that for sl_m this prefactor will be roughly $\sim 4^{10}m^{10}$. In particular the factor of 4 contributes a lot when m is small.
In conclusion, although the sl_3 invariant computes as n^{10} where n is the number of crossings, in practice, for $n = 6$, the prefactor will contribute a term n^{13}, making the effective complexity roughly $O(n^{23})$ for small knots up to 6 crossings. For knots up to 10 crossings this reduces to $O(10^{20})$. For bigger knots, the contribution will be smaller still. Only for knots of ~ 40 crossings, $n^{10} \sim 8(32)^{10}$.
We see that for the knots we are interested in the contribution of the number of

generators and the size of ${}^{\mathbf{t}}dm$ is more important than the number of crossings in the knots. This makes the computations very slow. It is therefore necessary to look at a way to reduce the number of monomials in the perturbation P by for example introducing a second parameter $\gamma^k = 0$ on the b^- side, to make the algebra nilpotent. γ is equivalent to ϵ, it is only present on the "other side" of the algerba $U_q(sl_3^\epsilon)$.
The effect will be that we compute terms of the expansion of Z_ϵ^3 in terms of γ, which are finite type invariants, which are less powerfull than the Z_3^ϵ invariant. It may be expected however, that when computed to a sufficient order of γ, these invariants will give enough information to for example prove that Z_3^ϵ distinguishes mutants.

Conclusion

Starting with the definition of knots, knots diagrams and the Alexander polynomial we have proven that the invariant Z_3^0 factorizes into Alexander polynomials. We used the Seifert surface to prove this fact. An algebraic implementation of the Seifert surface was used to compute the knot invariant. This was used, together with the action of the (co)multiplication and antipode on knot diagrams.
The main result of this chapter is the theorem that we can compute Z_3^ϵ in polynomial time, and in fact in $O(n^{10})$ computations, where n is the number of crossings in a given knot. The proof of this theorem was by considering the explicit zipping of R-matrices corresponding to a knot diagram of a knot K. We concluded wiith the observation that although this might seem a small cost, in practice this cost is much larger. This is mainly because of the number of generators of sl_3. For a general sl_n invariant this complexity will increase with $O(n^{12})$, where n is the rank of the algebra $U_q(sl_n)$.
More research is needed to bring this cost down. In particular the zipping of R-matrices could possibly be improved upon, to bring down the $O(n^{10})$ even further. It seems unlikely that the cost of the number of generators can be reduced significantly, but this is the most important contribution to the complexity of computing Z_3^ϵ. In comparison, for a knot of 6 crossings, this factor contributs roughly as $O(n^{20})$. This is much larger than the $O(n^{10})$ contribution of the zipping of the R-matrices. One way to reduce this cost greatly is by cutting off the multiplication in the quantum double. This can be done by introducing a second parameter dual to γ for example. Although in practice this reduces the strength of the knot invariant greatly, this seems to be the best bet to compute Z_n^ϵ for bigger knots.
Another possibility would be to reduce the number of variables involved in the zipping of the R-matrices. One way to do this, is to isolate the actual sl_3 invariant from the knot-polynomial $Z_3^\epsilon(K)$ of a knot K. For example, in the sl_2 case ane can isolate this part from a long expression in a central element. It is concievable that we can drop some zips by smartly zipping the R-matrices of a knot diagram and

still obtain the 'new part' of the sl_3 invariant. To obtain an idea of the general form of $Z_3^\epsilon(K)$ we need more data of course, but we can also look at the invariance of parts of $Z_3^\epsilon(K)$ under the q-Weyl group.
Another way to improve the computation speed would be to implement the computation of the determinants involved in the zipping of knots in C++ for example, or another efficient computer language. Since Mathematica is Python-based it may not be as efficient in computing and handling large expressions. Implementing the computation in C++ would also enable one to use larger computers for the computation of the knot diagrams.

4. Towards constructing $U_q(sl_n^\epsilon)$

Introduction

In this chapter we repeat the construction of $U_q(sl_3^\epsilon)$ of the first chapter for sl_n, for general n. In the first section we quantize the upper Borel subalgebra of sl_n, and we construct a basis by using the Weyl-group action. The Weyl group is constructed for $U_q(sl_2^\epsilon)$ in section 4.2, after which we continue with constructing the Weyl group for $U_q(sl_n^\epsilon)$. We assume that ϵ is invertible in this chapter. We calculate the algebra relations up to any order of ϵ by taking the power series expansion of an expression.
In this chapter we assume that ϵ is invertible, as in the non-invertible, $\epsilon^k = 0$ case the construction of the quantum Weyl group breaks down. It is not possible to construct the usual highest weight representations when $\epsilon^k = 0$. Taking ϵ invertible provides an isomorphism between sl_n^ϵ and sl_n. The usual automorphisms T_i that originate from the Weyl group are not algebra automorphisms when ϵ is not invertible. When working over $\mathbb{R}(\epsilon)$, the maps T_i turn out to be algebra automorphisms.
In the last section we prove that one can define algebra maps $\tilde{T}_i$ from the automorphisms T_i for non-invertible ϵ. However, the $\tilde{T}_i$ can only be applied to simple generators times a factor of ϵ. This is familiar from chapter one, where we saw a similar term ϵZ in the commutator.
A different set of symmetries has been found by Bar-Natan and Van der Veen when $\epsilon^k = 0$, or more generally for $\mathbb{R}[[\epsilon]]$. The set of symmetries for non-invertible ϵ is isomorphic to the dihedral group D_n for sl_n^ϵ. See [37] for details.
It remains to be seen if this means that the invariants arising are stronger, as they might have less symmetry, or if this means there are more hidden symmetries that arise in the invariants. This symmetries only differs for $n \geq 4$, as $D_3 = S_3$.
If $\epsilon \in \mathbb{R}[[\epsilon]]$, it has been noted that a quotient of an affine quantum group is obtained, see [37] and [5]. The Dynkin diagrams of affine Lie algebras have a circular form, so there are different symmetries than in the sl_n case. When $\epsilon^k = 0$ in an affine Lie algebra in some sense, these symmetries survive. See [37] and [5] for details.

The contents of this chapter is as follows. In the first section we provide the general sl_n^ϵ Lie algebra relations and its quantization $U_q(sl_n^\epsilon)$ for invertible ϵ. The construction of the $U_q(sl_n^\epsilon)$ is briefly covered. In the second section we cover the finite dimensional representation theory for $U_q(sl_2^\epsilon)$ for invertible ϵ, and an algebra automorphism is constructed. In the third section we proceed with the

$U_q(sl_n^\epsilon)$ case in the same way, following [29]. In the last section we scetch the connection between the Hopf algebras covered in this and the first chapter.

4.1. Quantizing a Lie subalgebra of sl_n

Let sl_n^ϵ be a Lie bialgebra over $\mathbb{R}(\epsilon)$ for an indeterminate ϵ with generators $H_i^\pm, X_i^\pm$, $i = 1 \cdots n$, Cartan-matrix a_{ij} and the relations (we introduce ϵ in the b^- side multiplication, as opposed to chapter 1)

$$[H_i^-, X_j^\pm] = \pm\epsilon a_{ij} X_j^\pm, [H_i^\pm, H_j^\mp] = 0, [H_i^+, X_j^\pm] = \pm a_{ij} X_j^\pm, \tag{4.1}$$

$$[X_i^+, X_i^-] = -\frac{1}{2}\delta_{i,j}(H_i^+ + \epsilon^{-1} H_i^-), (ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, (i \neq j), \tag{4.2}$$

$$\delta(X_i^+) = \epsilon X_i^+ \otimes H_i^+ - \epsilon H_i^+ \otimes X_i^+, \tag{4.3}$$

$$\delta(X_i^-) = X_i^- \otimes H_i^- - H_i^- \otimes X_i^-, \tag{4.4}$$

$$\delta(H_i^\pm) = 0. \tag{4.5}$$

In our convention, $a_{ii} = 2$, $a_{ij} = -1$ if $i = j \pm 1$ and else zero. We consider the double of the Lie algebra of upper triangular matrices $b^+ \subset gl_n$, so we will assume that the Cartan matrix has rank n. As noted in chapter 1, the Cartan matrix is well defined, even though the above algebra is not semisimple.
We observe furthermore that $H_i^\pm$ generate the Cartan subalgebra $\mathfrak{h}$, the biggest commutative subalgebra of $\mathfrak{g}$ (which is the case for semisimple Lie algebra's). Define $ad_X(Y) = [X, Y]$ as the adjoint action of $\mathfrak{g}$ on itself. Putting $\epsilon = 1$ and dividing out to $H_i^+ - H_i^- = 0$ yields the usual sl_n Lie bialgebra. In this chapter we consider the generalization of the classical double quasitriangular Lie bialgebra calculated in chapter 1, although with ϵ present in the b^- lower Borel subalgebra commutation relations.
The simple roots $\alpha_i : \mathfrak{h}^+ \to \mathbb{R}(\epsilon)$ are defined as the linear maps $\alpha_i(H_j^+) = a_{ij}$, and similarly for $\mathfrak{h}^- \subset \mathfrak{h}$, with an additional factor of ϵ. Since ϵ is invertible, the root space for $h^\pm$ are isomorphic and we may talk about the rootspace of sl_n^ϵ. As we are concerned with the quantization of b^+ in this section, we will use roots on $\mathfrak{h}^+$.
The fundamental reflections $s_i : \mathfrak{h} \to \mathfrak{h}$ are defined by $s_i(h) = h - \alpha_i(h)H_i^+$ for $h \in \mathfrak{h}$. The Weyl group of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h})$ generated by $s_1, \cdots, s_{n-1}$. The Lie algebra sl_n^ϵ is finite dimensional, so there is a unique element w of maximal length N. Write $w = s_{i_1} \cdots s_{i_N}$. Define the positive roots as the set $\Delta^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1}(\cdots s_{i_{N-1}}(\alpha_{i_N})\}$. Note that each element occurs exactly once.
Although sl_n^ϵ is not semisimple, the Killing form is nondegenerate on the subalgebras $\mathfrak{h}^\pm \subset \mathfrak{h}$ generated by $H_i^\pm$. The reason is that the upper an lower Borel subalgebras $b^\pm$ are embedded in sl_n. So the Cartan matrix and the root system corresponding to the $b^\pm$ algebra is well defined.
For a Lie algebra $\mathfrak{g}$ over a field, one can extend the fundamental reflections to act on $\mathfrak{g}$ instead of $\mathfrak{h} \subset \mathfrak{g}$, see [6], in which case we call the automorphism corre-

sponding to s_i T_i. In the case of sl_n^ϵ, we have the following definition of T_i. This definition is equivalent to the automorphisms T_i of sl_n, as one can check in for example [6], or [14].

Proposition 4.1.1. *Let sl_n^ϵ be the Lie algebra structure as specified above. The T_i defined in the following way*

$$T_i(X_i^\pm) = -X_i^\mp,\ T_i(H_j^+) = \epsilon^{-1}H_j^- - \epsilon^{-1}a_{ji}H_i^-, T_i(H_j^-) = \epsilon H_j^+ - \epsilon a_{ji}H_i^+, \quad (4.6)$$

$$T_i(X_j^+) = (-a_{ij})!^{-1}(ad_{X_i^+})^{-a_{ij}}(X_j^+), i \neq j \quad (4.7)$$

$$T_i(X_j^-) = (-1)^{a_{ij}}(-a_{ij})!^{-1}(ad_{X_i^-})^{-a_{ij}}(X_j^-), i \neq j. \quad (4.8)$$

are Lie algebra automorphisms of sl_n^ϵ.

Proof. The only relations that change in the presence of ϵ are the commutator $[H_i^-, X_j^\pm] = -\epsilon a_{ij}X_j^\pm$ and $[X_i^+, X_i^-] = -\frac{1}{2}(H_i^+ + \epsilon^{-1}H_i^-)$. Applying T_i and T_j on both sides of the first identity, we observe that T respects the relation. Here we make use of the Jacobi-identity to calculate commutators of commutators.
For the second identity, we observe that the right-hand side is invariant (modulo a global minus sign) under T_j if $i = j$ and if $j \neq i$ we gain a term $-a_{ij}(H_j^+ + \epsilon^{-1}H_j^-) = H_j^+ + \epsilon^{-1}H_j^-$, as $a_{ij} = -1$ if $i \neq j$. On the left hand side we obtain the term $[[X_j^+, X_i^+], [X_i^-, X_j^-]]$, which we can evaluate with applying the Jacobi identity twice. Note that $-a_{ij} = 1$, so $T_j(X_i^-) = [X_i^-, X_j^-]$. We see that $[[X_j^+, X_i^+], [X_i^-, X_j^-]] = -[[X_j^-, [X_i^-, X_i^+]], X_j^+] - [[[X_j^+, X_j^-], X_i^-], X_i^+]$. We only need to prove that $[X_j^+, [X_j^-, [X_i^+, X_i^-]]]$ yields a term $H_j^+ + \epsilon^{-1}H_j^-$, and similarly for the term $i \leftrightarrow j$. Using the commutator $[X^+, X^-]$ we obtain $-[[X_j^-, [X_i^-, X_i^+]], X_j^+] - [[[X_j^+, X_j^-], X_i^-], X_i^+] = \frac{1}{2}(-[[X_j^-, H_i^+ + \epsilon^{-1}H_i^-], X_j^+] + [[H_j^+ + \epsilon^{-1}H_j^-, X_i^-], X_i^+])$. With $[H_i^+, X_j^\pm] = \pm a_{ij}X_j^\pm$ and the relations for H_i^-, this yields the required result. This proves the theorem. □

These automorphisms obey the braid group relations $T_iT_jT_i = T_jT_iT_j$ for all $i \neq j$.

Proposition 4.1.2. *Let the T_i be as defined above, and let a_{ij} be the Cartan matrix corresponding to sl_n. Then $T_iT_jT_i = T_jT_iT_j$ for all $i \neq j$.*

Proof. As a_{ij} only takes nonzero values if i and j differ at most 1, we only need to check two non-trivial identities. The case for $H_i^\pm$ can be reduced to the sl_n case by counting the factors of ϵ on both sides and realizing T_i is linear in ϵ. We note that independently of the index i, T_i switches the sign of $H^\pm$. This is in fact the only thing that is different from sl_n for the Cartan subalgebra, together with ϵ we need to keep track of.
The case for $X_i^\pm$ is the same for both $b^\pm$, and can be reduced to the usual case by realizing that H^+ acts in the same way as $\epsilon^{-1}H^-$. We only need to count the factors of ϵ that are introduced when checking the Weyl condition on $X_j^\pm$. This is left as an exercise, as it follows by a straightforward calculation. □

We continue with constructing the quantization of the above sl_n^ϵ Lie bialgebra. We first quantize the upper triangular matrices Lie subalgebra b^+ of sl_n^ϵ. Consider the subalgebra b^+ generated by the simple root vectors X_i^+ and H_i^+ for all $i = 1 \cdots n-1$. remember that the cobracket on b^+ is multiplied with ϵ:

$$\delta(X_i^+) = \epsilon X_i^+ \wedge H_i^+. \tag{4.9}$$

The cobracket on the other positive root vectors is implicitly defined. To quantize a Lie bialgebra, only the cobracket on the simple generators are needed. We follow the usual construction of $U_q(sl_3)$ here to obtain a quantization of b^+. See chapter 6 and 8 of [6].
We are looking for a Hopf algebra with classical limit 4.9, so it is easiest to start with quantizing the cobracket. Firstly, let us take the trivial Hopf algebra structure on the universal enveloping algebra $U(b^+)$, as introduced earlier. $\delta(H_i^+) = 0$ yields

$$\Delta_h(H_i^+) = H_i^+ \otimes 1 + 1 \otimes H_i^+.$$

Continueing with X_i^+, we introduce a grading deg on b^+. Here $deg(H_i^+) = 0$ and $X_i^+ = 1$. In order to obtain a graded algebra, we need the (co)multiplication to preserve the grading, or at least not lowering the degree in the case of general positive roots. To this end, let us follow [6] and guess

$$\Delta_h(X_i^+) = X_i^+ \otimes e^{h\mu H_i^+} + e^{h\nu H_i^+} \otimes X_i^+, \tag{4.10}$$

where $\mu, \nu \in \mathbb{R}(\epsilon)[[h]]$, so that $e^{h\mu H_i^+}$ and $e^{h\nu H_i^+}$ are grouplike, meaning $\Delta_h(e^{h\mu H_i^+}) = e^{h\mu H_i^+} \otimes e^{h\mu H_i^+}$. Multiplying X_i^+ with $e^{-h\nu H_i^+}$ thus yields

$$\Delta_h(X_i^+) = X_i^+ \otimes e^{h\mu H_i^+} + 1 \otimes X_i^+,$$

so we can take $\nu = 0$ without loss of generality. We can use that the classical limit of $\Delta_h(X_i^+)$ equals 4.9, so we can take $\mu = \epsilon$ to obtain

$$\Delta_h(X_i^+) = X_i^+ \otimes e^{h\epsilon H_i^+} + 1 \otimes X_i^+. \tag{4.11}$$

Δ_h extends to an algebra homomorphism on the subalgebra generated by H_i^+ and X_i^+, since the H_i^+ has trivial comultiplication. Consequently, the multiplication (bracket) can be left unchanged. Hence we can directly write down the antipode for H_i^+ and X_i^+ from the calculated comultiplication.

$$S_h(H_i^+) = -H_i^+, S(X_i^+) = -X_i^+ e^{-h\epsilon H_i^+}. \tag{4.12}$$

We extend Δ_h to an algebra homomorphism on $U_h(b^+)$. Consider the classical

Serre-relations for $i \neq j$, that hold for the Lie algebra:

$$(ad_{X_i^{\pm}})^{1-a_{ij}}(X_j^{\pm}) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (X_i^{\pm})^k (X_j^{\pm})(X_i^{\pm})^{1-a_{ij}-k} = 0. \tag{4.13}$$

For sl_n, $1 - a_{ij} = 2$ for all the nontrivial relations (the case $i = j$ yields a vanishing commutator for X_i^+ and X_j^+). Note that in the case of sl_3, with the definition $[x, y] = z$ and $X_1^- = x$, $X_2^- = y$, gives $[z, x] = 0$.

In order for Δ_h to be an algebra homomorphism, 4.13 needs to be altered. Repeating the calculation we did in chapter 1, the correct form of the quantum Serre relations is obtained by replacing the binomial coefficients with quantum binomial coefficients, with $q = e^{\epsilon h} = 1 + \epsilon h$. For the calculation in the case of sl_n, see chapter 6 of [6]. In the presence of ϵ this calculation is the same.

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{e^{2\epsilon h}} (X_i^+)^k (X_j^+)(X_i^+)^{1-a_{ij}-k} = 0. \tag{4.14}$$

One of the ingredients for the proof that with these relations Δ_h does indeed become an algebra homomorphism is the commutation relation

$$e^{\epsilon h H_i^+} X_j^+ e^{-\epsilon h H_i^+} = e^{\epsilon h a_{ij}} X_j^+.$$

Together with the trivial counit, we have constructed the Hopf algebra structure on $U_q(b^+)$.

Theorem 4.1.1. *$U_h(b^+)$ is a quantization of the Lie bialgebra b^+, for invertible and non-invertible ϵ. Moreover, there exists an algebra isomorphism $U_q(b^+) \cong U(b^+)[[h]]$ in both cases.*

Proof. In order to prove that we have indeed found the quantization of b^+, observe that multiplication and comultiplication in $U_h(b^+)$ have b^+ as classical limit. It is also necessary to find a $\mathbb{R}(\epsilon)[[h]]$-module isomorphism between $U_h(b^+)$ and $U(b^+)[[h]]$. Since ϵ is invertible, this is equivalent to the sl_n case. This equivalence yields an $\mathbb{R}(\epsilon)[[h]]$-module isomorphism. Even stronger, we obtain an algebra isomorphism between $U_h(b^+)$ and $U(b^+)[[h]]$, as there exists an algebra isomorphism between $U_h(sl_n)$ and $U(sl_n)[[h]]$ by the rigidity theorem. See [6], chapter 6.1.
The case where ϵ is not invertible can be obtained from the first case by expanding the isomorphism in terms of ϵ. The fact that this can be done follows because the ϵ only occurs together with the h in the algebra relations on the b^+ side in the q-Serre relations. This implies that the isomorphism $U_h(b^+) \cong U(b^+)[[h]]$ is also defined over $\mathbb{R}[[\epsilon, h]]$. Moreover, ϵ is only present in q and q is invertible up to any order of ϵ^k. We note that after expansion of q there is no factor of ϵ^{-1} present in the relations of $U_h(b^+)$.

To prove that in finite order of ϵ we still have isomorphisms of $\mathbb{R}[\epsilon]/(\epsilon^k)[[h]]$-modules, note that injectivity follows from comparing the terms in each order of ϵ. The surjectivity follows in the same way. That we obtain an isomorphism of algebras follows from linearity of the isomorphism over ϵ. So we have a quantization of the Lie bialgebra b^+. □

We are now in a position to construct a PBW basis for $U_h(b^+)$, while also calculating the dual of $U_h(b^+)$, the partial R-matrices and the comultiplication on the PBW basis. These calculations are necessary in order to be able to write an algoritm that can calculate the algebra relations for general n. We could start with the dual basis to the simple roots X_i^+. Using these generators and the action of the Weyl group, we can calculate the necessary partial R-matrices. Using those (and their inverse), we can calculate the coproduct on basis elements associated with any positive root. Then we can find the (co)multiplication properties of the dual basis using the Hopf algebra pairing.

The action of the braid group can be defined straightforwardly on the algebra $U_h(n^+)$ spanned by X_i^+ in $U_h(b^+)$ through the Hopf algebra right-adjoint action Ad

$$Ad_x(y) = \sum x_{(1)} y S(x_{(2)})$$

of U_h and U_h^{cop}, the opposite coalgebra. Define $T_i(X_j^-) = Ad_{(-X_i^-)^{(-a_{ij})}}(X_j^-)$ and $T_i(X_j^+) = Ad^{cop}_{-(X_i^+)^{(-a_{ij})}}(X_j^+)$. As we will see, for invertible ϵ it is possible to define T_i on $U_q(sl_n^\epsilon)$. This will be covered in the next sections. For now we restrict ourselves to $U_q(n^+)$. Note that we define the T_i slightly different here than we will do in the next section. We leave out the central factor for cosmetic reasons. We note that this has no effect on the expression for $\Delta(X_\beta)$ we state here.
This action can be used to write down explicit generators of $U_q(b^+)$ for invertible ϵ. If $\beta = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta^+$, define $X_\beta^\pm = T_{i_1}\cdots T_{i_{k-1}}(X_{i_k}^\pm)$. Assuming that there are no redundant reflections in the notation for β, this is well defined, and yields generators $X_\beta^\pm$ for each positive root. The fact that this is well defined follows from the Weyl property for T_i. For the proof that T_i satisfy the Weyl property we refer to the next two sections.
Denote $X_\beta^\pm$ for the generators of $U_q(sl_n)$ corresponding to the root β, and denote $H_i^\pm$ for the generators of the quantized Cartan subalgebra of $U_q(sl_3^\epsilon)$, for $i = 1, \cdots n$. Let w be the longest root with decomposition $w = s_{i_1}\cdots s_{i_N}$. We denote the positive roots by $\beta_1, \cdots \beta_N$. Corresponding to this decomposition we have the non-simple generators $X_{\beta_i}^\pm$, where $i = 1, \cdots, N$
Monomials in X_β^+ and H_β^+ form a basis of $U_q(b^+)$. The folowing theorem is a generalization of the theorem we saw in chapter 1, however the proof is easier since we start with an algebra over $\mathbb{R}(\epsilon)$.

Theorem 4.1.2. *Let X_β^+ and H_β^+ be the generators corresponding to the positive roots*

β. Then the monomials $\prod_{i=1}^{N}(X_{\beta_i}^+)^{m_i}\prod_{i=1}^{n}(H_i^+)^{p_i}$ form a basis of $U_q(b^+)$.

Proof. From the classical PBW theorem it follows that ordered monomials in $X_\beta^\pm$ constitute a linear basis of $U_q(sl_n^\epsilon)$. Now we use the fact that ϵ is invertible, so that we have an isomorphism between $U_q(sl_n^\epsilon)$ and $U(sl_n^\epsilon)[[h]]$, by the rigidity theorem. □

This finishes the construction of $U_q(b^+)$. To obtain $U_q(sl_n^\epsilon)$ we need to calculate the QUE-dual of $U_q(b^+)$, which we refer to as $U_q(b^-)$, consistent with chapter 1. Then one can form the quantum double of $U_q(b^\pm)$ to form $U_q(sl_n^\epsilon)$. We skip this construction and state the relations of $U_q(sl_n^\epsilon)$. The proof that these relations form a Hopf algebra can be found in many sources, since ϵ is invertible.
Notice that $U_h(b^+)$ is also well defined over $\mathbb{R}[[\epsilon,h]]$ by expanding q. Over this ring, it is possible to divide out to ϵ^k for some k. When we take the quantum double of $U_h(b^+)$, we can no longer work over $\mathbb{R}[[\epsilon]]$, due to the factor $\frac{1}{q-q^{-1}}$ present. In the last section of this chapter we cover this issue.

Theorem 4.1.3. *Let ϵ be invertible, and let $q=e^{\epsilon h}$. The following relations*

$$[X_i^-,H_j^-]=\epsilon a_{ij}X_i^-,\ [X_i^+,H_j^+]=-a_{ij}X_i^+,\ [X_i^-,H_j^+]=a_{ij}X_i^-,\ [X_i^+,H_j^-]=-\epsilon a_{ij}X_i^+$$

$$[X_i^-,X_j^+]=\frac{q^{H_i^+}-q^{-\epsilon^{-1}H_i^-}}{q-q^{-1}}\delta_{ij},\quad \sum_{k=0}^{k=1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q^2}(X_i^\pm)^kX_j^\pm(X_i^\pm)^{1-a_{ij}-k}=0$$

$$\Delta(X_i^-)=X_i^-\otimes e^{hH_i^-/2}+e^{-hH_i^-/2}\otimes X_i^-,\Delta(X_i^+)=X_i^+\otimes e^{\epsilon hH_i^+/2}+e^{-\epsilon hH_i^+/2}\otimes X_i^+$$

$$\Delta(H_i^\pm)=H_i^\pm\otimes 1+1\otimes H_i^\pm,S(X_i^+)=-e^{\epsilon h}X_i^+,S(X_i^-)=-e^{-h}X_i^-,S(H_i)=-H_i,$$

define an Hopf algebra $U_q(sl_n^\epsilon)$ over $\mathbb{R}(\epsilon)$, which is the quantization of the Lie bialgebra sl_n^ϵ. The monomials

$$\prod_{i=1}^{N}(X_{\beta_i}^+)^{m_i}\prod_{i=1}^{n}(H_i^+)^{p_i}\prod_{i=1}^{n}(H_i^-)^{p_i'}\prod_{i=1}^{N}(X_{\beta_i}^-)^{m_i'}$$

form a basis of $U_q(sl_n^\epsilon)$.

In general, the action of the Braid group is not compatible with the coproduct when extended to the full Hopf algebra. This means that one has to compute the action of the braid group, before one can compute the coproduct. There is another option. One can express the coproduct of the generators in terms of the R-matrices corresponding to the $U_q(sl_2)_i$ subalgebras of $U_q(sl_n)$. We refer these R-matrices as partial R-matrices.
Let $A=a_{ij}$ be the Cartan matrix. Define $\zeta_i=\sum(A^{-1})_{ij}H_j^-$. For simple roots α_i, $i=1,\cdots,n-1$, associated with (dual) generators X_i^+,X_i^- and H_i^+,ζ_i, one has

the following pairing ($q = e^{\epsilon h}$):

$$\langle (H_i^+)^o (X_i^+)^t, (\zeta_i)^{o'} (X_i^-)^{t'} \rangle = \delta_{o,o'} \delta_{t,t'} h^{-o-t} o! [t]_q !. \tag{4.15}$$

Where $[s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}$ and $[n]_q ! = [n]_q [n-1]_q \cdots [1]_q$ is the quantum factorial. We wish to calculate the pairing on general monomials in $X^\pm)_i$ and H_j^+, ζ_j. To this end, define $\mathcal{R}_{h,i}$ as

$$\tilde{\mathcal{R}}_{h,i} = \sum_{k=0,l=0}^{\infty} \frac{h^{k+l} (X_i^+)^l (H_i^+)^k \otimes (\zeta_i)^k (X_i^-)^l}{k! [l]_q !}. \tag{4.16}$$

By the quantum double construction, this is the R-matrix for $U_h(sl_2)$ for the i-th simple root. Using the braid group action, one can define the R-matrix for general positive root $\beta_r = T_{i_1}(\cdots T_{i_r - 1}(\alpha_{i_r}))$ as follows. We note that the algebra automorphisms T_i can be defined on $U_q(sl_n^\epsilon)$. The definition can be found in section 4.3.

$$\tilde{\mathcal{R}}_{h,\beta_r} = (T_{i_1} \cdots T_{i_{r-1}} \otimes T_{i_1} \cdots T_{i_{r-1}})(\tilde{R}_{h,i_r}), \tag{4.17}$$

$$\tilde{\mathcal{R}}_{h,<\beta_r} = \tilde{\mathcal{R}}_{h,\beta_{r-1}} \cdots \tilde{\mathcal{R}}_{h,\beta_1}. \tag{4.18}$$

We have the following proposition, see section 4.3 for the proof.

Proposition 4.1.3. *(Comultiplication) For any $\beta \in \Delta^+$,*

$$\Delta_h(X_\beta^+) = \tilde{\mathcal{R}}_{h,<\beta}^{-1} (X_\beta^+ \otimes e^{\epsilon h H_\beta^-} + 1 \otimes X_\beta^+) \tilde{\mathcal{R}}_{h,<\beta}.$$

Note that if $\beta = \sum_i k_i \alpha_i$, then $H_\beta = \sum_i d_i k_i H_i$, where $d_i = 1$ are the Cartan integers, where we restrict ourselves to sl_n.

So it is possible to quantize the algebra on the simple generators and know the comultiplications on the non-simple generators. We turn the PBW basis consisting of monomials in generators into a dual basis to obtain the R-matrix. To this end, we define the following generators. Let $A = a_{ij}$ be the Cartan matrix. Remember $\zeta_i = \sum (A^{-1})_{ij} H_j^-$. The following pairing for monomials in the dual generators can be calculated by using the comultiplication.

Proposition 4.1.4.

$$\langle \prod_{i=1}^{N} (X_i^+)^{m_i} \prod_{i=1}^{n} (H_i^+)^{p_i}, \prod_{i=1}^{n} (\zeta_i)^{p_i'} \prod_{i=1}^{N} (X_i^-)^{m_i'} \rangle = \prod \delta_{m_i, m_i'} \prod \delta_{p_i, p_i'} \prod h^{m_i + p_i} \prod [m_i]_q !.$$

where $[n]_q = \frac{q^{-n} - q^n}{q^{-1} - q^1}$.

The proof makes use of proposition 8.3.7 in [6].

Proof. Let us scetch the proof of the general case. The proof is by induction, using proposition 8.3.7 in [6] that is proven in section 4.2.1. We apply Δ to the non-capital side, after which we only need to count the tensor-products that pair non-zero. Note that our algebra has the same pairing as the $U_q(b^\pm)$ dual pairing in [6], except for the factor of ϵ, and the correction with $q - q^{-1}$.
In particular, since ζ_i are dual to H_i^+, the basis of $U_q(b^-)$ corresponds to the $\{\zeta_i, \mu_{\beta_i}\}$ basis in [6]. The different conventions for the comultiplication of X_i^+ in [6] result in a factor of $q^{1/2t_r(t_r-1)}$ present in 8.3.7 in [6] that is absent here. Looking at our $\mathcal{R}_i$, in particular the prefactor R_{n_i} present in the sum, we get the required result.

□

Now this construction is finished, we can write down the universal R-matrix corresponding to $U_q(sl_n^\epsilon)$. The hardest part is calculating the PBW basis and the corresponding dual, and the multiplication relations between the generators. The quantum Serre relations together with the braid group action provide the multiplication relations between the PBW generators.

4.2. Representation theory of $U_q(sl_2^\epsilon)$

Let us proceed with calculating the comultiplication of the quantized Lie bialgebra $U_q(sl_2^\epsilon)$. Before we are able to properly calculate the comultiplication (in such a way that is generalizable, anyway), we need to look at the finite dimensional representations of $U_q(sl_2^\epsilon)$. In this section, we may write $U_q(sl_n^\epsilon)$, $U_h(sl_n^\epsilon)$ or $H_{n,\epsilon}$ for the quantization of sl_n^ϵ.
First we note that if $\epsilon^2 = 0$, it is impossible to define the q-Weyl group in the way it is usually done, since the finite dimensional highest weight representations cannot be constructed. A solution to this problem is to work over the field $\mathbb{R}(\epsilon)$, and prove afterwards that all components of the desired identity lie in $\mathbb{R}[[\epsilon]]$, so that we can divide out to (ϵ^2). For the remainder of this section we will work over the field $\mathbb{R}(\epsilon)[[h]]$.
If $\epsilon = 1$, and one divides out to $H^+ - H^-$ one gets $U_q(sl_2)$. This algebra is obtained by taking the quantum double of the upper triangular matrix subalgebra, with the Hopf structure calculated earlier. For a description of the representations of the regular $U_h(sl_2)$ see for example [6]. We will take the op quantum double construction in this section, instead of the cop construction.
We consider the algebra $U_q(sl_2^\epsilon)$, also denoted as $H_{2,\epsilon}$ for short, generated by X^+, X^-, H^+ and H^- and the following relations. Note that we introduce ϵ in the b^+ multiplication relations. Moreover, our conventions match the conventions used in [29]. In particular note the factor of $\frac{1}{q-q^{-1}}$ in the commutator between

$X^{\pm}$.

$$[X^-, H^-] = 2X^-,\ [X^+, H^+] = -2\epsilon X^+,\ [X^-, H^+] = 2\epsilon X^-,\ [X^+, H^-] = -2X^+$$
$$[X^-, X^+] = \frac{q^{H^-} - q^{-\epsilon^{-1}H^+}}{q - q^{-1}} \tag{4.19}$$

Note that we scaled the generators X^- by a factor of $\frac{1}{(q-q^{-1})}$ with respect to the algebra in the first chapter. Substituting $H^+ = 2A - B$ and $H^- = a$ yields the familiar algebra structure, where B is left out when considering only the $U_q(sl_2^\epsilon)$ subalgebra. We take $q = e^{-h\epsilon}$, which will be useful when constructing the universal R-matrix. This is a different from the previous section. Note that ϵ is invertible.
Multiplying X^- with $q - q^{-1}$ yields an algebra over $\mathbb{R}[[\epsilon]]$ (formally we also have the parameter h, so it is an algebra over $\mathbb{R}[[\epsilon, h]]$). The final results of the construction are valid for non-invertible ϵ over the ring $\mathbb{R}[[\epsilon]]$, as discussed in the last section of this chapter.
We can take $\tilde{H}^+ = \epsilon^{-1}H^+$ instead of H^+. In this case, we have an algebra homomorphism with the algebra in [29], by sending our $\tilde{H}^+$ to Reshetikhin's H, (as well as sending our H^- to H) and substituting $q^{\frac{1}{2}}$ for q. $U_q(sl_2^\epsilon)$ agrees with example 3.2.1 in [23] in the same way. The comultiplication, antipode and R-matrix are given by the following formulas.

$$\Delta(X^-) = X^- \otimes e^{\epsilon h H^-/2} + e^{-\epsilon h H^-/2} \otimes X^-, \Delta(X^+) = X^+ \otimes e^{hH^+/2} + e^{-hH^+/2} \otimes X^+ \tag{4.20}$$
$$\Delta(H^{\pm}) = H^{\pm} \otimes 1 + 1 \otimes H^{\pm}, S(X^+) = -e^h X^+, S(X^-) = -e^{-\epsilon h} X^-, S(H) = -H.$$

We also introduce the q-commutator as

$$[A, B]_q = qAB - q^{-1}AB. \tag{4.21}$$

In constructing the representations of $U_q(sl_2^\epsilon)$, we will follow [29] (see Reshetikhin's website for this paper). Denote the representation map by $\pi : H_{2,\epsilon} \to End(V)$, where V is the 2-dimensional vector space generated by $\{e_{\frac{1}{2}}, e_{-\frac{1}{2}}\}$. We obtain the following actions, denoted in matrix notation, where we use the order of the

basis as indicated.

$$\begin{aligned}\pi(H^+) &= \begin{bmatrix}\epsilon & 0\\ 0 & -\epsilon\end{bmatrix} \\ \pi(X^+) &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \\ \pi(H^-) &= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \\ \pi(X^-) &= \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.\end{aligned} \tag{4.22}$$

To prove this is a representation, one needs to prove that the maps given above are indeed algebra homomorphisms. This is a straightforward exercise. Remember that we take $q = e^{-\epsilon h}$. In the case of $j = \frac{1}{2}$ we obtain the simplest case of [29], if we identify $\tilde{H}^+$ and H^-.
Following [29], we can denote the representation in the following more general way. For this section, we use the convention that $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. Furthermore, $\begin{bmatrix}n\\ k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, as usual.
For a finite dimensional module V^j of dimension $2j+1$ we get, where j is a positive integer or half integer and V^j is generated by the basis vectors e^j_m, $-j \leq m \leq j$,

$$\pi^j : U_h(sl_2^\epsilon) \to End(V^j) \tag{4.23}$$
$$\pi^j(X^+)(e^j_m) = ([j-m]_q[j+m+1]_q)^{1/2} e^j_{m+1} \tag{4.24}$$
$$\pi^j(X^-)(e^j_m) = ([j+m]_q[j-m+1]_q)^{1/2} e^j_{m-1} \tag{4.25}$$
$$\pi^j(H^+)(e^j_m) = 2m\epsilon e^j_m \tag{4.26}$$
$$\pi^j(H^-)(e^j_m) = 2m e^j_m. \tag{4.27}$$

Again, checking that this yields a representation is straightforwardly writing out the relations 4.19. So 4.19 becomes a quasitriangular Hopf algebra with the following R-matrix. See [30], or [6], if you use the variables $E = e^{hH^+/2}X$ and $F = e^{-\epsilon h H^-/2}Y$, and identify $\epsilon^{-1}H^+$ with H^-. Note that the factor $1/2$ in the exponential comes from the pairing between H^+ and H^-. Ultimately this is due to a different definition of the comultiplication, which is used in calculating the pairing between monomials. We know that the R-matrix is defined as the universal R-matrix of the Drinfel'd double. Hence it must be of the form $\sum e_a \otimes f^a$,

where e and f are dual bases. We obtain after a correction with a factor $\frac{1}{2}$

$$\mathcal{R} = \exp(hH^+ \otimes H^-/2) \sum R_n(h)(e^{hH^+/2}X^+)^n \otimes (e^{-\epsilon hH^-/2}X^-)^n \tag{4.28}$$
$$R_n(h) = \frac{q^{1/2n(n-1)}(1-q^{-2})^n}{[n]_q!}.$$

An application of the Hopf algebra automorphism $S \otimes S$ to the R-matrix shows that we obtain (note the plus instead of the minus sign in $\tilde{R}_n$):

$$\mathcal{R} = \sum \tilde{R}_n(h)(e^{-hH^+/2}X^+)^n \otimes (e^{\epsilon hH^-/2}X^-)^n \exp(hH^+ \otimes H^-/2). \tag{4.29}$$
$$\tilde{R}_n(h) = \frac{q^{1/2n(n+1)}(1-q^{-2})^n}{[n]_q!}. \tag{4.30}$$

An essential element we can construct with the R-matrix is the ribbon element, or more importantly, the inverse of the ribbon element. See [23] for the precise definition. In particular we can write $\nu = pu$ for the ribbon element, where $u = \sum \mathcal{R}^{(2)} S(\mathcal{R}^{(1)})$, and $p^2 = u^{-1}v$, $u^{-1} = \sum \mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)})$, $v = S(u)$. Using these identities, we get

$$p^{-1} = e^{\frac{1}{2}(H^+ + \epsilon H^-)}.$$

The square root w of the inverse ribbon element is not a part of the algebra $H_{2,\epsilon}$ since it cannot be expressed in terms of $X^\pm$. Writing these out in matrix notation makes clear that $\pi(X^\pm)$ don't generate the entire $End(V^j)$, except in the case of $j = \frac{1}{2}$ and $j = 1$, the standard representation of $H_{2,\epsilon}$. We will write out the action of w in the representations V^j later in this section, but it will be the case that w sends a basis vector e^j_m to the vector e^j_{-m}. In matrix notation this is the element with only non-zero entries on the 'mirrored' diagonal. Since this is true for any m, $\pi(w)$ cannot be written as a linear combination of $(H^\pm)^b$ and $(X^\pm)^a$, a and b positive integers, as any combination of these will yield non-zero off-diagonal entries.
This means that we have to add w to the Hopf algebra. To prove that this makes sense, we can use proposition 6.3.12 and example 6.3.13 of [23]. It turns out that we obtain another Hopf algebra, called the quantum Weyl group ([6]), and denoted by $\overline{H}_{2,\epsilon}$, in some sense the completion of $H_{2,\epsilon}$[30]. Proposition 6.3.12 can be used for $U_q(sl_n^\epsilon)$ as well, but one obtains a Weyl element associated with the longest root in sl_n, so this lemma will be of less use there.

Proposition 4.2.1. *Define the algebra automorphism T by*

$$T(H^+) = -\epsilon H^-, T(\epsilon H^-) = -H^+,\ T(X^\pm) = -q^{\pm 1}X^\mp,$$

and let ν^{-1} be the inverse of the Ribbon element. Then these data together with the quasitriangular structure $\mathcal{R}$ on $H_{2,\epsilon}$ define the Hopf algebra $\overline{H_{2,\epsilon}}$, which is generated by

$H_{2,\epsilon}$ and w, obeying the relations

$$wgw^{-1} = T(g),\ w^2 = \nu, \Delta(w) = \mathcal{R}^{-1} w \otimes w, \varepsilon(w) = 1, S(w) = we^{\frac{h}{2}(\epsilon H^- + H^+)}.$$

Here $g \in H_{2,\epsilon}$, and ε is the counit.

Before proving the proposition, let us note that if we take $\epsilon = 1$, and we identify H^+ with H^-, taking the sl_2 limit, the antipode agrees with [30], remembering that we have introduced a factor of 2 in our conventions. The conventions here agree directly with [6], however they use an asymmetric comultiplication. The expressions for T can be checked in representations by explicitly taking the square root in representations of ν^{-1}.

Proof. We check the conditions for proposition 6.3.12 in [23].
We have $T^2 = id$ and $T \otimes T(\mathcal{R}) = \mathcal{R}_{21}$, which is a consequence of the expression for $\mathcal{R}$ given before, and can be checked by explicit calculation. This means that ν^{-1} has to obey

$$\nu^{-1} \text{ is central, } \Delta(\nu^{-1}) = ((\nu^{-1} \otimes \nu^{-1})\mathcal{R}_{21}\mathcal{R}), T(\nu^{-1}) = \nu^{-1}.$$

The first two conditions are satisfied by definition of the ribbon element. See prop. 2.1.8 in [23]. As noted before, $\nu^{-1} = p^{-2}u^{-1}$. Also it is useful to observe $T \circ S = S^{-1} \circ T$. Taking the automorphism T of this expression, we get $T(p^{-2})T(u^{-1}) = p^2 T(\mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)})) = p^2 \mathcal{R}^{(1)} S^{-2}(\mathcal{R}^2) = p^2 v^{-1} = \nu^{-1}$.
We have to prove that T defines an algebra map and an anti-coalgebra map, and we have to prove that $\mathcal{R}$ is a 2-cocycle. The last condition follows by definition of a quasitriangular structure, as usual. The fact that T is an algebra automorphism is checked by checking the algebra relations, as is the case with a anti-coalgebra map. This is a straightforward exercise and is left to the reader. This shows that our map T and the ribbon element obey the relations of proposition 6.3.12 in [23]. As a result we obtain a Hopf algebra $\overline{H}_{2,\epsilon}$ which is generated by $H_{2,\epsilon}$ and w^{-1}, which obeys the following relations

$$wgw^{-1} = T(g),\ w^2 = \nu, \Delta(w) = \mathcal{R}^{-1} w \otimes w, \varepsilon(w) = 1, S(w) = wuS(w^{-2})$$

We are left with the calculation of the antipode of the Weyl element.
For the calculation of the element $p^2 = u^{-1}v$ we used the Mathematica implementation, for which we refer to the appendix. In the program we used the *cop*-convention of the double construction instead of the *op*. Note that the antipode S on $U_q(b^+)$ provides an isomorhpism between the two double constructions. Under this isomorphism, the R-matrix $\mathcal{R}_{12}$ is taken to $\mathcal{R}_{12}^{-1}$, see exercise 7.1.2 in [23]. A simple calculation using proposition 2.1.8 in [23] shows that p is invariant under this isomorphism.
Using similar arguments together with the general expression $S(w) = wuS(w^{-2})$ from proposition 6.3.12 in [23], we can show that in general the antipode of the

Weyl group element is given by $S(w) = wp^{-1}$. Observe that the w we use here is the inverse of the w defined in [23], by definition. This ends the proof of the proposition. □

The T defined here for sl_2 is not a braid group generator. However, this is not important in the sl_2 case since there is only one simple root. It is possible to make T a braid group generator in the sl_n case by performing a simple transformation on w. For this construction we will have to do more work.
Let us now calculate the action of the inverse ribbon element on a basis-vector e_m^j of a module V^j. We leave the representation-map π^j out of the notation.

$$\begin{aligned}\nu^{-1}(e_m^j) &= u^{-1}p^{-1}(e_m^j) = \sum \mathcal{R}^{(2)}S^2(\mathcal{R}^{(1)})e^{-\frac{1}{2}(H^+ + \epsilon H^-)}(e_m^j) \\ &= exp(hH^-H^+/2)\cdot \\ &\sum R_n(h)(e^{-\epsilon h H^-/2}X^-)^n((-1)^2 e^{2h\epsilon}e^{hH^+/2}X^+)^n e^{-\frac{1}{2}(H^+ + \epsilon H^-)}(e_m^j)\end{aligned} \tag{4.31}$$

Observe that since X^+ acts as a raising operator, only the terms with $0 \leq n \leq j - m$ act nonzero on e_m^j. Hence we obtain for such an n-term in the R-matrix, which we will sum over afterwards,

$$\begin{aligned}&= exp(hH^-H^+/2)R_n(h)(e^{-\epsilon h H^-/2}X^-)^n((-1)^2 e^2 h e^{hH^+/2}X^+)^n e^{\epsilon(2m)}(e_m^j) \\ &= exp(hH^-H^+/2)R_n(h)(e^{-\epsilon h H^-/2}X^-)^n(-1)^{2n}e^{2\epsilon h n}(e^{h\epsilon(m+1+\cdots+m+n)})e^{\epsilon(2m)}(e_{m+n}^j) \\ &= exp(hH^-H^+/2)R_n(h)(-1)^{2n}e^{2\epsilon h n} \\ &e^{h\epsilon\frac{1}{2}((m+n)(m+n+1)-m(m+1))}(e^{\epsilon h H^-/2}X^-)^n e^{-\epsilon(2m)}e_{m+n}^j \\ &= exp(hH^-H^+/2)R_n(h)(-1)^{2n}e^{2\epsilon h n}\cdot \\ &e^{h\epsilon\frac{1}{2}((m+n)(m+n+1)-m(m+1))}e^{h\epsilon\frac{1}{2}(m(m-1)-(m+n)(m+n-1))}e^{\epsilon(2m)}e_m^j\end{aligned} \tag{4.32}$$

Remember that the module V^j is generated by the highest weight vector e_j^j. Since the inverse of of the ribbon element is central (see [23]), it is enough to check equality on the highest weight vector. Then only the $n = 0$ term contributes, and we get the following identity

$$\nu^{-1}e_m^j = e^{h\epsilon(2j(j+1))}e_m^j. \tag{4.33}$$

By Schur's lemma the action of ν^{-1} is proportional to the identity. Note in particular the contribution from p^{-1}. The following expression is a square root of ν^{-1}, and as it turns out the only one that meets the requirements of proposition 4.2.1. It can be proven by direct calculation that it satisfies $wgw^{-1} = T(g)$ and $w^2 = \nu$. We get the required result:

$$w^{-1}e_m^j = (-1)^{-j+m}e^{h\epsilon(j(j+1)-m)}e_{-m}^j.$$

We have the following lemma.

Lemma 4.2.1. *Let w^{-1} be as in proposition 4.2.1. Then the action of w^{-1} in the highest weight module V^j is given by*

$$w^{-1}e_m^j = (-1)^{-j+m}e^{h\epsilon(j(j+1)-m)}e_{-m}^j. \tag{4.34}$$

Proof. By definition, w is the non-central square root of the inverse ribbon element ν^{-1}. To calculate the action of w in the representation we have to know which square-root we have to use, given that we are working with matrices, so there are multiple options, a priori. We claim that the square root in the representation is uniquely determined by two equations:

$$wgw^{-1} = T(g) \text{ and } w^2 = \nu.$$

This is proved by looking at the action of ν in the representation, and is translated into the following lemma, which will not be proven here, but can be proved by looking at the Jordan decomposition of $w(j)$, or by counting the degrees of freedom, alternatively.

Lemma 4.2.2. *Suppose w(j) is a 2j+1 by 2j+1 invertible matrix, and let $xp(j) = \pi(X^+)$ and $xm(j) = \pi(X^-)$, the action of $X^\pm$ in the representation V^j. Let λ be any invertible element of the underlying ring. Then w(j) is uniquely defined by the following two equations*

$$\begin{aligned} w(j)^2 &= \lambda Id(j) \\ w(j)xp(j)w(j)^{-1} &= xm(j). \end{aligned} \tag{4.35}$$

Here Id(j) is the 2j+1 times 2j+1 identity matrix.

Proof. □

It is clear that the scuare root w given above satisfies

$$wgw^{-1} = T(g) \text{ and } w^2 = \nu.$$

This finishes the proof. □

We know the explicit action of w in any finite dimensional $H_{2,\epsilon}$ module, and we can compare it with other definitions. One can check that the given square root of the inverse ribbon element yields the correct T, as stated in proposition 4.2.1. Usually, w^{-1} is defined by its action on all finite representations V^j of $H_{2,\epsilon}$, for example in [29] and others. This is possible if $H_{2,\epsilon}$ is semisimple as an algebra. An algebra is said to be semisimple if the set of elements that act as zero in every irreducible representation contains only zero. We know that $H_{2,\epsilon}$ is not semisimple however, since $\epsilon^{-1}H^+ - H^-$ acts as zero in every representation. This element exactly generates the ideal we need to divide out to, in order to get the $U_q(sl_2)$, which is a semisimple algebra. The non-semisimplicity implies that w is well defined up to terms $H^-, \epsilon^{-1}H^+$, which have the same action in any representation,

if we would define w by its action in the representation. Of course, this is a consequence of the fact that $\epsilon^{-1}H^+ - H^-$ is central in $H_{2,\epsilon}$. This element will turn out to be the term we gain in our final expression, with respect to the sl_n case.
The w defined here agrees, after the 'semisimplification' of $H_{2,\epsilon}$, with the quantum Weyl element in [30], and is the inverse of the w defined in example 6.3.13 in [23], as mentioned before. This can be seen from the action of w in the representation. Note that $U_q(sl_2)$ in [23] agrees with the conventions of [30].

4.3. Constructing the q-Weyl group of $U_q(sl_n^\epsilon)$

We proceed with constructing quantum Weyl group of $U_q(sl_n^\epsilon)$, with Cartan matrix a_{ij} of sl_n. We follow [30]. Note that the relations here agree with [21] and [30], when we divide out to $\epsilon^{-1}H_i^+ - H_i^-$. Before reading this section it is advised to study [30] in full detail, since we copy a large part of his calculations.
This algebra is non semi-simple. The Weyl group elements used in the main part of this section come from the $H_{2,\epsilon}$ case, where we defined it to be the square root of ν^{-1}. Later in this section we introduce $\overline{w}_i$, where w_i and $\overline{w}_i$ are related by a simple transformation with $q^{-\epsilon^{-1}H^+H^-}$, like the $U_q(sl_n)$ case [30]. It turns out to be the case that the algebra automorphisms obtained this way will yield a braid group representation.
$U_q(sl_n^\epsilon)$ is generated by $\{X_i^\pm, H_i^\pm\}$ and the relations

$$[X_i^-, H_j^-] = a_{ij}X_i^-,\ [X_i^+, H_j^+] = -a_{ij}\epsilon X_i^+,\ [X_i^-, H_j^+] = a_{ij}\epsilon X_i^-,\ [X_i^+, H_j^-] = -a_{ij}X_i^+ \tag{4.36}$$

$$[X_i^-, X_j^+] = \frac{q^{H_i^-} - q^{-\epsilon^{-1}H_i^+}}{q - q^{-1}}\delta_{ij},\quad \sum_{k=0}^{k=1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^2} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-a_{ij}-k} = 0$$

$$\Delta(X_i^-) = X_i^- \otimes e^{\epsilon h H_i^-/2} + e^{-\epsilon h H_i^-/2} \otimes X_i^-, \Delta(X_i^+) = X_i^+ \otimes e^{hH_i^+/2} + e^{-hH_i^+/2} \otimes X_i^+$$
$$\Delta(H_i^\pm) = H_i^\pm \otimes 1 + 1 \otimes H_i^\pm, S(X_i^+) = -e^h X_i^+, S(X_i^-) = -e^{-\epsilon h} X_i^-, S(H_i) = -H_i.$$

Where $q = e^{-\epsilon h}$. Using proposition 4.2.1, we will write down the action of the Weyl group in $U_q(sl_n^\epsilon)$ for each of the $H_{2,\epsilon}$ subalgebras. The quantum Weyl elements are defined via the $H_{2,\epsilon}$ submodules of the representations of $U_q(sl_n^\epsilon)$, for each simple root α_i, $i = 1, \cdots, rank(sl_n) = n-1$ of sl_n.
$U_q(sl_n^\epsilon)$ is a quasitriangular Hopf algebra, as has been noted in the first section of this chapter. In this section we denote the R-matrix as $\mathcal{R}$, and its inverse as $\mathcal{R}^{-1}$. The notation $\overline{\mathcal{R}}$ is reserved for other purposes. The same notation will be used for the partial R-matrices of $U_q(sl_2^\epsilon)$.

In the notation of [30], let V^λ be a representation of $U_q(sl_n^\epsilon)$. We know that V^λ are highest weight representations [6]. $U_q(sl_n^\epsilon)$ is generated by $n-1$ copies of the subalgebras $H_{2,\epsilon,i}$ corresponding to the simple roots. In each of these copies

we can find a corresponding Weyl element w_i with proposition 4.2.1. We know that V^λ factorizes into irreducable $H_{2,\epsilon}$-submodules V^j. Checking this fact comes down to checking relations on simple generators. This is equivalent to lemma 2 in [30], which proves that if V is an $H_{2,\epsilon,i}$ module $X_j^\pm(V)$ is still an $H_{2,\epsilon,i}$ submodule. In general we have, for V^j irreducible $H_{2,\epsilon,i}$-submodules for the usual half integer values for j,

$$V^\lambda = \bigoplus_j (Hom(V^j, V^\lambda) \otimes V^j) = \bigoplus_j (W_j^\lambda \otimes V^j).$$

We define $W_j^\lambda = Hom_k(V^j, V^\lambda)$, with $k = \mathbb{R}(\epsilon)$. The isomorphism

$$f : \bigoplus_j (Hom(V^j, V^\lambda) \otimes V^j) \to V^\lambda$$

is given by $(\cdots, 0, \phi_j \otimes e_m^j, 0, \cdots) \mapsto \phi_j(e_m^j)$, evaluation. Since the $H_{2,\epsilon,i}$ modules V^j are irreducible submodules of V^λ, the homomorphisms $\phi_j \in Hom(V^j, V^\lambda)$ are, by Schurs lemma, the identity on V^j, scaling the constant to 1 without loss of generality.
Since $U_q(sl_n^\epsilon)$ is generated by the $H_{2,\epsilon,i}$ subalgebras corresponding to simple roots, we can assume that $Im(\phi_i)$ is a copy of V^i in V^λ as $H_{2,\epsilon,i}$-submodule, where $i = 1, \cdots n-1$. The action of $H_i^\pm$ on an element $\phi \otimes e_m^j$ is then given by $H_i^\pm \circ \phi \otimes e_m^j = H_i^\pm(\phi(e_m^j)) = -a_{ij}(\epsilon)^{(1\pm1)/2} m(\phi_j(e_m^j)$, since $[H_i^\pm, X_j^\pm] = \mp(\epsilon)^{(1\pm1)/2} a_{ij} X_j^\pm$, so the submodules $V^j \subset V^\lambda$ are invariant under the action of $H_i^\pm$. In general, the submodules V^j are not invariant under the action of $X_j^\pm$. This action is more complicated, and although it yields another $H_{2,\epsilon}$-submodule, it may not be the same submodule.
Define the elements w_i acting on $U_q(sl_n^\epsilon)$ via representations by

$$w_i = \bigoplus_j (I_{w_j^\lambda} \otimes (w_i)_j),$$

where $I_{w_j^\lambda}$ is the identity on W_j^λ. The $(w_i)_j$ are then defined via proposition 4.2.1, where $(w_i)_j$ acts on the $H_{2,\epsilon,i}$ submodule V^j by the action calculated in lemma 4.2.1. Note that in the semisimple case, like in [30], this definition would uniquely define w_i. In the non-semisimple case we need to require that the Weyl property holds for conjugation with w_i. Then the w_i become well-defined on the Cartan subalgebra.
Let us calculate $w_i H_j^+ w_i^{-1}$ from the definition of w_i, by comparing the action on irreducable modules. By the previous discussion we can check this for the factorization of V^λ into $H_{2,\epsilon,i}$-submodules, the i corresponding to the simple roots. We denote the vectors as e_m^n, omitting the ϕ. In this case, let $e_m^j \in V^n$ be any vector in any irreducible $H_{2,\epsilon,j}$-submodule $V^n \subset V^\lambda$ of highest weight n. Let $i \neq j$ (in the

case $i = j$ we get the action from proposition 4.2.1), then H_j acts only nonzero if $i = j \pm 1$.

$$w_i H_j^+ w_i^{-1}(e_m^n) = w_i H_j^+(e_{-m}^n) = a_{ij}\epsilon m w_i(e_{-m}^n) = a_{ij}\epsilon m e_m^n = -\epsilon m e_m^n.$$

Note that $a_{ij} = -1$. On the other hand we have

$$(\epsilon)^{-(-1\mp 1)/2}(H_j^\pm - a_{ij}H_i^\pm)(e_m^n) = (-a_{ji} + a_{ij}a_{ii})\epsilon m e_m^n = -m\epsilon e_m^n.$$

From this we can conclude that $w_i H_j^+ w_i^{-1}$ acts as $(\epsilon)^{-(-1\mp 1)/2}(H_j^\pm - a_{ij}H_i^\pm)$ in $H_{2,\epsilon,i}$-modules. From proposition 4.2.1 we obtain the $T_j(H_j)$ relations. Since the algebra is not semisimple, we can always add a term $H_i := \epsilon^{-1}H_i^+ - H_i^-$ (and idem for H_j) and get the same action in representations.
However, if we take the $T_j(H_j)$ relations together with the requirement that $T_iT_jT_i = T_jT_iT_j$ we get the following relations. The proof is by straightforwardly checking the Weyl-relation. We leave this to the reader. The requirement that the Weyl-property holds could be seen as a definition of the action of w_i on H_j, since it uniquely determines this action.

Lemma 4.3.1.

$$T_i(H_j^+) = \epsilon H_j^- - \epsilon a_{ij}H_i^-, T_i(\epsilon H_j^-) = H_j^+ - a_{ij}H_i^+,\ T_i(X_i^\pm) = -q^{\pm 1}X_i^\mp. \quad (4.37)$$

For sl_n, $a_{ij} = 2$ if $i = j$, $a_{ij} = -1$ if $i = j \pm 1$ and zero else. By proposition 4.2.1 we now have

$$T_i(e) = w_i e w_i^{-1}, \Delta(w_i) = \mathcal{R}(i)^{-1} w_i \otimes w_i, \quad (4.38)$$

where $\mathcal{R}(i)$ is the partial R-matrix on the i-th $H_{2,\epsilon}$-subalgebra defined by 4.28. Remember that the adjoint action of $U_q(sl_n^\epsilon)$ on itself is given by

$$ad_e(f) = \sum e_{(1)} f S(e_{(2)}).$$

We denote the adjoint action for short as $\circ$, in multiplicative notation. Using this definition, we calculate $w_i X_j^\pm w_i^{-1}$.

Let us define two sets of generators that make the comultiplication anti-symmetric, and correspond to the two ways to write the R-matrix in the $H_{2,\epsilon}$ case. Note that our definitions agree with the definitions of [30].

$$E_i = q^{\epsilon^{-1}H_i^+/2}X_i^+, F_i = q^{-H_i^-/2}X_i^-, K_i^+ = q^{\epsilon^{-1}H_i^+/2}, K_i^- = q^{H_i^-/2},$$
$$\overline{E}_i = q^{-\epsilon^{-1}H_i^+/2}X_i^+, \overline{F}_i = q^{H_i^-/2}X_i^-.$$

Via the adjoint action we have an action of the q-Weyl element on these generators. We now have the following lemmas, the proof of which are equivalent to

the proofs given in [30], except for a factor of two in the definitions of $H^\pm$ resp. H in [30], after reducing the Cartan subalgebra. The reason we cannot directly follow Reshetikhin and Kirrilov's proof is that we have to introduce the +/- back into the equations, so we have to check all the relations manually. It is insightful to study this proof, and the proof of proposition 2.2.1 in chapter 4 of [21], which are roughly the same. The proof of lemma 4.3.2 is by explicit calculation of the adjoint action of the Weyl group element.

Lemma 4.3.2.

$$\begin{aligned} w_i \circ \overline{E}_j &= w_i \overline{E}_j (K_i^+)^{a_{ij}} w_i^{-1} \\ w_i \circ F_j &= S(w_i^{-1})(K_i^-)^{-a_{ij}} F_j S(w_i). \end{aligned} \tag{4.39}$$

Proof. We will only scetch the proof here, since this proof is exactly the same as the proof of lemma 1 in [30], where we are keeping track of $H^\pm$ and the different factors 2. Note that we start with the expression 4.29 and let $\mathcal{R}^{-1} = S \otimes id(\mathcal{R})$. Then we use $\mathcal{R} = S \otimes S(\mathcal{R})$ to rewrite the adjoint action of w_i on $\overline{E}_j$ and F_j. After using the commutation relations, the final result is obtained by using the fact that $u^{-1} = \sum \mathcal{R}_2 S^2(\mathcal{R}_1)$, and $uhu^{-1} = S^2(h)$ for any $h \in H - 2, \epsilon$, so that u commutes with $K_i^\pm$. Note that $w^{-1} = (up)^{-1/2} = (up)^{-1} w = u^{-1} S(w)$. This ends the proof. □

Lemma 4.3.3. *The sets $V_{ij} = \{\overline{E}_j, \cdots, \overline{E}_i^{-a_{ij}} \circ \overline{E}_j\}$ and $\overline{V}_{ij} = \{F_j, \cdots, F_i^{-a_{ij}} \circ F_j\}$ are irreducible $H_{2,\epsilon,i}$-modules of weight $-a_{ij}$.*

Proof. Lemma 4.3.3 can be concluded directly from the algebra relations. Since we used the same conventions as [30], the relations are exactly the same. An explicit isomorphism between both sets and $V^{a_{ij}}$ is given by the maps

$$\begin{aligned} \phi(F_i^n \circ F_j) &= c_{ij}^- \sqrt{\frac{[n]_q!}{[-a_{ij}-n]_q!}} e^{\frac{-a_{ij}}{2}} \\ \psi(\overline{E}_i^n \circ \overline{E}_j) &= c_{ij}^+ \sqrt{\frac{[-a_{ij}-n]_q!}{[n]_q!}} e^{\frac{-a_{ij}}{2}} \end{aligned} \tag{4.40}$$

Note that because $\epsilon^{-1} H^+ - H^-$ acts as zero in V^j, we can make multiple choices for V_{ij} and $\overline{V}_{ij}$ that are isomorphic to V^j. We will parametrize these choices by the two parameters $c_{ij}^\pm$ in the future, where $c_{ij}^\pm$ stands for a central factor. □

Lemma 4.3.4.

$$\begin{aligned} \overline{E}_i^n \circ \overline{E}_j &= (K_i^+)^{-n} (K_j^+)^{-1} [X_i^+, \cdots [X_i^+, X_j^+]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{(a_{ij}+2n)/2-1}} \\ S(F_i^n \circ F_j) &= -q^{(-n-1)} [X_i^-, \cdots, [X_i^-, X_j^-]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{(a_{ij}+2n)/2-1}} (K_i^-)^n (K_j^-) \end{aligned} \tag{4.41}$$

Proof. The relations follow by induction to n from the algebra relations. We write down the case for $n = 1$ here by using the definition of the adjoint action.

$$\begin{aligned}\overline{E}_i \circ \overline{E}_j &= \overline{E}_i\overline{E}_j - e^{-H_i^+}\overline{E}_j e^{H_i^+}\overline{E}_i \\ &= \overline{E}_i\overline{E}_j - q\overline{E}_j\overline{E}_i = q^{1/2}(K_i^+)^{-1}(K_j^+)^{-1}q^{-1/2}(q^{-1/2}X_i^+X_j^+ - q^{1/2}X_j^+X_i^+) \\ &= (K_i^+)^{-1}(K_j^+)^{-1}[X_i^+, X_j^+]_{q^{-1/2}}.\end{aligned} \tag{4.42}$$

The case for higher n follows in the same fashion. The second formula follows in the same way, except we take the antipode on both sides afterwards. This proves the lemma. □

We can use the lemmas to calculate the algebra automorphisms associated with the quantum Weyl group explicitly. To this end let us introduce

$$\overline{w}_i = w_i q^{-\epsilon^{-1}H_i^+H_i^-/4}.$$

Lemma 4.3.5.

$$\Delta(\overline{w}_i) = \overline{\mathcal{R}}(i)^{-1}\overline{w}_i \otimes \overline{w}_i, \tag{4.43}$$

$$\overline{\mathcal{R}}(i) = q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum R_n(h)(e^{-hH_i^+/2}X_i^+)^n \otimes (e^{\epsilon hH_i^-/2}X_i^-)^n, \tag{4.44}$$

$$R_n(h) = \frac{q^{\frac{1}{2}n(n+1)}(1-q^{-2})^n}{[n]_q!}. \tag{4.45}$$

Proof. This can be calculated by straightforward computation from proposition 4.2.1, where it is important to remember that $q^{-\epsilon^{-1}H^+H^-/4}$ is not group-like, but that a correction appears when taking the coproduct. □

Note from the previous section that the partial R-matrix

$$\overline{\mathcal{R}}(i) = q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum \frac{(1-q^{-2})^n}{[n]_q!}E_i^n \otimes F_i^n.$$

Since the E, F have an antisymmetric coproduct, it is this form in which we will later recognize the algebra we are using, only with X, Y and Z instead of E_i and lowercase letters instead of F_i. Denote algebra automorphisms of $U_q(sl_n^\epsilon)$ as

$$T_i(h) = \overline{w}_i^{-1}h\overline{w}_i, \forall h \in U_q(sl_n^\epsilon). \tag{4.46}$$

We have the following formula, which is a direct consequence of the above lemma. (Denote $\mathcal{R}(i) = \mathcal{R}_i$).

$$\Delta(T_i(X_j^\pm)) = \overline{\mathcal{R}}_i T_i \otimes T_i(\Delta(X_j^\pm))\overline{\mathcal{R}}_i^{-1}.$$

The following theorem holds true and is proven by combining the above lem-

mas, together with the action of w_i in the irreducible modules. Note that the T_i are denoted in the opposite way as T in proposition 4.2.1. Observe that in the $U_q(sl_n)$ case, T_i are the quantization of the classical Weyl group action [6] in the sense that the action of T_i corresponds to the action of the simple reflections s_i corresponding to the simple roots α_i. Looking at the first order in h we obtain the same fact. One can compare the T_i with the action of the simple reflections given in the first section and convince oneself of this fact.

Theorem 4.3.1. *The T_i as defined above are well-defined up to central factors c_{ij} depending on $K_{i,j}^{\pm}$. Moreover, the T_i are $U_q(sl_n^\epsilon)$ algebra automorphisms, given by*

$$
\begin{aligned}
&T_i(K_j^+) = K_j^-(K_i^-)^{-a_{ij}}, T_i(K_j^-) = K_j^+(K_i^+)^{-a_{ij}}, \qquad (4.47)\\
&T_i(X_i^+) = -X_i^-(K_i^-)^{-1}(K_i^+)^{-1}, T_i(X_i^-) = -(K_i^-)(K_i^+)X_i^+,\\
&T_i(X_j^+) = c_{ij}(-1)^{a_{ij}}[-a_{ij}]_q!((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}},\\
&\qquad [X_i^+,\cdots,[X_i^+,X_j^+]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}((K_i^-)(K_i^+)^{-1})^{a_{ij}/2}, i\neq j\\
&T_i(X_j^-) = c_{ij}^{-1}\frac{1}{[-a_{ij}]_q!}((K_j^+)^{-1}(K_j^-))^{-1}((K_i^-)(K_i^+)^{-1})^{a_{ij}},\\
&\qquad [X_i^-,\cdots,[X_i^-,X_j^-]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}, i\neq j
\end{aligned}
$$

and when $a_{ij}=0$, $T_i(X_j^{\pm}) = X_j^{\pm}$.

Proof. The main objective is to prove that conjugation with the Weyl element respects the algebra structure. We will first prove that the T_i are of the form given above. This follows directly from lemmas 4.3.3 and 4.3.2 together with the action of w_i. We can then rewrite $\overline{E}_i^{-a_{ij}} \circ \overline{E}_j$ with lemma 4.3.4, taking $n = -a_{ij}$, since w_i takes the lowest weight vector $\overline{E}_j$ in the module homomorphic to $V^{a_{ij}}$ (and the same for F_j) to the highest weight vector, which is $\overline{E}_i^{-a_{ij}} \circ \overline{E}_j$, by the action of w_i in irreducible modules. This gives the desired relations for T_i applied on $X^{\pm}$ up to a central factor depending on i and j, due to the non-semisimplicity. Making this choice is equivalent to choosing a root of ν^{-1}. This shows that the T_i are not well defined, when defined from the quantum Weyl group, up to a central factor. Once we choose a root of ν^{-1}, we make a choice for c_{ij}. Concretely, this choice corresponds to choosing isomorphisms ϕ and ψ in lemma 4.3.3.
Let us get some specific values for c_{ij}. Firstly, the relations $T_i(K_j^{\pm}) = (K_j^{\mp})(K_i^{\mp})^{-a_{ij}}$ and $T_i(X_i^{\pm})$ follow from proposition 4.2.1 and lemma 4.3.1. Consider $T_i([X_j^{\pm}, X_j^{\mp}])$. Since we want T_i to be an algebra homomorphism, we get the requirement $c_{ij}^+ c_{ij}^- = 1$ for all i and j, since c_{ij} is central (it consists of powers of $\epsilon^{-1}H_i^+ - H_i^-$). In particular, c_{ij} must be invertible.

Hence we have the following relations.

$$
\begin{aligned}
&T_i(K_j^+) = K_j^-(K_i^-)^{-a_{ij}}, T_i(K_j^-) = K_j^+(K_i^+)^{-a_{ij}}, \\
&T_i(X_i^+) = -X_i^-(K_i^-)^{-1}(K_i^+)^{-1}, T_i(X_i^-) = -(K_i^-)(K_i^+)X_i^+ \\
&T_i(X_j^+) = c_{ij}(-1)^{a_{ij}}[-a_{ij}]_q!((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}} \\
&\qquad [X_i^+, \cdots, [X_i^+, X_j^+]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{-a_{ij}/2-1}}((K_i^-)(K_i^+)^{-1})^{a_{ij}/2}, i \neq j \\
&T_i(X_j^-) = \frac{1}{[-a_{ij}]_q!}c_{ij}^{-1}((K_j^+)^{-1}(K_j^-))^{-1}((K_i^-)(K_i^+)^{-1})^{a_{ij}} \\
&\qquad [X_i^-, \cdots, [X_i^-, X_j^-]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}} \cdots]_{q^{-a_{ij}/2-1}}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}, i \neq j.
\end{aligned}
\tag{4.48}
$$

Where we will write simply c_{ij} for c_{ij}^+. Comparing T_i with [30] we can conclude that the T_i must be algebra-automorphisms, since $(K_i^-)(K_i^+)^{-1}$ is a central element for all $i = 1, \cdots n-1$. The proof that T_i are automorphisms coincides then with the proof in [30]. This concludes the proof. □

From the lemmas in this section together with proposition 4.2.1 we obtain the following fact. It can be proven by direct verification, when we remember that $a_{ij} = -1$ for $i \neq j$, since we are working with the sl_n Cartan matrix. To prove it for a general Cartan matrix, it is enough to consider rank 1 and 2 cases [6], [30].

Theorem 4.3.2. *The c_{ij} in theorem 4.3.1 are group-like elements in H_i and H_j, with $H_i = \epsilon^{-1}H_i^+ - H_i^-$, $i = 1, \cdots n-1$.*

Proof. We know that c_{ij} are invertible from the proof of the last theorem. Note that for any invertible element c, we have $\Delta(c^{-1}) = \Delta(c)^{-1}$. Also, c_{ij} are central elements, and live in the Cartan subalgebra of $U_q(sl_n^\epsilon)$. As a consequence, c_{ij} must be a power-series in the elements $\epsilon^{-1}H_i^+ - H_i^- = H_i$, since these are the only central elements that are contained in the Cartan subalgebra. This follows from the relations $[(H_i^+)^k, X_j^\pm] = (H_i^+ \pm a_{ij})^k X_j^\pm - (H_i^\pm)^k X_j^\pm \neq X_j^\pm$.
We see that $\Delta(c_{ij})$ must also be central, since $\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i$. As a consequence of this, $\Delta^{op}(c_{ij}) = \mathcal{R}^{-1}\Delta(c_{ij})\mathcal{R} = \Delta(c_{ij})$. So if we can prove that $\Delta(c_{ij}) = (c_{ij} \otimes c_{ij})\Psi$ for some invertible 2-cocycle Ψ, then we know that Ψ is also a central element, with $\Psi_{21} = \Psi$ that lives in the tensor product of the Cartan-subalgebra with itself, since the tensor algebra of the Cartan part of $U_q(sl_n^\epsilon)$ is closed under taking the coproduct. That Ψ is a 2-cocycle means that $1 \otimes \Psi(id \otimes \Delta)(\Psi) = \Psi \otimes 1(\Delta \otimes id)(\Psi)$.
Since the c_{ij} are invertible, let us write $\Delta(c_{ij}) = (c_{ij} \otimes c_{ij})\Psi$. Since c_{ij} are central, so is Ψ. Let us write $\Psi = c \otimes c\Delta(c^{-1})$, where $c = c_{ij}^{-1}$ depends on H_i for some i without loss of generality. The fact that Ψ is central and symmetric and only dependent on elements of the Cartan subalgebra follows from the properties of

c_{ij}. We prove now that Ψ is a 2-cocycle.

$$\begin{aligned}1\otimes\Psi(id\otimes\Delta)(\Psi) &= 1\otimes c\otimes c(1\otimes\Delta(c^{-1}))c\otimes\Delta(c)id\otimes\Delta(\Delta(c^{-1}))\\ &= 1\otimes c\otimes c(c\otimes\Delta(cc^{-1}))id\otimes\Delta(\Delta(c^{-1}))\\ &= c\otimes c\otimes c(\Delta\otimes id)(\Delta(c^{-1}))\\ &= \Psi\otimes 1(\Delta\otimes id)(\Psi).\end{aligned}$$

The second-last equality follows from coassociativity and homomorphism property of Δ. The last equality follows from the definition of Ψ and the fact that Δ is a homomorphism.
Since Ψ is central it is a power series $\sum_{l_i,k_i\in\mathbb{N}}\prod_{i=1\cdots n-1} f_{\mathbf{k},\mathbf{l}}H_i^{k_i}\otimes H_i^{l_i}$, for central elements H_i and vectors $\mathbf{k}=(k_1,\cdots,k_{n-1})$ and $\mathbf{l}=(l_1,\cdots,l_{n-1})$, where $n-1$ indicates the rank of a_{ij}. However, the case where Ψ is dependend on multiple H_i can be reduced to the case of one variable.
The reason is that Ψ factorizes into terms corresponding to the simple roots. This fact follows from requiring that Ψ is a 2-cocycle and the fact that H_i^n are linearly independend for different i and n. This yields $f_{\mathbf{k},\mathbf{l}}=\prod_i f_{(0,\cdots,k_i,0,\cdots),(0,\cdots,0,l_i,0,\cdots,0)}$, by comparing different terms and using a straightforward induction argument. Conceptually, the general argument is equivalent to the case for two variables H_1 and H_2. This is left to the reader.
Let us note that we may take $f_{0,0}=1$ without loss of generality, as the invertibility of c_{ij} implies that Ψ is invertible. This is the case if and only if $f_0\neq 0$. So it is enough to look at the case where $\Psi=\sum_n f_{n_i}H_i^{n_i}\otimes H_i^{n_i}$.
Furthermore, we know that Ψ is symmetric, in the sense that when the tensor factors of Ψ are interchanged, we obtain Ψ. This forces

$$\Psi=\sum_{l_i,k_i\in\mathbb{N}}\prod_{i=1\cdots n-1} f_{\mathbf{k}+\mathbf{l},\mathbf{l}}(H_i^{k_i}\otimes 1+1\otimes H_i^{k_i})(H_i^{l_i}\otimes H_i^{l_i}).$$

We claim that $\mathbf{k}=0$ in the above expression. To see this it is enough to consider the case where Ψ only depends on one central variable, say H_i, by the previous discussion. By plugging in $\Psi=\sum_{l,k\in\mathbb{N}} f_{k+l,l}(H_i^k\otimes 1+1\otimes H_i^k)(H_i^l\otimes H_i^l)$ into the cocycle condition (putting $H_i=H$ for simplicity) we get

$$\begin{aligned}&\sum_{i,j,k,l\geq 0} f_{i+k,k}f_{j+l,l}(H^{i+k}\otimes H^k\otimes H^l+H^k\otimes H^{i+k}\otimes H^l)\\ &(\Delta(H)^{j+l}\otimes 1+\Delta(H)^l\otimes H^j)\\ &=\sum_{i,j,k,l\geq 0} f_{i+k,k}f_{j+l,l}(H^l\otimes H^{i+k}\otimes H^k+H^l\otimes H^k\otimes H^{i+k})\\ &(1\otimes\Delta(H)^{j+l}+H^j\otimes\Delta(H)^l).\end{aligned}$$

Let us look at the prefactor of the term $H\otimes 1\otimes 1$. Since we assumed $f_{0,0}=1$, we get $2f_{1,0}=3f_{1,0}$, so $f_{1,0}=0$. We use here that $\Delta(H)=H\otimes 1+1\otimes$

H. In general, we get as a prefactor of $H^m \otimes 1 \otimes 1$ the equation $2f_{n,0} = 3f_{n,0} + f_{n-1,0}f_{1,0} + f_{n-2,0}f_{2,0} + \cdots + f_{1,0}f_{n-1,0}$. Assuming $f_{i,0} = 0$ for $i < n$ as induction hypothesis, we get $f_{n,0} = 0$. This proves the claim. So Ψ can only be a power series in $H_i \otimes H_i$. Hence we can write $\Psi = \sum_{n_i,n_j} f_{n_i} H_i^{n_i} \otimes H_i^{n_i}$.
Without loss of generality we assume that Ψ depends only on $H_i = H$. We wish to prove that $\Psi = 1 \otimes 1$ if Ψ is an invertible, central, symmetric 2-cocycle that is an element of the Cartan subalgebra. Then we are finished, since the only elements that have a coproduct of the form $\Delta(c) = c \otimes c$ are elements $c = e^{eH}$. See for example lemma 6.4.1 in [6].
Let us now prove that $\Psi = 1 \otimes 1$ if Ψ is an invertible, central, symmetric 2-cocycle. We will do this by explicitly checking the 2-cocycle condition for $\Psi = \sum_n f_n H^n \otimes H^n$. From inserting the power series for Ψ into the 2-cocycle condition we get, with a straightforward substitution of summetion variables,

$$\begin{aligned} &\sum_{n,k,p=0}^{p=n} f_n f_k \binom{n}{p} H^{k+p} \otimes H^{k+n-p} \otimes H^n \\ &= \sum_{n,k,p=0}^{p=n} f_{k+p} f_{n-p} \binom{k+p}{p} H^{k+p} \otimes H^{k+n-p} \otimes H^n. \end{aligned}$$

We take $f_0 = 1$ without loss of generality. By linear independence of the generators we can compare term by term. We claim that this relation is satisfied only if $\Psi = \sum_n f_1 \frac{H^n \otimes H^n}{n!}$. We have to prove that the equation above holds only if $f_{n+1} = \frac{f_1^{n+1}}{n+1!}$. We observe that a given combination of exponents of the Hs appears only once in the equation above. The base case follows from the $k = 1$ and $n = 1$, $p = 1$-term: $f_1^2 = 2f_2 f_0$. Suppose that for some $l > 0$ the formula holds. Then we look at the terms with $n = l$ such that $H^{k+p} \otimes H^{k+n-p} \otimes H^n = H^{l+1} \otimes H \otimes H^l$. This implies that $k = 1$ and $p = l$. Writing down the coefficient of this term we immediately get $f_{l+1} = \frac{f_l f_1}{l+1}$, the desired result.
However, implementing $f_k = f_i^k/k!$ into the equation we see that (again comparing terms) for any pair non-negative integers n, k and $0 \leq p \leq n$, we have

$$\frac{f_1^{n+k}}{n!k!} \binom{n}{p} = \frac{f_1^{k+n}}{(k+p)!(n-p)!} \binom{k+p}{p}.$$

We quickly realize that $f_1 \neq 0$ implies $(n-p)!p! = n!$, which is something that people in highschool might wish is true, but fortunately for us, it is not. So $f_1 = 0$. But this means that c_{ij} is group-like. □

Theorem 4.3.3. ***(Weyl property)*** *Let the T_i be as in theorem 4.3.1, and let a_{ij} be the sl_n Cartan matrix. Denote $c_{ij} = c_i d_j$. Then $d_j = c_j^2$ for all $j = 1, \cdots n-1$ if and ony if for all $i, j = 1, \cdots n-1$,*

$$T_i T_j T_i = T_j T_i T_j. \tag{4.49}$$

We call this property the Weyl-property. The Weyl-property implies that we can use the T_i for defining Hopf algebra generators. In the case of sl_3 there are only two simple roots, so for a decomposition of the longest root vector into simple roots $w_0 = s_{i_1}s_{i_2}s_{i_1}$, where $i_1, i_2 = 1, 2, i_1 \neq i_2$, there is a unique way to write down the corresponding algebra generator: $X^\pm_{w_0} = T_{i_1}(T_{i_2}(X^\pm_{i_1}))$. Hence we do not need the Weyl property for sl_3^ϵ.
In general this is not the case. In that case, for each reduced decomposition of the longest classical Weyl group element, there exist multiple ways to write down the corresponding root-vector in the Hopf algebra. We need the Weyl-property in order for the construction of higher order generators to be well-defined. See for example proposition 8.1.3 in [6] and proposition 5, chapter 4, paragraph 1.5 in [3]. In general, the quantum Weyl group construction depends on a choice of the longest root decomposition. If the Weyl-property is not satisfied, then it is impossible to know which order of T_i belongs to a given decomposition, so the construction of higher order basis vectors is ill-defined. Let us prove the theorem.

Proof. First note that $T_k(T_j(X_i))$ is nonzero only if $i = j$ or $i = k$, so we only need to check one relation on the generators X_i^+ (the Weyl-relation follows for X_i^- from the X_i^+-relations, since the prefactors are inverted in this case). Secondly, if the automorphisms have no central prefactor, then the proof that they obey the Weyl property is similar to the proof in [30], by explicit verification. We leave this to the reader, since the computations are exactly the same. It is only needed to follow the additional central factors present in T_i as compared to the T_i with no central pre-factor. We include the central factor $((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}$ in c_{ij} in this proof for simplicity.
First note that c_{ij} are group-like by the previous theorem. Denote this central factor by $c_{ij} = c_i d_j$, where $c_i = e^{(\epsilon^{-1}H_i^+ - H_i^-)l}$, for some $l \in k$, since c_{ij} is group-like. Idem for d_i. We now prove

$$T_i(T_j(T_i(X_i^+))) = T_j(T_i(T_j(X_i^+))).$$

On the left hand side, no central factors are introduced: $T_i(X_i^+) = X_i^-(K_i^-)^2$. Since $i \neq j$, $a_{ij} = -1$. So $T_j(X_i^-) = c_{ji}^{-1}[X_j^- X_i^-]_{q^{-1/2}}$. Applying T_i to this expression yields a central factor of $c_{ij}^{-1}T_i(c_{ji}^{-1}) = c_i^{-1}d_j^{-1}T_i(c_{ji}^{-1}) = c_i^{-1}d_j^{-1}d_i^{-1}c_i c_j = d_j^{-1}d_i^{-1}c_j$.
On the right-hand side we get $T_j(X_i^+) = c_{ji}[X_j^+, X_i^+]_{q^{-1/2}}$. Applying T_i yields

$$T_i(c_{ji})c_{ij}[[X_i^+, X_j^+]_{q^{-1/2}}, X_i^-]_{q^{-1/2}},$$

where $T_i(c_{ji}) = T_i(c_j d_i) = d_i c_i^{-1}c_j^{-1}$. So as a central factor we have $d_i c_i^{-1}c_j^{-1}c_i d_j = d_i d_j c_j^{-1}$. Note that applying T_j to the commutator gives no additional central factor, since both X_i^+ and X_i^- are present. We have as a central factor $T_j(d_i d_j c_j^{-1}) = d_i^{-1}d_j^{-1}d_j c_j^{-1} = d_i^{-1}c_j^{-1} = c_{ji}^{-1}$. We see that the left and right handside are equal if

and only if $d_j = c_j^2$.

□

Note in particular that in 4.3.1 the central factor $((K_j^+)(K_j^-)^{-1})^{-1}((K_i^-)^{-1}(K_i^+))^{a_{ij}/2}$ obeys $d_j = c_j^2$, since for $i \neq j$ $a_{ij} = -1$ or $a_{ij} = 0$, in which case Weyl property is trivially satisfied. So choosing $c_{ij} = 1$ is admissable. One might want to make a different choice for the constant c_{ij} where $\Delta(z)$ is of a different form, more like that of the semisimple case. To this end consider an element $\check{w}_i$ with the properties

$$\begin{aligned} \Delta(\check{w}_i) &= \check{\mathcal{R}}(i)^{-1}\check{w}_i \otimes \check{w}_i, \\ \check{\mathcal{R}}(i) &= \sum R_n(h)(e^{-hH_i^+/2}X_i^+)^n \otimes (e^{\epsilon h H_i^-/2}X_i^-)^n, \\ \check{T}_i(h) &= \check{w}_i^{-1}h\check{w}_i, \end{aligned} \tag{4.50}$$

such that there is no central term is present in the T_i as compared to the T_i in [29]. This is possible if one introduces an abstract Weyl element obeying the above properties, however it is not clear if such an element exists at all. We could define the explicit automorphisms $\check{T}_i$ by requiring that under the identification of $\epsilon^{-1}H^+$ and H^-, $T_i = \check{T}_i$. In particular this means that

$$\Delta(\check{T}_i(X_j^\pm)) = \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_j^+))\check{\mathcal{R}}_i^{-1},$$

for some 2-cocycle $\check{\mathcal{R}}_i$. Note that $\check{\mathcal{R}}_i$ is an element of $U_q(sl_n^\epsilon) \otimes U_q(sl_n^\epsilon)$, so we can always write down this formula in the algebra $\overline{U_q(sl_n^\epsilon)}$. Note that $\check{T}_i$ by construction correspond to a choice for c_{ij}, since we want $\check{T}_i$ to obey the Weyl-property. We now prove the following fact.

Proposition 4.3.1. *The automorphisms T_i as defined in theorem4.3.1 obey the Weyl property and are of the form $T_i(h) = \overline{w}_i^{-1}h\overline{w}_i$, where $\overline{w}_i$ are defined as in lemma* **??**, *if and only if $c_{ij} = 1$.*

Proof. Under the identification of $\epsilon^{-1}H^+$ and H^-, in $U_q(sl_n)$, we have $T_i = \check{T}_i$, so $\Psi = \mathcal{R}_i^{-1}\check{\mathcal{R}}_i$ is equal to $1 \otimes 1$ under this identification, by semisimplicity of $U_q(sl_n)$. We claim that Ψ is a central element.
We now use the fact that $T_i(X_j^+) = c_{ij}\check{T}_i(X_j^+)$ for some group-like element c_{ij} if $i \neq j$. From the identities $\Delta(\check{T}_i(X_j^\pm)) = \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_j^+))\check{\mathcal{R}}_i^{-1}$ and idem for T_i it follows that (noting that T_i and $\check{T}_i$ agree on $H_{2,\epsilon,i}$ and on the Cartan subalgebra of

$U_q(sl_n^\epsilon) = H_{n,\epsilon}$ by definition, as this action was calculated from the $H_{2,\epsilon}$ action)

$$\begin{aligned}
&(c_{ij} \otimes c_{ij})(T_i(X_j^+) \otimes T_i(e^{H_j^+}) + T_i(e^{-H_j^+}) \otimes T_i(X_j^+)) \\
&= R_i^{-1}\check{\mathcal{R}}_i(c_{ij}T_i(X_j^+) \otimes T_i(e^{H_j^+}) + T_i(e^{-H_j^+}) \otimes c_{ij}T_i(X_j^+))\check{\mathcal{R}}_i^{-1}\mathcal{R}_i \\
&= c_{ij}T_i(X_j^+) \otimes T_i(e^{H_j^+}) + T_i(e^{-H_j^+}) \otimes c_{ij}T_i(X_j^+).
\end{aligned}$$

But this implies $T_i(X_j^+) \otimes c_{ij}^{-1} = T_i(X_j^+) \otimes 1$, by invertability of c_{ij}, and linear independence of the terms involved. So $c_{ij} = 1$. It should be remarked that
We now prove the claim. Therefore look at a similar expression, $\Delta(T_i(X_i^+))$. We know that $\check{T}_i(X_i^\pm) = T_i(X_i^\pm)$. Furthermore $\Psi / \sim = 1 \otimes 1$, and $\mathcal{R}_i$ and $\check{\mathcal{R}}_i$ are elements of $H_{\epsilon,2,i}$. Let $H_i = \epsilon^{-1}H_i^+ - H_i^-$. Then Ψ must be a power series in $H_i \otimes H_i$, in $\epsilon^{-1}(H_i^+ \otimes H_i^- - H_i^- \otimes H_i^+)$ and $H_i^+ \otimes H_i^+ - H_i^- \otimes H_i^-$. We have

$$e^{H_i^\pm \otimes H_i^\pm} e^{H_i^+} \otimes X_i^+ = e^{H_i^+} \otimes X_i^+ e^{H_i^\pm \otimes 1} e^{H_i^\pm \otimes H_i^\pm}.$$

This observation is generalizable to general power series in $H^\pm \otimes H^\pm$, and holds also for the opposite case. Moreover, we observe that

$$\begin{aligned}
\Delta(\check{T}_i(X_i^+)) &= \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_i^+))\check{\mathcal{R}}_i^{-1} \\
&= \check{\mathcal{R}}_i T_i \otimes T_i(\Delta(X_i^+))\check{\mathcal{R}}_i^{-1} \\
&= R_i T_i \otimes T_i(\Delta(X_i^+))\mathcal{R}_i^{-1}.
\end{aligned}$$

Note that $H^\pm$ only introduce an aditional term when commutated with X_i^+, so the terms of $\Delta(X_i^+) = X_i^+ \otimes e^{H_i^+/2} + e^{H_i^+/2} \otimes X_i^+$ do not get mixed by commutating with Ψ, so by linear independence of the two terms in the coproduct we can compare them term by term. Since the $H^\pm$ commute with each other and Ψ is a power series in tensor products of $H_i^\pm$, we get the following identities

$$\begin{aligned}
\Psi X_i^- \otimes 1\Psi^{-1} &= X_i^- \otimes 1, \\
\Psi 1 \otimes X_i^-\Psi^{-1} &= 1 \otimes X_i^-.
\end{aligned}$$

This means that Ψ commutes with $X_i^- \otimes 1$ and $1 \otimes X_i^+$, and with the above observation this means that Ψ must be a power series in $H_i \otimes H_i$. However, this means that Ψ is central in $U_q(sl_n^\epsilon) \otimes U_q(sl_n^\epsilon)$. This ends the proof. □

So there is no choice in c_{ij} if we want to have a quantum Weyl group. Such an algebra would have the above mentioned property by construction, since $\Delta(w_i)$ is definend explicitly in the $H_{2,\epsilon}$ case. Since we have an Hopf algebra we automatically obtain for any generalization of $\overline{H_{2,\epsilon}}$,

$$\Delta(\check{T}_i(X_j^\pm)) = \check{\mathcal{R}}_i\check{T}_i \otimes \check{T}_i(\Delta(X_j^+))\check{\mathcal{R}}_i^{-1}.$$

We now have

$$\begin{aligned}
\Delta(T_i(X_j^+)) &= \Delta(\overline{w}^{-1}X_j^+\overline{w}_i)\\
&= \overline{\mathcal{R}}_i T_i \otimes T_i(\Delta(X_j^+))\overline{\mathcal{R}}_i^{-1},\\
\overline{\mathcal{R}}_i &= q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum R_n(h)(e^{-hH_i^+/2}X_i^+)^n\otimes(e^{\epsilon h H_i^-/2}X_i^-)^n,\\
R_n &= \frac{q^{1/2n(n-1)}(1-q^{-2})^n}{[n]_q!}
\end{aligned}$$

This is the result with which we can calculate the comultiplication for $U_q(sl_n^\epsilon)$ for non-simple generators $T_i(X_j^\pm)$, corresponding to the roots $\alpha_i+\alpha_j$. Of course this can be generalized to higher order generators for $U_q(sl_n^\epsilon)$. This is something straightforward and will not be done here. An example of the more general construction can be found in [6].
We rewrite the automorphisms T_i to apply them to the generators E_i and F_i with non-symmetric comultiplication. This yields the algebra we use in chapter 1, when $S\otimes id$ is applied to the quantum double. We rewrite the expressions in theorem 4.3.1, with $c_{ij}=1$.

$$\begin{aligned}
\Delta(T_i(E_j)) &= \mathcal{R}_i T_i\otimes T_i(\Delta(E_j))\mathcal{R}_i^{-1},\\
\mathcal{R}_i &= q^{\epsilon^{-1}(H_i^-\otimes H_i^+ - H_i^+\otimes H_i^-)/4}\sum\frac{(1-q^{-2})^n}{[n]_q!}(E_i)^n\otimes(F_i)^n,\\
T_i(K_j^+) &= K_j^-(K_i^-)^{-a_{ij}}, T_i(K_j^-) = K_j^+(K_i^+)^{-a_{ij}},\\
T_i(E_i) &= -F_i(K_i^-)^{-1}(K_i^+)^{-1}, T_i(F_i) = -(K_i^-)(K_i^+)E_i,\\
T_i(E_j) &= (-1)^{a_{ij}}[-a_{ij}]_q!((K_i^-)^{-1}(K_i^+))^{-a_{ij}/2}\\
&\quad [E_i,\cdots,[E_i,E_j]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}, i\neq j,\\
T_i(F_j) &= \frac{1}{[-a_{ij}]_q!}((K_i^-)(K_i^+)^{-1})^{-a_{ij}/2}\\
&\quad [F_i,\cdots,[F_i,F_j]_{q^{a_{ij}/2}}]_{q^{a_{ij}/2+1}}\cdots]_{q^{-a_{ij}/2-1}}, i\neq j.
\end{aligned}$$

In particular, this formula is also valid for non-invertible ϵ, when we remember that $q^{-\epsilon h}$. The terms present are elements of the ring of power-series of ϵ. This allows for use in chapter 1.
To match the expressions with the algebra in chapter 1, note that we are using a different convention for $[n]_q$ in the last two sections of this chapter, then the factor $(1-q^{-2})^n$ is absorbed. The factor of $q^{1/2n(n-1)}$ is absorbed by the commutating of the group-like factor $e^{H^-/2}\otimes e^{H^-/2}$, with index i and j and the R-matrix. Furthermore, we change the definition of q to $e^{\epsilon h}$, and substitute $H_1^+\mapsto 2A-B$ and $H_2^+\mapsto 2B-A$. The change in q implies that we should keep track of all exponentials e^h and change the sign to e^{-h} and vice versa. Any factor q is substi-

tuted.
After performing these substitutions, there is one fundamental difference between the Hopf algebra used in this chapter and the one used in chapter 1. The Hopf algebra $U_q(sl_n^\epsilon)$ as defined in this chapter cannot be expanded modulo ϵ, as can be seen from the factor ϵ^{-1} present in the commutator between $X^\pm$. To solve this problem, we have to scale the X^- generator. This is the subject of the next section.
Once the conventions are correct, we are well on our way to implementing $U_q(sl_n^\epsilon)$ in the tensor formalism however. Especially when the program is more optimized, this is a very interesting topic of research. Theoretically, it is also interesting to be able to do calculations in $U_q(sl_n^\epsilon)$ for any n, and any order of ϵ, even when the program is not much faster than it is now.

4.4. Epilogue

Define gl_n^ϵ as the Lie bialgebra over $\mathbb{R}(\epsilon)$ with generators $X_i^\pm$, $H_i^\pm$, $i = 1, \cdots, n-1$ and the relations

$$[H_i^-, X_j^\pm] = \pm a_{ij} X_j^\pm, [H_i^\pm, H_j^\mp] = 0, [H_i^+, X_j^\pm] = \pm a_{ij} X_j^\pm, \tag{4.51}$$

$$[X_i^+, X_i^-] = -\frac{1}{2}\delta_{i,j}(H_i^+ + H_i^-), (ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, (i \neq j), \tag{4.52}$$

$$\delta(X_i^+) = X_i^+ \otimes H_i^+ - H_i^+ \otimes X_i^+, \tag{4.53}$$

$$\delta(X_i^-) = X_i^- \otimes H_i^- - H_i^- \otimes X_i^-, \tag{4.54}$$

$$\delta(H_i^\pm) = 0. \tag{4.55}$$

a_{ij} is the usual sl_n Cartan matrix.
In gl_n^ϵ, ϵ is an invertible indeterminate. The Lie bialgebra gl_n^ϵ is a quasitriangular Lie algebra that can be obtained through the classical double on the Lie bialgebras of upper and lower triangular matrices $b^\pm \subset gl_n^\epsilon$, generated by $\{H_i^{pm}, X_j^\pm\}$ respectively. This procedure is described in many standard sources, and follows the same procedure as described in chapter 1.
To obtain sl_n^ϵ from gl_n^ϵ, multiply H_i^- in the relations 4.51 and 4.52 with $\epsilon\epsilon^{-1}$. We define $\epsilon H_i^- =: \tilde{H}_i^-$, in the spirit of the Wigner contraction described in appendix A.4. When the redundant factors of ϵ^{-1} in $\epsilon^{-1}[\tilde{H}_i^-, X_j^\pm] = \pm a_{ij} X_j^\pm$ are transferred by multiplying both sides with ϵ, one obtains the familiar Lie algebra relations, and the slightly different cobracket

$$[\tilde{H}_i^-, X_j^\pm] = \pm\epsilon a_{ij} X_j^\pm, [H_i^+, X_j^\pm] = \pm a_{ij} X_j^\pm,$$

$$[X_i^+, X_i^-] = -\frac{1}{2}\delta_{i,j}(H_i^+ + \epsilon^{-1}\tilde{H}_i^-), (ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, (i \neq j),$$

$$\delta(X_i^+) = X_i^+ \otimes H_i^+ - H_i^+ \otimes X_i^+,$$

$$\delta(X_i^-) = \epsilon^{-1}(X_i^- \otimes \tilde{H}_i^- - \tilde{H}_i^- \otimes X_i^-).$$

We can multiply δ with any constant in $\mathbb{R}(\epsilon)$, this will yield a cobracket on the same Lie algebra. To this end, consider the b^- Lie bialgebra where we multiply δ with ϵ. We obtain

$$\delta(X_i^-) = X_i^- \otimes H_i^- - H_i^- \otimes X_i^-.$$

Let us introduce the dual Lie algebra b^+ of b^- with generators $\{X_i^+, \tilde{H}_i^+\}$ by $\langle X_i^+, X_j^- \rangle = \delta_{ij}$ and $\langle \tilde{H}_i^+, \tilde{H}_j^- \rangle = a_{ij}$. The algebra relations of b^+ are defined through the cobracket of b^-, and so we obtain the relations (note that ϵ is invertible, to obtain the Serre relation)

$$[\tilde{H}_i^+, X_j^+] = +a_{ij} X_j^+, (ad_{X_i^+})^{1-a_{ij}}(X_j^+) = 0, (i \neq j).$$

The cobracket is defined through the Lie algebra relations of b^- and takes the form

$$\delta(X_i^+) = \epsilon(X_i^+ \otimes \tilde{H}_i^+ - \tilde{H}_i^+ \otimes X_i^+).$$

Taking the classical double of b^+ and b^- we obtain the Lie bialgebra sl_n^ϵ with relations 4.1. As noted in the first section of this chapter, we have a set of Lie algebra automorphisms T_i on sl_n^ϵ, which are defined with the adjoint action on sl_n^ϵ.
We can do a Wigner contraction on sl_n^ϵ by multiplying $X_i^- \in sl_n^\epsilon$ with $\epsilon\epsilon^{-1}$ and defining $\tilde{X}_i^- := \epsilon X_i^-$. This has no effect on the Lie algebra relations of $b^\pm$ and the cobracket of $b^\pm$, as can be seen by multiplying the relations with ϵ on both sides. It has an effect on the pairing between $\tilde{X}_i^+$ and $\tilde{X}_i^-$, which yields ϵ.
In sl_n^ϵ, this changes the bracket between $\tilde{X}_i^\pm$ to $[\tilde{X}_i^-, \tilde{X}_i^+] = \frac{1}{2}(\epsilon\tilde{H}_i^+ + \tilde{H}_i^-)$. This relation should remind the reader of the definition of sl_n^ϵ in chapter one. For now, let us denote this algebra as $\tilde{sl}_n^\epsilon$. In particular we observe that with these Lie bialgebra relations, it is possible to divide out to ϵ^k, as there are no explicit factors of ϵ^{-1} present in the algebra relations of $\tilde{sl}_n^\epsilon$.
However, another effect of this Wigner contraction is that the bracket $[,]$ no longer defines a set of automorphisms T_i of $\tilde{sl}_n^\epsilon$. When writing out the requirement that T_j is a Lie algebra map for the $[\tilde{X}_i^+, \tilde{X}_i^-]$ relation, one finds that T_j is an algebra map only when one multiplies with a factor $\epsilon^{a_{ij}} = \epsilon^{-1}$ when T_j is applied to $\tilde{X}_i^-$. So we have to define $\tilde{T}_j(\tilde{X}_i^-) = \epsilon^{-1}T_j(\tilde{X}_i^-)$ when $i = j \pm 1$ and $\tilde{T}_j(u) = T_j(u)$ else.
This means that the Lie algebra automorphisms $\tilde{T}_i$ of sl_n^ϵ, when defined on $\tilde{sl}_n^\epsilon$ gain a factor ϵ^{-1} with respect to the Lie algebra automorphisms on sl_n^ϵ that are used in this chapter. We note that in the definition of non-simple generators this yields also a power of ϵ^{-1}. In particular, for $\tilde{sl}_3^\epsilon$ this yields $\tilde{X}_{\alpha_1+\alpha_2}^- = \tilde{T}_1(\tilde{X}_2^-) = \epsilon^{-1}T_1(\tilde{X}_2^-)$. Multiplying both sides wth ϵ now gives $\epsilon X_{\alpha_1+\alpha_2}^- = [X_1^-, X_2^-]$, which is the relation familiar from chapter one.
A complication from this specific Wigner contraction on sl_n^ϵ is that the automorphisms $\tilde{T}_i$ do not obey the Weyl-property. Writing out $\tilde{T}_i\tilde{T}_j\tilde{T}_i(X_i^\pm) = \tilde{T}_j\tilde{T}_i\tilde{T}_j(X_i^\pm)$

gives a different factor of ϵ on both sides. Since the $\tilde{T}_i$ reduce to the usual automorphisms on sl_n when ϵ is put to one, one can only compensate by introducing a factor of ϵ^p in $\tilde{T}_i$. However, one quickly sees that it is not possible to make such a choice such that $\tilde{T}_i$ obey the Weyl property.
As has been noted before, in the case of sl_3 this is not a problem, since there is a unique way to decompose the reflection corresponding to the longest root. For higher $n > 3$, one has to make a choice for a decomposition of the longest Weyl group element and live with this. The result for different choices of decomposition yields different Lie algebras that are presumably isomorphic, although this is not directly clear. This is an interesting subject of future study.

When one quantizes sl_n^ϵ, this yields the Hopf algebra $U_q(sl_n^\epsilon)$ of theorem 4.1.3. The Hopf algebra $U_q(\tilde{sl}_3^\epsilon)$ can be obtained from $U_q(sl_3^\epsilon)$ by multiplying X_i^- with $q - q^{-1}\frac{1}{q-q^{-1}}$ and defining $\tilde{X}_i^- = (q - q^{-1})X_i^-$. This scaling only infuenzes the relations between $X_i^\pm$. The comultiplication, antipode and the other relations stay the same.
Let T_i be as in equation 4.48 with $c_{ij} = 1$. As proved in the third section of this chapter, T_i obey the Weyl property and are automorphisms of $U_q(sl_3^\epsilon)$ when $c_{ij} = 1$. Note that ϵ was introduced on the b^+ side in the previous sections. This does not change the properties of the T_i, since the relation between the two algebras is a scaling of $H_i^\pm$. We have encountered this fact in the classical case, and the quantum case follows in exactly the same way.
If we wish to define automorphisms $\tilde{T}_i$ on $U_q(\tilde{sl}_n^\epsilon)$, we have to correct in the same way as in the Lie algebra case, by introducing an additional factor of $\frac{1}{q-q^{-1}}$ when T_i is applied to $\tilde{X}_i^-$. We define $\tilde{T}_j(\tilde{X}_-^i) = \frac{1}{q-q^{-1}}T_j(\tilde{X}_i^-)$ for $i = j \pm 1$ and $\tilde{T}_j(u) = T_j(u)$ in any other case, for an elementary generator $u \in U_q(\tilde{sl}_n^\epsilon)$. In exactly the same way as the classical $\tilde{T}_i$ failed to have the Weyl-property, so do the quantum Weyl group automorphisms $\tilde{T}_i$. This can be seen by applying $\tilde{T}_j$ to $\tilde{X}_i^-$ and counting the terms $\frac{1}{q-q^{-1}}$ that are introduced.
Equivalent to the classical case, when defining non-simple generators in $\tilde{sl}_n^\epsilon$, one has to choose a decomposition of the longest Weyl group element. Moreover, on the $U_q(b^-) \subset U_q(\tilde{sl}_n^\epsilon)$ side, a number of factors $\frac{1}{q-q^{-1}}$ are introduced in the definition of nonsimple generators. This yields relations like $(q^{-1} - q)\tilde{X}_{\alpha_1+\alpha_2}^- = T_1(X_2^-)$ in the case of $U_q(\tilde{sl}_3^\epsilon)$. Here, α_i are the simple roots of $\tilde{sl}_3^\epsilon$. This should remind the reader of the relations in chapter 1, although the $b^\pm$ algebras switched place there, among some other details. For a generator corresponding to a root of length $k = 1, \cdots, n-1$, we obtain $k-1$ factors of $q - q^{-1}$. This yields $\tilde{T}_{\alpha_{i_1}}(\cdots(\tilde{T}_{\alpha_{i_{k-1}}}((q - q^{-1})^{k-1}\tilde{X}_{\alpha_{k-1}}^-) = (q - q^{-1})^{k-1}\tilde{X}_\beta^-$, where $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$.
We can use the results of the previous section to obtain an expression of the comultiplication of nonsimple elements such as $\tilde{X}_{\alpha_1+\alpha_2}^-$ in $U_q(\tilde{sl}_3^\epsilon)$. In the case of $\tilde{X}_{\alpha_1+\alpha_2}^-$ for example we get $(q^{-1} - q)\tilde{X}_{\alpha_1+\alpha_2}^- = \overline{\mathcal{R}}_1 T_1 \otimes T_1(\Delta((q - q^{-1})\tilde{X}_2^-))\overline{\mathcal{R}}_i^{-1}$ in

the notation of the previous section.
When we wish to consider $U_q(\tilde{sl}_n^\epsilon)$ over the ring $R_{\epsilon^k} = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$, this implies that we do not get a direct expression for $\Delta(\tilde{X}^-_{\alpha_1+\alpha_2})$ for example, but only for $\Delta((q-q^{-1})\tilde{X}^-_{\alpha_1+\alpha_2})$, since $q-q^{-1} = 2\epsilon h + \cdots$, and ϵ is not invertible. So to obtain the comultiplication of $\tilde{X}^-_{\alpha_1+\alpha_2}$ modulo ϵ^{k+1}, to take a specific example, we need to consider $U_q(\tilde{sl}_3^\epsilon)$ over the ring $R_{\epsilon^{k+1}}$, since over R_{ϵ^k} (which is working mod ϵ^{k+1}), terms proportional to ϵ^{k+1} vanish. So working over R_{ϵ^k} would only give us $\Delta(\tilde{X}^-_{\alpha_1+\alpha_2})$ up to and including order ϵ^{k-1}, since $(q-q^{-1})X^-_{\alpha_1+\alpha_2} = [X_1^-, X_2^-]$. One sees that one can work mod ϵ^{k+1} when k is even, as the ϵ^{2k} term vanish in the expansion of $q-q^{-1}$.
This observation is particularly useful when attempting to construct a general $U_q(sl_n^{\epsilon^k})$ knot invariant. Using formula 4.48 of the previous section, we can obtain expressions for the comultiplication of the non-simple generators. We have to work modulo ϵ^{k+n-1} to obtain the comultiplication of every non-simple generator, since for an element of maximal length $n-1$, there are $n-2$ factors of ϵ introduced, yielding a prefactor of ϵ^{k+n-2} for the comultiplication of the longest Weyl group element generator. When computing the knot invariant itself, so for the multiplication of R-matrices, one can then work modulo ϵ^{k+1} again.
A particular surprise when specializing to $\epsilon^k = 0$, is that the $\tilde{T}_i$ are not algebra automorphisms of $U_q(\tilde{sl}_n^\epsilon)$, due to the noninvertible factor of $q-q^{-1}$ present. In particular, we cannot apply $\tilde{T}_i$ to a non-simple generator such as $\tilde{X}_3^-$, but only to $(q-q^{-1})\tilde{X}_3^-$.
In general, the exact properties of T_i become more complicated as more factors of ϵ are introduced in the definition of the generators associated with positive non-simple roots. For $\tilde{sl}_3^\epsilon$ there is one non-invertible factor introduced when working over $\epsilon^k = 0$, but for $\tilde{sl}_n^\epsilon$ there are $n-2$ factors introduced in the definition of the element corresponding to the longest classical Weyl element. So if we wish to calculate the comultiplication of this generator in the first order of ϵ, say $\tilde{X}^-_\beta \in U_q(\tilde{sl}_n^\epsilon)$, we have to work modulo ϵ^n.
This explains why the usual symmetries $U_q(\tilde{sl}_n^\epsilon)$ for invertible ϵ are not symmetries of $U_q(\tilde{sl}_n^\epsilon)$ for $\epsilon^k = 0$. Some symmetries of $\tilde{sl}_n^\epsilon$ for non-invertible ϵ were found by Roland van der Veen and Dror Bar-Natan in [37], the classical case. We do not know if the symmetries in [37] provide a full discription of the symmetries of $\tilde{sl}_n^\epsilon$, or if there is a bigger set of hidden symmetries. This remains an interesting topic of research. It is also interesting to find the explicit quantum group analogue of the $\tilde{sl}_n^\epsilon$ symmetries.

Conclusion

In this chapter we constructed $U_q(b^-)$ and $U_q(sl_n\epsilon)$ from $b^+ \subset sl_n$ for invertible ϵ. We observed that in the algebra relations ϵ occured only in $q = e^{\epsilon h}$, and hence

one can take the expansion to the k-th order in ϵ for any k when the algebra generators are rescaled with a suitable factor of $(q-q^{-1})^m$, for some positive m.
The fact that $U_q(sl_n^\epsilon)$ is not semisimple does not change the symmetries of $U_q(sl_n^\epsilon)$ for invertible ϵ. They are equal to the symmetries of $U_q(sl_n)$ for invertible epsilon. In the last section we found that only when we specialize to $\epsilon^k = 0$ for $k > 0$ we lose most of the usual symmetries. It turns out that in this case, instead of S_n, we obtain D_n as the group of automorphisms of $U_q(sl_n^\epsilon)$. See [37].
However, important equalities to calculate the comultiplication remain true when ϵ is not invertible, and even when one specializes to $\epsilon^k = 0$. The main purpuse of the last sections of this chapter was to prove these formula for the coproduct in terms of partial R-matrices. We observed that after rescaling, this formula can be expanded in terms of ϵ, so that it is also valid for non invertible ϵ.
Using this formula, we constructed a dual PBW basis of $U_q(sl_n^\epsilon)$ in the first section of this chapter and we gave the pairing between monomials. This enables one to construct the universal R-matrix of $U_q(sl_n^\epsilon)$. We observed that when one wants to know the coproduct (and antipode) of $U_q(\tilde{sl}_n^\epsilon)$ modulo ϵ^{k+1}, one has to work modulo ϵ^{k+n-1}.
In the previous chapter we gave an upper bound for the computational complexity of the $U_q(sl_n^\epsilon)$ invariant. In short, this provided the insight that for small knots (i.e. less than say 20 crossings) the contribution of the number of crossings is smaller than the contribution of rank of sl_n. It remains to be seen if this problem can be overcome. On the other hand, it is interesting to gain insight in the symmetries of $U_q(sl_n^\epsilon)$ in the case where $\epsilon^k = 0$. This may reduce the number of computations one might have to do. This will also give insight in the quantum invariants that are obtained.
A concrete topic of future research is the implementation of $U_q(sl_n^\epsilon)$ using the comultiplication calculated in this chapter. As a first step, we wish to implement these formulas for the case $U_q(sl_3^\epsilon)$, to be able to match the conventions we used earlier with the ones used by [29] and [6] and others. Especially with the last section in mind, this should be possible in the current implementation of $U_q(sl_3^\epsilon)$, when one defines the (co)multiplication tensors for general order of ϵ. In the last section we concluded that the higher n, the higher the order of ϵ one needs to work over in order to obtain the full Hopf algebra structure. Once the Hopf algebra structure has been found, one can restrict oneself to any (lower) order of ϵ to calculate the invariant for actual knots.

A. Appendices

A.1. Mathematica

In this section we present the implementation of the zipping formalism used to do calculations with the Hopf algebra $U_q(sl_3^\epsilon)$, and to calculate the quantum double explicitly. The proof of the zipping theorem can be found in chapter 2. The program labeled *sl3invariant.nb* is an implementation of this theorem. The program *sl3invariant.nb* is based on the program *sl2invariant.nb* developed by Bar-Natan and Van der Veen. This program can be found on

$$http://drorbn.net/AcademicPensieve/Projects/SL2Invariant/index.html.$$

In *sl2invariant.nb* one can find an implementation of the invariant based on the quantum group $U_q(sl_2^\epsilon)$. The knot invariant presented in this thesis is based on the $U_q(sl_2^\epsilon)$ construction by Bar-Natan and Van der Veen.
The difference between *sl2invariant.nb* and *sl3invariant.nb* is the use of the three-stage zip. This is an essential difference, since it provides a convergent implementation of the zipping theorem for the $U_q(sl_3^\epsilon)$ Hopf algebra. The proof that this implementation is convergent can be found in chapter 2.
In this program we implement the quantum group $U_q(sl_3)$ constructed in chapter 1. We check (co)associativity, if Δ is a homomorphism, the pairing axioms, the antipode axioms, associativity and the Turaev moves. The knot invariant is computed for the Trefoil, the mirror Trefoil, the figure eight and the 6-3 knot in the Rolfson knot table.

The full sl_3 invariant using the Drinfel'd double. For compatibility reasons, we use XX instead of the generator X. This program continues sl2invariant.nb by Dror Bar-Natan and Roland van der Veen.

Profiling

```
(*BeginProfile[];*)
```

External Utilities

In[3]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

Program

Program

Internal Utilities

In[4]:=
```
MaxBy[list_, fun_, n_] := list[[Ordering[fun /@ list, -n]]];
```

Canonical Form:

Program

In[5]:=
```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
          Expand[ℰ] //. e^x_ e^y_ :↦ e^(x+y) /. e^x_ :↦ e^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (XX | Y | Z | A | B | b | s | t | a | x | y | z | XX* | Y* | Z* |
            A* | B* | s* | t* | b* | a* | x* | y* | z*)_, ∞] ⋃ {XX, Y, Z, A, B, b,
          s, t, a, x, y, z, XX*, Y*, Z*, A*, B*, s*, t*, b*, a*, x*, y*, z*},
     Total[CoefficientRules[Expand[ℰ], vs] /.
       (ps_ → c_) :↦ CCF[c] (Times @@ vs^ps)]
    ];
```

Program

The Kronecker δ:

Program

In[9]:=
```
Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
```

Program

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

In[10]:=
```
E /: E[L1_, Q1_, P1_] ≡ E[L2_, Q2_, P2_] :=
  CF[L1 == L2] ⋀ CF[Q1 == Q2] ⋀ CF[Normal[P1 - P2] == 0];
E /: E[L1_, Q1_, P1_] E[L2_, Q2_, P2_] := E[L1 + L2, Q1 + Q2, P1 * P2];
E[L_, Q_, P_]_$k_ := E[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

In[13]:=
```
E3@E_sp__[ω_, L_, Q_, P_] := Module[
    {NP = Normal[P]},
    E_sp[L, ω^-1 Q, (ω^-1 NP /. ϵ → ω^-4 ϵ) + O[ϵ]^($k+1)] // CF
   ];
E4@E_sp__[L_, Q_, P_] := Module[
    {NP = Normal[P], ω},
    ω = (NP /. ϵ → 0)^-1;
    E_sp[ω, L, ω Q, (ω NP /. ϵ → ω^4 ϵ) + O[ϵ]^($k+1)] // CF
   ];
```

Program

Zip and Bind

Program

Variables and their duals:

Program

In[15]:=
```
(u_i_)* := (u*)_i;
((u_i_)*)* := u_i;
(((u_)*)_i_)* := u_i;
((u_)*)* := u;
```

Program

Finite Zips:

Program

In[19]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ε_, ζ_] := PP_Collect@Collect[ε, ζ];
Zip_{}[P_] := P; Zip_{ζ_, ζs__}[P_] := PP_Zip[
  (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :↦ ∂_{ζ*, d} f) /. ζ* → 0]
```

Program

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars. $\mathbb{E}[L, Q, P]$ means $e^{\hbar(L+Q)} P$, where L is linear in the a,b's, Q is a combination of $x_i X_j$ (possibly starred and/or mixed with other variables), and P is a perturbation polynomial. It should be interpreted via $\mathbb{O}[\mathbb{E}[\ldots], \{X_1, Y_1, Z_1, A_1, B_1, b_1, a_1, z_1, y_1, x_1\}_i, \ldots]$, with an assumed standard ordering on the generators for an interpretation of the tensor as an expression in $U_q(\mathsf{sl}_3^{\epsilon})$.

Program

```
QZip_ζs_List@E[L_, Q_, P_] := PP_QZip@Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ E[L, c + ηs.qt.ys, Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]] ];
```

Program

Upper to lower and lower to Upper:

Program

In[23]:=
```
U2l = {𝕓_i^p_· → 𝕖^-p b_i, 𝕓^p_· → 𝕖^-p b, 𝕒_i^p_· → 𝕖^-p a_i, 𝕒^p_· → 𝕖^-p a,
    𝕋_i^p_· → 𝕖^p t_i, 𝕋^p_· → 𝕖^p t, 𝕊_i^p_· → 𝕖^p s_i, 𝕊^p_· → 𝕖^p s, 𝔸_i^p_· → 𝕖^p a*_i, 𝔸^p_· → 𝕖^p a*,
    𝔹_i^p_· → 𝕖^p b*_i, 𝔹^p_· → 𝕖^p b*, 𝔹_i^p_· → 𝕖^-p B_i, 𝔹^p_· → 𝕖^-p B, 𝔸_i^p_· → 𝕖^-p A_i,
    𝔸^p_· → 𝕖^-p A, 𝒜_i^p_· → 𝕖^p A*_i, 𝒜^p_· → 𝕖^p A*, ℬ_i^p_· → 𝕖^p B*_i, ℬ^p_· → 𝕖^p B*};
l2U = {𝕖^c_·b_i_+d_· :→ 𝕓_i^-c 𝕖^d, 𝕖^c_·b+d_· :→ 𝕓^-c 𝕖^d, 𝕖^c_·a_i_+d_· :→ 𝕒_i^-c 𝕖^d, 𝕖^c_·a+d_· :→ 𝕒^-c 𝕖^d,
    𝕖^c_·t_i_+d_· :→ 𝕋_i^c 𝕖^d, 𝕖^c_·t+d_· :→ 𝕋^c 𝕖^d, 𝕖^c_·s_i_+d_· :→ 𝕊_i^c 𝕖^d, 𝕖^c_·s+d_· :→ 𝕊^c 𝕖^d,
    𝕖^c_·a*_i_+d_· :→ 𝔸_i^c 𝕖^d, 𝕖^c_·a*+d_· :→ 𝔸^c 𝕖^d, 𝕖^c_·b*_i_+d_· :→ 𝔹_i^c 𝕖^d, 𝕖^c_·b*+d_· :→ 𝔹^c 𝕖^d,
    𝕖^c_·B_i_+d_· :→ 𝔹_i^-c 𝕖^d, 𝕖^c_·B+d_· :→ 𝔹^-c 𝕖^d, 𝕖^c_·A_i_+d_· :→ 𝔸_i^-c 𝕖^d, 𝕖^c_·A+d_· :→ 𝔸^-c 𝕖^d,
    𝕖^c_·A*_i_+d_· :→ 𝒜_i^c 𝕖^d, 𝕖^c_·A*+d_· :→ 𝒜^c 𝕖^d, 𝕖^c_·B*_i_+d_· :→ ℬ_i^c 𝕖^d, 𝕖^c_·B*+d_· :→ ℬ^c 𝕖^d,
    𝕖^ℰ_ :→ 𝕖^Expand@ℰ};
```

Program

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^{Q}$ as a single"P". Here the z's are A, B and a^*, b^* and the ζ's are A^*, B^* and a, b. s and t are not regarded as scalars for zip-technicalities. DB and STB are variations of B with a different choice for ζs in LZip, to speed up the zipping of tensors with s and t instead of A and B.

Program

```
LZip_ζs_List@𝔼[L_, Q_, P_] :=
  PP_LZip@Module[{ζ, z, zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    c = L /. Alternatives@@(ζs ⋃ zs) → 0;
    ys = Table[∂_ζ (L /. Alternatives@@zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives@@ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_z,ζ* - ∂_z,ζ L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. U2l /. (zrule⋃ζrule);
    EEQ[ps___] :=
     EEQ[ps] = PP_"EEQ"@(CF[𝕖^-Q1 D[𝕖^Q1, Sequence@@Thread[{zs, {ps}}]]] /.
          Alternatives@@zs → 0 //. l2U);
    CF /@ ((*CF/@*)𝔼[
        c + ηs.lt.ys, Q1 /. Alternatives@@zs → 0,
        Det[lt] (Zip_ζs[(EQ@@zs) (P /. U2l /. (zrule⋃ζrule))] /.
            Derivative[ps___][EQ][___] :→ EEQ[ps] /. _EQ → 1)
      ] //. l2U)
    ];
```

Program

In[26]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_Bind@Module[{n},
    Times[
        L /. Table[(v : XX | Y | Z | A | 𝔸 | B | 𝔹 | s | t | S | T |
                b | (*𝕓|*)a(*|𝕒*) | x | y | z)_i → v_n@i, {i, {is}}],
        R /. Table[(v : XX* | Y* | Z* | A*(*|𝒜*) | B* | (*ℬ| *)b* | s* |
                t* | a* | 𝕒 | b | x* | y* | z*)_i → v_n@i, {i, {is}}]
      ] // LZip_Flatten@Table[{A*_n@i,B*_n@i,(s*)_n@i,(t*)_n@i,b_n@i,a_n@i},{i,{is}}] //
     QZip_Flatten@Table[{XX_n@i,Y*_n@i,y_n@i,x*_n@i},{i,{is}}] // QZip_Flatten@Table[{Z*_n@i,z_n@i},{i,{is}}]];
B_is__[L_, R_] := B_{is}[
    L,
    R];
```

```
DB{}[L_, R_] := L R;
DB{is__}[L_𝔼, R_𝔼] := PPDBind@Module[{n},
    Times[
      L /. Table[(v : XX | Y | Z | s | t | b | 𝕓 | a | 𝕒 | x | y | z)i → vn@i, {i, {is}}],
      R /. Table[(v : XX* | Y* | Z* | s* | t* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)i → vn@i, {i, {is}}]
     ] // LZipFlatten@Table[{(s*)n@i, (t*)n@i, bn@i, an@i}, {i, {is}}] //
   QZipFlatten@Table[{XXn@i, Y*n@i, Yn@i, x*n@i}, {i, {is}}] // QZipFlatten@Table[{Z*n@i, zn@i}, {i, {is}}]];
DBis__[L_, R_] := DB{is}[L, R];
```

In[29]:=
```
STB{}[L_, R_] := L R;
STB{is__}[L_𝔼, R_𝔼] := PPSTBind@Module[{n},
    Times[
      L /. Table[(v : XX | Y | Z | b | 𝕓 | a | 𝕒 | x | y | z)i → vn@i, {i, {is}}],
      R /. Table[(v : XX* | Y* | Z* | b* | a* | 𝒶 | 𝒷 | x* | y* | z*)i → vn@i, {i, {is}}]
     ] // LZipFlatten@Table[{bn@i, an@i}, {i, {is}}] //
   QZipFlatten@Table[{XXn@i, Y*n@i, Yn@i, x*n@i}, {i, {is}}] // QZipFlatten@Table[{Z*n@i, zn@i}, {i, {is}}]];
STBis__[L_, R_] := STB{is}[L, R];
```

Program

𝔼 morphisms with domain and range.

Program

In[32]:=
```
Bis_List[𝔼d1_→r1_[L1_, Q1_, P1_], 𝔼d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ Bis[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
STBis_List[𝔼d1_→r1_[L1_, Q1_, P1_], 𝔼d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ STBis[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] // 𝔼d2_→r2_[L2_, Q2_, P2_] :=
  Br1⋂d2[𝔼d1→r1[L1, Q1, P1], 𝔼d2→r2[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ⋀ (r1 == r2) ⋀ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼d1_→r1_[L1_, Q1_, P1_] 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼d_→r_[L_, Q_, P_]$k_ := 𝔼d→r @@ 𝔼[L, Q, P]$k;
𝔼_[𝓔___][i_] := {𝓔}⟦i⟧;
```

Program

“Define” code

Program

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

```
In[39]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is_ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
       SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
       SD[op_{isp}, op_{is},$k]; SD[op_{sis__}, op_{sis}];
      ] /. {SD → SetDelayed,
       isp → {is} /. {i → i_, j → j_, k → k_},
       nis → {is} /. {i → ii, j → jj, k → kk},
       nisp → {is} /. {i → ii_, j → jj_, k → kk_}
      }] ]]
```

Program

Booting Up

Program

```
$k = 1;
```

The multiplication tensors on both halves of the double are defined here. $k indicates the degree $\epsilon^{\$k+1} = 0$ we are working over. The am and bm tensors given here only work for $k=1 and $k=0.

```
In[43]:= Define[am_{i,j→k} =
    𝔼_{{i,j}→{k}}[ (a*_i + a*_j) a_k + (b*_i + b*_j) b_k, (ⅇ^((b*)_j+(a*)_j) z*_i + z*_j + ⅇ^(-(b*)_j+2 (a*)_j) x*_i (y*)_j) z_k +
       (ⅇ^(-(a*)_j+2 (b*)_j) y*_i + y*_j) y_k + (ⅇ^(-(b*)_j+2 (a*)_j) x*_i + x*_j) x_k,
      1 + ϵ (ħ (z*)_j (x*)_i ⅇ^(-(b*)_j+2 (a*)_j) z_k x_k + ħ (z*)_j (x*)_i (y*)_j ⅇ^(-(b*)_j+2 (a*)_j) z_k z_k -
         ħ (z*)_j (y*)_i ⅇ^(2 (b*)_j-(a*)_j) z_k y_k - ħ (y*)_i (x*)_i (y*)_j ⅇ^((a*)_j+(b*)_j) z_k y_k - ħ (x*)_i
          ⅇ^(-(b*)_j+2 (a*)_j) (y*)_j (y*)_j z_k y_k - ħ (x*)_i ⅇ^(-(b*)_j+2 (a*)_j) (y*)_j y_k x_k) + O[ϵ]^2]_$k,
   bm_{i,j→k} = 𝔼_{{i,j}→{k}}[A_k A*_i + A_k A*_j + B_k B*_i + B_k B*_j, XX_k XX*_i + XX_k XX*_j + Y_k Y*_i +
       Y_k Y*_j + Z_k Z*_i + Z_k Z*_j, 1 + (-XX_k (A*)_i (XX*)_j + ħ XX_k Y_k (XX*)_j (Y*)_i -
         Y_k (B*)_i (Y*)_j + 2 Z_k (XX*)_i (Y*)_j - ħ XX_k Z_k (XX*)_j (Z*)_i +
         ħ Y_k Z_k (Y*)_j (Z*)_i - Z_k (A*)_i (Z*)_j - Z_k (B*)_i (Z*)_j) ϵ + O[ϵ]^2]_$k];
```

The R-matrix is defined with the Faddeev-Quesne formula.

Program

```
Define[R_{i,j} =
  E_{}→{i,j}[ħ A_i a_j + ħ B_i b_j, ħ XX_i x_j + ħ Y_i y_j + ħ Z_i z_j, e^(Sum_{k=2}^{$k+1} (e^(-2 ϵ ħ) - 1)^k (ħ XX_i x_j)^k / (k (1 - e^(-2 k ϵ ħ)))
    e^(Sum_{l=2}^{$k+1} (e^(-2 ϵ ħ) - 1)^l (ħ Y_i y_j)^l / (l (1 - e^(-2 l ϵ ħ)))) e^(Sum_{m=2}^{$k+1} (e^(-2 ϵ ħ) - 1)^m (ħ Z_i z_j)^m / (m (1 - e^(-2 m ϵ ħ))))]_$k,
 R̄_{i,j} = E_{}→{i,j}[-ħ a_j A_i - ħ b_j B_i, -ħ x_j XX_i A_i^(2 ħ) B_i^(-ħ) + ħ^2 XX_i Y_i z_j A_i^ħ B_i^ħ -
    ħ z_j Z_i A_i^ħ B_i^ħ - ħ y_j Y_i A_i^(-ħ) B_i^(2 ħ), 1 + If[$k == 0, 0, (R̄_{i,j},$k-1)_$k[3] -
      (((R̄_{i,j},0)_$k R_{1,2} (R̄_{3,4},$k-1)_$k) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
 P_{i,j} = E_{i,j}→{}[1/ħ A*_i a*_j + 1/ħ B*_i (b*)_j,
    1/ħ XX*_i x*_j + 1/ħ Y*_i y*_j + 1/ħ Z*_i z*_j, 1 + If[$k == 0, 0, (P_{i,j},$k-1)_$k[3] -
      (R_{1,2} // ((P_{1,j},0)_$k (P_{i,2},$k-1)_$k))[3]]]
```

Program

In[45]:=

```
Define[aS_j = R̄_{i,j} ~B_i~ P_{i,j},
 āS_i = E_{i}→{i}[-a*_i a_i - b*_i b_i, -e^(-(b*)_i-(a*)_i) z*_i z_i + e^(-(b*)_i-(a*)_i) y*_i (x*)_i z_i -
    e^(-2 (b*)_i+(a*)_i) y*_i y_i - e^((b*)_i-2 (a*)_i) x*_i x_i, 1 + If[$k == 0, 0, (āS_{i},$k-1)_$k[3] -
      ((āS_{i},0)_$k ~B_i~ aS_i ~B_i~ (āS_{i},$k-1)_$k)[3]]]
```

Program

In[46]:=

```
Define[bS_i = R_{i,1} ~B_1~ aS_1 ~B_1~ P_{i,1},
 b̄S_i = R_{i,1} ~B_1~ āS_1 ~B_1~ P_{i,1},
 aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
 bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

Program

In[47]:=

```
Define[dm_{i,j→k} = (E_{i,j}→{i,j}[(A*)_i A_i + (B*)_i B_i + b*_j b_j + a*_j a_j,
      y*_j y_j + x*_j x_j + z*_j z_j + (XX*)_i XX_i + (Y*)_i Y_i + Z*_i Z_i, 1] (aΔ_{i→1,2} // aΔ_{2→2,3} // āS_3)
    (bΔ_{j→-1,-2} // bΔ_{-2→-2,-3})) // (P_{-1,3} P_{-3,1} am_{2,j→k} bm_{i,-2→k}),
 dS_i = E_{i}→{1,2}[(A*)_i A_1 + (B*)_i B_1 + b*_i b_2 + a*_i a_2,
    y*_i y_2 + x*_i x_2 + z*_i z_2 + (XX*)_i XX_1 + (Y*)_i Y_1 + Z*_i Z_1, 1] // (b̄S_1 aS_2) // dm_{2,1→i},
 dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j}),
 d̄S_i = E_{i}→{1,2}[(A*)_i A_1 + (B*)_i B_1 + b*_i b_2 + a*_i a_2,
    y*_i y_2 + x*_i x_2 + z*_i z_2 + (XX*)_i XX_1 + (Y*)_i Y_1 + Z*_i Z_1, 1] // (bS_1 āS_2) // dm_{2,1→i}]
```

Program

In[48]:=

```
Define[C_i = E_{}→{i}[0, 0, 1/(A_i^ħ B_i^ħ) - ħ (a_i + b_i) ϵ / (A_i^ħ B_i^ħ) + O[ϵ]^2]_$k,
 C̄_i = E_{}→{i}[0, 0, (A_i^ħ B_i^ħ + (ħ a_i A_i^ħ B_i^ħ + ħ b_i A_i^ħ B_i^ħ) ϵ) + O[ϵ]^2]_$k,
 Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
 K̄ink_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
```

Program

Note: s=2A-B+ϵa, t=2B-A+ϵb. This substitution is implemented in the following tensors.

Program

In[49]:=
```
Define[AB2st_i = 𝔼_{i}→{i}[b*_i b_i + a*_i a_i + (A*)_i (2/3 (s_i + t_i/2)) + (B*)_i (2/3 (t_i + s_i/2)),
    (XX*)_i XX_i + (Y*)_i Y_i + Z*_i Z_i + y*_i y_i + x*_i x_i + z*_i z_i,
    1 - ϵ (A*)_i 1/3 (2 a_i + b_i) - ϵ (B*)_i 1/3 (2 b_i + a_i) + O[ϵ]^2]_$k,
  st2AB_i = 𝔼_{i}→{i}[b*_i b_i + a*_i a_i + (s*)_i (2 A_i - B_i) + (t*)_i (2 B_i - A_i), (XX*)_i XX_i +
      (Y*)_i Y_i + Z*_i Z_i + y*_i y_i + x*_i x_i + z*_i z_i, 1 + ϵ (s*)_i a_i + ϵ (t*)_i b_i + O[ϵ]^2]_$k]
```

The following definitions are used for a slightly faster implementation of the quantum-double that leaves out the s and the t from the zip. Since s and t are central, this is well defined and yields the same result. These tensors should be zipped using the STB and DB zip function. As such, we also check the axioms for these tensors.

In[50]:=
```
Define[stR_i,j = (R_i,j ~ B_{i,j} ~ (AB2st_i AB2st_j)) // Simplify,
  ‾stR‾_i,j = (R̄_i,j ~ B_{i,j} ~ (AB2st_i AB2st_j)) // Simplify,
  ‾stC‾_i = (C̄_i ~ B_{i} ~ (AB2st_i)) // Simplify,
  stC_i = (C_i ~ B_{i} ~ (AB2st_i)) // Simplify,
  ‾stKink‾_i = (K̄ink̄_i ~ B_{i} ~ (AB2st_i)) // Simplify,
  stKink_i = (Kink_i ~ B_{i} ~ (AB2st_i)) // Simplify,
  stdm_i,j→k = ((st2AB_i st2AB_j) ~ B_i,j ~ dm_i,j→k ~ B_k ~ AB2st_k),
  stdΔ_i→j,k = (st2AB_i // dΔ_i→j,k // (AB2st_j AB2st_k)),
  stdS_i = (st2AB_i // dS_i // AB2st_i),
  stP_i,j = ((st2AB_i st2AB_j) // P_i,j),
  ddΔ_i→j,k = (st2AB_i // dΔ_i→j,k // (AB2st_j AB2st_k)) /.
    {S_j|k → S, T_j|k → T, t_j|k → t, s_j|k → s, (s*)_i → 0, (t*)_i → 0},
  ddS_i = (st2AB_i // dS_i // AB2st_i) /. {S_i → S, T_i → T,
     s_i → s, t_i → t, (s*)_i → 0, (t*)_i → 0},
  PP_i,j = ((st2AB_i st2AB_j) // P_i,j) /. {(s*)_i|j → 0, (t*)_i|j → 0},
  RR_i,j = (R_i,j // (AB2st_i AB2st_j)) /. {t_i|j → t, s_i|j → s},
  ‾RR‾_i,j = (R̄_i,j // (AB2st_i AB2st_j)) /. {t_i|j → t, s_i|j → s, S_i|j → S, T_i|j → T},
  ‾CC‾_i = (C̄_i ~ B_{i} ~ (AB2st_i)) /. {S_i|j → S, T_i|j → T} // Simplify,
  CC_i = (C_i ~ B_{i} ~ (AB2st_i)) /. {S_i|j → S, T_i|j → T} // Simplify,
  ‾KKink‾_i =
   (K̄ink̄_i ~ B_{i} ~ (AB2st_i)) /. {t_i|j → t, s_i|j → s, S_i|j → S, T_i|j → T} // Simplify,
  KKink_i = (Kink_i ~ B_{i} ~ (AB2st_i)) /. {t_i|j → t, s_i|j → s, S_i|j → S, T_i|j → T} //
    Simplify,
  ddm_i,j→k = ((st2AB_i st2AB_j) // dm_i,j→k // AB2st_k) /.
    {S_k → S, T_k → T, t_k → t, s_k → s, (s*)_i|j → 0, (t*)_i|j → 0}];

Define[BB_i, j→k = ((dΔ_i→1,r1 dΔ_j→2,r2) // dS_r1 // ‾dS‾_r2 // dm_r1,r2→k // dm_k,1→k // dm_k,2→k)]
```

Testing

```
Block[{$k = 1},
  {am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j}, aS → aS_i,
   āS → āS_i, bS → bS_i, b̄S → b̄S_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k}, dΔ → dΔ_{i→j,k},
   C → C_i, C̄ → C̄_i, Kink → Kink_i, K̄ink → K̄ink_i, AB2st → AB2st_i, st2AB → st2AB_i}] //
 Column
```

Check that on the generators this agrees with our conventions in the handout:

```
Timing@{{"[x,a]" →
     ((𝔼_{}→{1,2}[0, 0, a_2 x_1] // am_{1,2→1})[3] - (𝔼_{}→{1,2}[0, 0, a_1 x_2] // am_{1,2→1})[3]),
    "[A,X]" → ((𝔼_{}→{1,2}[0, 0, XX_2 A_1] // bm_{1,2→1})[3] -
       (𝔼_{}→{1,2}[0, 0, XX_1 A_2] // bm_{1,2→1})[3]), "[x,y]" →
     ((𝔼_{}→{1,2}[0, 0, y_2 x_1] // am_{1,2→1})[3] - (𝔼_{}→{1,2}[0, 0, y_1 x_2] // am_{1,2→1})[3]),
    "[Y,X]" → ((𝔼_{}→{1,2}[0, 0, XX_2 Y_1] // bm_{1,2→1})[3] -
       (𝔼_{}→{1,2}[0, 0, XX_1 Y_2] // bm_{1,2→1})[3])} /. z_1 → z,
  {"Δ[X]" → Last[𝔼_{}→{1}[0, 0, XX_1] ~B_1~ bΔ_{1→1,2}],
   "Δ[A]" → Last[𝔼_{}→{1}[0, 0, A_1] ~B_1~ bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼_{}→{1}[0, 0, a_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[z]" → Last[𝔼_{}→{1}[0, 0, z_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼_{}→{1}[0, 0, x_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[Z]" → Last[𝔼_{}→{1}[0, 0, Z_1] ~B_1~ bΔ_{1→1,2}]},
  {
    "S(a)" → ((𝔼_{}→{1}[0, 0, a_1] ~B_1~ aS_1)[3]),
    "S(z)" → ((𝔼_{}→{1}[0, 0, z_1] ~B_1~ aS_1)[3]),
    "S(x)" → ((𝔼_{}→{1}[0, 0, x_1] ~B_1~ aS_1)[3]),
    "S(A)" → ((𝔼_{}→{1}[0, 0, A_1] ~B_1~ bS_1)[3]),
    "S(X)" → ((𝔼_{}→{1}[0, 0, XX_1] ~B_1~ bS_1)[3]),
    "S(Z)" → ((𝔼_{}→{1}[0, 0, Z_1] ~B_1~ bS_1)[3])
   } /. z_1 → z}
```

```
{1.265625, {{[x,a] → 2 x + O[ϵ]^2, [A,X] → -XX ϵ + O[ϵ]^2,
    [x,y] → z - x y ħ ϵ + O[ϵ]^2, [Y,X] → (-2 Z + XX Y ħ) ϵ + O[ϵ]^2},
   {Δ[X] → (XX_2 + XX_1 𝔸_2^{-2 ħ} 𝔹_2^{ħ}) + O[ϵ]^2, Δ[A] → (A_1 + A_2) + O[ϵ]^2,
    Δ[a] → (a_1 + a_2) + O[ϵ]^2, Δ[z] → (z_1 + z_2) + (2 ħ x_1 y_2 - ħ a_1 z_2 - ħ b_1 z_2) ϵ + O[ϵ]^2,
    Δ[x] → (x_1 + x_2) - ħ a_1 x_2 ϵ + O[ϵ]^2, Δ[Z] → (Z_2 + Z_1 𝔸_2^{-ħ} 𝔹_2^{-ħ} + ħ XX_1 Y_2 𝔸_2^{-2 ħ} 𝔹_2^{ħ}) + O[ϵ]^2},
   {S(a) → -a + O[ϵ]^2, S(z) → -z + (2 x y ħ + 2 z ħ - a z ħ - b z ħ) ϵ + O[ϵ]^2,
    S(x) → -x - a x ħ ϵ + O[ϵ]^2, S(A) → -A + O[ϵ]^2, S(X) → -XX 𝔸^{2 ħ} 𝔹^{-ħ} + O[ϵ]^2,
    S(Z) → (-Z 𝔸^{ħ} 𝔹^{ħ} + XX Y 𝔸^{ħ} 𝔹^{ħ} ħ) + (2 Z 𝔸^{ħ} 𝔹^{ħ} ħ - XX Y 𝔸^{ħ} 𝔹^{ħ} ħ^2) ϵ + O[ϵ]^2}}}
```

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
Timing@Block[{$k = 1},
  HL /@
   {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
 ]
```

```
{0.437500, {True, True}}
```

R and *P* are inverses:

```
Timing@
 Block[{$k = 1}, {HL[(R_{i,j} // P_{i,k}) ≡ 𝔼_{{k}→{j}}[a_j a*_k + b_j b*_k, x_j x*_k + y_j y*_k + z_j z*_k, 1]]}]
```

```
{0.031250, {True}}
```

as and $\overline{aS}$ are inverses, bs and $\overline{bS}$ are inverses:

```
Timing[HL /@ {(\overline{aS}_1 // aS_1) ≡ 𝔼_{{1}→{1}}[a_1 a*_1 + b_1 b*_1, x_1 x*_1 + y_1 y*_1 + z_1 z*_1, 1],
    (\overline{bS}_1 // bS_1) ≡ 𝔼_{{1}→{1}}[A_1 A*_1 + B_1 B*_1, XX_1 XX*_1 + Y_1 Y*_1 + Z_1 Z*_1, 1]}]
```

```
{0.406250, {True, True}}
```

(co)-associativity on both sides

```
Timing[HL /@
  {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
   (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
```

```
{1.078125, {True, True, True, True}}
```

Δ is an algebra morphism

```
Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
    (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
```

```
{1.312500, {True, True}}
```

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@ {
    (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{1,2→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{1,2→1},
    (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{1,2→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{1,2→1}}]
```

```
{1.015625, {True, True, True, True}}
```

S is an algebra anti-(co)morphism

```
Timing[HL /@
  {am_{1,2→1} ~ B_1 ~ aS_1 ≡ (aS_1 aS_2) ~ B_{1,2} ~ am_{2,1→1}, bm_{1,2→1} ~ B_1 ~ bS_1 ≡ (bS_1 bS_2) ~ B_{1,2} ~ bm_{2,1→1},
   aS_1 ~ B_1 ~ aΔ_{1→1,2} ≡ aΔ_{1→2,1} ~ B_{1,2} ~ (aS_1 aS_2), bS_1 ~ B_1 ~ bΔ_{1→1,2} ≡ bΔ_{1→2,1} ~ B_{1,2} ~ (bS_1 bS_2)}]
```

```
{2.500000, {True, True, True, True}}
```

R-matrix and antipode

```
R_{1,2} ~ B_1 ~ (bS_1) ≡ \overline{R}_{1,2}
```

```
True
```

Pairing axioms

```
Timing[HL /@ {(bm_{1,2→1} 𝔼_{{3}→{3}}[b*_3 b_3 + a*_3 a_3, y*_3 y_3 + x*_3 x_3 + z*_3 z_3, 1]) ~ B_{1,3} ~ P_{1,3} ≡
     (𝔼_{{1}→{1}}[(A*)_1 A_1 + (B*)_1 B_1, (XX*)_1 XX_1 + (Y*)_1 Y_1 + Z*_1 Z_1, 1]
        𝔼_{{2}→{2}}[(A*)_2 A_2 + (B*)_2 B_2, (XX*)_2 XX_2 + (Y*)_2 Y_2 + Z*_2 Z_2, 1] aΔ_{3→4,5}) ~ B_{1,4} ~
      P_{1,4} ~ B_{2,5} ~ P_{2,5}, (bΔ_{1→1,2} 𝔼_{{3}→{3}}[b*_3 b_3 + a*_3 a_3, y*_3 y_3 + x*_3 x_3 + z*_3 z_3, 1]
        𝔼_{{4}→{4}}[b*_4 b_4 + a*_4 a_4, y*_4 y_4 + x*_4 x_4 + z*_4 z_4, 1]) ~ B_{1,3} ~ P_{1,3} ~ B_{2,4} ~ P_{2,4} ≡
     (𝔼_{{1}→{1}}[(A*)_1 A_1 + (B*)_1 B_1, (XX*)_1 XX_1 + (Y*)_1 Y_1 + Z*_1 Z_1, 1] am_{3,4→3}) ~ B_{1,3} ~ P_{1,3}}]
```

```
{0.34375, {True, True}}
```

```
Timing[HL /@ {((bS1 𝔼{2}→{2}[b*2 b2 + a*2 a2, y*2 y2 + x*2 x2 + z*2 z2, 1]) // P1,2) ≡
     ((𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] aS2) // P1,2),
    (bS1 𝔼{2}→{2}[b*2 b2 + a*2 a2, y*2 y2 + x*2 x2 + z*2 z2, 1]) ~B1,2~P1,2 ≡
     (𝔼{1}→{1}[(A*)1 A1 + (B*)1 B1, (XX*)1 XX1 + (Y*)1 Y1 + Z*1 Z1, 1] aS2) ~B1,2~P1,2}]
{0.28125, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
{
    "[a,y]" → ((𝔼{}→{1,2}[0, 0, y2 a1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, y1 a2] ~B1,2~ dm1,2→1)[[3]]),
    "[b,x]"  → ((𝔼{}→{1,2}[0, 0, x2 b1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, x1 b2] ~B1,2~ dm1,2→1)[[3]]),
    "[b,y]" → ((𝔼{}→{1,2}[0, 0, y2 b1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, y1 b2] ~B1,2~ dm1,2→1)[[3]]),
    "[a,x]" → ((𝔼{}→{1,2}[0, 0, x2 a1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, x1 a2] ~B1,2~ dm1,2→1)[[3]]),
    "[a,z]" → ((𝔼{}→{1,2}[0, 0, z2 a1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, z1 a2] ~B1,2~ dm1,2→1)[[3]]),
    "[b,z]" → ((𝔼{}→{1,2}[0, 0, z2 b1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, z1 b2] ~B1,2~ dm1,2→1)[[3]]),
    "[x,z]" → ((𝔼{}→{1,2}[0, 0, z2 x1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, z1 x2] ~B1,2~ dm1,2→1)[[3]]),
    "[y,z]" → ((𝔼{}→{1,2}[0, 0, z2 y1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, z1 y2] ~B1,2~ dm1,2→1)[[3]]),
    "[x,y]" → ((𝔼{}→{1,2}[0, 0, y2 x1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, y1 x2] ~B1,2~ dm1,2→1)[[3]]),
    "[Y,y]" → ((𝔼{}→{1,2}[0, 0, y2 Y1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, y1 Y2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[Y,x]" → ((𝔼{}→{1,2}[0, 0, x2 Y1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, x1 Y2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[X,y]" → ((𝔼{}→{1,2}[0, 0, y2 XX1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, y1 XX2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[XX,x]" → ((𝔼{}→{1,2}[0, 0, x2 XX1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, x1 XX2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[Z,z]" → ((𝔼{}→{1,2}[0, 0, z2 Z1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, z1 Z2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[Z,y]" → ((𝔼{}→{1,2}[0, 0, y2 Z1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, y1 Z2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[Z,x]" → ((𝔼{}→{1,2}[0, 0, x2 Z1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, x1 Z2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[XX,z]" → ((𝔼{}→{1,2}[0, 0, z2 XX1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, z1 XX2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[Y,z]" → ((𝔼{}→{1,2}[0, 0, z2 Y1] ~B1,2~ dm1,2→1)[[3]] -
         (𝔼{}→{1,2}[0, 0, z1 Y2] ~B1,2~ dm1,2→1)[[3]]) // Expand // Simplify,
    "[a,Z]" → ((𝔼{}→{1,2}[0, 0, Z2 a1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, Z1 a2] ~B1,2~ dm1,2→1)[[3]]),
    "[a,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 a1] ~B1,2~ dm1,2→1)[[3]] -
       (𝔼{}→{1,2}[0, 0, XX1 a2] ~B1,2~ dm1,2→1)[[3]]),
    "[a,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 a1] ~B1,2~ dm1,2→1)[[3]] -
```

```
        (𝔼{}→{1,2}[0, 0, Y1 a2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[b,Z]" → ((𝔼{}→{1,2}[0, 0, Z2 b1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Z1 b2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[b,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 b1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, XX1 b2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[b,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 b1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Y1 b2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[A,Z]" → ((𝔼{}→{1,2}[0, 0, Z2 A1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Z1 A2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[A,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 A1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, XX1 A2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[A,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 A1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Y1 A2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[B,Z]" → ((𝔼{}→{1,2}[0, 0, Z2 B1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Z1 B2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[B,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 B1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, XX1 B2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[B,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 B1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Y1 B2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[XX,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 XX1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Y1 XX2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[Z,Y]" → ((𝔼{}→{1,2}[0, 0, Y2 Z1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, Y1 Z2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[Z,XX]" → ((𝔼{}→{1,2}[0, 0, XX2 Z1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, XX1 Z2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[A,x]" → ((𝔼{}→{1,2}[0, 0, x2 A1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, x1 A2] ~B1,2~ dm1,2→1)⟦3⟧),
      "[A,y]" → ((𝔼{}→{1,2}[0, 0, y2 A1] ~B1,2~ dm1,2→1)⟦3⟧ -
        (𝔼{}→{1,2}[0, 0, y1 A2] ~B1,2~ dm1,2→1)⟦3⟧)
    } /. {v_1 → v} // Expand // Factor
{
  "Δ(a)" → ((𝔼{1,2}→{}[0, 0, a1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(x)" → ((𝔼{1,2}→{}[0, 0, x1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(b)" → ((𝔼{1,2}→{}[0, 0, b1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(y)" → ((𝔼{1,2}→{}[0, 0, y1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(z)" → ((𝔼{1,2}→{}[0, 0, z1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(XX)" → ((𝔼{1,2}→{}[0, 0, XX1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(Y)" → ((𝔼{1,2}→{}[0, 0, Y1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(Z)" → ((𝔼{1,2}→{}[0, 0, Z1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(A)" → ((𝔼{1,2}→{}[0, 0, A1] ~B1~ dΔ1→1,2)⟦3⟧),
  "Δ(B)" → ((𝔼{1,2}→{}[0, 0, B1] ~B1~ dΔ1→1,2)⟦3⟧)} // Simplify
{
   "S(a)" → ((𝔼{1}→{1}[0, 0, a1] ~B1~ dS1)⟦3⟧),
   "S(x)" → ((𝔼{1}→{1}[0, 0, x1] ~B1~ dS1)⟦3⟧),
   "S(b)" → ((𝔼{1}→{1}[0, 0, b1] ~B1~ dS1)⟦3⟧),
   "S(y)" → ((𝔼{1}→{1}[0, 0, y1] ~B1~ dS1)⟦3⟧),
   "S(z)" → ((𝔼{1}→{1}[0, 0, z1] ~B1~ dS1)⟦3⟧),
   "S(XX)" → ((𝔼{1}→{1}[0, 0, XX1] ~B1~ dS1)⟦3⟧),
   "S(Y)" → ((𝔼{1}→{1}[0, 0, Y1] ~B1~ dS1)⟦3⟧),
   "S(Z)" → ((𝔼{1}→{1}[0, 0, Z1] ~B1~ dS1)⟦3⟧),
   "S(A)" → ((𝔼{1}→{1}[0, 0, A1] ~B1~ dS1)⟦3⟧),
   "S(B)" → ((𝔼{1}→{1}[0, 0, B1] ~B1~ dS1)⟦3⟧)
```

$$\Big\{[a,y] \to y + O[\epsilon]^2,\ [b,x] \to x + O[\epsilon]^2,\ [b,y] \to -2\,y + O[\epsilon]^2,$$
$$[a,x] \to -2\,x + O[\epsilon]^2,\ [a,z] \to -z + O[\epsilon]^2,\ [b,z] \to -z + O[\epsilon]^2,$$
$$[x,z] \to x\,z\,\hbar\,\epsilon + O[\epsilon]^2,\ [y,z] \to -y\,z\,\hbar\,\epsilon + O[\epsilon]^2,\ [x,y] \to z - x\,y\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[Y,y] \to \frac{-1 + \mathbb{A}^{\hbar}\,\mathbb{B}^{-2\,\hbar}}{\hbar} + \left(-b\,\mathbb{A}^{\hbar}\,\mathbb{B}^{-2\,\hbar} + 2\,y\,Y\,\hbar\right)\epsilon + O[\epsilon]^2,\ [Y,x] \to -x\,Y\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[X,y] \to -XX\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ [XX,x] \to \frac{-1 + \mathbb{A}^{-2\,\hbar}\,\mathbb{B}^{\hbar}}{\hbar} + \left(-a\,\mathbb{A}^{-2\,\hbar}\,\mathbb{B}^{\hbar} + 2\,x\,XX\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$[Z,z] \to \frac{-1 + \mathbb{A}^{-\hbar}\,\mathbb{B}^{-\hbar}}{\hbar} + \mathbb{A}^{-\hbar}\,\mathbb{B}^{-\hbar}\left(-a - b + 2\,z\,Z\,\mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$[Z,y] \to -XX + y\,Z\,\hbar\,\epsilon + O[\epsilon]^2,\ [Z,x] \to Y\,\mathbb{A}^{-2\,\hbar}\,\mathbb{B}^{\hbar} + \mathbb{A}^{-2\,\hbar}\left(x\,Z\,\mathbb{A}^{2\,\hbar} - (-1+a)\,Y\,\mathbb{B}^{\hbar}\right)\hbar\,\epsilon + O[\epsilon]^2,$$
$$[XX,z] \to (-2\,y + XX\,z\,\hbar)\,\epsilon + O[\epsilon]^2,\ [Y,z] \to \left(2\,x\,\mathbb{A}^{\hbar}\,\mathbb{B}^{-2\,\hbar} + Y\,z\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$[a,Z] \to Z + O[\epsilon]^2,\ [a,XX] \to 2\,XX + O[\epsilon]^2,\ [a,Y] \to -Y + O[\epsilon]^2,\ [b,Z] \to Z + O[\epsilon]^2,$$
$$[b,XX] \to -XX + O[\epsilon]^2,\ [b,Y] \to 2\,Y + O[\epsilon]^2,\ [A,Z] \to -Z\,\epsilon + O[\epsilon]^2,$$
$$[A,XX] \to -XX\,\epsilon + O[\epsilon]^2,\ [A,Y] \to O[\epsilon]^2,\ [B,Z] \to -Z\,\epsilon + O[\epsilon]^2,\ [B,XX] \to O[\epsilon]^2,$$
$$[B,Y] \to -Y\,\epsilon + O[\epsilon]^2,\ [XX,Y] \to (2\,Z - XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2,\ [Z,Y] \to Y\,Z\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$[Z,XX] \to -XX\,Z\,\hbar\,\epsilon + O[\epsilon]^2,\ [A,x] \to x\,\epsilon + O[\epsilon]^2,\ [A,y] \to O[\epsilon]^2\Big\}$$

$$\Big\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^2,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + O[\epsilon]^2,$$
$$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^2,\ \Delta(y) \to (y_1 + y_2) - \hbar\,b_1\,y_2\,\epsilon + O[\epsilon]^2,$$
$$\Delta(z) \to (z_1 + z_2) + \hbar\,(2\,x_1\,y_2 - (a_1 + b_1)\,z_2)\,\epsilon + O[\epsilon]^2,\ \Delta(XX) \to \left(XX_1 + XX_2\,\mathbb{A}_1^{-2\,\hbar}\,\mathbb{B}_1^{\hbar}\right) + O[\epsilon]^2,$$
$$\Delta(Y) \to \left(Y_1 + Y_2\,\mathbb{A}_1^{\hbar}\,\mathbb{B}_1^{-2\,\hbar}\right) + O[\epsilon]^2,\ \Delta(Z) \to \left(Z_1 + Z_2\,\mathbb{A}_1^{-\hbar}\,\mathbb{B}_1^{-\hbar} + \hbar\,XX_2\,Y_1\,\mathbb{A}_1^{-2\,\hbar}\,\mathbb{B}_1^{\hbar}\right) + O[\epsilon]^2,$$
$$\Delta(A) \to (A_1 + A_2) + O[\epsilon]^2,\ \Delta(B) \to (B_1 + B_2) + O[\epsilon]^2\Big\}$$

$$\Big\{S(a) \to -a + O[\epsilon]^2,\ S(x) \to -x - a\,x\,\hbar\,\epsilon + O[\epsilon]^2,\ S(b) \to -b + O[\epsilon]^2,$$
$$S(y) \to -y - b\,y\,\hbar\,\epsilon + O[\epsilon]^2,\ S(z) \to -z + (2\,x\,y - (-2 + a + b)\,z)\,\hbar\,\epsilon + O[\epsilon]^2,$$
$$S(XX) \to -XX\,\mathbb{A}^{2\,\hbar}\,\mathbb{B}^{-\hbar} - 2\left(XX\,\mathbb{A}^{2\,\hbar}\,\mathbb{B}^{-\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$S(Y) \to -Y\,\mathbb{A}^{-\hbar}\,\mathbb{B}^{2\,\hbar} - 2\left(Y\,\mathbb{A}^{-\hbar}\,\mathbb{B}^{2\,\hbar}\,\hbar\right)\epsilon + O[\epsilon]^2,$$
$$S(Z) \to \mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,(-Z + XX\,Y\,\hbar) + \mathbb{A}^{\hbar}\,\mathbb{B}^{\hbar}\,\hbar\,(-2\,Z + 3\,XX\,Y\,\hbar)\,\epsilon + O[\epsilon]^2,$$
$$S(A) \to -A + O[\epsilon]^2,\ S(B) \to -B + O[\epsilon]^2\Big\}$$

(co)-associativity

```
Timing[HL /@
  {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2->1)}]
```

```
{14.218750, {True, True}}
```

```
Timing[HL /@ {(stdΔ1→1,2 // stdΔ2→2,3) ≡ (stdΔ1→1,3 // stdΔ1→1,2),
    (stdm1,2→1 // stdm1,3→1) ≡ (stdm2,3→2 // stdm1,2->1)}]
```

```
{7.531250, {True, True}}
```

```
Timing[HL /@ {(ddm1,2→1 ~ STB1 ~ ddm1,3→1) ≡ (ddm2,3→2 ~ STB2 ~ ddm1,2->1)}]
```

```
{2.125000, {True}}
```

Δ is an algebra morphism

```
Timing@HL[dm1,2→1 ~ B1 ~ dΔ1→1,2 ≡ (dΔ1→1,3 dΔ2→2,4) ~ B1,2,3,4 ~ (dm3,4→2 dm1,2→1)]
```

```
{7.296875, True}
```

S_2 inverts R, but not S_1:

```
In[65]:= Timing@{HL[R1,2 ~ B2 ~ dS2 ≡ R̄1,2]}
Out[65]= {0.796875, {True}}
```

S is convolution inverse of id

```
Timing[HL[# ≡ 𝔼{1}→{1}[0, 0, 1]] & /@
  {(dΔ1→1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2→1, (dΔ1→1,2 ~ B2 ~ dS2) // dm1,2→1}]
{4.703125, {True, True}}
```

S is a (co)-algebra anti-morphism

```
Timing[HL /@ Expand /@
   {dm1,2→1 ~ B1 ~ dS1 ≡ (dS1 dS2) ~ B1,2 ~ dm2,1→1, dS1 ~ B1 ~ dΔ1→1,2 ≡ dΔ1→2,1 ~ B1,2 ~ (dS1 dS2)}]
{22.718750, {True, True}}
```

Quasi-triangular axiom 1:

```
Timing@HL[R1,2 ~ B1 ~ dΔ1->1,3 ≡ (R1,4 R3,2) ~ B2,4 ~ dm2,4→2]
{0.375000, True}
```

Quasi-triangular axiom 2:

```
Timing@HL[
  ((dΔ1->1,2 R3,4) ~ B1,2,3,4 ~ (dm1,3→1 dm2,4→2)) ≡ ((dΔ1→2,1 R3,4) ~ B1,2,3,4 ~ (dm3,1→1 dm4,2→2))]
{2.359375, True}
```

The Drinfel'd element inverse property, $(u_1\,\bar{u}_2)\sim B_{1,2}\sim \mathrm{dm}_{1,2\to 1}\equiv \mathbb{E}[0,\,0,\,1]$:

```
Timing@
 HL[((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1→i) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1→j)) ~ Bi,j ~ dmi,j→i ≡
   𝔼{}→{i}[0, 0, 1]]
{2.453125, True}
```

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy. Taking the square root of z and multiplying it with S(u) yields the ribbon element v.

```
Timing@Block[{$k = 1},
  ((R1,2 ~ B1 ~ dS1 ~ B1,2 ~ dm2,1→i) ~ Bi ~ dSi) (R1,2 ~ B2 ~ dS2 ~ B2 ~ dS2 ~ B1,2 ~ dm2,1→j)) ~
   Bi,j ~ dmi,j→i]
{8.062500, 𝔼{}→{i}[0, 0, 𝔸i^(2ħ) 𝔹i^(2ħ) + (2ħ ai 𝔸i^(2ħ) 𝔹i^(2ħ) + 2ħ bi 𝔸i^(2ħ) 𝔹i^(2ħ)) ϵ + O[ϵ]^2]}
```

Turaev moves are checked here.

T-4:

```
Timing@Block[{$k = 1},
  HL /@ {(C̄1 C̄2 R3,4 C5 C6) ~ B1,3 ~ dm1,3→1 ~ B1,5 ~ dm1,5→1 ~ B2,4 ~ dm2,4→2 ~ B2,6 ~ dm2,6→2 ≡ R1,2}]
{3.796875, {True}}
```

T-5, T-6

In[63]:=
```
Timing@Block[{$k = 1},
  HL /@ {(Ci C̄j) ~ B_{i,j} ~ dm_{i,j→i} ≡ 𝔼_{}→{i}[0, 0, 1], (Ci C̄j) ~ B_{i,j} ~ dm_{j,i→i} ≡ 𝔼_{}→{i}[0, 0, 1],
    (C̄i C̄j) ~ B_{i,j} ~ dm_{i,j→i} ≡ (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i)
      (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}}]
```

Out[63]=
```
{16.296875, {True, True, True}}
```

Reidemeister 2 or T-3:

```
Timing[HL[# ≡ 𝔼_{}→{1,2}[0, 0, 1]] & /@
  {(R̄1,2 R3,4) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2}), (R1,2 R̄3,4) ~ B_{1,2,3,4} ~ (dm_{1,3→1} dm_{2,4→2})}]
{2.500000, {True, True}}
```

```
Timing[HL[# ≡ 𝔼_{}→{1,2}[0, 0, 1]] & /@ {(((R̄R̄1,2 RR3,4)) ~ B_{1,2,3,4} ~ ((ddm_{1,3→1} ddm_{2,4→2}))),
    ((RR1,2 R̄R̄3,4) ~ STB_{1,2,3,4} ~ (ddm_{1,3→1} ddm_{2,4→2}))}]
{2.718750, {True, True}}
```

Cyclic Reidemeister 2 or T-2:

```
Timing@HL[(R1,4 R̄5,2 C̄3) ~ B_{2,4} ~ dm_{2,4→2} ~ B_{1,3} ~ dm_{1,3→1} ~ B_{1,5} ~ dm_{1,5→1} ≡ C̄1 𝔼_{}→{2}[0, 0, 1]]
{7.765625, True}
```

Reidemeister 3 or T-1:

```
Timing@HL[((R_{1,2} R_{4,3} R_{5,6}) ~ B_{1,4} ~ dm_{1,4→1} ~ B_{2,5} ~ dm_{2,5→2} ~ B_{3,6} ~ dm_{3,6→3}) ≡
    ((R_{1,6} R_{2,3} R_{4,5}) ~ B_{1,4} ~ dm_{1,4→1} ~ B_{2,5} ~ dm_{2,5→2} ~ B_{3,6} ~ dm_{3,6→3})]
{5.343750, True}
```

```
Timing@HL[((RR_{1,2} RR_{4,3} RR_{5,6}) ~ STB_{1,4} ~ ddm_{1,4→1} ~ STB_{2,5} ~ ddm_{2,5→2} ~ STB_{3,6} ~ ddm_{3,6→3}) ≡
    ((RR_{1,6} RR_{2,3} RR_{4,5}) ~ STB_{1,4} ~ ddm_{1,4→1} ~ STB_{2,5} ~ ddm_{2,5→2} ~ STB_{3,6} ~ ddm_{3,6→3})]
{1.656250, True}
```

Relations between the four kinks or T-7

In[64]:=
```
Timing[HL /@ {Kink_i ≡ (R_{3,1} C_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→i},
    K̄ink̄_j ≡ (R̄_{3,1} C̄_2) ~ B_{1,2} ~ dm_{1,2→1} ~ B_{1,3} ~ dm_{1,3→j}, (Kink_i K̄ink̄_j) ~ B_{i,j} ~ dm_{i,j→1} ≡
    𝔼_{}→{1}[0, 0, 1], (Kink_i K̄ink̄_j) ~ B_{i,j} ~ dm_{j,i→1} ≡ 𝔼_{}→{1}[0, 0, 1]}]
```

Out[64]=
```
{9.187500, {True, True, True, True}}
```

The Trefoil

```
Timing@Block[{$k = 1}, ZZ = RR_{1,5} RR_{6,2} RR_{3,7} C̄C̄4 K̄K̄ink̄8 K̄K̄ink̄9 K̄K̄ink̄10;
  Do[ZZ = (ZZ /. {ϵ → 0}) ~ B_{1,r} ~ (ddm_{1,r→1} /. {ϵ → 0}), {r, 2, 10}];
  {Simplify /@ (ZZ /. {ϵ → 0})}]
```

$$\left\{1.250000,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR_{1,5} RR_{6,2} RR_{3,7} C̄C̄4 K̄K̄ink̄8 K̄K̄ink̄9 K̄K̄ink̄10);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~ B_{1,r} ~ (ddm_{1,r→1}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{28.890625,$$

$$\Big\{\mathbf{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{S^2\,T^2}{\left(1-S+S^2\right)\,\left(1-T+T^2\right)\,\left(1-S\,T+S^2\,T^2\right)}+\Big(2\,S^2\,T^2\,\Big(S-2\,S^2+3\,S^3-2\,S^4+T-$$
$$2\,S\,T+S^2\,T+2\,S^3\,T-5\,S^4\,T+6\,S^5\,T-2\,T^2+S\,T^2-S^3\,T^2-4\,S^4\,T^2+5\,S^5\,T^2-11\,S^6\,T^2+$$
$$3\,T^3+2\,S\,T^3-S^2\,T^3+4\,S^3\,T^3+6\,S^4\,T^3-S^5\,T^3+7\,S^6\,T^3+10\,S^7\,T^3-2\,T^4-5\,S\,T^4-$$
$$4\,S^2\,T^4+6\,S^3\,T^4-24\,S^4\,T^4+10\,S^5\,T^4-12\,S^6\,T^4-11\,S^7\,T^4-6\,S^8\,T^4+6\,S\,T^5+$$
$$5\,S^2\,T^5-S^3\,T^5+10\,S^4\,T^5+12\,S^5\,T^5-S^6\,T^5+10\,S^7\,T^5+13\,S^8\,T^5-11\,S^2\,T^6+7\,S^3\,T^6-$$
$$12\,S^4\,T^6-S^5\,T^6-16\,S^6\,T^6+11\,S^7\,T^6-20\,S^8\,T^6+10\,S^3\,T^7-11\,S^4\,T^7+10\,S^5\,T^7+$$
$$11\,S^6\,T^7-14\,S^7\,T^7+15\,S^8\,T^7-6\,S^4\,T^8+13\,S^5\,T^8-20\,S^6\,T^8+15\,S^7\,T^8-8\,S^8\,T^8+$$
$$\left(1-S+S^2\right)\,\left(1-T+T^2\right)^2\,\left(-2+S+3\,S\,T+2\,S^6\,T^4+S^4\,T^2\,(3+T)-S^5\,T^3\,(3+T)-\right.$$
$$\left.S^2\,T\,(1+3\,T)+S^3\,T\,\left(-1+T^2\right)\right)\,a_1+\left(1-S+S^2\right)^2\,\left(1-T+T^2\right)\,\left(-2+T+3\,S\,T-\right.$$
$$\left.S\,(1+3\,S)\,T^2+S\,\left(-1+S^2\right)\,T^3+S^2\,(3+S)\,T^4-S^3\,(3+S)\,T^5+2\,S^4\,T^6\right)\,b_1+$$
$$2\,x_1\,XX_1+2\,S^3\,x_1\,XX_1-5\,T\,x_1\,XX_1-2\,S\,T\,x_1\,XX_1-3\,S^2\,T\,x_1\,XX_1-2\,S^3\,T\,x_1\,XX_1-$$
$$5\,S^4\,T\,x_1\,XX_1+9\,T^2\,x_1\,XX_1+5\,S\,T^2\,x_1\,XX_1+9\,S^2\,T^2\,x_1\,XX_1+9\,S^3\,T^2\,x_1\,XX_1+$$
$$5\,S^4\,T^2\,x_1\,XX_1+9\,S^5\,T^2\,x_1\,XX_1-7\,T^3\,x_1\,XX_1-12\,S\,T^3\,x_1\,XX_1-12\,S^2\,T^3\,x_1\,XX_1-$$
$$14\,S^3\,T^3\,x_1\,XX_1-12\,S^4\,T^3\,x_1\,XX_1-12\,S^5\,T^3\,x_1\,XX_1-7\,S^6\,T^3\,x_1\,XX_1+4\,T^4\,x_1\,XX_1+$$
$$10\,S\,T^4\,x_1\,XX_1+18\,S^2\,T^4\,x_1\,XX_1+14\,S^3\,T^4\,x_1\,XX_1+14\,S^4\,T^4\,x_1\,XX_1+18\,S^5\,T^4\,x_1\,XX_1+$$
$$10\,S^6\,T^4\,x_1\,XX_1+4\,S^7\,T^4\,x_1\,XX_1-7\,S\,T^5\,x_1\,XX_1-12\,S^2\,T^5\,x_1\,XX_1-12\,S^3\,T^5\,x_1\,XX_1-$$
$$14\,S^4\,T^5\,x_1\,XX_1-12\,S^5\,T^5\,x_1\,XX_1-12\,S^6\,T^5\,x_1\,XX_1-7\,S^7\,T^5\,x_1\,XX_1+9\,S^2\,T^6\,x_1\,XX_1+$$
$$5\,S^3\,T^6\,x_1\,XX_1+9\,S^4\,T^6\,x_1\,XX_1+9\,S^5\,T^6\,x_1\,XX_1+5\,S^6\,T^6\,x_1\,XX_1+9\,S^7\,T^6\,x_1\,XX_1-$$
$$5\,S^3\,T^7\,x_1\,XX_1-2\,S^4\,T^7\,x_1\,XX_1-3\,S^5\,T^7\,x_1\,XX_1-2\,S^6\,T^7\,x_1\,XX_1-5\,S^7\,T^7\,x_1\,XX_1+$$
$$2\,S^4\,T^8\,x_1\,XX_1+2\,S^7\,T^8\,x_1\,XX_1+2\,y_1\,Y_1-5\,S\,y_1\,Y_1+9\,S^2\,y_1\,Y_1-7\,S^3\,y_1\,Y_1+$$
$$4\,S^4\,y_1\,Y_1-2\,S\,T\,y_1\,Y_1+5\,S^2\,T\,y_1\,Y_1-12\,S^3\,T\,y_1\,Y_1+10\,S^4\,T\,y_1\,Y_1-7\,S^5\,T\,y_1\,Y_1-$$
$$3\,S\,T^2\,y_1\,Y_1+9\,S^2\,T^2\,y_1\,Y_1-12\,S^3\,T^2\,y_1\,Y_1+18\,S^4\,T^2\,y_1\,Y_1-12\,S^5\,T^2\,y_1\,Y_1+$$
$$9\,S^6\,T^2\,y_1\,Y_1+2\,T^3\,y_1\,Y_1-2\,S\,T^3\,y_1\,Y_1+9\,S^2\,T^3\,y_1\,Y_1-14\,S^3\,T^3\,y_1\,Y_1+$$
$$14\,S^4\,T^3\,y_1\,Y_1-12\,S^5\,T^3\,y_1\,Y_1+5\,S^6\,T^3\,y_1\,Y_1-5\,S^7\,T^3\,y_1\,Y_1-5\,S\,T^4\,y_1\,Y_1+$$
$$5\,S^2\,T^4\,y_1\,Y_1-12\,S^3\,T^4\,y_1\,Y_1+14\,S^4\,T^4\,y_1\,Y_1-14\,S^5\,T^4\,y_1\,Y_1+9\,S^6\,T^4\,y_1\,Y_1-$$
$$2\,S^7\,T^4\,y_1\,Y_1+2\,S^8\,T^4\,y_1\,Y_1+9\,S^2\,T^5\,y_1\,Y_1-12\,S^3\,T^5\,y_1\,Y_1+18\,S^4\,T^5\,y_1\,Y_1-$$
$$12\,S^5\,T^5\,y_1\,Y_1+9\,S^6\,T^5\,y_1\,Y_1-3\,S^7\,T^5\,y_1\,Y_1-7\,S^3\,T^6\,y_1\,Y_1+10\,S^4\,T^6\,y_1\,Y_1-$$
$$12\,S^5\,T^6\,y_1\,Y_1+5\,S^6\,T^6\,y_1\,Y_1-2\,S^7\,T^6\,y_1\,Y_1+4\,S^4\,T^7\,y_1\,Y_1-7\,S^5\,T^7\,y_1\,Y_1+$$
$$9\,S^6\,T^7\,y_1\,Y_1-5\,S^7\,T^7\,y_1\,Y_1+2\,S^8\,T^7\,y_1\,Y_1-2\,XX_1\,Y_1\,z_1+7\,S\,XX_1\,Y_1\,z_1-$$
$$9\,S^2\,XX_1\,Y_1\,z_1+7\,S^3\,XX_1\,Y_1\,z_1-2\,S^4\,XX_1\,Y_1\,z_1-5\,S\,T\,XX_1\,Y_1\,z_1-5\,S^4\,T\,XX_1\,Y_1\,z_1+$$
$$12\,S\,T^2\,XX_1\,Y_1\,z_1-9\,S^2\,T^2\,XX_1\,Y_1\,z_1+21\,S^3\,T^2\,XX_1\,Y_1\,z_1-9\,S^4\,T^2\,XX_1\,Y_1\,z_1+$$
$$12\,S^5\,T^2\,XX_1\,Y_1\,z_1-2\,T^3\,XX_1\,Y_1\,z_1-5\,S\,T^3\,XX_1\,Y_1\,z_1-9\,S^2\,T^3\,XX_1\,Y_1\,z_1-$$
$$7\,S^3\,T^3\,XX_1\,Y_1\,z_1-7\,S^4\,T^3\,XX_1\,Y_1\,z_1-9\,S^5\,T^3\,XX_1\,Y_1\,z_1-5\,S^6\,T^3\,XX_1\,Y_1\,z_1-$$
$$2\,S^7\,T^3\,XX_1\,Y_1\,z_1+7\,S\,T^4\,XX_1\,Y_1\,z_1+21\,S^3\,T^4\,XX_1\,Y_1\,z_1-7\,S^4\,T^4\,XX_1\,Y_1\,z_1+$$
$$21\,S^5\,T^4\,XX_1\,Y_1\,z_1+7\,S^7\,T^4\,XX_1\,Y_1\,z_1-9\,S^2\,T^5\,XX_1\,Y_1\,z_1-9\,S^4\,T^5\,XX_1\,Y_1\,z_1-$$
$$9\,S^5\,T^5\,XX_1\,Y_1\,z_1-9\,S^7\,T^5\,XX_1\,Y_1\,z_1+7\,S^3\,T^6\,XX_1\,Y_1\,z_1-5\,S^4\,T^6\,XX_1\,Y_1\,z_1+$$
$$12\,S^5\,T^6\,XX_1\,Y_1\,z_1-5\,S^6\,T^6\,XX_1\,Y_1\,z_1+7\,S^7\,T^6\,XX_1\,Y_1\,z_1-2\,S^4\,T^7\,XX_1\,Y_1\,z_1-$$
$$2\,S^7\,T^7\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-7\,S\,z_1\,Z_1+9\,S^2\,z_1\,Z_1-5\,S^3\,z_1\,Z_1+2\,S^4\,z_1\,Z_1-7\,T\,z_1\,Z_1+$$
$$10\,S\,T\,z_1\,Z_1-12\,S^2\,T\,z_1\,Z_1+5\,S^3\,T\,z_1\,Z_1-2\,S^4\,T\,z_1\,Z_1+9\,T^2\,z_1\,Z_1-12\,S\,T^2\,z_1\,Z_1+$$
$$18\,S^2\,T^2\,z_1\,Z_1-12\,S^3\,T^2\,z_1\,Z_1+9\,S^4\,T^2\,z_1\,Z_1-3\,S^5\,T^2\,z_1\,Z_1-5\,T^3\,z_1\,Z_1+$$
$$5\,S\,T^3\,z_1\,Z_1-12\,S^2\,T^3\,z_1\,Z_1+14\,S^3\,T^3\,z_1\,Z_1-14\,S^4\,T^3\,z_1\,Z_1+9\,S^5\,T^3\,z_1\,Z_1-$$
$$2\,S^6\,T^3\,z_1\,Z_1+2\,S^7\,T^3\,z_1\,Z_1+2\,T^4\,z_1\,Z_1-2\,S\,T^4\,z_1\,Z_1+9\,S^2\,T^4\,z_1\,Z_1-14\,S^3\,T^4\,z_1\,Z_1+$$
$$14\,S^4\,T^4\,z_1\,Z_1-12\,S^5\,T^4\,z_1\,Z_1+5\,S^6\,T^4\,z_1\,Z_1-5\,S^7\,T^4\,z_1\,Z_1-3\,S^2\,T^5\,z_1\,Z_1+$$
$$9\,S^3\,T^5\,z_1\,Z_1-12\,S^4\,T^5\,z_1\,Z_1+18\,S^5\,T^5\,z_1\,Z_1-12\,S^6\,T^5\,z_1\,Z_1+9\,S^7\,T^5\,z_1\,Z_1-$$
$$2\,S^3\,T^6\,z_1\,Z_1+5\,S^4\,T^6\,z_1\,Z_1-12\,S^5\,T^6\,z_1\,Z_1+10\,S^6\,T^6\,z_1\,Z_1-7\,S^7\,T^6\,z_1\,Z_1+$$
$$2\,S^3\,T^7\,z_1\,Z_1-5\,S^4\,T^7\,z_1\,Z_1+9\,S^5\,T^7\,z_1\,Z_1-7\,S^6\,T^7\,z_1\,Z_1+4\,S^7\,T^7\,z_1\,Z_1\Big)\,\epsilon\Big)\Big/$$

$$\left(\left(1-S+S^2\right)^3\left(1-T+T^2\right)^3\left(1-S\,T+S^2\,T^2\right)^3\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

The Figure Eight knot

```
Timing@Block[{$k = 1},
  ZZ = (RR8,1 RR2,6 RR5,9 RR10,3 CC7 CC4);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~STB1,r~ (ddm1,r→1 /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{2.125000,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\left(1-3\,T+T^2\right)\left(1-3\,S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR8,1 RR2,6 RR5,9 RR10,3 CC7 CC4);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{532.593750,$$

$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ -\frac{S^2\,T^2}{\left(1-3\,S+S^2\right)\left(1-3\,T+T^2\right)\left(1-3\,S\,T+S^2\,T^2\right)}-\Big(2\,\Big(S^2\,T^2\,\Big(4-9\,S+2\,S^2-9\,T+$$
$$12\,S\,T+6\,S^2\,T+2\,T^2+6\,S\,T^2-6\,S^3\,T^2-2\,S^4\,T^2-6\,S^2\,T^3-12\,S^3\,T^3+9\,S^4\,T^3-$$
$$2\,S^2\,T^4+9\,S^3\,T^4-4\,S^4\,T^4+\left(1-3\,T+T^2\right)\left(-2+2\,S^4\,T^2+3\,S\,(1+T)-3\,S^3\,T\,(1+T)\right)$$
$$a_1+\left(1-3\,S+S^2\right)\left(-2+3\,(1+S)\,T-3\,S\,(1+S)\,T^3+2\,S^2\,T^4\right)b_1+$$
$$2\,x_1\,XX_1+2\,S\,x_1\,XX_1-9\,T\,x_1\,XX_1-3\,S\,T\,x_1\,XX_1-9\,S^2\,T\,x_1\,XX_1+4\,T^2\,x_1\,XX_1+$$
$$16\,S\,T^2\,x_1\,XX_1+16\,S^2\,T^2\,x_1\,XX_1+4\,S^3\,T^2\,x_1\,XX_1-9\,S\,T^3\,x_1\,XX_1-3\,S^2\,T^3\,x_1\,XX_1-$$
$$9\,S^3\,T^3\,x_1\,XX_1+2\,S^2\,T^4\,x_1\,XX_1+2\,S^3\,T^4\,x_1\,XX_1+2\,y_1\,Y_1-9\,S\,y_1\,Y_1+4\,S^2\,y_1\,Y_1+$$
$$2\,T\,y_1\,Y_1-3\,S\,T\,y_1\,Y_1+16\,S^2\,T\,y_1\,Y_1-9\,S^3\,T\,y_1\,Y_1-9\,S\,T^2\,y_1\,Y_1+16\,S^2\,T^2\,y_1\,Y_1-$$
$$3\,S^3\,T^2\,y_1\,Y_1+2\,S^4\,T^2\,y_1\,Y_1+4\,S^2\,T^3\,y_1\,Y_1-9\,S^3\,T^3\,y_1\,Y_1+2\,S^4\,T^3\,y_1\,Y_1-$$
$$2\,XX_1\,Y_1\,z_1+11\,S\,XX_1\,Y_1\,z_1-2\,S^2\,XX_1\,Y_1\,z_1-2\,T\,XX_1\,Y_1\,z_1-8\,S\,T\,XX_1\,Y_1\,z_1-$$
$$8\,S^2\,T\,XX_1\,Y_1\,z_1-2\,S^3\,T\,XX_1\,Y_1\,z_1+11\,S\,T^2\,XX_1\,Y_1\,z_1-8\,S^2\,T^2\,XX_1\,Y_1\,z_1+11\,S^3\,T^2$$
$$XX_1\,Y_1\,z_1-2\,S^2\,T^3\,XX_1\,Y_1\,z_1-2\,S^3\,T^3\,XX_1\,Y_1\,z_1+4\,z_1\,Z_1-9\,S\,z_1\,Z_1+2\,S^2\,z_1\,Z_1-$$
$$9\,T\,z_1\,Z_1+16\,S\,T\,z_1\,Z_1-3\,S^2\,T\,z_1\,Z_1+2\,S^3\,T\,z_1\,Z_1+2\,T^2\,z_1\,Z_1-3\,S\,T^2\,z_1\,Z_1+$$
$$16\,S^2\,T^2\,z_1\,Z_1-9\,S^3\,T^2\,z_1\,Z_1+2\,S\,T^3\,z_1\,Z_1-9\,S^2\,T^3\,z_1\,Z_1+4\,S^3\,T^3\,z_1\,Z_1\Big)\Big)\,\epsilon\Big)\Big/$$
$$\left(\left(1-3\,S+S^2\right)^2\left(1-3\,T+T^2\right)^2\left(1-3\,S\,T+S^2\,T^2\right)^2\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

The 6-3 knot

```
Timing@Block[{$k = 1},
  ZZ = (RR6,12 RR10,14 RR13,7 RR1,5 RR8,2 RR3,9 CC4 CC11);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 14}];
  {Simplify@ZZ}]
```

$$\Big\{337.828125,\ \Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \left(S^4\,T^4\right)\Big/\Big(\left(1-3\,S+5\,S^2-3\,S^3+S^4\right)$$
$$\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)\left(1-3\,S\,T+5\,S^2\,T^2-3\,S^3\,T^3+S^4\,T^4\right)\Big)+$$
$$\Big(2\,S^4\,T^4\,\Big(8-21\,S+30\,S^2-15\,S^3+4\,S^4-21\,T+36\,S\,T-30\,S^2\,T-24\,S^3\,T+18\,S^4\,T-$$
$$6\,S^5\,T+30\,T^2-30\,S\,T^2+12\,S^2\,T^2+45\,S^3\,T^2+6\,S^4\,T^2-6\,S^5\,T^2-15\,T^3-24\,S\,T^3+$$
$$45\,S^2\,T^3-48\,S^3\,T^3-24\,S^4\,T^3+6\,S^6\,T^3+6\,S^7\,T^3+4\,T^4+18\,S\,T^4+6\,S^2\,T^4-$$
$$24\,S^3\,T^4+24\,S^5\,T^4-6\,S^6\,T^4-18\,S^7\,T^4-4\,S^8\,T^4-6\,S\,T^5-6\,S^2\,T^5+24\,S^4\,T^5+$$
$$48\,S^5\,T^5-45\,S^6\,T^5+24\,S^7\,T^5+15\,S^8\,T^5+6\,S^3\,T^6-6\,S^4\,T^6-45\,S^5\,T^6-12\,S^6\,T^6+$$

```
30 S^7 T^6 - 30 S^8 T^6 + 6 S^3 T^7 - 18 S^4 T^7 + 24 S^5 T^7 + 30 S^6 T^7 - 36 S^7 T^7 + 21 S^8 T^7 -
4 S^4 T^8 + 15 S^5 T^8 - 30 S^6 T^8 + 21 S^7 T^8 - 8 S^8 T^8 + (1 - 3 T + 5 T^2 - 3 T^3 + T^4)
  (-4 + 4 S^8 T^4 + 9 S (1 + T) - 9 S^7 T^3 (1 + T) - 2 S^2 (5 + 9 T + 5 T^2) +
    2 S^6 T^2 (5 + 9 T + 5 T^2) + 3 S^3 (1 + 5 T + 5 T^2 + T^3) - 3 S^5 T (1 + 5 T + 5 T^2 + T^3)) a1 +
(1 - 3 S + 5 S^2 - 3 S^3 + S^4) (-4 + 9 (1 + S) T - 2 (5 + 9 S + 5 S^2) T^2 +
    3 (1 + 5 S + 5 S^2 + S^3) T^3 - 3 S (1 + 5 S + 5 S^2 + S^3) T^5 + 2 S^2 (5 + 9 S + 5 S^2) T^6 -
    9 S^3 (1 + S) T^7 + 4 S^4 T^8) b1 + 4 x1 XX1 - 2 S x1 XX1 - 2 S^2 x1 XX1 + 4 S^3 x1 XX1 -
15 T x1 XX1 + 3 S T x1 XX1 - 3 S^2 T x1 XX1 + 3 S^3 T x1 XX1 - 15 S^4 T x1 XX1 +
30 T^2 x1 XX1 + 6 S T^2 x1 XX1 + 12 S^2 T^2 x1 XX1 + 12 S^3 T^2 x1 XX1 + 6 S^4 T^2 x1 XX1 +
30 S^5 T^2 x1 XX1 - 21 T^3 x1 XX1 - 51 S T^3 x1 XX1 - 6 S^2 T^3 x1 XX1 - 30 S^3 T^3 x1 XX1 -
6 S^4 T^3 x1 XX1 - 51 S^5 T^3 x1 XX1 - 21 S^6 T^3 x1 XX1 + 8 T^4 x1 XX1 + 44 S T^4 x1 XX1 +
56 S^2 T^4 x1 XX1 + 8 S^3 T^4 x1 XX1 + 8 S^4 T^4 x1 XX1 + 56 S^5 T^4 x1 XX1 + 44 S^6 T^4 x1 XX1 +
8 S^7 T^4 x1 XX1 - 21 S T^5 x1 XX1 - 51 S^2 T^5 x1 XX1 - 6 S^3 T^5 x1 XX1 - 30 S^4 T^5 x1 XX1 -
6 S^5 T^5 x1 XX1 - 51 S^6 T^5 x1 XX1 - 21 S^7 T^5 x1 XX1 + 30 S^2 T^6 x1 XX1 + 6 S^3 T^6 x1 XX1 +
12 S^4 T^6 x1 XX1 + 12 S^5 T^6 x1 XX1 + 6 S^6 T^6 x1 XX1 + 30 S^7 T^6 x1 XX1 - 15 S^3 T^7 x1 XX1 +
3 S^4 T^7 x1 XX1 - 3 S^5 T^7 x1 XX1 + 3 S^6 T^7 x1 XX1 - 15 S^7 T^7 x1 XX1 + 4 S^4 T^8 x1 XX1 -
2 S^5 T^8 x1 XX1 - 2 S^6 T^8 x1 XX1 + 4 S^7 T^8 x1 XX1 + 4 y1 Y1 - 15 S y1 Y1 + 30 S^2 y1 Y1 -
21 S^3 y1 Y1 + 8 S^4 y1 Y1 - 2 T y1 Y1 + 3 S T y1 Y1 + 6 S^2 T y1 Y1 - 51 S^3 T y1 Y1 +
44 S^4 T y1 Y1 - 21 S^5 T y1 Y1 - 2 T^2 y1 Y1 - 3 S T^2 y1 Y1 + 12 S^2 T^2 y1 Y1 -
6 S^3 T^2 y1 Y1 + 56 S^4 T^2 y1 Y1 - 51 S^5 T^2 y1 Y1 + 30 S^6 T^2 y1 Y1 + 4 T^3 y1 Y1 +
3 S T^3 y1 Y1 + 12 S^2 T^3 y1 Y1 - 30 S^3 T^3 y1 Y1 + 8 S^4 T^3 y1 Y1 - 6 S^5 T^3 y1 Y1 +
6 S^6 T^3 y1 Y1 - 15 S^7 T^3 y1 Y1 - 15 S T^4 y1 Y1 + 6 S^2 T^4 y1 Y1 - 6 S^3 T^4 y1 Y1 +
8 S^4 T^4 y1 Y1 - 30 S^5 T^4 y1 Y1 + 12 S^6 T^4 y1 Y1 + 3 S^7 T^4 y1 Y1 + 4 S^8 T^4 y1 Y1 +
30 S^2 T^5 y1 Y1 - 51 S^3 T^5 y1 Y1 + 56 S^4 T^5 y1 Y1 - 6 S^5 T^5 y1 Y1 + 12 S^6 T^5 y1 Y1 -
3 S^7 T^5 y1 Y1 - 2 S^8 T^5 y1 Y1 - 21 S^3 T^6 y1 Y1 + 44 S^4 T^6 y1 Y1 - 51 S^5 T^6 y1 Y1 +
6 S^6 T^6 y1 Y1 + 3 S^7 T^6 y1 Y1 - 2 S^8 T^6 y1 Y1 + 8 S^4 T^7 y1 Y1 - 21 S^5 T^7 y1 Y1 +
30 S^6 T^7 y1 Y1 - 15 S^7 T^7 y1 Y1 + 4 S^8 T^7 y1 Y1 - 4 XX1 Y1 z1 + 19 S XX1 Y1 z1 -
32 S^2 XX1 Y1 z1 + 19 S^3 XX1 Y1 z1 - 4 S^4 XX1 Y1 z1 + 2 T XX1 Y1 z1 - 22 S T XX1 Y1 z1 +
16 S^2 T XX1 Y1 z1 + 16 S^3 T XX1 Y1 z1 - 22 S^4 T XX1 Y1 z1 + 2 S^5 T XX1 Y1 z1 +
2 T^2 XX1 Y1 z1 + 35 S T^2 XX1 Y1 z1 - 28 S^2 T^2 XX1 Y1 z1 + 34 S^3 T^2 XX1 Y1 z1 -
28 S^4 T^2 XX1 Y1 z1 + 35 S^5 T^2 XX1 Y1 z1 + 2 S^6 T^2 XX1 Y1 z1 - 4 T^3 XX1 Y1 z1 -
22 S T^3 XX1 Y1 z1 - 28 S^2 T^3 XX1 Y1 z1 - 4 S^3 T^3 XX1 Y1 z1 - 4 S^4 T^3 XX1 Y1 z1 -
28 S^5 T^3 XX1 Y1 z1 - 22 S^6 T^3 XX1 Y1 z1 - 4 S^7 T^3 XX1 Y1 z1 + 19 S T^4 XX1 Y1 z1 +
16 S^2 T^4 XX1 Y1 z1 + 34 S^3 T^4 XX1 Y1 z1 - 4 S^4 T^4 XX1 Y1 z1 + 34 S^5 T^4 XX1 Y1 z1 +
16 S^6 T^4 XX1 Y1 z1 + 19 S^7 T^4 XX1 Y1 z1 - 32 S^2 T^5 XX1 Y1 z1 + 16 S^3 T^5 XX1 Y1 z1 -
28 S^4 T^5 XX1 Y1 z1 - 28 S^5 T^5 XX1 Y1 z1 + 16 S^6 T^5 XX1 Y1 z1 - 32 S^7 T^5 XX1 Y1 z1 +
19 S^3 T^6 XX1 Y1 z1 - 22 S^4 T^6 XX1 Y1 z1 + 35 S^5 T^6 XX1 Y1 z1 - 22 S^6 T^6 XX1 Y1 z1 +
19 S^7 T^6 XX1 Y1 z1 - 4 S^4 T^7 XX1 Y1 z1 + 2 S^5 T^7 XX1 Y1 z1 + 2 S^6 T^7 XX1 Y1 z1 -
4 S^7 T^7 XX1 Y1 z1 + 8 z1 Z1 - 21 S z1 Z1 + 30 S^2 z1 Z1 - 15 S^3 z1 Z1 + 4 S^4 z1 Z1 -
21 T z1 Z1 + 44 S T z1 Z1 - 51 S^2 T z1 Z1 + 6 S^3 T z1 Z1 + 3 S^4 T z1 Z1 - 2 S^5 T z1 Z1 +
30 T^2 z1 Z1 - 51 S T^2 z1 Z1 + 56 S^2 T^2 z1 Z1 - 6 S^3 T^2 z1 Z1 + 12 S^4 T^2 z1 Z1 -
3 S^5 T^2 z1 Z1 - 2 S^6 T^2 z1 Z1 - 15 T^3 z1 Z1 + 6 S T^3 z1 Z1 - 6 S^2 T^3 z1 Z1 + 8 S^3 T^3 z1 Z1 -
30 S^4 T^3 z1 Z1 + 12 S^5 T^3 z1 Z1 + 3 S^6 T^3 z1 Z1 + 4 S^7 T^3 z1 Z1 + 4 T^4 z1 Z1 +
3 S T^4 z1 Z1 + 12 S^2 T^4 z1 Z1 - 30 S^3 T^4 z1 Z1 + 8 S^4 T^4 z1 Z1 - 6 S^5 T^4 z1 Z1 +
6 S^6 T^4 z1 Z1 - 15 S^7 T^4 z1 Z1 - 2 S T^5 z1 Z1 - 3 S^2 T^5 z1 Z1 + 12 S^3 T^5 z1 Z1 -
6 S^4 T^5 z1 Z1 + 56 S^5 T^5 z1 Z1 - 51 S^6 T^5 z1 Z1 + 30 S^7 T^5 z1 Z1 - 2 S^2 T^6 z1 Z1 +
3 S^3 T^6 z1 Z1 + 6 S^4 T^6 z1 Z1 - 51 S^5 T^6 z1 Z1 + 44 S^6 T^6 z1 Z1 - 21 S^7 T^6 z1 Z1 +
4 S^3 T^7 z1 Z1 - 15 S^4 T^7 z1 Z1 + 30 S^5 T^7 z1 Z1 - 21 S^6 T^7 z1 Z1 + 8 S^7 T^7 z1 Z1) ϵ) /
```

$$\left(\left(1-3\,S+5\,S^2-3\,S^3+S^4\right)^2\left(1-3\,T+5\,T^2-3\,T^3+T^4\right)^2\left(1-3\,S\,T+5\,S^2\,T^2-3\,S^3\,T^3+S^4\,T^4\right)^2\right)+O[\epsilon]^2\Big]\Big\}\Big\}$$

Mirror Trefoil

```
Timing@Block[{$k = 1},
  ZZ = (RR1,5 RR6,2 RR3,7 CC4 KKink8 KKink9 KKink10);
  Do[Print["Doing ", r];
   ZZ = (ZZ /. {ϵ → 0}) ~STB1,r~ (ddm1,r→1 /. {ϵ → 0}), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\left\{3.562500,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}\right]\right\}\right\}$$

```
Timing@Block[{$k = 1},
  ZZ = (RR1,5 RR6,2 RR3,7 CC4 KKink8 KKink9 KKink10);
  Do[Print["Doing ", r]; ZZ = (ZZ) ~STB1,r~ (ddm1,r→1), {r, 2, 10}];
  {Simplify@ZZ}]
```

$$\Big\{49.390625,$$

$$\Big\{\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{S^2\,T^2}{\left(1-S+S^2\right)\left(1-T+T^2\right)\left(1-S\,T+S^2\,T^2\right)}+\Big(2\,S^2\,T^2\Big(8-15\,S+20\,S^2-13\,S^3+$$
$$6\,S^4-15\,T+14\,S\,T-11\,S^2\,T-10\,S^3\,T+11\,S^4\,T-10\,S^5\,T+20\,T^2-11\,S\,T^2+16\,S^2\,T^2+$$
$$S^3\,T^2+12\,S^4\,T^2-7\,S^5\,T^2+11\,S^6\,T^2-13\,T^3-10\,S\,T^3+S^2\,T^3-12\,S^3\,T^3-10\,S^4\,T^3+$$
$$S^5\,T^3-5\,S^6\,T^3-6\,S^7\,T^3+6\,T^4+11\,S\,T^4+12\,S^2\,T^4-10\,S^3\,T^4+24\,S^4\,T^4-6\,S^5\,T^4+$$
$$4\,S^6\,T^4+5\,S^7\,T^4+2\,S^8\,T^4-10\,S\,T^5-7\,S^2\,T^5+S^3\,T^5-6\,S^4\,T^5-4\,S^5\,T^5+S^6\,T^5-$$
$$2\,S^7\,T^5-3\,S^8\,T^5+11\,S^2\,T^6-5\,S^3\,T^6+4\,S^4\,T^6+S^5\,T^6-S^7\,T^6+2\,S^8\,T^6-6\,S^3\,T^7+$$
$$5\,S^4\,T^7-2\,S^5\,T^7-S^6\,T^7+2\,S^7\,T^7-S^8\,T^7+2\,S^4\,T^8-3\,S^5\,T^8+2\,S^6\,T^8-S^7\,T^8+$$
$$\left(1-S+S^2\right)\left(1-T+T^2\right)^2\left(-2+S+3\,S\,T+2\,S^6\,T^4+S^4\,T^2\,(3+T)-S^5\,T^3\,(3+T)-\right.$$
$$\left.S^2\,T\,(1+3\,T)+S^3\,T\left(-1+T^2\right)\right)a_1+\left(1-S+S^2\right)^2\left(1-T+T^2\right)\left(-2+T+3\,S\,T-\right.$$
$$\left.S\,(1+3\,S)\,T^2+S\left(-1+S^2\right)T^3+S^2\,(3+S)\,T^4-S^3\,(3+S)\,T^5+2\,S^4\,T^6\right)b_1+$$
$$2\,x_1\,XX_1+2\,S^3\,x_1\,XX_1-5\,T\,x_1\,XX_1-2\,S\,T\,x_1\,XX_1-3\,S^2\,T\,x_1\,XX_1-2\,S^3\,T\,x_1\,XX_1-$$
$$5\,S^4\,T\,x_1\,XX_1+9\,T^2\,x_1\,XX_1+5\,S\,T^2\,x_1\,XX_1+9\,S^2\,T^2\,x_1\,XX_1+9\,S^3\,T^2\,x_1\,XX_1+$$
$$5\,S^4\,T^2\,x_1\,XX_1+9\,S^5\,T^2\,x_1\,XX_1-7\,T^3\,x_1\,XX_1-12\,S\,T^3\,x_1\,XX_1-12\,S^2\,T^3\,x_1\,XX_1-$$
$$14\,S^3\,T^3\,x_1\,XX_1-12\,S^4\,T^3\,x_1\,XX_1-12\,S^5\,T^3\,x_1\,XX_1-7\,S^6\,T^3\,x_1\,XX_1+4\,T^4\,x_1\,XX_1+$$
$$10\,S\,T^4\,x_1\,XX_1+18\,S^2\,T^4\,x_1\,XX_1+14\,S^3\,T^4\,x_1\,XX_1+14\,S^4\,T^4\,x_1\,XX_1+18\,S^5\,T^4\,x_1\,XX_1+$$
$$10\,S^6\,T^4\,x_1\,XX_1+4\,S^7\,T^4\,x_1\,XX_1-7\,S\,T^5\,x_1\,XX_1-12\,S^2\,T^5\,x_1\,XX_1-12\,S^3\,T^5\,x_1\,XX_1-$$
$$14\,S^4\,T^5\,x_1\,XX_1-12\,S^5\,T^5\,x_1\,XX_1-12\,S^6\,T^5\,x_1\,XX_1-7\,S^7\,T^5\,x_1\,XX_1+9\,S^2\,T^6\,x_1\,XX_1+$$
$$5\,S^3\,T^6\,x_1\,XX_1+9\,S^4\,T^6\,x_1\,XX_1+9\,S^5\,T^6\,x_1\,XX_1+5\,S^6\,T^6\,x_1\,XX_1+9\,S^7\,T^6\,x_1\,XX_1-$$
$$5\,S^3\,T^7\,x_1\,XX_1-2\,S^4\,T^7\,x_1\,XX_1-3\,S^5\,T^7\,x_1\,XX_1-2\,S^6\,T^7\,x_1\,XX_1-5\,S^7\,T^7\,x_1\,XX_1+$$
$$2\,S^4\,T^8\,x_1\,XX_1+2\,S^7\,T^8\,x_1\,XX_1+2\,y_1\,Y_1-5\,S\,y_1\,Y_1+9\,S^2\,y_1\,Y_1-7\,S^3\,y_1\,Y_1+$$
$$4\,S^4\,y_1\,Y_1-2\,S\,T\,y_1\,Y_1+5\,S^2\,T\,y_1\,Y_1-12\,S^3\,T\,y_1\,Y_1+10\,S^4\,T\,y_1\,Y_1-7\,S^5\,T\,y_1\,Y_1-$$
$$3\,S\,T^2\,y_1\,Y_1+9\,S^2\,T^2\,y_1\,Y_1-12\,S^3\,T^2\,y_1\,Y_1+18\,S^4\,T^2\,y_1\,Y_1-12\,S^5\,T^2\,y_1\,Y_1+$$
$$9\,S^6\,T^2\,y_1\,Y_1+2\,T^3\,y_1\,Y_1-2\,S\,T^3\,y_1\,Y_1+9\,S^2\,T^3\,y_1\,Y_1-14\,S^3\,T^3\,y_1\,Y_1+$$
$$14\,S^4\,T^3\,y_1\,Y_1-12\,S^5\,T^3\,y_1\,Y_1+5\,S^6\,T^3\,y_1\,Y_1-5\,S^7\,T^3\,y_1\,Y_1-5\,S\,T^4\,y_1\,Y_1+$$
$$5\,S^2\,T^4\,y_1\,Y_1-12\,S^3\,T^4\,y_1\,Y_1+14\,S^4\,T^4\,y_1\,Y_1-14\,S^5\,T^4\,y_1\,Y_1+9\,S^6\,T^4\,y_1\,Y_1-$$
$$2\,S^7\,T^4\,y_1\,Y_1+2\,S^8\,T^4\,y_1\,Y_1+9\,S^2\,T^5\,y_1\,Y_1-12\,S^3\,T^5\,y_1\,Y_1+18\,S^4\,T^5\,y_1\,Y_1-$$
$$12\,S^5\,T^5\,y_1\,Y_1+9\,S^6\,T^5\,y_1\,Y_1-3\,S^7\,T^5\,y_1\,Y_1-7\,S^3\,T^6\,y_1\,Y_1+10\,S^4\,T^6\,y_1\,Y_1-$$
$$12\,S^5\,T^6\,y_1\,Y_1+5\,S^6\,T^6\,y_1\,Y_1-2\,S^7\,T^6\,y_1\,Y_1+4\,S^4\,T^7\,y_1\,Y_1-7\,S^5\,T^7\,y_1\,Y_1+$$

```
9 S^6 T^7 y1 Y1 - 5 S^7 T^7 y1 Y1 + 2 S^8 T^7 y1 Y1 - 2 XX1 Y1 z1 + 7 S XX1 Y1 z1 -
9 S^2 XX1 Y1 z1 + 7 S^3 XX1 Y1 z1 - 2 S^4 XX1 Y1 z1 - 5 S T XX1 Y1 z1 - 5 S^4 T XX1 Y1 z1 +
12 S T^2 XX1 Y1 z1 - 9 S^2 T^2 XX1 Y1 z1 + 21 S^3 T^2 XX1 Y1 z1 - 9 S^4 T^2 XX1 Y1 z1 +
12 S^5 T^2 XX1 Y1 z1 - 2 T^3 XX1 Y1 z1 - 5 S T^3 XX1 Y1 z1 - 9 S^2 T^3 XX1 Y1 z1 -
7 S^3 T^3 XX1 Y1 z1 - 7 S^4 T^3 XX1 Y1 z1 - 9 S^5 T^3 XX1 Y1 z1 - 5 S^6 T^3 XX1 Y1 z1 -
2 S^7 T^3 XX1 Y1 z1 + 7 S T^4 XX1 Y1 z1 + 21 S^3 T^4 XX1 Y1 z1 - 7 S^4 T^4 XX1 Y1 z1 +
21 S^5 T^4 XX1 Y1 z1 + 7 S^7 T^4 XX1 Y1 z1 - 9 S^2 T^5 XX1 Y1 z1 - 9 S^4 T^5 XX1 Y1 z1 -
9 S^5 T^5 XX1 Y1 z1 - 9 S^7 T^5 XX1 Y1 z1 + 7 S^3 T^6 XX1 Y1 z1 - 5 S^4 T^6 XX1 Y1 z1 +
12 S^5 T^6 XX1 Y1 z1 - 5 S^6 T^6 XX1 Y1 z1 + 7 S^7 T^6 XX1 Y1 z1 - 2 S^4 T^7 XX1 Y1 z1 -
2 S^7 T^7 XX1 Y1 z1 + 4 z1 Z1 - 7 S z1 Z1 + 9 S^2 z1 Z1 - 5 S^3 z1 Z1 + 2 S^4 z1 Z1 - 7 T z1 Z1 +
10 S T z1 Z1 - 12 S^2 T z1 Z1 + 5 S^3 T z1 Z1 - 2 S^4 T z1 Z1 + 9 T^2 z1 Z1 - 12 S T^2 z1 Z1 +
18 S^2 T^2 z1 Z1 - 12 S^3 T^2 z1 Z1 + 9 S^4 T^2 z1 Z1 - 3 S^5 T^2 z1 Z1 - 5 T^3 z1 Z1 +
5 S T^3 z1 Z1 - 12 S^2 T^3 z1 Z1 + 14 S^3 T^3 z1 Z1 - 14 S^4 T^3 z1 Z1 + 9 S^5 T^3 z1 Z1 -
2 S^6 T^3 z1 Z1 + 2 S^7 T^3 z1 Z1 + 2 T^4 z1 Z1 - 2 S T^4 z1 Z1 + 9 S^2 T^4 z1 Z1 - 14 S^3 T^4 z1 Z1 +
14 S^4 T^4 z1 Z1 - 12 S^5 T^4 z1 Z1 + 5 S^6 T^4 z1 Z1 - 5 S^7 T^4 z1 Z1 - 3 S^2 T^5 z1 Z1 +
9 S^3 T^5 z1 Z1 - 12 S^4 T^5 z1 Z1 + 18 S^5 T^5 z1 Z1 - 12 S^6 T^5 z1 Z1 + 9 S^7 T^5 z1 Z1 -
2 S^3 T^6 z1 Z1 + 5 S^4 T^6 z1 Z1 - 12 S^5 T^6 z1 Z1 + 10 S^6 T^6 z1 Z1 - 7 S^7 T^6 z1 Z1 +
2 S^3 T^7 z1 Z1 - 5 S^4 T^7 z1 Z1 + 9 S^5 T^7 z1 Z1 - 7 S^6 T^7 z1 Z1 + 4 S^7 T^7 z1 Z1) e) /
((1 - S + S^2)^3 (1 - T + T^2)^3 (1 - S T + S^2 T^2)^3) + O[e]^2]}}}

(*EndProfile[]; *)
```

The double multiplication tensor

For the sake of completeness, we give the explicit formula for ${}^{\mathbf{t}}dm^k_{ij}$. We denote by $\mathbf{a} = exp[-a^*]$, and similarly for b. As we did before, $\mathbb{A} = exp[-A]$, and the same convention holds for B.

$$
\begin{aligned}
{}^{\mathbf{t}}dm^k_{ij} = \mathbb{E}\Big[& a_k a^*{}_i + a_k a^*{}_j + A_k A^*{}_i + A_k A^*{}_j + b_k b^*{}_i + b_k b^*{}_j + B_k B^*{}_i + B_k B^*{}_j, \\
& \frac{x_k \mathbf{a}_j^2 x^*{}_i}{\mathbf{b}_j} + x_k x^*{}_j + X_k X^*{}_i + \frac{X_k \mathbf{a}_i^2 X^*{}_j}{\mathbf{b}_i} + \frac{\mathbb{A}_k^{-2\hbar}\left(\mathbb{A}_k^{2\hbar} - \mathbb{B}_k^{\hbar}\right) x^*{}_i X^*{}_j}{\hbar} + \frac{y_k \mathbf{b}_j^2 y^*{}_i}{\mathbf{a}_j} + \\
& y_k y^*{}_j + \frac{z_k \mathbf{a}_j^2 x^*{}_i y^*{}_j}{\mathbf{b}_j} + Y_k Y^*{}_i + \frac{Y_k \mathbf{b}_i^2 Y^*{}_j}{\mathbf{a}_i} + \frac{\mathbb{B}_k^{-2\hbar}\left(-\mathbb{A}_k^{\hbar} + \mathbb{B}_k^{2\hbar}\right) y^*{}_i Y^*{}_j}{\hbar} + z_k \mathbf{a}_j \mathbf{b}_j z^*{}_i + \\
& z_k z^*{}_j + Z_k Z^*{}_i + Z_k \mathbf{a}_i \mathbf{b}_i Z^*{}_j - \frac{Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i Z^*{}_j}{\mathbf{a}_i} + \frac{X_k \mathbf{a}_i^2 y^*{}_i Z^*{}_j}{\mathbf{b}_i} + \\
& \frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbb{A}_k^{\hbar} - \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Z^*{}_j}{\hbar} + \frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(-1 + \mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right) z^*{}_i Z^*{}_j}{\hbar}, \\
& 1 + \\
& \Bigg(-\frac{x_k \mathbf{a}_j^2 A^*{}_j x^*{}_i}{\mathbf{b}_j} - \frac{X_k \mathbf{a}_i^2 A^*{}_i X^*{}_j}{\mathbf{b}_i} - \frac{2\hbar x_k X_k \mathbf{a}_i^2 \mathbf{a}_j^2 x^*{}_i X^*{}_j}{\mathbf{b}_i \mathbf{b}_j} + a_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j + \\
& \frac{x_k \mathbb{A}_k^{-2\hbar}\left(-\mathbf{a}_j^2 \mathbb{A}_k^{2\hbar} + 3\mathbf{a}_j^2 \mathbb{B}_k^{\hbar}\right)\left(x^*\right)_i^2 X^*{}_j}{\mathbf{b}_j} + \frac{X_k \mathbb{A}_k^{-2\hbar}\left(-\mathbf{a}_i^2 \mathbb{A}_k^{2\hbar} + 3\mathbf{a}_i^2 \mathbb{B}_k^{\hbar}\right) x^*{}_i \left(X^*\right)_j^2}{\mathbf{b}_i} + \\
& \frac{\mathbb{A}_k^{-4\hbar}\left(-\mathbb{A}_k^{4\hbar} + 4\mathbb{A}_k^{2\hbar}\mathbb{B}_k^{\hbar} - 3\mathbb{B}_k^{2\hbar}\right)\left(x^*\right)_i^2 \left(X^*\right)_j^2}{2\hbar} - \frac{y_k \mathbf{b}_j^2 B^*{}_j y^*{}_i}{\mathbf{a}_j} + \frac{\hbar X_k y_k \mathbf{a}_i^2 \mathbf{b}_j^2 X^*{}_j y^*{}_i}{\mathbf{a}_j \mathbf{b}_i} - \\
& \frac{2 y_k \mathbf{b}_j^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j y^*{}_i}{\mathbf{a}_j} - \frac{\hbar x_k y_k \mathbf{a}_j^2 x^*{}_i y^*{}_j}{\mathbf{b}_j} - \frac{z_k \mathbf{a}_j^2 A^*{}_j x^*{}_i y^*{}_j}{\mathbf{b}_j} - \\
& \frac{2\hbar X_k z_k \mathbf{a}_i^2 \mathbf{a}_j^2 x^*{}_i X^*{}_j y^*{}_j}{\mathbf{b}_i \mathbf{b}_j} + \frac{z_k \mathbb{A}_k^{-2\hbar}\left(-\mathbf{a}_j^2 \mathbb{A}_k^{2\hbar} + 3\mathbf{a}_j^2 \mathbb{B}_k^{\hbar}\right)\left(x^*\right)_i^2 X^*{}_j y^*{}_j}{\mathbf{b}_j} - \\
& \hbar y_k z_k \mathbf{a}_j \mathbf{b}_j x^*{}_i y^*{}_i y^*{}_j - \frac{\hbar y_k z_k \mathbf{a}_j^2 x^*{}_i \left(y^*\right)_j^2}{\mathbf{b}_j} + \frac{\hbar X_k Y_k \mathbf{a}_i^2 X^*{}_j Y^*{}_i}{\mathbf{b}_i} - \\
& \frac{Y_k \mathbf{b}_i^2 B^*{}_i Y^*{}_j}{\mathbf{a}_i} + \frac{\hbar x_k Y_k \mathbf{a}_j^2 \mathbf{b}_i^2 x^*{}_i Y^*{}_j}{\mathbf{a}_i \mathbf{b}_j} + \frac{2 Z_k \mathbf{b}_i^2 X^*{}_i Y^*{}_j}{\mathbf{a}_i} - \frac{2\hbar y_k Y_k \mathbf{b}_i^2 \mathbf{b}_j^2 y^*{}_i Y^*{}_j}{\mathbf{a}_i \mathbf{a}_j} + \\
& b_k \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} y^*{}_i Y^*{}_j + \frac{x_k \mathbb{B}_k^{-2\hbar}\left(-\mathbf{a}_j^2 \mathbb{A}_k^{\hbar} + \mathbf{a}_j^2 \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Y^*{}_j}{\mathbf{b}_j} + \\
& \frac{X_k \mathbb{B}_k^{-2\hbar}\left(-\mathbf{a}_i^2 \mathbb{A}_k^{\hbar} - \mathbf{a}_i^2 \mathbb{B}_k^{2\hbar}\right) X^*{}_j y^*{}_i Y^*{}_j}{\mathbf{b}_i} + \frac{y_k \mathbb{B}_k^{-2\hbar}\left(3\mathbf{b}_j^2 \mathbb{A}_k^{\hbar} - \mathbf{b}_j^2 \mathbb{B}_k^{2\hbar}\right)\left(y^*\right)_i^2 Y^*{}_j}{\mathbf{a}_j} +
\end{aligned}
$$

$$\frac{\hbar Y_k z_k \mathbf{a}_j^2 \mathbf{b}_i^2 x^*{}_i y^*{}_j Y^*{}_j}{\mathbf{a}_i \mathbf{b}_j} + \frac{z_k \mathbb{B}_k^{-2\hbar} \left(-\mathbf{a}_j^2 \mathbb{A}_k^{\hbar} + \mathbf{a}_j^2 \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i y^*{}_j Y^*{}_j}{\mathbf{b}_j} +$$

$$\frac{Y_k \mathbb{B}_k^{-2\hbar} \left(3\mathbf{b}_i^2 \mathbb{A}_k^{\hbar} - \mathbf{b}_i^2 \mathbb{B}_k^{2\hbar}\right) y^*{}_i \left(Y^*\right)_j^2}{\mathbf{a}_i} + \frac{\mathbb{B}_k^{-4\hbar} \left(-3\mathbb{A}_k^{2\hbar} + 4\mathbb{A}_k^{\hbar}\mathbb{B}_k^{2\hbar} - \mathbb{B}_k^{4\hbar}\right) \left(y^*\right)_i^2 \left(Y^*\right)_j^2}{2\hbar} -$$

$$z_k \mathbf{a}_j \mathbf{b}_j A^*{}_j z^*{}_i - z_k \mathbf{a}_j \mathbf{b}_j B^*{}_j z^*{}_i - \frac{\hbar X_k z_k \mathbf{a}_i^2 \mathbf{a}_j \mathbf{b}_j X^*{}_j z^*{}_i}{\mathbf{b}_i} + \frac{2 y_k \mathbf{b}_j^2 X^*{}_j z^*{}_i}{\mathbf{a}_j} +$$

$$2 z_k \mathbf{a}_j \mathbf{b}_j \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j z^*{}_i - \frac{\hbar Y_k z_k \mathbf{a}_j \mathbf{b}_i^2 \mathbf{b}_j Y^*{}_j z^*{}_i}{\mathbf{a}_i} - \frac{2 x_k \mathbf{a}_j^2 \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} Y^*{}_j z^*{}_i}{\mathbf{b}_j} +$$

$$\frac{\mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-2\hbar} \left(-2\mathbb{A}_k^{2\hbar} + 2\mathbb{B}_k^{\hbar}\right) X^*{}_j Y^*{}_j z^*{}_i}{\hbar} - \frac{2 z_k \mathbf{a}_j^2 \mathbb{A}_k^{\hbar} \mathbb{B}_k^{-2\hbar} y^*{}_j Y^*{}_j z^*{}_i}{\mathbf{b}_j} +$$

$$\frac{\hbar x_k z_k \mathbf{a}_j^2 x^*{}_i z^*{}_j}{\mathbf{b}_j} - \frac{\hbar y_k z_k \mathbf{b}_j^2 y^*{}_i z^*{}_j}{\mathbf{a}_j} + \frac{\hbar z_k^2 \mathbf{a}_j^2 x^*{}_i y^*{}_j z^*{}_j}{\mathbf{b}_j} - \frac{\hbar X_k Z_k \mathbf{a}_i^2 X^*{}_j Z^*{}_i}{\mathbf{b}_i} +$$

$$\frac{\hbar Y_k Z_k \mathbf{b}_i^2 Y^*{}_j Z^*{}_i}{\mathbf{a}_i} - Z_k \mathbf{a}_i \mathbf{b}_i A^*{}_i Z^*{}_j - Z_k \mathbf{a}_i \mathbf{b}_i B^*{}_i Z^*{}_j - \frac{\hbar x_k Z_k \mathbf{a}_i \mathbf{a}_j^2 \mathbf{b}_i x^*{}_i Z^*{}_j}{\mathbf{b}_j} -$$

$$\frac{\hbar Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i Z^*{}_j}{\mathbf{a}_i} + \frac{\hbar a_k Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i Z^*{}_j}{\mathbf{a}_i} + \frac{Y_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} B^*{}_i x^*{}_i Z^*{}_j}{\mathbf{a}_i} +$$

$$\frac{2\hbar x_k Y_k \mathbf{a}_j^2 \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} \left(x^*\right)_i^2 Z^*{}_j}{\mathbf{a}_i \mathbf{b}_j} - \frac{2 Z_k \mathbf{b}_i^2 \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_i Z^*{}_j}{\mathbf{a}_i} +$$

$$2\hbar X_k Y_k \mathbf{a}_i \mathbf{b}_i \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{\hbar} x^*{}_i X^*{}_j Z^*{}_j + \frac{Y_k \mathbb{A}_k^{-4\hbar} \left(\mathbf{b}_i^2 \mathbb{A}_k^{2\hbar} \mathbb{B}_k^{\hbar} - \mathbf{b}_i^2 \mathbb{B}_k^{2\hbar}\right) \left(x^*\right)_i^2 X^*{}_j Z^*{}_j}{\mathbf{a}_i} -$$

$$\frac{\hbar y_k Z_k \mathbf{a}_i \mathbf{b}_i \mathbf{b}_j^2 y^*{}_i Z^*{}_j}{\mathbf{a}_j} - \frac{X_k \mathbf{a}_i^2 A^*{}_i y^*{}_i Z^*{}_j}{\mathbf{b}_i} - \frac{\hbar x_k X_k \mathbf{a}_i^2 \mathbf{a}_j^2 x^*{}_i y^*{}_i Z^*{}_j}{\mathbf{b}_i \mathbf{b}_j} -$$

$$b_k \mathbb{A}_k^{-\hbar} \mathbb{B}_k^{-\hbar} x^*{}_i y^*{}_i Z^*{}_j + \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(\mathbb{A}_k^{\hbar} - \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Z^*{}_j +$$

$$a_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(-\mathbb{A}_k^{\hbar} + \mathbb{B}_k^{2\hbar}\right) x^*{}_i y^*{}_i Z^*{}_j + \frac{x_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(-2\mathbf{a}_j^2 \mathbb{A}_k^{\hbar} + 2\mathbf{a}_j^2 \mathbb{B}_k^{2\hbar}\right) \left(x^*\right)_i^2 y^*{}_i Z^*{}_j}{\mathbf{b}_j} +$$

$$\frac{\hbar X_k^2 \mathbf{a}_i^4 X^*{}_j y^*{}_i Z^*{}_j}{\mathbf{b}_i^2} + \frac{X_k \mathbb{A}_k^{-2\hbar} \mathbb{B}_k^{-\hbar} \left(-\mathbf{a}_i^2 \mathbb{A}_k^{\hbar} + 3\mathbf{a}_i^2 \mathbb{B}_k^{2\hbar}\right) x^*{}_i X^*{}_j y^*{}_i Z^*{}_j}{\mathbf{b}_i} +$$

$$\frac{\mathbb{A}_k^{-4\hbar}\mathbb{B}_k^{-\hbar}\left(-\mathbb{A}_k^{3\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}+\mathbb{A}_k^{2\hbar}\mathbb{B}_k^{2\hbar}-\mathbb{B}_k^{3\hbar}\right)(x^*)_i^2\,X^*{}_j y^*{}_i Z^*{}_j}{\hbar}+$$
$$\frac{y_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-\mathbf{b}_j^2\mathbb{A}_k^{\hbar}-\mathbf{b}_j^2\mathbb{B}_k^{2\hbar}\right)x^*{}_i\,(y^*)_i^2\,Z^*{}_j}{\mathbf{a}_j}-\frac{\hbar z_k Z_k\mathbf{a}_i\mathbf{a}_j^2\mathbf{b}_i x^*{}_i y^*{}_j Z^*{}_j}{\mathbf{b}_j}+$$
$$\frac{2\hbar Y_k z_k\mathbf{a}_j^2\mathbf{b}_i^2\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}(x^*)_i^2\,y^*{}_j Z^*{}_j}{\mathbf{a}_i\mathbf{b}_j}-\frac{\hbar X_k z_k\mathbf{a}_i^2\mathbf{a}_j^2 x^*{}_i y^*{}_i y^*{}_j Z^*{}_j}{\mathbf{b}_i\mathbf{b}_j}+$$
$$\frac{z_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-2\mathbf{a}_j^2\mathbb{A}_k^{\hbar}+2\mathbf{a}_j^2\mathbb{B}_k^{2\hbar}\right)(x^*)_i^2\,y^*{}_i y^*{}_j Z^*{}_j}{\mathbf{b}_j}+\frac{\hbar X_k Y_k\mathbf{a}_i^2 y^*{}_i Y^*{}_i Z^*{}_j}{\mathbf{b}_i}-$$
$$\frac{\hbar Y_k^2\mathbf{b}_i^4\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i Y^*{}_j Z^*{}_j}{\mathbf{a}_i^2}-\hbar X_k Y_k\mathbf{a}_i\mathbf{b}_i y^*{}_i Y^*{}_j Z^*{}_j+2Z_k\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{\hbar}\mathbb{B}_k^{-2\hbar}y^*{}_i Y^*{}_j Z^*{}_j-$$
$$\frac{4Y_k\mathbf{b}_i^2\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}x^*{}_i y^*{}_i Y^*{}_j Z^*{}_j}{\mathbf{a}_i}+\frac{X_k\mathbb{B}_k^{-2\hbar}\left(\mathbf{a}_i^2\mathbb{A}_k^{\hbar}-\mathbf{a}_i^2\mathbb{B}_k^{2\hbar}\right)(y^*)_i^2\,Y^*{}_j Z^*{}_j}{\mathbf{b}_i}+$$
$$\frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-3\hbar}\left(2\mathbb{A}_k^{\hbar}-2\mathbb{B}_k^{2\hbar}\right)x^*{}_i\,(y^*)_i^2\,Y^*{}_j Z^*{}_j}{\hbar}-2\hbar z_k Z_k\mathbf{a}_i\mathbf{a}_j\mathbf{b}_i\mathbf{b}_j z^*{}_i Z^*{}_j+$$
$$a_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}z^*{}_i Z^*{}_j+b_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}z^*{}_i Z^*{}_j+\frac{3\hbar Y_k z_k\mathbf{a}_j\mathbf{b}_i^2\mathbf{b}_j\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i z^*{}_i Z^*{}_j}{\mathbf{a}_i}+$$
$$\frac{x_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_j^2-\mathbf{a}_j^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i z^*{}_i Z^*{}_j}{\mathbf{b}_j}+\frac{X_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbf{a}_i^2+\mathbf{a}_i^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)X^*{}_j z^*{}_i Z^*{}_j}{\mathbf{b}_i}+$$
$$\frac{\mathbb{A}_k^{-3\hbar}\mathbb{B}_k^{-\hbar}\left(2\mathbb{A}_k^{2\hbar}-2\mathbb{B}_k^{\hbar}\right)x^*{}_i X^*{}_j z^*{}_i Z^*{}_j}{\hbar}-\frac{\hbar X_k z_k\mathbf{a}_i^2\mathbf{a}_j\mathbf{b}_j y^*{}_i z^*{}_i Z^*{}_j}{\mathbf{b}_i}+$$
$$\frac{y_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(\mathbf{b}_j^2+\mathbf{b}_j^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)y^*{}_i z^*{}_i Z^*{}_j}{\mathbf{a}_j}+$$
$$z_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-2\mathbf{a}_j\mathbf{b}_j\mathbb{A}_k^{\hbar}+2\mathbf{a}_j\mathbf{b}_j\mathbb{B}_k^{2\hbar}\right)x^*{}_i y^*{}_i z^*{}_i Z^*{}_j+$$
$$\frac{z_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_j^2-\mathbf{a}_j^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i y^*{}_j z^*{}_i Z^*{}_j}{\mathbf{b}_j}+\frac{Y_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{b}_i^2-\mathbf{b}_i^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)Y^*{}_j z^*{}_i Z^*{}_j}{\mathbf{a}_i}+$$
$$\frac{\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-3\hbar}\left(-2\mathbb{A}_k^{\hbar}+2\mathbb{B}_k^{2\hbar}\right)y^*{}_i Y^*{}_j z^*{}_i Z^*{}_j}{\hbar}+z_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_j\mathbf{b}_j-\mathbf{a}_j\mathbf{b}_j\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)(z^*)_i^2\,Z^*{}_j-$$
$$\frac{\hbar Y_k Z_k\mathbf{b}_i^2\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i Z^*{}_i Z^*{}_j}{\mathbf{a}_i}-\frac{\hbar X_k Z_k\mathbf{a}_i^2 y^*{}_i Z^*{}_i Z^*{}_j}{\mathbf{b}_i}+\hbar Y_k Z_k\mathbf{b}_i^3\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i\,(Z^*)_j^2-$$
$$\hbar X_k Z_k\mathbf{a}_i^3 y^*{}_i\,(Z^*)_j^2+2\hbar X_k Y_k\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{\hbar}x^*{}_i y^*{}_i\,(Z^*)_j^2+$$
$$Z_k\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-3\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{\hbar}+\mathbf{a}_i\mathbf{b}_i\mathbb{B}_k^{2\hbar}\right)x^*{}_i y^*{}_i\,(Z^*)_j^2+$$

$$\frac{Y_k \mathbb{A}_k^{-4\hbar}\left(3\mathbf{b}_i^2\mathbb{A}_k^{\hbar}-\mathbf{b}_i^2\mathbb{B}_k^{2\hbar}\right)(x^*)_i^2\, y^*{}_i\,(Z^*)_j^2}{\mathbf{a}_i}+$$
$$\frac{X_k \mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-\hbar}\left(-2\mathbf{a}_i^2\mathbb{A}_k^{\hbar}+2\mathbf{a}_i^2\mathbb{B}_k^{2\hbar}\right)x^*{}_i\,(y^*)_i^2\,(Z^*)_j^2}{\mathbf{b}_i}+$$
$$\frac{\mathbb{A}_k^{-4\hbar}\mathbb{B}_k^{-2\hbar}\left(-3\mathbb{A}_k^{2\hbar}+4\mathbb{A}_k^{\hbar}\mathbb{B}_k^{2\hbar}-\mathbb{B}_k^{4\hbar}\right)(x^*)_i^2\,(y^*)_i^2\,(Z^*)_j^2}{2\hbar}+$$
$$Z_k\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}\left(3\mathbf{a}_i\mathbf{b}_i-\mathbf{a}_i\mathbf{b}_i\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)z^*{}_i\,(Z^*)_j^2+$$
$$\frac{Y_k\mathbb{A}_k^{-3\hbar}\left(-4\mathbf{b}_i^2+2\mathbf{b}_i^2\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}\right)x^*{}_i z^*{}_i\,(Z^*)_j^2}{\mathbf{a}_i}+\frac{2X_k\mathbf{a}_i^2\mathbb{A}_k^{-\hbar}\mathbb{B}_k^{-\hbar}y^*{}_i z^*{}_i\,(Z^*)_j^2}{\mathbf{b}_i}+$$
$$\frac{1}{\hbar}\mathbb{A}_k^{-3\hbar}\mathbb{B}_k^{-2\hbar}\left(3\mathbb{A}_k^{\hbar}-\mathbb{A}_k^{2\hbar}\mathbb{B}_k^{\hbar}-3\mathbb{B}_k^{2\hbar}+\mathbb{A}_k^{\hbar}\mathbb{B}_k^{3\hbar}\right)x^*{}_i y^*{}_i z^*{}_i\,(Z^*)_j^2+$$
$$\left.\frac{\mathbb{A}_k^{-2\hbar}\mathbb{B}_k^{-2\hbar}\left(-3+4\mathbb{A}_k^{\hbar}\mathbb{B}_k^{\hbar}-\mathbb{A}_k^{2\hbar}\mathbb{B}_k^{2\hbar}\right)(z^*)_i^2\,(Z^*)_j^2}{2\hbar}\right)\epsilon+O[\epsilon]^2\Bigg]$$

A.2. Poisson-Lie groups

In this section we will describe the connection between Poisson-Lie groups and Lie-bi algebras. A large part of this appendix is taken from the masterthesis "The two dimensional Ising Model" by the author. In this appendix we will introduce the notion of a Lie group, followed by the definition of a Poisson Lie group. We follow the construction of Lee [22] and [6]. A general knowledge about smooth manifolds is required.

Definition A.2.1. *A Lie group is a smooth manifold G without boundary that is a group with a smooth multiplication map $m : G \times G \to G$ and a smooth inversion map $i : G \to G$. Let $g, h \in G$, then $i(g) = g^{-1}$ is called the inverse of g and m(g,h)=gh. Denote with $L_g(h) = gh$ left translation and with $R_g(h) = hg$ right translation.*

Definition A.2.2. *Let G and H be Lie groups, then a Lie group homomorphism F from G to H is a map $F : G \to H$ that is a group homomorphism. It is called a Lie group isomorphism if it is a diffeomorphism.*

Definition A.2.3. *Let M be a smooth manifold, and let TM be the tangent bundle of M. A vectorfield X on M is a section of the map $\pi : TM \to M$. THat is, X is a map $X : M \to TM$, such that $\pi \circ X = Id_M$.*

One can add vector fields pointwise. If (U, x^i) is a chart of M, and $p \in M$, then $p \to \frac{\partial}{\partial x^i}|_p$ is a vector field on U, which we will call the i-th coördinate vector field, and it will be denoted by $\partial/\partial x^i$. A vector field X can be written out on chart as a linear combination of coördinate vector fields, and this will be denoted with $X = X^i \frac{\partial}{\partial x^i}$, where the summation symbol over i is omitted.

Definition A.2.4. *Let X and Y be smooth vector fields on a smooth manifold M. Let $f : M \to \mathbb{R}$ be a smooth function. Then the Lie bracket of X and Y is given by $[Y, X]f = XYf - YXf$.*

Given a smooth function $f : M \to \mathbb{R}$, it is possible to apply X and Y to f to obtain new smooth vector fields fX and fY respectively. On the other hand, by differentiation, a vector field can act on a function. To show that the Lie bracket is well defined, one has to show that $[X, Y]$ is again a vectorfield. This is equivalent to showing that it obeys the product rule, which will be omitted here.
From now on we will mean with M a smooth manifold with Lie bracket $[\cdot, \cdot]$, and with X, Y, Z smooth vectorfields on M. The space of smooth vector fields on M is denoted by $\mathcal{X}(M)$ and the space of smooth functions on M is denoted by $C^\infty(M)$.

Proposition A.2.1. *The Lie bracket satisfies the following identities:*

(a) (linearity) Let $a, b \in \mathbb{R}$. Then

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]. \tag{A.1}$$

(b) (anti-symmetry)

$$[X, Y] = -[Y, X] \tag{A.2}$$

(c) (Jacobi identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{A.3}$$

(d) Let $f, g \in C^\infty(M)$, then

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X. \tag{A.4}$$

Definition A.2.5. *Let V be a finite dimensional vector space, and denote with $GL(V)$ the group of invertible linear transformations on V, which is isomorphic to a Lie Group GL_n for some n. If G is a Lie group, then a finite dimensional representation of G is a Lie group homomorphism from G to $GL(V)$ seen as Lie group for some V. if a representation $\rho : G \to GL(V)$ is injective, then the representation is said to be faithful.*

Definition A.2.6. *Let G be a Lie group. The Lie algebra of G is the set of all smooth left-invariant vector fields, and it is denoted by Lie(G).*

The Lie algebra of G is well defined because the Lie bracket of two left invariant vector fields (invariant under L_g for all g) is again left invariant. It turns out that $Lie(G)$ is finite dimensional and that the dimension of $Lie(G)$ is equal to $dim(G)$. [22] The representation of a Lie group yields a representation of the corresponding Lie algebra by taking the tangent map. We proceed with the definition of a Poisson manifold.

Definition A.2.7. *(Poisson Structure) Let M be a smooth manifold of finite dimension m, and denote with $C(M)$ the algebra of smooth real valued functions on M. A Poisson structure on M is an $\mathbb{R}$ bilinear map $\{,\} : C(M) \times C(M) \to C(M)$ (the Poisson bracket) satisfying for all $f_1, f_2, f_3 \in C(M)$:*

1. $\{f_1, f_2\} = -\{f_2, f_1\}$
2. $\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$
3. $\{f_1 f_2, f_3\} = \{f_1, f_3\} f_2 + f_1 \{f_2, f_3\}$

One needs to consider maps between Poisson structures as well.

Definition A.2.8. *(Poisson Maps) A smooth map $F : M \to N$ between Poisson manifolds is a Poisson map if it preserves the Poisson brackets of M and N: $\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N$.*
(Product Poisson structure) The Product Poisson structure is given by

$$\{f_1(x,y), f_2\}_{M\times N}(x,y) = \{f_1(.,y), f_2(.,y)\}_M(x) + \{f_1(x,.), f_2(x,.)\}_N(y),$$

where $f_1, f_2 \in C(M \times N)$.

Finally we are able to define Poisson-Lie groups.

Definition A.2.9. *A Poisson-Lie group G is a Lie group which also has a Poisson structure that is compatible with the Lie structure, i.e. the multiplication map $\mu : G \times G \to G$ is a Poisson map. A homomorphism of Poisson Lie groups is a homomorphism of Lie groups that is also a Poisson map.*

Now let us go into the relation between Poisson-Lie groups and Lie bialgebras.

Theorem A.2.1. *Define on a Poisson-Lie group G $Ad(x)(y) = xyx^{-1}$ for all $x, y \in G$. Then the tangent space at the unit element e of G is a Liealgebra $\mathfrak{g}$ with Lie bracket $[X, Y] = T_e Ad(X)(Y)$. Define the cobracket δ by the relation*

$$\langle X, d\{f_1, f_2\}_e \rangle = \langle \delta(X), (df_1)_1 \otimes (df_2)_e \rangle.$$

Then $(T_eG, [,], \delta)$ is a Lie bialgebra.

The proof consists of checking the definitions. (See [6], page 25.) Note that if a Lie algebra corresponding to a Lie group G (not necessarily a Poisson-Lie group) is quasitriangular, i.e. if δ is a coboundary, then one can use the classical r-matrix to define the Poisson bracket on G. See proposition 2.2.2 on page 61 of [6]. On the other hand one can define from a classical r-matrix $r \in \mathfrak{g} \times \mathfrak{g}$ a corresponding R-matrix $\mathcal{R} : G \times G \to G \times G$ which is a solution of the quantum Yang Baxter equation: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$. See page 67 of [6] for more details. Confusingly, $\mathcal{R}$ is called a classical R-matrix in [6].
The dual of the universal enveloping algebra of a semisimple Lie algebra corresponds to the function algebra on its corresponding Poisson-Lie group. See

chapter 7 of [6]. This is not the case for $U_q(sl_3^\epsilon)$, since this algebra is not semisimple. Suppose this were the case, then the space of functions on the quantum group $U_q(sl_3^\epsilon)$ would be spanned by the representation-matrices of finite dimensional representations, and each function would be fully determined by its action on finite dimensional representations. We know that this is not the case by looking at central elements in $U_q(sl_3^\epsilon)$, so the dual of $U_q(sl_3^\epsilon)$ cannot correspond to the function algebra of a Poisson-Lie group.
It would be interesting to consider the corresponding construction of $\mathcal{F}(G)$ with a non-invertible term epsilon, and quantize it. This might give insight in $U_q(sl_3^\epsilon)$. When we consider ϵ in the ring $\mathbb{R}[[\epsilon]]$ it turns out to be equivalent to the quantization of a quotient of an affine Lie algebra where the central extension is quotiented out, see [37] and [5]. This suggests that a geometric interpretation of the dual of $U_q(sl_3^\epsilon)$ over the ring $\mathbb{R}[[\epsilon]]$ is possible.

A.3. Lie algebras and root systems

In this section we will give the definitions of a root system corresponding to a Lie algebra. This appendix is taken from the master thesis "The two dimensional Ising Model" by the author. It is not our aim to introduce the reader to Lie theory, so we will only state a few definitions and results. For a good introduction in Lie algebras and finite dimensional representation of Lie algebras, see for example [14].

Definition A.3.1. *(Lie algebra) Let L be a vector space over a commutative ring R, with a bracket operation $[\cdot,\cdot] : L \times L \to L$ with the following properties:*

(L1) The bracket operation is bilinear.

(L2) [xx]=0 for all $x \in L$.

(L3) The Jacobi identity is satisfied: [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.

Then L is called a Lie algebra.

From now on, when we write L, we will always mean a Lie algebra L.

Definition A.3.2. *A derivation of L is a linear map $\delta : L \to L$ satisfying the product rule: $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in L$. The collection of all derivation on L is denoted by Der(L).*

Since $Der(L) \subset End(L)$, we can define a representation on L by sending an element $x \in L$ to its derivation $ad(x) = [x, \cdot]$. This representation (a representation of a Lie algebra L is a linear map tp $\mathfrak{gl}(L)$ respecting the bracket operation) is called the adjoint representation, and plays an important role. Using this representation, we can define a symmetric, bilinear form on L.

Definition A.3.3. *(Killing Form) For $x, y \in L$, define the Killing form $\kappa(x, y) = Tr(ad(x)ad(y))$, where Tr denotes the trace.*

A special class of Lie algebras are the so called semisimple Lie algebras. This class has certain nice properties, which we will need.

Definition A.3.4. *Let $L^{(i)}$ be the sequence obtained by $L^{(0)} = L$ and $L^{(i+1)} = [L^{(i)}, L^{(i)}]$. We call L sovable if $L^{(n)} = 0$ for some n.*

The unique maximal solvable ideal of L is called the radical of L and is denoted by $Rad(L)$. Its existence follows from the property that if I and J are solvable ideals, then so is $I + J$.

Definition A.3.5. *(semisimple Lie algebra) Let L be a Lie algebra such that rad(L)=0. Then L is called semisimple.*

For semisimple Lie algebras, the Killing form is nondegenerate (i.e. the adjoint representation is faithful, i.e. 1 to 1). This is also true for a general faithful representation ϕ of L. Define a symmetric, bilinear form $\beta(x, y) = Tr(\phi(x)\phi(y))$. If ϕ is faithful and L is semisimple, then β is nondegenerate and associative. For a proof of this, see [14].
It can be checked, by using the Jacobi identity, that the Killing form is invariant under the adjoint action of L on itself, defined by $ad : L \times L \to L : (x, y) \mapsto [x, y]$. So the Killing form satisfies: $\kappa(ad_x(y), ad_x(z)) = \kappa(y, z)$, for all x, y, z in L. It is interesting to look at a general adjoint action invariant, bilinear form β. One can define the Casimir element associated to this form the following way.

Definition A.3.6. *(Casimir element) Let L be semisimple, with basis $(x_1, x_2, \cdots, x_n)$. Let β be an adjoint invariant bilinear form on L, and let $(y_1, \cdots, y_n)$ be the dual basis with respect to this two form: $\delta_{ij} = \beta(x_i, y_j)$. Then define the Casimir element associated with β as follows:*

$$c_\beta = \sum_{i=1}^{n} y_i \otimes x_i \in \mathfrak{U}(L), \tag{A.5}$$

where $\mathfrak{U}(L)$ is the universal enveloping algebra of L.

The construction of the Casimir element can be generalized, at least in theory, for any semisimple Lie algebra to higher degree Casimir elements. This might be trivial in some cases, whereas in other cases it might not be.

Definition A.3.7. *(generalized Casimir element) Let L be semisimple, and let $(x_{\alpha_1}), \cdots, (x_{\alpha_n})$ be bases of L. Define the multilinear form $\beta(x_1, \cdots, x_n) = Tr(ad(x_1) \cdots ad(x_n))$. Then define the generalized casimir element c_β by*

$$c_\beta = \sum_{\alpha_1, \cdots \alpha_n} \frac{x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}}{\beta(x_{\alpha_1}, \cdots, x_{\alpha_n})}. \tag{A.6}$$

The degrees for which these generalized Casimir elements exist minus one are called the exponents of the Lie algebra. The next concept we want to define is the Coxeter number. In order to define this concept, we need to introduce roots and the Weyl group.

Definition A.3.8. *Let L be semisimple, and let κ be the killing form on L. Let H be the maximal subalgebra of L consisting of elements x for which $ad(x)$ is diagonalizable (such an element x is called semisimple, and and algebra consisting of such elements is called Toral). Let $\alpha, \beta \in H^*$, such that $L_\alpha = \{x \in L | [hx] = \alpha(h)x \text{ for all } h \in H\} \neq 0$ (such α are called roots, the set of roots is denoted by Φ). Denote by $P_\alpha = \{\beta \in H^* | (\beta, \alpha) = 0\}$ the reflecting hyperplane of α (here $(\cdot, \cdot)$ denotes the Killing form transferred from H to H^*, which we may do since the killing form is nondegenerate on H, see [14]), and define $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.*

As it turns out, the set Φ of all roots of L obeys the axioms of a root system.

Definition A.3.9. *(Root system) A subset Φ of an euclidean space E is called a root system in E if the following axioms are satisfied:*

R1 *Φ is finite, spans E and does not contain 0.*

R2 *If $\alpha \in \Phi$, then the only multiples of α contained in Φ are $\pm\alpha$.*

R3 *If $\alpha \in \Phi$, then σ_α leaves Φ invariant.*

R4 *If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \langle \alpha, \beta \rangle \in \mathbb{Z}$.*

Here, σ_α is defined similarly as the case in which $E = H^*$, since any Euclidian space is equipped with a nondegenerate, positive definite symmetric, bilinear form. Let us now define the notion of a coroot $\alpha^\vee$ for a root α as follows

$$\alpha^\vee = \frac{2\alpha}{|\alpha|^2}. \tag{A.7}$$

We need the definition of simple roots.

Definition A.3.10. *Let Δ be a subset of a root system Φ of a Euclidian space E such that*

B1 *Δ is a basis of E,*

B2 *Each root can be written as a linear combination of elements of Δ, such that the coefficients are all nonnegative or all nonpositive.*

Then Δ is called a base, and its elements are called simple roots.

Fix a base $\{\alpha_1, \cdots, \alpha_r\}$ for the roots of L, and let θ be the highest root of L, in the sense that the sum of the coefficients a_i, when θ is written out as a linear combination of simple roots is maximized. The coeffients a_i are called marks. The coefficients $a_i^\vee$, when θ is decomposed in terms of $\alpha_i^\vee$ are called comarks.
With a base fixed for L, we can define the Cartan matrix as $A_ij = \kappa(\alpha_i, \alpha_j^\vee)$, where i and j run between 1 and r. Now let us define the Weyl group.

Definition A.3.11. *(Weyl group) Let Φ be a root system, and let $\mathcal{W}$ be the group generated the reflections σ_α, for $\alpha \in \Phi$. We call $\mathcal{W}$ the Weyl group of Φ.*

From the definition of a root system, it is clear that $\mathcal{W}$ permutes the roots, and hence can be seen as a subgroup of the symmetric group on Φ. To define the Coxeter element and the Coxeter number, we need a few more definitions.

Definition A.3.12. *(Base) A subset $\Delta \subset \Phi$ is called a base if Δ is a basis of Φ and if each root β can be written as $\beta = \sum k_\alpha \alpha$ with the integral coefficients k_α all nonnegative or nonpositive. The roots in Δ are called simple roots. The reflections corresponding to these roots are called simple reflections.*

Now we can define the Coxeter element.

Definition A.3.13. *(Coxeter element) Let Φ be a root system of a semisimple Lie algebra L with a fixed base $\Delta = (\alpha_1, \cdots, \alpha_n)$. Then $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ is called a Coxeter element of L. The order of w is called the Coxeter number.*

Note that one can define several Coxeter elements in given group, so it is important to prove that these elements have the same order. This will not be done here, but the proof that all Coxeter elements are conjugate to each other can for example be found in for example [14].

A.4. Wigner group contraction

In this appendix we describe the process of Wigner group contraction. In 1953 Wigner et al. came up with this method to transform Lie groups and their corresponding Lie algebras into different Lie groups. This is accomplished by a continuous transformation with a function $t(\epsilon)$ on the generators of which the limit $\epsilon \to 0$ is taken. Wigner proved that this limit exists under certain conditions. We follow [12]. We will use Wigner group contraction for the construction of the Lie algebra $sl_2^{\epsilon=0}$. This gives some inspiration for the origin of the parameter ϵ.
Let $\epsilon \in [0,1]$, and let $\mathfrak{g}, \mathfrak{f}$ be Lie algebras. Let $t_\epsilon : \mathfrak{g} \to \mathfrak{f}$ be a one to one Lie algebra map for all $\epsilon \neq 0$ such that $t_1 = id$ and $\det(t_0) = 0$. Let $a, b, c \in \mathfrak{g}$. Then we have

$$t_\epsilon^{-1}[t_\epsilon(a), t_\epsilon(b)] = c.$$

We may now take the limit $\epsilon \to 0$. If this limit exists, this results in a Lie algebra $\mathfrak{g}'$ for any $\epsilon \in [0,1]$. For $\epsilon = 0$ the result is nonisomorphic to $\mathfrak{g}$, with bracket $[a,b] = \lim_{\epsilon \to 0} t_\epsilon^{-1}[t_\epsilon(a), t_\epsilon(b)]$. In this case we call $\mathfrak{g}'$ the contraction of $\mathfrak{g}$, and we say that $\mathfrak{g}$ is contracted with respect to t_ϵ. Suppose we have a basis a_i of $\mathfrak{g}$. When the contraction of $\mathfrak{g}$ exists, define the basis a_i' of $\mathfrak{g}'$ as $a_i' = t_\epsilon(a_i)$.
The following theorem is taken from [12], we will not prove it here.

Theorem A.4.1. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ be a Lie algebra and t_ϵ a transformation as specified above such that*

$$t_0(\mathfrak{h}) = \mathfrak{h},$$
$$t_0(\mathfrak{h}') = 0.$$

Then $\mathfrak{g}$ can be contracted with respect to $\mathfrak{h}$ if and only if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, in this case $\mathfrak{h}$ is a subalgebra of the contraction $\mathfrak{g}'$ of $\mathfrak{g}$, and $\mathfrak{h}'$ is a commutative subalgebra of $\mathfrak{g}'$. In particular $\mathfrak{g}'$ is not semisimple.

We will now treat the example relevant for us, the case where $\mathfrak{g} = \mathfrak{gl}_2$.

Example A.4.1. *Define $\mathfrak{gl}_2$ as the Lie algebra with generators $\{X, A, a, x\}$ and the relations*

$$\begin{aligned}[A, X] = X, [x, a] = x,\\ [a, X] = X, [x, A] = x,\\ [X, x] = A + a, [a, A] = 0.\end{aligned}$$

Define the Lie algebra map t_ϵ as $t_\epsilon(a) = a, t_\epsilon(x) = x$ on the subalgebra $\mathfrak{h}$, and as $t_\epsilon(X) = \epsilon X, t_\epsilon(A) = \epsilon A$. We define the elements $A' = \epsilon A, X' = \epsilon X, x' = x, a' = a$. Then we find the following relations for $\{X', A', a', x'\}$:

$$[x', a'] = [x, a] = x \tag{A.8}$$
$$[x', A'] = [x, \epsilon A] = \epsilon x' \tag{A.9}$$
$$[X', A'] = [\epsilon X, \epsilon A] = -\epsilon X' \tag{A.10}$$
$$[X', a'] = [\epsilon X, a] = X' \tag{A.11}$$
$$[X', x'] = [\epsilon X, x] = \epsilon(A + a) = A' + \epsilon a'. \tag{A.12}$$

In these relations we already recognize a subalgebra of the Lie algebra constructed in section 1.1 of chapter 1, in the case where $\epsilon^2 \neq 0$. This is also the sl_2^ϵ algebra as constructed in [35]. Since the elements $\{X, A\}$ generate a subalgebra of $\mathfrak{gl}_2$, by theorem A.4.1 we can take the limit of $\epsilon \to 0$. The result is the Lie algebra $sl_2^{\epsilon=0}$.

It is possible to do the same thing for the sl_n case, covered in chapter 4. In this case, one could start with the algebra of section 4.4.2 in [10] to obtain the quasi-triangular Lie bialgebra covered in chapter 1, which one would need to quantize in the manner of chapter 4. This is a straightforward exercise for the reader.

A.5. Rings

In this appendix we follow [19]. By a ring R we always mean a commutative ring with identity 1 and of characteristic zero. The characteristic of a ring is the smallest number such that $1^n = 1 + 1 + \cdots + 1 = 0$.
An element $r \in R$ is called a zero devisor of R if there exists a nonzero element $x \in R$ such that $rx = 0$. An element of R is called regular if it is not a zero devisor. We define an integral domain as a ring without zero devisors.
An ideal $I \subset R$ of R is a set I containing 0 such that I is closed under addition, and such that if $i \in I$ and $r \in R$, $ir \in I$. $\mathfrak{m}$ is the maximal ideal $\mathfrak{m}$ of a ring R if $\mathfrak{m} \neq R$ and if for any ideal $I \subset R$ such that $\mathfrak{m} \subset I$, either $I = \mathfrak{m}$ or $I = R$.

Definition A.5.1. *An ideal $I \subset R$ is called a prime ideal if for any $a, b \in R$ such that $ab \in I$, $a \in I$ or $b \in I$, and if $I \neq R$. Define the spectrum $Spec(R)$ of R as the set of prime ideals of R.*

Denote by $R[x_1, \cdots, x_n]$ the ring of polynomials in n indeterminates with coefficients in R.

Definition A.5.2. *Let k be a field, let $S \subset k[x_1, \cdots, x_n]$. Define the affine variety of S as $\mathcal{V}_{k^n}(S) := \{(\xi_1, \cdots, \xi_n) \in k^n | f(\xi_1, \cdots, \xi_n) = 0 \, \forall f \in S\}$. For $X \in k^n$, define the ideal of X as $\mathcal{I}(X) = \mathcal{I}_{k[x_1, \cdots x_n]}(X) := \{f \in k[x_1, \cdots x_n] | f(\xi_1, \cdots, \xi_n) = 0$ for all $(\xi_1, \cdots, \xi_n) \in X\}$.*

We can now define the coordinate ring of a set $X \subset k^n$.

Definition A.5.3. *Let $X \subset k^n$ be an affine variety. Define the coordinate ring of X as $k[X] := K[x_1, \cdots, x_n]/\mathcal{I}(X)$.*

We define a module over a ring R as one defines a vector space over a field k.

Definition A.5.4. *Let R be a ring. A (left-)module M over R is an abelian group $(M, +)$ together with an operation $\cdot : R \times M \to M$ such that for $r, s \in R$ and $x, y \in M$,*

- $r \cdot (x + y) = r \cdot x + r \cdot y$
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(rs) \cdot x = r \cdot (s \cdot x)$
- $1_R \cdot x = x$.

A module over R is called free if it has an R-basis. An R-basis of a module M is a generating set of M that is linearly independent over R. Denote for a subset $S \subset M$ of an R-module M, (S) for the submodule of M generated by S. By definition this is equal to the set of all linear combinations of S. If $S = \{m_1, cdotsm_n\}$, we may write $(S) = (m_1, \cdots m_n)$. In the same way we may define an ideal $(m_1, \cdots m_n) \subset R$ generated by the set $\{m_1, \cdots m_n\} \subset R$.
Define the formal power series ring in the variable x over a ring R as $R[[x]] := \{\sum_{i=0}^{\infty} a_i x^i | a_i \in R\}$, and similarly for any finite number of indeterminates x_i, $i \in I$.

Definition A.5.5. *A ring R is called local if it has precisely one maximal ideal. R is called Noetherian if for every strictly ascending chain of subideals $I_i \subset M$ such that $I_i \subset I_{i+1}$ there exists an integer n such that $I_i = I_n$ for all $i \geq n$.*

If R is a local Noetherian ring with maximal ideal $\mathfrak{m}$, we can define the residual class field $K := R/\mathfrak{m}$. Furthermore if $\mathcal{M}$ is a set of sets, we define a chain in $\mathcal{M}$ as a subset $\mathcal{C} \subset \mathcal{M}$ that is totally ordered by inclusion. The length of a chain $\mathcal{C}$ is defined as $length(\mathcal{C}) := |\mathcal{C}| - 1 \in \mathbb{N}_0 \cup \{-1, \infty\}$. We then define

$$length(\mathcal{M}) := \sup\{length(\mathcal{C}) | \mathcal{C} \text{ is a chain in } \mathcal{M}\}.$$

Define the dimension of R as $dim(R) = length(Spec(R))$.
It turns out that $dim_K(\mathfrak{m}/\mathfrak{m}^2) \geq dim(R)$.

Definition A.5.6. *A local ring R is called regular if $dim_K(\mathfrak{m}/\mathfrak{m}^2) = dim(R)$.*

For a ring R, we define an R-algebra A to be a ring A with a homomorphism $\alpha : R \to A$. So an R-algebra is a commutative, associative algebra with unit. For a field k, an affine k-algebra is a finitely generated k-algebra. A k-algebra is finitely generated if it is isomorphic to the ring $k[x_1, \cdots, x_n]/I$, where $I \subset k[x_1, \cdots, x_n]$ is an ideal. It is clear that $R_\epsilon = \mathbb{R}[\epsilon]/(\epsilon^2)$ is an affine $\mathbb{R}$-algebra.

Definition A.5.7. *Let A be an algebra over a field k. Define the transcendence degree of A as $sup\{|T||T \subset A$ is finite and algebraically independent$\}$.*

For a k-algebra A, we define a set $a = \{a_1, \cdots, a_n\}$ to be algebraically independent if for all $f \in k[x_1, \cdots, x_n], f(a) \neq 0$. An example is the ring R_ϵ. We see that R_ϵ has transcendence degree 0 over $\mathbb{R}$, as $\epsilon^2 = 0$. Moreover, R_ϵ is local, with unique maximal ideal (ϵ) (observe that any regular element is invertible).
For affine k-algebras, $dim(A) = trdeg(A)$. The proof can be found in e.g. [19], chapter 5. So $dim(R_\epsilon) = 0$. However, in R_ϵ, (ϵ) is the maximal ideal. Since $(\epsilon)^2 = 0$, $dim_{\mathbb{R}}((\epsilon)/(\epsilon)^2) = dim_{\mathbb{R}}((\epsilon)) = 1$. So we see that R_ϵ is not regular.

Definition A.5.8. *Let M be an R-module and let $m \in M$. m is called a torsion element of M if there exists a regular $r \in R$ such that $rm = 0$. M is called torsion-free if 0 is the only torsion element of M.*

In the ring R_ϵ, the set of regular elements is given by $\{r = a + \epsilon b \in R_\epsilon | a \neq 0\}$. Let R be any ring, and let M be a free R-module. It is clear from the definition of linear independence that M is torsion-free. Let M be a free R-module. Define the dual M^* of M as $M^* = Hom_R(M, R)$. Observe that M^* has a natural R-module structure. Let $\phi \in M^*$ and $r \in R, m \in M$, then $r\phi(m) = \phi(rm)$. Let r be a regular element of R, $r\phi = 0$ implies that $\phi(rm) = 0$ for all $m \in M$. However, since r is regular and M is torsion free, $rm \neq 0$ if $m \neq 0$. It is easy to show (by induction, for example) that if $R = K[X]/(X^n)$ for a field K and an integer $n > 0$, and if r is regular, $\{rm | m \in M\} = M$. This implies that $\phi = 0$. So M^* is torsion free.
We continue with the discription of freeness and flatness of a module M over the ring R_ϵ.

Definition A.5.9. *Let R be a ring, and let $M_1, M2, M$ be R-modules. Let $f_i : M_1 \to M_2$ be an injective map. Define the map $\phi_f : M_1 \otimes M \to M_2 \otimes M : x \otimes m \mapsto f(x) \otimes m$. We call M flat if for any injective map f, ϕ_f is injective.*

A consequence of this definition is that if $M_1 \to M_2 \to M_3$ is an exact sequence, then $M_1 \otimes M \to M_2 \otimes M \to M_3 \otimes M$ is also an exact sequence. We will now prove that over R_ϵ, the notions of flatness and freeness coincide.

Proposition A.5.1. *Let M be an R_ϵ-module. Then M is flat if and only if M is free.*

Before proving the proposition, observe that for modules over any ring it is true that free modules are also flat. The converse is not always the case. When M is finitely generated, the conditions of flatness and freeness are identical. We will not prove these facts here. See for example [8], chapter 6. We will prove these facts for the ring R_ϵ here.

Proof. We first prove freeness $\implies$ flatness for an R_ϵ-module M. This is a well known fact, but it is proven here nonetheless. Suppose that M has an R_ϵ-basis $\{m_i\}_{i\in I}$. Let K and L be two R_ϵ-modules, let $\phi : K \to L$ be an injective map, and let $k, k' \in K, m \in M$, and $m \neq 0, k \neq 0$. Let $l \in L$. Tensor products are over R_ϵ.
We wish to prove that $\phi' : K \otimes M \to L \otimes M, k \otimes m \mapsto \phi(k) \otimes m$ is an injective map. Assume $\phi(k) \otimes m = 0$. We want to prove that $\phi(k) = 0$. Let us write $m = \sum_i c_i m_i$, to obtain $\phi(k) \otimes m = 0$ if and only if $\sum_i c_i \phi(k) \otimes m_i = 0$. Since the m_i form a basis of M, we can define $\pi^{-1} : L \otimes M \to L \times M$ on elements of the form $l \otimes m_i$ by sending $l \otimes m_i \mapsto (l, m_i) \in L \times M$. Now we define $\psi : L \times M \to K \otimes M : \psi(l, m_i) = k' \otimes m_i$, where k' is chosen such that $\phi(k') = \phi(k)$. Since ϕ is injective, $k = k'$, so this map is well-defined. We extend the map $\psi \circ \pi^{-1} : L \otimes M \to K \otimes M$, which is only defined on the set $\{l \otimes m_i | l \in \phi(K), i \in I\} \subset L \otimes M$ as a linear map. By construction $\psi \circ \pi^{-1}(\phi(k) \otimes m) = k \otimes m$, and $\phi' \circ \psi \circ \pi^{-1}(l \otimes m) = l \otimes m$, where $l \otimes m \in \phi'(K \otimes M)$. This implies that ϕ' is injective.
For the other implication, we assume that M is flat. We use the fact that any module over a field (i.e. a vector space) is free. This can be proven by using the maximal principle on a chain of linealry independent sets to construct a maximal linearly independent subset. Taking the union of all the sets in this chain provides a maximal element in this chain. Its elements are linearly independent, and it must span the vector space by maximality. We refer to other sources for the extended proof.
Proceding with a flat R_ϵ-module M, we observe that $M/\epsilon M$ is an $\mathbb{R}$-module. Concretely, if $(\epsilon) \subset R_\epsilon$ is the ideal generated by ϵ, we observe that $\mathbb{R} = \frac{R_\epsilon}{(\epsilon)}$. Taking the tensor product with M yields $M/\epsilon M$ as an $\mathbb{R}$-module.
Consider the short exact sequence

$$0 \to \epsilon \cdot \mathbb{R} \overset{f}{\hookrightarrow} R_\epsilon \overset{g}{\twoheadrightarrow} \mathbb{R} = \frac{R_\epsilon}{(\epsilon)} \to 0.$$

g is given by $a + \epsilon b \mapsto a$, and f is the inclusion. All spaces are considered as R_ϵ-modules. Since M is flat we can take the tensor product with $\otimes_{R_\epsilon} M$ to obtain

$$0 \to \epsilon \cdot M \hookrightarrow M \overset{\pi}{\twoheadrightarrow} \frac{M}{\epsilon M} \to 0.$$

We can form another exact sequence $R_\epsilon \overset{\cdot\epsilon}{\to} R_\epsilon \to R_\epsilon/(\epsilon)$. This implies $M \overset{\cdot\epsilon}{\to} M \to M/\epsilon M$ is exact, since M is flat. So $\epsilon M = \ker(M \overset{\epsilon\cdot}{\to} M)$, so we obtain an injective map $M/\epsilon M \overset{h}{\hookrightarrow} \epsilon M \subset M$, that is also surjective. So $M/\epsilon M \cong \epsilon M$.

Suppose that $\{\bar{m}_i\}_{i\in I}$ is an $\mathbb{R}$-basis of $M/\epsilon M$. Choose a set $\{m_i\}_{i\in I} \subset M$ such that $\pi(m_i) = \bar{m}_i$ for all $i \in I$. We claim that $\{m_i\}_{i\in I}$ is an R_ϵ-basis of M. To see that $\{m_i\}_{i\in I}$ spans M, we consider an element $m \in M$, then $\pi(m) = \sum c_i \bar{m}_i$, where $c_i \in \mathbb{R}$. Then we know that $m = \sum c_i m_i + \epsilon n$, for some $\epsilon n \in \ker(M \xrightarrow{\cdot\epsilon} M) = \epsilon M$. Because there is an isomorphism $M/\epsilon M \cong \epsilon M$, we can express n as a linear combination $n = \sum \epsilon \bar{c}_i m_i$, for $\bar{c}_i \in \mathbb{R}$. This proves that $\{m_i\}_{i\in I}$ spans M.
To prove linear independence of $\{m_i\}_{i\in I}$, we proceed in a similar fashion. Suppose $\sum_{i\in I'} c_i m_i =$ for $c_i \in R_\epsilon$, where i runs over a finite set I'. We wish to prove that $c_i = 0$ for all i. We know that $g(c_i) = 0$ for all i, as $\pi(m_i)$ is an $\mathbb{R}$-basis of $M/\epsilon M$. We interpret $\pi(c_i m_i) = \pi(c_i \otimes m_i) = g(c_i) \otimes m_i$. This implies that $c_i \in \epsilon M$, so $c_i = \epsilon d_i$ for some $d_i \in \mathbb{R}$. Denote $\bar{m}_i = \pi(m_i)$. Since $\epsilon M \cong M/\epsilon M$ through multiplication with ϵ, we know that $\epsilon \bar{m}_i$ form an $\mathbb{R}$-basis of ϵM as an $\mathbb{R}$-module. Hence $d_i = 0$ for all i, and we have proven linear independence. This finishes the proof. □

As a concrete application we wish to extend an $\mathbb{R}$-basis of $M/\epsilon M$ to an R_ϵ-basis of M. We will use this construction in the thesis, for example in chapter 1.

Corollary A.5.1. *Let M be a free (and flat) R_ϵ-module. Let $\{\tilde{m}_i\}_{i\in I}$ be an $\mathbb{R}$-basis of $M/\epsilon M$. Let $\{m_i\}_{i\in I}$ be such that under the projection $\pi : M \to M/\epsilon M$, $\pi(m_i) = \tilde{m}_i$, for all $i \in I$. Then $\{m_i\}_{i\in I}$ is an R_ϵ-basis of M.*

This finishes the discussion of the ring R_ϵ.

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Summary

The subject of this thesis is a certain set of quantum groups, and the knot invariants arising from these quantum groups. A knot is an embedding of the circle S^1 into the three-dimensional space $\mathbb{R}^3$. Two knots are equivalent if they can be transformed into each other in a continuous way. Such a transformation is called an isotopy. A knot invariant is a map that maps a knot to a set S such that the image is invariant under isotopies of knots. S can be any set. In our case S will be the space of polynomials in two variables.
A quantum group is, contrary to what the name sugests, not a group. A quantum group is a Hopf algebra that originates from the functions on a Lie group. A Hopf algebra is a vector space equipped with a (co)product and a (co)unit and an antipode. In particular, a Hopf algebra is an algebra with unit. The multiplication in an algebra A can be seen as a map $\mu : A \otimes A \to A$. The dual space of an algebra is a coalgebra (ignoring infinite dimensionality issues). A coalgebra is a vector space equipped with a coproduct and a counit. The dual map of the multiplication map is a map $\mu^* : A^* \to A^* \otimes A^*$, where the tensor product is completed in the appropriate sense. This construction can be applied to the infinite dimensional case by appropriately defining the dual space of A.
A Hopf algebra is both an algebra and a coalgebra which has an antipode S. S plays the role of the inverse, but is only a convolution inverse of the coproduct. This means that when $Id \otimes S$ (or $S \otimes id$) is applied to the coproduct, and both tensor factors are multiplied, this should yield zero. Like the inverse, S is an anti-homomorphism. Some Hopf algebras can be equipped with a quasitriangular structure. These are the Hopf algebras that will be considered in this thesis. A quasitriangular Hopf algebra enables us to define a knot invariant from the Hopf algebra.
A quasitriangular structure $\mathcal{R}$ on a Hopf algebra H is called an R-matrix. An R-matrix satisfies the Yang-Baxter equation. When we write $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$, and denote $\mathcal{R}_{ij}$ for an element in $H^{\otimes n}$, $i, j \leq n, i \neq j$, where $\mathcal{R}_1$ is in the i-th factor and $\mathcal{R}_2$ is in the j-th factor. The Yang-Baxter equation is then written as

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

A way to construct a quasitriangular Hopf algebra is given by the Drinfel'd double construction. A standard example is given by the quantum group $U_q(sl_2)$. From this example famous invariants such as the Jones polynomial and the Alexander polynomial can be cosntructed. This algebra originates from functions on the Lie group SL(2). The $U_q(sl_2)$ Hopf algebra can be considered as the quotient of

the Drinfel'd double of a Hopf subalgebra $U_q(b^-)$ of $U_q(sl_2)$ and its dual $U_q(b^+)$. It is possible to define a variation of the $U_q(sl_2)$ quantum group by deforming the comultiplication on $U_q(b^-)$ with a parameter ϵ such that $\epsilon^2 = 0$. This deformation is equivalent to multiplying the Lie bialgebra cobracket of the Lie bialgebra b^- with ϵ, and quantizing this Lie bi algebra to obtain $U_q(b^-_\epsilon)$. A quasitriangular Hopf algebra can then be obtained by applying the Drinfel'd double construction to $U_q(b^-_\epsilon)$.
In this thesis, this construction is applied to the Lie bialgebra sl_3 to obtain the corresponding quantum group $U_q(sl^\epsilon_3)$. The knot invariant that is obtained from this quasitriangular structure is studied and calculated for a few knots. For these calculations, a new formalism is needed, and this formalism is introduced in this thesis, along with the proof of convergence. In particular it is proven that the knot invariant corresponding to $U_q(sl^\epsilon_3)$ can be calculated in polynomial time. An attempt is made to generalize this construction to sl_n.

Samenvatting

Dit proefschrift gaat over zogenaamde kwantumgroepen en hun knoopinvarianten. Een knoop is een inbedding van de cirkel S^1 in een driedemensionale ruimte $\mathbb{R}^3$. We beschouwen twee knopen als equivalent als ze op een continue wijze in elkaar vervormd kunnen worden. Zo'n continue vervorming tussen twee knopen noemen we een isotopie. Een knoopinvariant is een afbeelding van de ruimte van knopen naar een verzameling S, zodanig dat deze afbeelding invariant is onder isotopieën. S kan van alles zijn, en in ons geval zal S een polynoomruimte zijn in twee variabelen.
Een kwantumgroep is in tegenstelling tot wat de naam doet vermoeden, geen groep. Een kwantumgroep is een Hopf-algebra die afkomstig is van functies op een Lie-groep. Een Hopf-algebra heeft een (co)vermenigvuldiging en een (co)eenheidselement, en kan dus gezien worden als een algebra. Een vermenigvuldigingsafbeelding op een algebra A is een afbeelding van het tensor product $A \otimes A$ naar A. De duale van een algebra A is een zogenaamde coalgebra. Een coalgebra is een vectorruimte uitgerust met een coproduct en een coeenheid. De duale afbeelding van de vermenigvuldiging in A is dan een afbeelding van A^* naar $A^* \otimes A^*$, grof gezegd. Deze constructie geeft problemen als A oneindig dimensionaal is. Dit kan opgelost worden door de completering van het tensor product te beschouwen, en door de duale op een andere manier te construeren.
Een Hopf-algebra is dan een algebra en een coalgebra die uitgerust is met een antipode S. De antipode speelt de rol van inverse, maar is dat net niet. De antipode is de inverse van het coproduct, in de zin dat als beide tensor-factoren met elkaar vermenigvuldigd worden nadat $S \otimes id$ (of andersom) is toegepast op de covermenigvuldiging Δ_h, dit nul geeft.
In deze scriptie worden quasi-triangulaire Hopf-algebras beschouwd. Een quasitriangulaire structuur voor een Hopf-algebra H wordt genoteerd met R, de R-matrix. Een R-matrix voldoet aan de Yang-Baxter vergelijking. Een quasitriangulaire structuur op Hopf-algebras zorgt ervoor dat we een knoopinvariant kunnen definieren met behulp van de Hopf-algebra.
Laat $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$, en noteer $\mathcal{R}_{i,j}$ voor een element in $H^{\otimes n}$, $i, j \leq n, i \neq j$ waarbij $\mathcal{R}_1$ in de i-de factor staat en $\mathcal{R}_2$ in de j-de factor te vinden is. Dan is de Yang-Baxter vergelijking geschreven als

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Een manier om quasitriangulaire Hopf-algebras te construeren is de Drinfel'd dubbel constructie, ook wel de quantum-dubbelconstructie genoemd.

Een standaard voorbeeld waarmee bekende knoopinvarianten zoals het Alexander en het Jones polynoom geconstrueerd kunnen worden is de kwantumgroep $U_q(sl_2)$. Deze algebra is afkomstig van functies op de Lie groep SL(2). De $U_q(sl_2)$ Hopf-algebra kan beschouwd worden als het quotient van de Drinfel'd dubbel van een deel Hopf-algebra $U_q(b^-)$ en zijn duale $U_q(b^+)$.
Het is mogelijk om een variatie op de $U_q(sl_2)$ te definieren door de covermenigvuldiging op $U_q(b^-)$ te vervormen met een parameter ϵ met de eigenschap $\epsilon^2 = 0$. Dit is equivalent met het vermenigvuldigen van het co-haakje in de Lie bialgebra b^- en die te kwantiseren tot de Hopf-algebra $U_q(b^-_\epsilon)$. Vervolgens verkrijgen we een quasitriangulaire Hopf-algebra door de Drinfel'd dubbel constructie toe te passen.
In dit proefschrift word deze constructie toegepast op de Lie algebra sl_3, en de daarmee corresponderende kwantumgroep $U_q(sl_3)$. De resulterende knoopinvariant wordt bestudeerd en uitgerekend voor enkele knopen. Een belangrijk resultaat is dat dit in polynomiale tijd in het aantal kruisingen van de knoop kan plaatsvinden. Om de berekeningen uit te voeren, is het nodig om een formalisme te gebruiken dat in dit proefschrift wordt ingevoerd. In het laatste hoofdstuk wordt deze constructie veralgemeniseerd naar sl_n.

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Curriculum Vitae

Sjabbo Schaveling was born in 1992 in Amsterdam, and was raised in Alkmaar, the Netherlands. He attended the Stedelijk Dalton College Alkmaar from 2004 until 2010, after which he proceeded with studying physics and mathematics at the Universiteit van Amsterdam. After completing the double bachelor programme, he did a masters in mathematical physics at the UvA. He wrote his master thesis on the two dimensional Ising model in march 2016 under the supervision of Eric Opdam and Bernard Nienhuis at the UvA. After graduating in March 2016, he became a PhD student under the supervision of Roland van der Veen at the Universiteit Leiden. In his PhD thesis he studies certain solutions of the Yang-Baxter equations arising from quantum groups.
Besides mathematics, Sjabbo has a broad interest in music and has played in various bands as a guitarist and vocalist. Since 2015 he has played in Mr Stone and the Black Dogs, which released its second EP in February 2020.